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# Option pricing for stochastic volatility models : Vol-of-Vol expansion

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## **Abstract**

In this article, we propose an analytical approximation for the pricing of European options for some lognormal stochastic volatility models. This approximation is a second-order Taylor series expansion of the Fourier transform with respect to the "volatility of volatility". We give, using these formulas, a new method of variance reduction for the Monte-Carlo simulation of the trajectories of the underlying.

## **1 Introduction**

Following the rejection of the assumption of constant volatility by empirical studies, a large number of models have been proposed in the literature such as generalized Lévy processes, fractional Brownian motion, the diffusions with jumps and stochastic volatility models. Melino and Turnbull have shown in [22] that the assumption of stochastic volatility leads to a distribution of the underlying asset that is closer to the empirical observations than the log-normal distribution.

The issue of stochastic volatility and its effects on prices of options have been widely studied in the literature (see, e.g., Johnson and Shanno [18], Hull and White [17], Scott [25], Wiggins [27]). The models developed by these authors require either the use of Monte Carlo simulation or the numerical solution of partial differential equation (parabolic in most cases) with dimension larger than 2 for the price of conventional options. Some of these models require the "questionable" as assumption of zero correlation between the underlying and its volatility. Under this assumption, Hull and White [17] gave an approximation of the price of a European option as a Taylor series expansion, Stein and Stein [26] provide a solution for option prices that can be obtained using

numerical integration in two variables. Heston [16] developed a semi-analytical solution for pricing European options in a model with correlated stochastic volatility.

More recently, there has been an explosion of literature on approximation methods for the pricing of European options. Alós et al. [1] studied the short-time behavior of implied volatility in a stochastic volatility model using an extension of Ito's formula. Some authors have proposed analytical techniques to calculate the asymptotic expansion of implied volatility for very short or very long maturities, include Hagan et al. [13], Berestycki et al. [3], [4], Henry Labordère [14], [15] and Laurence [20], or more recently, the work of Forde, Jacquier and Mijatovic [9], [10] on the asymptotic expansion of implied volatility in the Heston model for short and long maturities. Other studies are based on an asymptotic expansion of prices with respect to the Drift of the volatility process for short maturities. For example, in [11], Fouque et al propose an asymptotic expansion with respect to the parameter of mean reversion. Lewis [21], Benhamou et al [2] propose approximation methods based on the asymptotic expansion with respect to the volatility of volatility. In all the works cited above, the scope is often limited and the results can only be applied in a specific context, outside which we lose either the analytical formulas or the quality (accuracy) of these formulas.

This work belongs to the latter category. We first consider a lognormal stochastic volatility model (Scott model [25] with time-dependent parameters). We propose an approximation method based on a second-order Taylor expansion of Fourier transform of the joint distribution of the underlying and its variance by solving the Fokker-Planck equation with respect to the "volatility of volatility" parameter. We obtain an approximation of the density of the underlying as a sum of successive derivatives of the Gaussian density. In addition, thanks to the particular shape of the approximate density, we obtain easily a similar formula for prices of European options as well as that of the implied volatility. Our approach extends to models where the variance is an exponential of a sum of Ornstein-Uhlenbeck processes. It therefore allows to approach the density of the underlying asset in Bergomi's model(s) ([5], [6], [7]). Our results are similar to those obtained recently Bergomi and Guyon [8] who derive the second order Taylor expansion of European options in some stochastic volatility models, including 2 factor Bergomi's model, using some perturbation techniques.

The formulas we obtain, especially those giving an approximation of the density function of the underlying asset, can also be used to reduce the variance of a Monte-Carlo simulation. In [12], Fouque and Han use the approximations of European option prices obtained by perturbation techniques to build variance reduction methods such as "importance sampling" and Control Variate. In this work we propose a new method of variance reduction of type "Control Variate" which uses the explicit approximation we obtain for the distribution function of the underlying. The idea is to build a new process whose the density is given by the approximated density of

the underlying and strongly correlated with the trajectories of the underlying asset. We give an estimate of the gain in variance in terms of the volatility of volatility. Several numerical experiments are provided to prove the performance of this method of variance reduction.

This paper is organized as follows: Section 2 deals with the case of a one factor lognormal stochastic volatility model (time-dependent Scott model). We give the second-order Taylor expansion of the Fourier transform of log-returns with respect to the volatility of volatility parameters. We deduce the second-order Taylor expansion of the density of log-returns with respect to the volatility of volatility as well as the implied volatility and the skew. In section 3 we generalize the results of Section 2 to the multidimensional case. In section 4 we present a new method of variance reduction of the Monte-Carlo simulation. In section 5 we give some numerical results.

## 2 One Factor Case : Time Dependent Scott Model

In this section we consider a generalization of the model proposed by Scott [25]. Under this model, the dynamics of the underlying is given by the SDE

$$\begin{cases} \frac{dS_t}{S_t} = f(t, V_t) dW_t^S, \\ dV_t = -bV_t dt + \omega \sigma_t dW_t^V, \quad d\langle W^S, W^V \rangle_t = \rho_t dt, \end{cases} \quad (2.1)$$

where  $f^2(t, v) = m_t e^v$ ,  $m$ ,  $\rho$  and  $\sigma$  are deterministic continuous functions of time. Assume  $\omega, \sigma \geq 0$ . Assume also that  $V_0 = 0$  (Otherwise, we replace  $m_t$  by  $m_t e^{V_0 e^{-bt}}$  and  $V$  by  $V - V_0$ ).

Denote by  $p(t, x, v)$  the joint density of the pair of random variables  $(X_t = \log(S_t/S_0), V_t)$ . This density satisfies the following Fokker-Planck equation :

$$\partial_t p = \partial_v [bvp] + \frac{(\omega \sigma_t)^2}{2} \frac{\partial^2 p}{\partial v^2} + \frac{1}{2} f^2(t, v) \frac{\partial p}{\partial x} + \frac{1}{2} f^2(t, v) \frac{\partial^2 p}{\partial x^2} + \rho \omega \sigma_t f(t, v) \frac{\partial^2 p}{\partial x \partial v} \quad (2.2)$$

with the boundary condition  $p(0, x, v|0) = \delta_0(x) \delta_0(v)$ .

Consider the Fourier transform of  $p$  defined as

$$\varphi(t, \xi, \zeta; \omega) := \int_{-\infty}^{\infty} dx e^{ix\xi} \int_{-\infty}^{\infty} dv e^{iv\zeta} p(t, x, v), \quad \zeta, \xi \in \mathbb{R}.$$

The function  $\varphi$  is solution of the equation

$$\mathcal{L}\varphi(t, \xi, \zeta; \omega) = 0, \quad \forall (t, \xi, \zeta) \in ]0, \infty[ \times \mathbb{R} \times \mathbb{R}, \quad (2.3)$$

where  $\mathcal{L}$  is the operator defined by

$$\begin{aligned}\mathcal{L}h(t, \xi, \zeta; \omega) &= \partial_t h(t, \xi, \zeta; \omega) + \omega^2 \frac{\sigma_t^2}{2} \zeta^2 h(t, \xi, \zeta; \omega) + b\zeta \partial_\zeta h(t, \xi, \zeta; \omega) + \\ & m_t \frac{\xi^2 + i\xi}{2} h(t, \xi, \zeta - i; \omega) + \xi \zeta \omega \rho_t \sigma_t \sqrt{m_t} h(t, \xi, \zeta - \frac{i}{2}; \omega).\end{aligned}\quad (2.4)$$

In this section, we propose to approximate this Fourier transform by its second order Taylor expansion with respect to the parameter  $\omega$ , which measures the volatility of volatility. Our approach is inspired by the work of Perelló et al [24] giving an approach to approximate the Fourier transform in the case of constant parameters. Note that our method is not a generalization of [24] to the case of time-dependent parameters. Indeed, our approach is based on solving the equation  $\varphi(t, \xi, \zeta) \equiv \varphi(t, \xi, \zeta, \omega)$  for all  $\xi, \zeta \in \mathbb{R}$  by writing its second order Taylor expansion with respect to  $\omega$ , while the approximation of [24] is obtained by rescaling the equation and then truncating the terms of order higher than 2 with respect to  $\frac{m}{\sigma}$ . We will compare both methods by giving some numerical examples.

To calculate the Taylor expansion of the solution of (2.4) with respect to  $\omega$ , we differentiate it successively with respect to  $\omega$ . We obtain that  $\varphi$  can be written in terms of its Taylor series expansion with respect to  $\omega$  as follows:

$$\varphi(t, \xi, \zeta; \omega) = \varphi_0(t, \xi, \zeta) + \omega \varphi_1(t, \xi, \zeta) + \frac{\omega^2}{2} \varphi_2(t, \xi, \zeta) + \dots, \quad (2.5)$$

where  $\varphi_i$  is solution of

$$\mathcal{L}_i \varphi_i(t, \xi, \zeta) = 0, \quad \forall (t, \xi, \zeta) \in ]0, \infty[ \times \mathbb{R} \times \mathbb{R},$$

and

$$\mathcal{L}_0 h(t, \xi, \zeta) = \partial_t h(t, \xi, \zeta) + b\zeta \partial_\zeta h(t, \xi, \zeta) + m_t \frac{\xi^2 + i\xi}{2} h(t, \xi, \zeta - i), \quad (2.6)$$

$$\mathcal{L}_1 h(t, \xi, \zeta) = \mathcal{L}_0 h(t, \xi, \zeta) + \xi \zeta \rho_t \sigma_t \sqrt{m_t} \varphi_0(t, \xi, \zeta - \frac{i}{2}), \quad (2.7)$$

$$\mathcal{L}_2 h(t, \xi, \zeta) = \mathcal{L}_0 h(t, \xi, \zeta) + 2\xi \zeta \rho_t \sigma_t \sqrt{m_t} \varphi_1(t, \xi, \zeta - \frac{i}{2}) + \sigma_t^2 \zeta^2 \varphi_0(t, \xi, \zeta). \quad (2.8)$$

**Definition 2.1.** Let  $h \in \mathcal{C}^{1,2,2,n+1}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+)$ . We say that  $h$  is a  $n$ -order  $\omega$ -equivalent of  $\varphi$ , we denote  $h \equiv \varphi[n]$ , if for all  $(t, \xi, \zeta) \in ]0, \infty[ \times \mathbb{R} \times \mathbb{R}$ , we have

$$\frac{\partial^i h}{\partial \omega^i}(t, \xi, \zeta; 0) = \varphi_i(t, \xi, \zeta), \quad \forall i \leq n. \quad (2.9)$$

The main result of this section is the following result, whose proof can be found in the appendix. It gives the second order  $w$ -equivalence of the Fourier transform of the joint density of  $(X_t, V_t)$

**THEOREM 2.2.** *Let's denote by*

$$\hat{\varphi}(t, \xi, \zeta) = e^{-(A_0(t, \xi; \omega) + \omega A_1(t, \xi; \omega) \zeta + \omega^2 A_2(t, \xi) \zeta^2)}, \quad (2.10)$$

where  $A_0$ ,  $A_1$  and  $A_2$  are given by

$$\begin{aligned} A_2(t, \xi) &= \frac{1}{2} \int_0^t \sigma_s^2 e^{-2b(t-s)} ds, \\ A_1(t, \xi; \omega) &= \int_0^t (\eta_s(\xi) + 2i\omega \mu_s A_2(s, \xi)) e^{-(\gamma_t(\xi; \omega) - \gamma_s(\xi; \omega))} ds, \\ A_0(t, \xi; \omega) &= \int_0^t \mu_s(\xi) (1 + \omega^2 A_2(s) + (i\omega A_1(s, \xi; \omega) - \frac{\omega^2}{2} A_1^2(s, \xi; 0))) ds, \end{aligned}$$

with  $\mu_t(\xi) := m_t \frac{\xi^2 + i\xi}{2}$ ,  $\eta_t(\xi) = \xi \rho_t \sigma_t \sqrt{m_t}$  and

$$\gamma_t(\xi; \omega) = bt - i\frac{\omega}{2} \int_0^t \eta_s(\xi) ds.$$

Then, we have

$$\hat{\varphi} \equiv \varphi \quad [2]. \quad (2.11)$$

### Approximating the density of $X_t$

We now consider the approximation of the Fourier transform of the joint distribution of  $(X_t, V_t)$  given by (2.10). Taking the particular case  $\zeta = 0$ , we deduce an approximation of the Fourier transform of  $X_t$ . This approximation is given as

$$\hat{\varphi}(t, \xi, 0; \omega) = e^{-A_0(t, \xi; \omega)}.$$

We can write the second order Taylor series expansion of  $A_0(t, \xi, \omega)$  with respect to  $\omega$ , which is obtained by calculating the first order Taylor expansion of  $A_1(t, \xi, \omega)$  with respect to  $\omega$ :

$$\begin{aligned} A_1(t, \xi; \omega) &= \int_0^t \eta_s(\xi) e^{-b(t-s)} (1 + i\frac{\omega}{2} \int_s^t \eta_u(\xi) du) ds + i\omega \int_0^t \mu_s(\xi) \int_0^s \sigma_u^2 e^{-b(t+s-2u)} du ds \\ &\quad + \mathcal{O}(\omega^2) \end{aligned}$$

and  $A_1^2(t, \xi; 0)$ :

$$A_1^2(t, \xi; 0) = \left( \int_0^t \eta_s(\xi) e^{-b(t-s)} ds \right)^2.$$

It follows that  $\hat{\varphi}(t, \xi, 0; \omega)$  can be written as

$$\hat{\varphi}(t, \xi, 0; \omega) = \exp \left( -i\mu_1(t; \omega)\xi - \mu_2(t; \omega)\xi^2 + i\mu_3(t; \omega)\xi^3 + \mu_4(t; \omega)\xi^4 + \mathcal{O}(\omega^3) \right),$$

where

$$\begin{aligned} \mu_1(t; \omega) &= \frac{1}{2} \int_0^t m_s ds + \frac{\omega^2}{4} \int_0^t m_s \int_0^s \sigma_u^2 e^{-2b(s-u)} dud s, \\ \mu_2(t; \omega) &= \mu_1(t; \omega) + \frac{\omega^2}{4} \int_0^t m_\tau \int_0^\tau m_s \int_0^s \sigma_u^2 e^{-b(\tau+s-2u)} dud s d\tau - \frac{\omega}{2} \int_0^t m_\tau \int_0^\tau \eta_s(1) e^{-b(\tau-s)} ds d\tau, \\ \mu_3(t; \omega) &= -\frac{\omega}{2} \int_0^t m_\tau \int_0^\tau \eta_s(1) e^{-b(\tau-s)} ds d\tau + \frac{\omega^2}{4} \int_0^t m_\tau \int_0^\tau \eta_s(1) e^{-b(\tau-s)} \int_s^\tau \eta_u(1) dud s d\tau \\ &\quad + \frac{\omega^2}{4} \int_0^t m_\tau \left( \int_0^\tau \eta_s(1) e^{-b(\tau-s)} ds \right)^2 d\tau + \frac{\omega^2}{2} \int_0^t m_\tau \int_0^\tau m_s \int_0^s \sigma_u^2 e^{-b(\tau+s-2u)} dud s d\tau, \\ \mu_4(t; \omega) &= \frac{\omega^2}{4} \int_0^t m_\tau \int_0^\tau m_s \int_0^s \sigma_u^2 e^{-b(\tau+s-2u)} dud s d\tau + \frac{\omega^2}{4} \int_0^t m_\tau \left( \int_0^\tau \eta_s(1) e^{-b(\tau-s)} ds \right)^2 d\tau \\ &\quad + \frac{\omega^2}{4} \int_0^t m_\tau \int_0^\tau \eta_s(1) e^{-b(\tau-s)} \int_s^\tau \eta_u(1) dud s d\tau. \end{aligned}$$

Similarly, writing that  $e^x = 1 + x + x^2/2 + \dots$ , we can write  $\hat{\varphi}(t, \xi, 0; \omega)$  as

$$\hat{\varphi}(t, \xi, 0; \omega) = e^{(-i\mu_1(t; \omega)\xi - \frac{\nu(t)}{2}\xi^2)} \left( 1 + \sum_{n=1}^6 (-i)^n \nu_n(t; \omega) \xi^n + \mathcal{O}(\omega^3) \right), \quad (2.12)$$

where  $\nu(t) = \int_0^t m_s ds$  and

$$\begin{aligned} \nu_1(t; \omega) &= \frac{\omega^2}{4} \int_0^t m_s \int_0^s \sigma_u^2 e^{-2b(s-u)} dud s, \\ \nu_2(t; \omega) &= \nu_1 + \frac{\omega^2}{4} \int_0^t m_\tau \int_0^\tau m_s \int_0^s \sigma_u^2 e^{-b(\tau+s-2u)} dud s d\tau - \frac{\omega}{2} \int_0^t m_\tau \int_0^\tau \eta_s(1) e^{-b(\tau-s)} ds d\tau, \\ \nu_3(t; \omega) &= \mu_3(t; \omega), \quad \nu_4(t; \omega) = \mu_4(t; \omega) + \frac{\omega^2}{8} \left( \int_0^t m_\tau \int_0^\tau \eta_s(1) e^{-b(\tau-s)} ds d\tau \right)^2, \\ \nu_5(t; \omega) &= \frac{\omega^2}{4} \left( \int_0^t m_\tau \int_0^\tau \eta_s(1) e^{-b(\tau-s)} ds d\tau \right)^2, \quad \nu_6(t; \omega) = \frac{\omega^2}{8} \left( \int_0^t m_\tau \int_0^\tau \eta_s(1) e^{-b(\tau-s)} ds d\tau \right)^2. \end{aligned}$$

We then obtain the next result

**THEOREM 2.3.** Let's denote  $\varphi_X(t, \xi; \omega) := \mathbb{E} e^{i\xi X_t}$ . Consider the function  $\psi$  defined by

$$\psi(t, \xi; \omega) := e^{(-i\mu_1(t)\xi - \frac{\nu(t)}{2}\xi^2)} \left( 1 + \sum_{n=1}^6 (-i)^n \nu_n(t; \omega) \xi^n \right).$$

Then we have

$$\psi \equiv \varphi_X \quad [2].$$

Furthermore, if we denote by  $p_X(t, x; \omega) = \mathbb{P}(X_t \in dx)$  and by  $\hat{p}_X(t, x; \omega) := \mathcal{F}^{-1}(\psi)(t, x; \omega)$ , then we have

$$\hat{p}_X \equiv p_X \quad [2], \quad (2.13)$$

where  $\hat{p}_X$  is "explicitly" given by

$$\hat{p}_X(t, x; \omega) = \frac{1}{\sqrt{2\pi\nu(t)}} e^{-\frac{(x+\mu_1(t; \omega))^2}{2\nu(t)}} \left( 1 + \sum_{n=1}^6 (-1)^n \frac{\nu_n(t; \omega)}{\nu^{\frac{n}{2}}(t)} H_n\left(\frac{x+\mu_1(t; \omega)}{\sqrt{\nu(t)}}\right) \right), \quad (2.14)$$

where the  $H_n$ 's are the Hermite polynomials.

**Proof:** First, from the way we obtained the expression of  $\psi(t, \xi; \omega)$  from that of  $\hat{\varphi}(t, \xi, 0; \omega)$ , it is easy to see that  $\psi \equiv \hat{\varphi}_X \quad [2]$ , where  $\hat{\varphi}_X(t, \xi; \omega) := \hat{\varphi}(t, \xi, 0; \omega)$ . It follows that  $\psi \equiv \varphi_X \quad [2]$  (because  $\hat{\varphi}_X \equiv \varphi_X \quad [2]$ ). Now, to show that  $\mathcal{F}^{-1}(\varphi_X) \equiv \mathcal{F}^{-1}(\psi) \quad [2]$ , it suffices to show that we can differentiate under  $\int$  with respect to  $\omega$ . i.e.

$$\partial_\omega^j \left[ \int_{-\infty}^{\infty} e^{-ix\xi} \psi(t, \xi; \omega) d\xi \right]_{\omega=0} = \int_{-\infty}^{\infty} \left( e^{-ix\xi} \partial_\omega^j \psi(t, \xi; \omega) \Big|_{\omega=0} \right) d\xi, \quad j = 1, 2. \quad (2.15)$$

and

$$\partial_\omega^j \left[ \int_{-\infty}^{\infty} e^{-ix\xi} \varphi_X(t, \xi; \omega) d\xi \right]_{\omega=0} = \int_{-\infty}^{\infty} \left( e^{-ix\xi} \partial_\omega^j \varphi_X(t, \xi; \omega) \Big|_{\omega=0} \right) d\xi, \quad j = 1, 2. \quad (2.16)$$

The first equality holds since  $\psi$  is given as

$$\psi(t, \xi; \omega) := e^{(-i\mu_1(t)\xi - \frac{\nu(t)}{2}\xi^2)} \left( 1 + \sum_{n=2}^6 (-i)^n \nu_n(t) \xi^n \right),$$

where  $\nu(t) > 0$  and the  $\nu_n(t)$ 's are polynomial functions of  $\omega$ . We write that for any  $\xi \in \mathbb{R}$  and



for any  $j \in \mathbb{N}$ ,  $\omega \mapsto \psi(t, \xi; \omega)$  is differentiable and its derivative can be written as

$$\frac{\partial^j}{\partial \omega^j} \psi(t, \xi; \omega) = Q_\omega^j(t, \xi) e^{(-i\mu_1(t)\xi - \frac{\nu(t)}{2}\xi^2)}, \quad \forall \omega \geq 0,$$

where  $Q_\omega^j(t, \cdot)$  is a polynomial function. It follows that we can write

$$\frac{\partial^j}{\partial \omega^j} \int_{-\infty}^{\infty} e^{-ix\xi} \psi(t, \xi; \omega) d\xi = \int_{-\infty}^{\infty} e^{-ix\xi} \frac{\partial^j}{\partial \omega^j} \psi(t, \xi; \omega) d\xi, \quad \forall j \in \mathbb{N}.$$

On the other hand, we have

$$\frac{\partial^j \varphi_X(t, \xi; \omega)}{\partial \omega^j} \Big|_{\omega=0} := \frac{\partial^j \mathbb{E} e^{i\xi X_t}}{\partial \omega^j} \Big|_{\omega=0} = \frac{\partial^j \psi(t, \xi; \omega)}{\partial \omega^j} \Big|_{\omega=0}, \quad \forall j = 1, 2.$$

So for any  $M > 0$  there exists  $\delta > 0$  such that

$$\sup_{\xi \in [-M, M]} \left| e^{\xi^2} \left( \frac{\partial^j \varphi_X(t, \xi; \omega)}{\partial \omega^j} - \frac{\partial^j \psi(t, \xi; \omega)}{\partial \omega^j} \right) \right| < 1, \quad \forall \omega < \delta, \quad \forall j = 1, 2.$$

For the big values of  $\xi$ , we apply Hölder's inequality as follows

$$|\partial_\omega \varphi_X(t, \xi; \omega)| \leq |\xi| \|\partial_\omega X_t\|_p \left| \varphi_X(t, \frac{p}{p-1}\xi; \omega) \right|^{\frac{p-1}{p}}, \quad \forall p > 1.$$

It follows that, for any  $p > 1$ , we have

$$|\partial_\omega^j \varphi_X(t, \xi; \omega)| \leq c_p |\xi|^j \left| \varphi_X(t, \frac{p}{p-1}\xi; \omega) \right|^{\frac{p-1}{p}}, \quad j = 1, 2.$$

As  $|\varphi_X(t, \xi; \omega)| \rightarrow 0$ , we get  $\sup_{|\xi| > M} |\varphi_X(t, \xi; \omega)| \leq |\varphi_X(t, M_0; \omega)|$ , where  $M_0 \geq M$ . On the other hand, we have

$$|\varphi_X(t, M_0; \omega)| \leq |\varphi_X(t, M_0; 0)| + \omega |\partial_\omega \varphi_X(t, M_0; 0)|, \quad \forall \omega \in [0, \omega_0].$$

We finally obtain that

$$\left| \frac{\partial^j}{\partial \omega^j} \varphi_X(t, \xi; \omega) \right| \leq P_\omega^j(t, \xi) e^{(-i\mu_1(t)\xi - \frac{\nu(t)}{2}\xi^2)}, \quad \forall \xi \in \mathbb{R}, \quad \forall \omega \leq \omega_0, \quad j = 1, 2,$$

where  $P_\omega^j(t, \cdot)$  is a polynomial function. This allows us to write

$$\frac{\partial^j}{\partial \omega^j} \int_{-\infty}^{\infty} e^{-ix\xi} \varphi_X(t, \xi; \omega) d\xi = \int_{-\infty}^{\infty} e^{-ix\xi} \frac{\partial^j}{\partial \omega^j} \varphi_X(t, \xi; \omega) d\xi, \quad \forall j \leq 2.$$

Thus

$$\frac{\partial^j p_X}{\partial \omega^j}(t, x, 0) = \mathcal{F}^{-1}\left(\frac{\partial^j \varphi_X}{\partial \omega^j}\right)(t, x, 0) = \mathcal{F}^{-1}\left(\frac{\partial^j \psi}{\partial \omega^j}\right)(t, x, 0) = \hat{p}(t, x, 0), \quad \forall j \leq 2 \quad \square$$

**Corollary 2.4.** *Consider the function  $C : (t, K; \omega) \in R_+^3 \mapsto \mathbb{E}(e^{X_t} - K)_+$ . Then, we have*

$$\hat{C} \equiv C \quad [2],$$

where  $\hat{C}(t, K; \omega) = \int_{\mathbb{R}} (e^x - K)_+ \hat{p}_X(t, x; \omega) dx$ . Furthermore,  $\hat{C}$  is explicitly given by

$$\hat{C}(t, K; \omega) = C_{BS}(1, K, \sqrt{\nu}) + \frac{K}{\sqrt{\nu}} N'(d_2) \left( \sum_{n=0}^4 \frac{z_n}{\nu^{\frac{n}{2}}} H_n(-d_2) \right), \quad (2.17)$$

where  $C_{BS}(t, K, I) = N(d_1) - KN(d_2)$ , with  $d_1 = \frac{\log(\frac{1}{K}) + \frac{t}{2}I^2}{I}$ ,  $d_2 = \frac{\log(\frac{1}{K}) - \frac{t}{2}I^2}{I}$ ,  $z_4 = \nu_6$ ,  $z_3 = -\nu_5 + z_4$ ,  $z_2 = \nu_4 + z_3$ ,  $z_1 = -\nu_3 + z_2$ ,  $z_0 = \nu_2 + z_1 = \nu_1$ .<sup>1</sup>

## Implied volatility

Denote by  $\Sigma$  the implied volatility defined as

$$C(t, k; \omega) := \mathbb{E}(e^{X_t} - e^k)_+ = N\left(\frac{-k + \frac{t}{2}\Sigma^2(t, k; \omega)}{\sqrt{t}\Sigma(t, k; \omega)}\right) - e^k N\left(\frac{-k - \frac{t}{2}\Sigma^2(t, k; \omega)}{\sqrt{t}\Sigma(t, k; \omega)}\right). \quad (2.18)$$

Note that the dependence of  $C$  with respect to  $\omega$  is only through  $\Sigma$ . So the second order Taylor series expansion of  $C$  with respect to  $\omega$  is given in terms of the Taylor expansion of  $\Sigma$  with respect to  $\omega$  as

$$C(t, k; \omega) = C_{BS}(t, k, \Sigma_0) + e^k N'(d) \sqrt{t} \Sigma_1 \omega + e^k N'(d) \left( \left( \frac{d^2}{\Sigma_0} - d \right) t \Sigma_1^2 + \sqrt{t} \Sigma_2 \right) \frac{\omega^2}{2} + \mathcal{O}(\omega^3), \quad (2.19)$$

where  $d = (k + \frac{t}{2}\Sigma_0^2(t, k)) / (\sqrt{t}\Sigma_0(t, k))$  and

$$\Sigma_i \equiv \Sigma_i(t, k) = \frac{\partial^i \Sigma(t, k; \omega)}{\partial \omega^i} \Big|_{\omega=0}. \quad (2.20)$$

---

<sup>1</sup>Note that  $\nu_1 - \nu_2 + \nu_3 - \nu_4 + \nu_5 - \nu_6 = 0$ .

Now, we can easily check that (2.17) can be written as

$$\begin{aligned}\hat{C} &= C_{BS} + \frac{KN'(d)}{2\sqrt{\nu}}\chi_1 d\omega + \frac{KN'(d)}{\sqrt{\nu}} \times \\ &\quad \left(2\chi_0 - \chi_2 \frac{d}{\sqrt{\nu}} + (\chi_2 + \chi_3 + \chi_4) \frac{d^2 - 1}{\nu} + \frac{1}{2}(\chi_1)^2 \left( \frac{d^4 - 6d^2 + 3}{\nu^2} - \frac{d^3 - 3d}{\nu\sqrt{\nu}} \right) \right) \frac{\omega^2}{4}\end{aligned}\quad (2.21)$$

where

$$\begin{aligned}\chi_0 &:= \int_0^t m_s \int_0^s \sigma_u^2 e^{-2b(s-u)} du ds, \quad \chi_1 := \int_0^t m_\tau \int_0^\tau \eta_s(1) e^{-b(\tau-s)} ds d\tau, \\ \chi_2 &:= \int_0^t m_\tau \int_0^\tau m_s \int_0^s \sigma_u^2 e^{-b(\tau+s-2u)} du ds d\tau, \quad \chi_3 := \int_0^t m_\tau \left( \int_0^\tau \eta_s(1) e^{-b(\tau-s)} ds \right)^2 d\tau, \\ \chi_4 &:= \int_0^t m_\tau \int_0^\tau \eta_s(1) e^{-b(\tau-s)} \int_s^\tau \eta_u(1) du ds d\tau.\end{aligned}\quad (2.22)$$

Identifying (2.21) and (2.19) we obtain the following result

**Corollary 2.5.** *For  $t > 0$  and  $k \in \mathbb{R}$ , we have*

$$\Sigma_0(t, k) = \sqrt{\frac{1}{t} \int_0^t m_s ds}, \quad (2.23)$$

$$\Sigma_1(t, k) = \frac{1}{2\sqrt{t}} \frac{x + \frac{1}{2}\Sigma_0^2}{\Sigma_0^3} \int_0^t m_\tau \int_0^\tau \rho_s \sigma_s \sqrt{m_s} e^{-b(\tau-s)} ds d\tau, \quad (2.24)$$

$$\Sigma_2(t, k) = \frac{\chi_0}{\sqrt{t}\Sigma_0} - \chi_2 \frac{d}{2t\Sigma_0^2} + (\chi_2 + \chi_3 + \chi_4) \frac{d^2 - 1}{t^{\frac{3}{2}}\Sigma_0^3} + (\chi_1)^2 \left( \frac{3d}{4t^2\Sigma_0^4} + \frac{3 - 6d^2}{4t^{\frac{5}{2}}\Sigma_0^5} \right). \quad (2.25)$$

where the  $\chi$ 's are given by (2.22) and  $d = \frac{k + \frac{t}{2}\Sigma_0^2(t, k)}{\sqrt{t}\Sigma_0(t, k)}$ .

In particular, we have the following expansion of the skew with respect to the volatility of volatility (see [8] for similar result)

$$\mathcal{S}_T = \frac{\omega}{2t^2\Sigma_0^3}\chi_1 + \frac{\omega^2}{2} \left( \frac{\frac{1}{2}\chi_2 + \chi_3 + \chi_4}{t^2\Sigma_0^3} - \frac{3(\chi_1)^2}{4t^3\Sigma_0^5} \right). \quad (2.26)$$

### 3 Multi-factor case : Bergomi's model

In this section we consider a N-dimensional model defined by the stochastic differential equations

$$\begin{cases} \frac{dS_t}{S_t} = rdt + f(t, V_t^1, \dots, V_t^N) dW_t^S, \\ dV_t^n = (\alpha_n(t) - \kappa_n V_n)dt + \sigma_n(t) dW_t^n, \quad d\langle W^S, W^n \rangle_t = \rho_n^S dt, \quad n = 1, \dots, N, \end{cases} \quad (3.1)$$

where  $d\langle W^n, W^m \rangle_t = \rho_{n,m} dt = 0$ ,  $\forall m, n \leq N$ ,  $\alpha_n, \sigma_n$ ,  $n \leq N$ , are deterministic continuous functions of time and the function  $f$  is defined by

$$f^2(t, V_t^1, \dots, V_t^N) = m_t \exp \left( \omega \sum_{n=1}^N \theta_n V_t^n \right). \quad (3.2)$$

This model is a generalization of the Bergomi model ([5], [6], [7]). It corresponds also to another version of Bergomi's model we proposed in [23]. These models are driven from a Markovian modeling of the forward variance curve. The "factors"  $V^1, \dots, V^N$  allow to control the volatility dynamics by calibrating the so-called volatility derivatives (futures and options on VIX in the case of the S&P 500, options on realized variance ...etcetera). The number of factors  $N$  is the number of degrees of freedom we need to calibrate both the "VIX smiles" and the implied volatility of the underlying. Although these models allow a good calibration of the volatility of volatility, they have a large defect that is the cost of the evaluation of European options since the only available method is the Monte Carlo simulation.

In this section, we extend the results of the previous section to the multidimensional case. We will give an approximation of the Fourier transform of the joint distribution of  $(X_t := \log S_t/S_0, V_t^1, \dots, V_t^N)$ , which appears as its second order Taylor series expansion with respect to  $\omega$ . Let's set

$$\varphi(t, \xi, \zeta_1, \dots, \zeta_N; \omega) := \mathbb{E} \exp \left( i\xi X_t + i \sum_{n=1}^N \zeta_n V_t^n \right).$$

We keep the notations of the previous section. The following result gives an approximation of  $\varphi$

**Proposition 3.1.** *Assume for all  $n, m \leq N$ ,  $\rho_{n,m} = 0$ . Consider the function  $\hat{\varphi}$  defined as*

$$\hat{\varphi}(t, \xi, \zeta_1, \dots, \zeta_N; \omega) := \exp \left( -A_0(t, \xi; \omega) - \sum_1^N A_n(t, \xi; \omega) \zeta_n - \sum_1^N C_n(t, \xi; \omega) \zeta_n^2 \right), \quad (3.3)$$

where, for  $n = 1, \dots, N$ ,

$$\left\{ \begin{array}{l} C_n(t, \xi; \omega) = \frac{1}{2} \int_0^t \sigma_n^2(s) e^{-2\kappa_n(t-s)} ds, \\ A_n(t, \xi; \omega) = \int_0^t (\xi \rho_n^S \sigma_n(s) \sqrt{m_s} - i\alpha_s + 2i\omega \mu_s C_n(s, \xi)) e^{-(\gamma_n(t, \xi) - \gamma_n(s, \xi))} ds, \\ A_0(t, \xi; \omega) = \int_0^t \mu_s \left( 1 + i\omega \sum_{n=1}^M A_n(s, \xi) - \frac{\omega^2}{2} (\sum_{n=1}^M A_n(s, \xi))^2 + \omega^2 \sum_{n=1}^M C_n(s, \xi) \right) ds, \\ \mu_t(\xi) := m_t \frac{\xi^2 + i\xi}{2}, \quad \gamma_n(t, \xi) = \kappa_n t - i\xi \rho_n^S \frac{\omega}{2} \int_0^t \sigma_n(s) \sqrt{m_s} ds. \end{array} \right.$$

Then, we have

$$\hat{\varphi} \equiv \varphi \quad [2]. \quad (3.4)$$

The proof of this proposition is very similar to the 1 factor case. As a result of this, if we set

$$\varphi_X(t, \xi; \omega) := \mathbb{E} e^{i\xi X_t}. \quad (3.5)$$

and  $\hat{\varphi}_X(t, \xi; \omega) := e^{-A_0(t, \xi; \omega)}$ , we obtain

$$\hat{\varphi}_X \equiv \varphi_X \quad [2]. \quad (3.6)$$

Writing  $C(t, \xi; \omega) = \sum_{n=1}^N C_n(t, \xi; \omega)$  and

$$A(t, \xi; \omega) = \sum_{n=1}^N \int_0^t (\xi \rho_n^S \sigma_n(s) \sqrt{m_s} - i\alpha_n(s) + i\omega \mu_s \int_0^s \sigma_n^2(r, \xi) e^{-2\kappa_n(s-r)} dr) e^{-(\gamma_n(t, \xi) - \gamma_n(s, \xi))} ds,$$

we obtain that  $A_0$  can be written as

$$A_0(t, \xi; \omega) = \int_0^t \mu_s(\xi) \left( 1 + i\omega A(s, \xi) - \frac{\omega^2}{2} A^2(s, \xi) + \omega^2 C(s, \xi) \right) ds.$$

Now writing the second order Taylor expansion of  $A(t, \xi; \omega)$  with respect to  $\omega$ , we have

$$\begin{aligned} A &= \sum_{n=1}^N \int_0^t (x \rho_n^S \sigma_n(s) \sqrt{m_s} - i\alpha_n(s)) e^{-\kappa_n(t-s)} \left( 1 - \frac{i}{2} \omega \rho_n^S x \int_s^t \sigma_n(u) \sqrt{m_u} du \right) ds \\ &\quad + i\omega \sum_{n=1}^N \int_0^t \mu_s \int_0^s \sigma_n^2(r, x) e^{-\kappa_n(t+s-2r)} dr ds + \mathcal{O}(\omega^2). \end{aligned}$$

It follows that  $\hat{\varphi}_X(t, x; \omega)$  can be written as

$$\hat{\varphi}_X(t, x; \omega) = \exp \left( -i\mu_1(t, \omega)x - \mu_2(t, \omega)x^2 + i\mu_3(t, \omega)x^3 + \mu_4(t, \omega)x^4 + \mathcal{O}(\omega^3) \right),$$

where the  $m\mu_i$ 's are given by (C.3). Also, we can write  $\hat{\varphi}_X$  as

$$\hat{\varphi}_X(t, x) = e^{(-i\mu_1 x - \frac{\nu(t)}{2} x^2)} \left( 1 + \sum_{n=1}^6 (-i)^n \nu_n(t; \omega) x^n + \mathcal{O}(\bar{\omega}^3) \right), \quad (3.7)$$

where  $\nu(t) := 2\mu_1(t)$  and

$$\begin{aligned}\nu_2(t; \omega) &= \mu_2(t; \omega) - \mu_1(t; \omega), \quad \nu_3(t; \omega) = \mu_3(t; \omega), \\ \nu_4(t; \omega) &= \mu_4(t; \omega) + \frac{\omega^2}{8} \left( \sum_{n=1}^N \rho_n^S \int_0^t m_u \int_0^u \sigma_n(s) \sqrt{m_s} e^{-\kappa_n(u-s)} ds du \right)^2, \\ \nu_5(t; \omega) &= \frac{\omega^2}{4} \left( \sum_{n=1}^N \rho_n^S \int_0^t m_u \int_0^u \sigma_n(s) \sqrt{m_s} e^{-\kappa_n(u-s)} ds du \right)^2, \\ \nu_6(t; \omega) &= \frac{\omega^2}{8} \left( \sum_{n=1}^N \rho_n^S \int_0^T m_t \int_0^t \sigma_n(s) \sqrt{m_s} e^{-\kappa_n(t-s)} ds \right)^2.\end{aligned}$$

We obtain the next result

**Proposition 3.2.** *Let's set  $p_X(t, x; \omega) = \mathbb{P}(X_t \in dx)$ . We have*

$$p_X \equiv P \quad [2], \quad (3.8)$$

where  $P$  is defined as

$$P(t, x; \omega) = \frac{1}{\sqrt{2\pi\nu(t)}} e^{-\frac{(x+\mu_1)^2}{2\nu(t)}} \left( 1 + \sum_{n=2}^6 (-1)^n \frac{\nu_n(t)}{\nu^{\frac{n}{2}}} H_n\left(\frac{x + \mu_1(t)}{\sqrt{\nu(t)}}\right) \right) \quad (3.9)$$

and the  $H_n$ 's are the Hermite polynomials. In particular, if we set  $C(t, K; \omega) := \mathbb{E}(e^{X_t} - K)_+$ , we obtain

$$C \equiv \tilde{C} \quad [2], \quad (3.10)$$

where  $\tilde{C}$  is defined by

$$\tilde{C}(t, K; \omega) = C_{BS}(t, K, \sqrt{\nu}) + \left( \sum_{n=2}^6 \nu_n \right) N(d_1) + \frac{K}{\sqrt{\nu}} N'(d_2) \left( \sum_{n=0}^4 \frac{z_n}{\nu^{\frac{n}{2}}} H_n(-d_2) \right), \quad (3.11)$$

with  $C_{BS}(t, K, \Sigma) = N(d_1) - KN(d_2)$ ,  $d_1 = \frac{\log(\frac{1}{K}) + \frac{t}{2}\Sigma^2}{\sqrt{t}\Sigma}$ ,  $d_2 = \frac{\log(\frac{1}{K}) - \frac{t}{2}\Sigma^2}{\sqrt{t}\Sigma}$  and for  $n = 0, \dots, 4$ ,  $z_n = \sum_{i=n+2}^6 (-1)^i \nu_i$ .

## 4 Application : Variance Reduction

The formulas given in the previous section, including those that give the approximation of the density and the price of European call options, can also be used otherwise. We can use these formulas as a tool to reduce the variance for the simulation of the trajectories of the underlying

asset via the Monte Carlo method. In this section, we propose a control variate for the variance reduction of Monte Carlo simulations for pricing European options. Our method is based on the use of the approximation of the distribution function of  $\log S_t$  (when it is available) in a general stochastic volatility model. The idea is to construct a process  $\hat{S}$  having as density (or distribution) the approximate density and strongly correlated with the trajectories of the underlying asset, then we write for a bound function  $H$

$$\mathbb{E} H(S_t) = \mathbb{E} H(S_t) - \mathbb{E} H(\hat{S}_t) + \mathbb{E} H(\hat{S}_t).$$

As the law of  $\hat{S}_t$  is known, we have just to simulate  $H(S_t) - H(\hat{S}_t)$  whose variance will be very small compared to the variance of  $H(S_t)$ .

#### 4.1 The Method

We consider a general one factor stochastic volatility model, where we assume that the dynamics of the underlying is modeled by the SDE

$$\frac{dS_t^\epsilon}{S_t^\epsilon} = \sqrt{f(V_t^\epsilon)} \left( \rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right), \quad (4.1)$$

where  $V^\epsilon$  is the unique strong solution to the SDE

$$dV_t^\epsilon = \alpha(t, V_t^\epsilon) dt + \epsilon \sigma(t, V_t^\epsilon) dW_t^1, \quad (4.2)$$

with  $d\langle W^1, W^2 \rangle_t = 0$  and  $\alpha, \sigma$  are satisfying sufficient conditions which ensure the existence of a strong solution to the stochastic differential equation in (4.1).

**Proposition 4.1.** *For  $t > 0$ , denote by  $F_t^\epsilon(\cdot)$  the distribution function of  $X_t^\epsilon := \log(S_t^\epsilon)$ . Denote by  $V^0$  the unique solution starting from  $v$  of the ODE*

$$dV_t^0 = \alpha(t, V_t^0) dt.$$

*Then the random variable*

$$\hat{X}_t^\epsilon = (F_t^\epsilon)^{-1} \circ N \left( \frac{\sqrt{1 - \rho^2}}{\sqrt{\int_0^t f(V_s^\epsilon) ds}} \int_0^t \sqrt{f(V_s^\epsilon)} dW_s^2 + \frac{\rho}{\sqrt{\int_0^t f(V_s^0) ds}} \int_0^t \sqrt{f(V_s^0)} dW_s^1 \right) \quad (4.3)$$

*has the same law as  $X_t^\epsilon$ .*

**Proof:** As  $V^0$  is deterministic, the random variable  $\int_0^t \sqrt{f(V_s^0)} dW_s^1$  is Gaussian. On the other

hand, since  $V^\epsilon$  is a strong solution of (4.2), then  $V_s^\epsilon$  is  $\mathcal{F}_s^{W^1}$  measurable, with  $\mathcal{F}_s^{W^1}$  is the filtration generated by the Brownian motion  $W^1$ . Thus  $V_s^\epsilon$  is independent of  $W^2$  and therefore the random variable  $\frac{1}{\sqrt{\int_0^t f(V_s^\epsilon)ds}} \int_0^t \sqrt{f(V_s^\epsilon)}dW_s^2$  is a standard normal Random Variable  $\mathcal{N}(0,1)$ . Now we can easily check that, conditionally on  $\int_0^t \sqrt{f(V_s^0)}dW_s^1$ , the random variable  $\frac{1}{\sqrt{\int_0^t f(V_s^\epsilon)ds}} \int_0^t \sqrt{f(V_s^\epsilon)}dW_s^2$  is  $\mathcal{N}(0,1)$ . Then, the random variable  $\frac{\sqrt{1-\rho^2}}{\sqrt{\int_0^t f(V_s^\epsilon)ds}} \int_0^t \sqrt{f(V_s^\epsilon)}dW_s^2 + \frac{\rho}{\sqrt{\int_0^t f(V_s^0)ds}} \int_0^t \sqrt{f(V_s^0)}dW_s^1$  is  $\mathcal{N}(0,1)$ , which implies that  $N\left(\frac{\sqrt{1-\rho^2}}{\sqrt{\int_0^t f(V_s^\epsilon)ds}} \int_0^t \sqrt{f(V_s^\epsilon)}dW_s^2 + \frac{\rho}{\sqrt{\int_0^t f(V_s^0)ds}} \int_0^t \sqrt{f(V_s^0)}dW_s^1\right)$  is uniformly distributed on  $[0,1]$ . Thus  $\hat{X}_t^\epsilon$  has the same law as  $X_t^\epsilon$ .

**Proposition 4.2.** *Let  $G$  be  $\mathcal{C}^2$ , with bounded derivatives. We have*

$$\mathbb{E} \left[ \left( G(\hat{X}_t^\epsilon) - G(X_t^\epsilon) - \mathbb{E}\{G(\hat{X}_t^\epsilon) - G(X_t^\epsilon)\} \right)^2 \right] = \mathcal{O}(\epsilon^2), \quad (4.4)$$

**Remark 4.1.** Note that the variance of  $G(X_t^\epsilon)$  is of order 0 in  $\epsilon$ . This method of variance reduction makes possible to get rid of terms of order 0 and 1 in  $\epsilon$ . As we can see in the next numerical examples, the second order Taylor series expansion of price with respect to  $\epsilon$  was very close to the real price, we believe that the remaining variance will be very negligible compared to the total variance.

**Proof:** We first note that  $X_t^0 = \hat{X}_t^0$ . Indeed,  $V_t^0$  is deterministic, which means that  $X_t^0$  is Gaussian with mean  $-\frac{1}{2} \int_0^t f(V_s^0)ds$  and variance  $\int_0^t f(V_s^0)ds$ . So

$$F_t^0(x) = N \left( \frac{x + \frac{1}{2} \int_0^t f(V_s^0)ds}{\sqrt{\int_0^t f(V_s^0)ds}} \right).$$

Its inverse is given by

$$(F_t^0)^{-1}(y) = N^{-1}(y) \times \sqrt{\int_0^t f(V_s^0)ds} - \frac{1}{2} \int_0^t f(V_s^0)ds.$$

Therefore

$$\begin{aligned} \hat{X}_t^0 &= (F_t^0)^{-1} \circ N \left( \frac{\sqrt{1-\rho^2}}{\sqrt{\int_0^t f(V_s^0)ds}} \int_0^t \sqrt{f(V_s^0)}dW_s^2 + \frac{\rho}{\sqrt{\int_0^t f(V_s^0)ds}} \int_0^t \sqrt{f(V_s^0)}dW_s^1 \right) \\ &= \sqrt{1-\rho^2} \int_0^t \sqrt{f(V_s^0)}dW_s^2 + \rho \int_0^t \sqrt{f(V_s^0)}dW_s^1 - \frac{1}{2} \int_0^t f(V_s^0)ds = X_t^0. \end{aligned}$$



We now write the Taylor expansion of  $X_t$  and  $\hat{X}_t$  with respect to  $\epsilon$ . We obtain

$$\begin{cases} X_t^\epsilon = X_t^0 + \epsilon X_t^1 + \frac{\epsilon^2}{2} X_t^2 + \dots, \\ \hat{X}_t^\epsilon = \hat{X}_t^0 + \epsilon \hat{X}_t^1 + \frac{\epsilon^2}{2} \hat{X}_t^2 + \dots \end{cases}$$

where  $X_t^0 = \hat{X}_t^0$ . Similarly, we can write

$$\begin{cases} G(X_t^\epsilon) = G(X_t^0) + \epsilon G'(X_t^0) X_t^1 + \dots, \\ G(\hat{X}_t^\epsilon) = G(\hat{X}_t^0) + \epsilon G'(\hat{X}_t^0) \hat{X}_t^1 + \dots \end{cases}$$

Note that since  $X_t^\epsilon$  and  $\hat{X}_t^\epsilon$  have the same law, we have

$$\mathbb{E} \left[ \left( G(\hat{X}_t^\epsilon) - G(X_t^\epsilon) - \mathbb{E}\{G(\hat{X}_t^\epsilon) - G(X_t^\epsilon)\} \right)^2 \right] = 2\mathbb{E} \left[ G^2(X_t^\epsilon) - G(X_t^\epsilon) G(\hat{X}_t^\epsilon) \right]$$

On the other hand, by writing that  $\mathbb{E} [G^2(X_t^\epsilon)] = \frac{1}{2} \mathbb{E} [G^2(X_t^0) + G^2(\hat{X}_t^0)]$ , we obtain

$$\begin{aligned} \mathbb{E} [G^2(X_t^\epsilon) - G(X_t^\epsilon) G(\hat{X}_t^\epsilon)] &= \frac{1}{2} \mathbb{E} [G^2(X^0 + \epsilon X^1) + G^2(\hat{X}^0 + \epsilon \hat{X}^1) + \mathcal{O}(\epsilon^2)] - \\ &\quad \mathbb{E} [(G(X_t^0) + \epsilon G'(X^0) X^1) (G(\hat{X}^0) + \epsilon G'(\hat{X}^0) \hat{X}^1) + \mathcal{O}(\epsilon^2)] \\ &= \frac{1}{2} \mathbb{E} [G^2(X^0) + 2\epsilon G G'(X^0) X^1 + G^2(\hat{X}^0) + 2\epsilon G G'(\hat{X}^0) \hat{X}^1] \\ &\quad - \mathbb{E} [(G(X^0) G(\hat{X}^0) + \epsilon G'(X^0) G(\hat{X}^0) X^1 + \epsilon G(X^0) G'(\hat{X}^0) \hat{X}^1)] \\ &\quad + \mathcal{O}(\epsilon^2) \\ &= \mathcal{O}(\epsilon^2). \end{aligned}$$

Because  $X^0 = \hat{X}^0$ . We have finally

$$\mathbb{E} [G^2(X_t^\epsilon) - G(X_t^\epsilon) G(\hat{X}_t^\epsilon)] = \frac{\epsilon^2}{2} \mathbb{E} [(G')^2(X^0) (X^1 - \hat{X}^1)^2] + \mathcal{O}(\epsilon^3).$$

## 4.2 Application

In the case where the instantaneous variance is given as a sum of exponential Ornstein-Uhlenbeck processes, we can use the approximation of the distribution function of the underlying obtained in the previous section to construct the control variate. For example, consider the case of one factor model with parameters  $t = 1$  month,  $1$  year,  $r = 0$ ,  $m = 0.1 \times e^{-\frac{\omega^2}{2}t}$ ,  $b = 3$ ,  $\sigma = 1$ , and  $\rho = -0.75$ . In the following tables, we compare the variance of the price of  $(K - e^{X_t})_+$  and  $(K - e^{X_t})_+ - (K - e^{\hat{X}_t})_+$  for several strikes. We also compare the variance of  $e^{X_t}$  with the

variance of  $e^{X_t} - e^{\hat{X}_t}$  for  $\omega = 100\%$

	$t = 1 \text{ month}$			$t = 1 \text{ year}$		
Strike	Var	VarDif	$\frac{Var}{VarDif}$	Var	VarDif	$\frac{Var}{VarDif}$
var $S$	0,8%	0,002%	344	6,17 %	0,10%	59
90%	0,04%	0,001%	54	1,05%	0,02%	45
100%	0,28%	0,002%	176	1,90%	0,03%	65
110%	0,64%	0,002%	394	2,92%	0,03%	85
120%	0,79%	0,002%	407	3,94%	0,04%	98

**Remark 4.2.** Note that the random variable  $e^{X_t}$  is square integrable, since  $\rho = -0.75 < -\frac{1}{\sqrt{2}}$  (cf. [19]) and  $e^{X_t}$  defines a true martingale.

## 5 Numerical results: Comparison with Perelló et al (cf [24])

We study the error of this approximation by comparing our results with those obtained with the Monte Carlo and Perelló, Sircar and Masoliver [24]. We consider the case of a European Put with several strikes and several maturities. We consider the case of constant parameters and we choose the following values for the model parameters:  $S_0 = 100$ ,  $r = 0$ ,  $m = 0.1 \times e^{-\frac{\omega^2}{2}T}$ ,  $b = 3$ ,  $\sigma = 1$ ,  $\rho = -0.6$  and compare the results for  $\omega = 100\%$ ,  $200\%$  and  $T = 1 \text{ month}$ ,  $2 \text{ months}$ ,  $1 \text{ year}$ . We see that for short maturities and low levels of volatility the volatility, both methods (our approximation method and PSM) are almost identical and give the real price, while for long maturities and/or high levels of  $\omega$  the error in our method is stable and does not exceed 1-2 %.

$\omega = 100\%$

	2 months			1 year		
Strike	MC	Proxy	PSM	MC	Proxy	PSM
80	0,26 $\pm 0.003$	0,26	0,23	2,65 $\pm 0.009$	2,65	2,28
90	1,46 $\pm 0.005$	1,46	1,38	5,53 $\pm 0.011$	5,53	5,03
100	4,96 $\pm 0.005$	4,97	4,86	9,96 $\pm 0.013$	9,96	9,42
110	11,52 $\pm 0.005$	11,52	11,45	15,96 $\pm 0.012$	15,96	15,47
120	20,32 $\pm 0.006$	20,31	20,28	23,33 $\pm 0.013$	23,33	22,94
130	30,05 $\pm 0.005$	30,05	30,03	31,74 $\pm 0.014$	31,75	31,46

In terms of implied volatility

	2 months			1 year		
Strike	MC	Proxy	PSM	MC	Proxy	PSM
80	33,5 $\pm 0.004$	33,45	32,66	26,6 $\pm 0.010$	26,57	25,13
90	31,89 $\pm 0.005$	31,89	31,23	25,76 $\pm 0.012$	25,76	24,34
100	30,50 $\pm 0.005$	30,51	29,83	25,02 $\pm 0.014$	25,03	23,66
110	29,35 $\pm 0.000$	29,34	28,73	24,39 $\pm 0.000$	24,40	23,09
120	28,43 $\pm 0.000$	28,40	27,68	23,85 $\pm 0.000$	23,85	22,58
130	27,76 $\pm 0.000$	27,83	25,93	23,38 $\pm 0.000$	23,39	22,05

$\omega = 200\%$

	1 month			1 year		
Strike	MC	Proxy	PSM	MC	Proxy	PSM
80	0,05 $\pm 0.002$	0,05	0,03	0,05 $\pm 0.009$	0,45	0,3
90	0,59 $\pm 0.005$	0,59	0,53	1,8 $\pm 0.013$	1,76	1,33
100	3,41 $\pm 0.006$	3,41	3,27	5,08 $\pm 0.016$	5,06	4,35
110	10,45 $\pm 0.006$	10,45	10,39	11,40 $\pm 0.014$	11,39	10,92
120	20,02 $\pm 0.006$	20,03	20,00	20,29 $\pm 0.016$	20,27	20,07
130	30 $\pm 0.007$	30,01	30	30,04 $\pm 0.018$	30,10	19,99

In terms of implied volatility :

	1 month			1 year		
Strike	MC	Proxy	PSM	MC	Proxy	PSM
80	36,2 $\pm 0.004$	35,31	33,36	16 $\pm 0.012$	15,36	14,12
90	32,69 $\pm 0.006$	32,64	31,62	14,25 $\pm 0.014$	14,11	12,55
100	29,59 $\pm 0.007$	29,6	28,39	12,73 $\pm 0.018$	12,70	10,9
110	27,29 $\pm 0.000$	27,26	26,14	11,59 $\pm 0.014$	11,55	9,88
120	26,19 $\pm 0.000$	27,13	21,74	11 $\pm 0.000$	11,24	8,74
130	29,35 $\pm 0.000$	31,62	30	11,11 $\pm 0.000$	12,58	-

**Remark 5.1.** In Monte Carlo simulations, we used the method of variance reduction introduced in section 3.4.

**Remark 5.2.** One must be careful when we simulated pay-off that are unbounded (Calls for example) because the random variable  $e^{X_t}$  is not necessarily square-integrable, in which case the variance is infinite. However, there are cases where we are sure that  $e^{X_t}$  is square-integrable, for

example when  $\rho < -\frac{1}{\sqrt{2}}$  (see Jordain [19]). On the other hand, according to [19], the process  $e^X$  is a martingale if and only if  $\rho \leq 0$ . In particular, the Call-Put parity is not satisfied when  $\rho > 0$ .

## A Scott model with constant parameters:

We give here the results of our approximation for Scott model when parameters are constant. It might be useful for practical purpose.

Let's denote by

$$\begin{aligned}
I_1 &:= \int_0^t m_s \int_0^s \sigma_u^2 e^{-2b(s-u)} du ds = m\sigma^2 \frac{2bt - 1 - e^{-2bt}}{4b^2}, \\
I_2 &:= \int_0^t m_\tau \int_0^\tau m_s \int_0^s \sigma_u^2 e^{-b(\tau+s-2u)} du ds d\tau = m^2\sigma^2 \frac{2bt - 3 + 4e^{-bt} - e^{-2bt}}{2b^3}, \\
I_3 &:= \int_0^t m_\tau \int_0^\tau \eta_s(1) e^{-b(\tau-s)} ds d\tau = \rho\sigma m\sqrt{m} \frac{bt - 1 + e^{-bt}}{b^2}, \\
I_4 &:= \int_0^t m_\tau \int_0^\tau \eta_s(1) e^{-b(\tau-s)} \int_s^\tau \eta_u(1) du ds d\tau = \rho^2\sigma^2 m^2 \frac{bt(1 + e^{-bt}) - 2(1 - e^{-bt})}{b^3}, \\
I_5 &:= \int_0^t m_\tau \left( \int_0^\tau \eta_s(1) e^{-b(\tau-s)} ds \right)^2 d\tau = \rho^2\sigma^2 m^2 \frac{2bt - 3 + 4e^{-bt} - e^{-2bt}}{2b^3}. \tag{A.1}
\end{aligned}$$

The  $\nu$ 's are given by

$$\begin{aligned}
\nu_1(t; \omega) &= \frac{\omega^2}{4} I_1, \\
\nu_2(t; \omega) &= \nu_1 + \frac{\omega^2}{4} I_2 - \frac{\omega}{2} I_3, \\
\nu_3(t; \omega) &= -\frac{\omega}{2} I_3 + \frac{\omega^2}{4} I_4 + \frac{\omega^2}{4} I_5 + \frac{\omega^2}{2} I_2 \\
\nu_4(t; \omega) &= \frac{\omega^2}{4} I_2 + \frac{\omega^2}{4} I_4 + \frac{\omega^2}{4} I_4 + \frac{\omega^2}{8} I_3^2, \\
\nu_5(t; \omega) &= \frac{\omega^2}{4} I_3^2, \quad \nu_6(t; \omega) = \frac{\omega^2}{8} I_3^2.
\end{aligned}$$

## B Proof of Theorem 2.2

We can easily proof the theorem by writing  $\hat{\varphi}$  as  $\hat{\varphi}(t, \xi, \zeta; \omega) = \hat{\varphi}_0(t, \xi, \zeta) + \omega \hat{\varphi}_1(t, \xi, \zeta) + \frac{\omega}{2} \hat{\varphi}_2(t, \xi, \zeta) + \mathcal{O}(\omega^3)$  and showing that for  $i = 0, 1, 2$ ,

$$\mathcal{L}_i \hat{\varphi}_i(t, \xi, \zeta) = 0, \quad \forall (t, \xi, \zeta) \in ]0, \infty[ \times \mathbb{R} \times \mathbb{R}.$$

This approximation is obtained as follows:

We set  $\tilde{V}_t = \frac{1}{\omega} V_t$ . In particular,  $\tilde{V}$  is the unique solution (starting from 0) to the SDE

$$d\tilde{V}_t = -b\tilde{V}_t dt + \sigma_t dW_t^V$$

Denote by  $\tilde{p}(t, x, v)$  the joint density of the pair of random variables  $(X_t = \log(S_t/S_0), \tilde{V}_t)$ . This density satisfies the following Fokker-Planck equation :

$$\partial_t \tilde{p} = \partial_v [bv \tilde{p}] + \frac{\sigma_t^2}{2} \frac{\partial^2 \tilde{p}}{\partial v^2} + \frac{1}{2} f^2(t, \omega v) \frac{\partial \tilde{p}}{\partial x} + \frac{1}{2} f^2(t, \omega v) \frac{\partial^2 \tilde{p}}{\partial x^2} + \rho_t \sigma_t f(t, \omega v) \frac{\partial^2 \tilde{p}}{\partial x \partial v}. \quad (\text{B.1})$$

With  $\tilde{p}(0, x, v|0) = \delta_0(x) \delta_0(v)$ .

Consider the Fourier transform of  $\tilde{p}$ , defined as

$$\tilde{\varphi}(t, \xi, \zeta) := \int_{-\infty}^{\infty} dx e^{ix\xi} \int_{-\infty}^{\infty} dv e^{iv\zeta} \tilde{p}(t, x, v).$$

Note that  $\tilde{\varphi}(t, \xi, \zeta) := \mathbb{E} e^{i(\xi X_t + \zeta \tilde{V}_t)}$  and since  $V$  is Gaussian (in particular  $V_t$  admits exponential moments of all orders), then  $\varphi(t, \xi, \cdot)$  is well defined and analytic over  $\mathbb{C}$ . The function  $\tilde{\varphi}$  is then solution of the following equation

$$\begin{aligned} -\partial_t \tilde{\varphi} &= \frac{\sigma_t^2}{2} \zeta^2 \tilde{\varphi}(t, \xi, \zeta) + b\zeta \partial_\zeta \tilde{\varphi} + m_t \frac{\xi^2 + i\xi}{2} \varphi(t, \xi, \zeta - i\omega) + \\ &\quad \xi \zeta \rho_t \sigma_t \sqrt{m_t} \tilde{\varphi}(t, \xi, \zeta - i\frac{\omega}{2}). \end{aligned} \quad (\text{B.2})$$

With initial condition  $\tilde{\varphi}(0, \xi, \zeta) = 1$ .

We seek a solution of (B.2) given as

$$\tilde{\varphi}(t, \xi, \zeta) = \exp \left( - \sum_{n \geq 0} A_n(t, \xi) \zeta^n \right).$$

So, the equation (B.2) becomes

$$\begin{aligned}
\sum_{n \geq 0} \partial_t A_n(t, \xi) \zeta^n &= \frac{\sigma_t^2}{2} \zeta^2 - b \sum_{n \geq 1} n A_n(t, \xi) \zeta^n + \\
&\quad m_t \frac{\xi^2 + i\xi}{2} \exp \left( \sum_{n \geq 0} A_n(t, \xi) [\zeta^n - (\zeta - i\omega)^n] \right) + \\
&\quad \xi \zeta \rho_t \sigma_t \sqrt{m_t} \exp \left( \sum_{n \geq 0} A_n(t, \xi) \left[ \zeta^n - \left( \zeta - i \frac{\omega}{2} \right)^n \right] \right). \quad (\text{B.3})
\end{aligned}$$

As we focus only on the distribution of  $X_t$  and not necessarily that of pair  $(X_t, V_t)$ , we will solve this equation for  $\zeta$  around 0. So we perform the scaling  $\zeta = \omega\lambda$ , where we assume that  $\lambda \in \mathbb{R}$ . In particular, we have

$$\sum_{n \geq 0} A_n(t, \xi) [\zeta^n - (\zeta - i\omega)^n] = i\omega A_1(t, \xi) + \omega^2 A_2(t, \xi) [2i\lambda + 1] + \mathcal{O}(\omega^3). \quad (\text{B.4})$$

Also, we obtain a similar development for  $\sum_{n \geq 0} A_n(t, \xi) [\zeta^n - (\zeta - i\frac{\omega}{2})^n]$ . Therefore, the equation (B.2) can be written as

$$\begin{aligned}
\partial_t A_0 + \omega \partial_t A_1 \lambda + \omega^2 \partial_t A_2 \lambda^2 &= -b\omega A_1 \lambda + \omega^2 \left( \frac{\sigma_t^2}{2} - 2bA_2 \right) \lambda^2 + \mathcal{O}(\omega^3) + \\
&\quad \mu_t(\xi) \exp \left( i\omega A_1 + \omega^2 (1 + 2i\lambda) A_2 + \mathcal{O}(\omega^3) \right) + \\
&\quad v\eta_t(\xi) \exp \left( i\frac{\omega}{2} A_1 + \omega^2 \left( \frac{1}{4} + i\lambda \right) A_2 + \mathcal{O}\left(\frac{\omega^3}{8}\right) \right),
\end{aligned}$$

where  $\mu_t(\xi) := m_t \frac{\xi^2 + i\xi}{2}$  and  $\eta_t(\xi) = \xi \rho_t \sigma_t \sqrt{m_t}$ . So the previous system becomes, by writing  $e^x = 1 + x + \frac{x^2}{2} + \dots$ ,

$$\begin{aligned}
\partial_t A_0 + \omega \partial_t A_1 \lambda + \omega^2 \partial_t A_2 \lambda^2 &= -b\omega A_1 \lambda + \omega^2 \left( \frac{\sigma_t^2}{2} - 2bA_2 \right) \lambda^2 + \\
&\quad \mu_t \left( 1 + i\omega A_1 + \omega^2 A_2 - \frac{\omega^2}{2} A_1^2 + 2i\omega^2 A_2 \lambda \right) + \\
&\quad \eta_t \left( 1 + i\frac{\omega}{2} A_1 \right) \omega \lambda + \tilde{\mathcal{O}}(\omega^3).
\end{aligned}$$

The solution of the truncated system (without  $\tilde{\mathcal{O}}(\omega^3)$ ) is given by the triplet  $(A_0, A_1, A_2)$ , solution

of the system

$$\begin{cases} \partial_t A_2(t, \xi) = \frac{\sigma_t^2}{2} - 2bA_2(t, \xi), \\ \partial_t A_1(t, \xi) = -(b - i\frac{\omega}{2}\eta_t(\xi))A_1(t, \xi) + 2i\omega\mu_t(\xi)A_2(t, \xi) + \eta_t(\xi), \\ \partial_t A_0(t, \xi) = \mu_t(\xi) \left( 1 + i\omega A_1(t, \xi) + \omega^2 A_2(t, \xi) - \frac{\omega^2}{2} A_1^2(t, \xi) \right), \\ A_0(0, \xi) = A_1(0, \xi) = A_2(0, \xi) = 0. \end{cases} \quad (\text{B.5})$$

The solution of this system is given by

$$\begin{aligned} A_2(t, \xi) \equiv A_2(t, \xi; \omega) &= \frac{1}{2} \int_0^t \sigma_s^2 e^{-2b(t-s)} ds, \\ A_1(t, \xi) \equiv A_1(t, \xi; \omega) &= \int_0^t (\eta_s + 2i\omega\mu_s A_2(s, \xi)) e^{-(\gamma(t, \xi) - \gamma(s, \xi))} ds, \\ A_0(t, \xi) \equiv A_0(t, \xi; \omega) &= \int_0^t \mu_s (1 + i\omega A_1 + \omega^2 A_2 - \frac{\omega^2}{2} A_1^2)(s, \xi) ds, \end{aligned} \quad (\text{B.6})$$

where

$$\gamma(t, \xi) = bt - i\frac{\omega}{2}\xi \int_0^t \rho_s \sigma_s \sqrt{m_s} ds.$$

We then obtain an approximation of  $\tilde{\varphi}(t, \xi, \omega\lambda)$  which can be written as

$$\tilde{\varphi}(t, \xi, \omega\lambda) \approx \exp \left( - \left( A_0(t, \xi; \omega) + A_1(t, \xi; \omega)(\omega\lambda) + A_2(t, \xi; \omega)(\omega\lambda)^2 \right) \right). \quad (\text{B.7})$$

We now consider the Fourier transform of the pair  $(X_t, V_t)$ , defined as

$$\varphi(t, \xi, \zeta) := \mathbb{E} e^{i\xi X_t + i\zeta V_t} = \tilde{\varphi}(t, \xi, \omega\zeta),$$

then  $\varphi$  can be approximated by

$$\hat{\varphi}(t, \xi, \zeta) := e^{-\left( A_0(t, \xi; \omega) + \omega A_1(t, \xi; \omega)\zeta + \omega^2 A_2(t, \xi; \omega)\zeta^2 \right)}. \quad (\text{B.8})$$

## C Proof of Proposition 3.1

Let  $p(t, x, v_1, \dots, v_N)$  be the density of the vector  $(X_t, V_1(t), \dots, V_N(t))$ . This density satisfies the following Fokker-Planck equation:

$$\partial_t p = \sum_n \partial_{v_n} (v_n p) + \frac{\sigma_n^2}{2} \sum_n \frac{\partial^2 p}{\partial v_n^2} + \frac{1}{2} f^2(t; v) (\partial_x + \partial_{xx}) p + \sum_n \rho_n^S \sigma_n f(t; v) \frac{\partial^2 p}{\partial x \partial v_n}.$$

Consider the Fourier transform defined as

$$\varphi(t, \xi, \zeta_1, \dots, \zeta_N) := \int_{-\infty}^{\infty} dx e^{ix\xi} \int_{-\infty}^{\infty} d\zeta_1 e^{i\zeta_1 v_1} \dots \int_{-\infty}^{\infty} d\zeta_N e^{i\zeta_N v_N} p(t, x, v_1, \dots, v_N).$$

The function  $\varphi$  is solution of the following equation

$$\begin{aligned} -\partial_t \varphi &= \sum_n \left( \frac{\sigma_n^2}{2} \zeta_n^2 - i\alpha_n(t) \zeta_n \right) \varphi(t, \xi, \zeta) + \sum_n \kappa_n \zeta_n \partial_{\zeta_n} \varphi \\ &\quad + m_t \frac{\xi^2 + i\xi}{2} \varphi(t, \xi, \zeta_1 - i\omega, \dots, \zeta_N - i\omega) + \\ &\quad \xi \sqrt{m_t} \sum_n \rho_n^S \sigma_n(t) \zeta_n \varphi(t, \xi, \zeta_1, \dots, \zeta_{n-1}, \zeta_n - i\frac{\omega}{2}, \zeta_{n+1}, \dots, \zeta_N). \end{aligned} \quad (\text{C.1})$$

Note that we are only interested in the Fourier transform of  $X_t$  corresponding to  $\varphi(t, \xi, 0, \dots, 0)$ . One then proceeds as in one-dimensional : at the order 3 with respect to  $\omega$  and for  $(\zeta_1, \dots, \zeta_N) \in \mathcal{D}(0, \omega)^N$ , we have

$$\varphi(t, x; \zeta_1, \dots, \zeta_N) = \exp \left( -A_0 - \sum_{n=1}^M A_n(t, \xi) \zeta_n - \sum_{n \neq m=1}^M B_{n,m}(t, \xi) \zeta_n \zeta_m - \sum_{n=1}^M C_n(t, \xi) \zeta_n^2 \right).$$

In this case, the equation (C.1) becomes

$$\begin{aligned} A'_0 &+ \sum_{n=1}^M A'_n \zeta_n + \sum_{n \neq m=1}^M B'_{n,m} \zeta_n \zeta_m + \sum_{n=1}^M C'_n \zeta_n^2 = \mathcal{O}(\omega^3) + \sum_n \left( \frac{\sigma_n^2}{2} \zeta_n^2 - i\alpha_n(t) \zeta_n \right) \\ &- \sum_n \kappa_n A_n \zeta_n - \sum_n \kappa_n \zeta_n \left( \sum_{m=1, m \neq n}^N B_{n,m} \zeta_m \right) - 2 \sum_{n=1}^M \kappa_n C_n(t, \xi) \zeta_n^2 \\ &+ \mu_t \exp \left( i\omega \sum_{n=1}^M A_n + \sum_{n \neq m=1}^M B_{n,m} (\omega^2 + i\omega(\zeta_n + \zeta_m)) + \sum_{n=1}^M C_n (\omega^2 + 2i\omega \zeta_n) \right) \\ &+ \xi \sqrt{m_t} \sum_n \rho_n^S \sigma_n(t) \zeta_n \exp \left( i\frac{\omega}{2} A_n + i\frac{\omega}{2} \sum_{m=1, m \neq n}^N B_{n,m} \zeta_m + C_n \left( \frac{\omega^2}{4} + i\omega \zeta_n \right) \right). \end{aligned}$$



On the other hand, we have

$$\exp \left( i\omega \sum_{n=1}^M A_n + \sum_{n \neq m=1}^M B_{n,m}(\omega^2 + i\omega(\zeta_n + \zeta_m)) + \sum_{n=1}^M C_n(\omega^2 + 2i\omega\zeta_n) \right) = \mathcal{O}(\omega^3) +$$

$$1 + i\omega \sum_{n=1}^M A_n - \frac{\omega^2}{2} \left( \sum_{n=1}^M A_n \right)^2 + \sum_{n \neq m=1}^M B_{n,m}(\omega^2 + i\omega(\zeta_n + \zeta_m)) + \sum_{n=1}^M C_n(\omega^2 + 2i\omega\zeta_n)$$

and

$$\sum_n \rho_n^S \zeta_n \exp \left( i\frac{\omega}{2} A_n + i\frac{\omega}{2} \sum_{m=1, m \neq n}^N B_{n,m} \zeta_m + C_n \left( \frac{\omega^2}{4} + i\omega\zeta_n \right) \right) = \sum_n \rho_n^S \zeta_n \left( 1 + i\frac{\omega}{2} A_n \right) + \mathcal{O}(\omega^3).$$

Solving equation (C.1) is equivalent to solve the system

$$\begin{cases} \sum_{n=1}^M C'_n - \frac{\sigma_n^2}{2} + 2\kappa_n C_n = 0, \\ \sum_{n \neq m=1}^M B'_{n,m} + \kappa_n B_{n,m} = 0, \\ \sum_{n=1}^M (A'_n + i\alpha_n(t) + \kappa_n A_n + 2i\mu_t \omega C_n - x\sqrt{m_t} \rho_n^S \sigma_n(t)(1 + i\frac{\omega}{2} A_n)) = 0, \\ A'_0 = \mu_t \left( 1 + i\omega \sum_{n=1}^M A_n - \frac{\omega^2}{2} (\sum_{n=1}^M A_n)^2 + \omega^2 \sum_{n=1}^M C_n \right). \end{cases}$$

We can immediately deduce that for all  $n, m$ , we have  $B_{n,m} = 0$ . This gives the following system

$$\begin{cases} C'_n - \frac{\sigma_n^2}{2} + 2\kappa_n C_n = 0, \quad \forall n = 1, \dots, N, \\ A'_n + i\alpha_n(t) + (\kappa_n - i\frac{\omega}{2} \xi \sqrt{m_t} \rho_n^S \sigma_n(t)) A_n - 2i\mu_t \omega C_n - \xi \sqrt{m_t} \rho_n^S \sigma_n(t) = 1, \quad \forall n \geq 1, \\ A'_0 = \mu_t \left( 1 + i\omega \sum_{n=1}^M A_n - \frac{\omega^2}{2} (\sum_{n=1}^M A_n)^2 + \omega^2 \sum_{n=1}^M C_n \right). \end{cases}$$

This is equivalent to

$$\begin{cases} C_n(t) = \frac{1}{2} \int_0^t \sigma_n^2(s) e^{-2\kappa_n(t-s)} ds, \quad \forall n = 1, \dots, N, \\ A_n = \int_0^t (\xi \rho_n^S \sigma_n(s) \sqrt{m_s} - i\alpha_s + 2i\omega \mu_s C_n(s, \xi)) e^{-(\gamma_n(t, \xi) - \gamma_n(s, \xi))} ds, \\ A_0(t, \xi) = \int_0^t \mu_s \left( 1 + i\omega \sum_{n=1}^M A_n(s, \xi) - \frac{\omega^2}{2} (\sum_{n=1}^M A_n(s, \xi))^2 + \omega^2 \sum_{n=1}^M C_n(s, \xi) \right) ds, \end{cases}$$

where

$$\gamma_n(t, \xi) = \kappa_n t - i\xi \rho_n^S \frac{\omega}{2} \int_0^t \sigma_n(s) \sqrt{m_s} ds. \quad (\text{C.2})$$

It follows that

$$\varphi_X(T, \xi) := \mathbb{E} e^{i\xi X_T} = e^{-A_0(t, \xi)}.$$

Let's set  $A(t, x) := \sum_{n=1}^M A_n(s, \xi)$ . Writing  $A$  as a first order Taylor series expansion with respect to  $\omega$ , we obtain

$$\begin{aligned} A(t, \xi) &= \sum_{n=1}^N \int_0^t (x \rho_n^S \sigma_n(s) \sqrt{m_s} - i\alpha_n(s) + i\omega \mu_s \int_0^s \sigma_n^2(r, \xi) e^{-2\kappa_n(s-r)} dr) e^{-(\gamma_n(t, \xi) - \gamma_n(s, \xi))} ds, \\ &= \sum_{n=1}^N \int_0^t (\xi \rho_n^S \sigma_n(s) \sqrt{m_s} - i\alpha_n(s)) e^{-\kappa_n(t-s)} \left(1 - \frac{i}{2} \omega \rho_n^S x \int_s^t \sigma_n(u) \sqrt{m_u} du\right) ds, \\ &\quad + i\omega \sum_{n=1}^N \int_0^t \mu_s \int_0^s \sigma_n^2(r, \xi) e^{-\kappa_n(t+s-2r)} dr ds + \mathcal{O}(\omega^2). \end{aligned}$$

Similarly, we have

$$A^2(t, \xi) = \left( \sum_{n=1}^N \int_0^t (\xi \rho_n^S \sigma_n(s) \sqrt{m_s} - i\alpha_n(s)) e^{-\kappa_n(t-s)} ds \right)^2 + \mathcal{O}(\omega).$$

It follows that  $\varphi_X(T, \xi)$  can be written as

$$\varphi_X(T, \xi) = \exp \left( -i\mu_1(T)\xi - \mu_2(T)\xi^2 + i\mu_3(T)\xi^3 + \mu_4(T)\xi^4 + \mathcal{O}(\omega^3) \right),$$

where

$$\begin{aligned}
\mu_1(T) &= \frac{1}{2} \int_0^T m_s ds + \frac{\omega^2}{4} \sum_{n=1}^N \int_0^T m_t \int_0^t \sigma_n^2(s) e^{-2\kappa_n(t-s)} ds + \frac{\omega}{2} \sum_{n=1}^N \int_0^T m_t \int_0^t \alpha_n(s) e^{-\kappa_n(t-s)} ds \\
&\quad + \frac{\omega^2}{4} \int_0^T m_t \left( \sum_{n=1}^N \int_0^t \alpha_n(s) e^{-\kappa_n(t-s)} ds \right)^2 dt, \\
\mu_2(T) &= \mu_1(T) - \frac{\omega}{2} \sum_{n=1}^N \rho_n^S \int_0^T m_t \int_0^t \sigma_n(s) \sqrt{m_s} e^{-\kappa_n(t-s)} ds + \\
&\quad \frac{\omega^2}{4} \sum_{n=1}^N \rho_n^S \int_0^T m_t \int_0^t \alpha_n(s) \int_s^t \sigma_n(r) \sqrt{m_r} dr ds + \\
&\quad \frac{\omega^2}{4} \sum_{n=1}^N \int_0^T m_t \int_0^t m_s \int_0^s \sigma_n^2(r, x) e^{-\kappa_n(t+s-2r)} dr ds \\
&\quad + \frac{\omega^2}{2} \sum_{n,m=1}^N \int_0^T m_t \int_0^t \rho_n^S \sigma_n(s) \sqrt{m_s} \int_0^t \alpha_m(s) e^{-\kappa_m(t-s)} ds dt \\
\mu_3(T) &= -\frac{\omega}{2} \sum_{n=1}^N \rho_n^S \int_0^T m_t \int_0^t \sigma_n(s) \sqrt{m_s} e^{-\kappa_n(t-s)} ds + \\
&\quad \frac{\omega^2}{4} \sum_{n=1}^N (\rho_n^S)^2 \int_0^T m_t \int_0^t \sigma_n(s) \sqrt{m_s} e^{-\kappa_n(t-s)} \int_s^t \sigma_n(r) \sqrt{m_r} dr ds \\
&\quad + \frac{\omega^2}{2} \sum_{n=1}^N \int_0^T m_t \int_0^t m_s \int_0^s \sigma_n^2(s) e^{-\kappa_n(t+s-2u)} du ds dt \\
&\quad + \frac{\omega^2}{4} \int_0^T m_t \left( \sum_{n=1}^N \int_0^t \rho_n^S \sigma_n(s) \sqrt{m_s} e^{-\kappa_n(t-s)} ds \right)^2 dt \\
&\quad + \frac{\omega^2}{2} \sum_{n,m=1}^N \int_0^T m_t \int_0^t \rho_n^S \sigma_n(s) \sqrt{m_s} \int_0^t \alpha_m(s) e^{-\kappa_m(t-s)} ds dt, \\
\mu_4(T) &= \frac{\omega^2}{4} \sum_{n=1}^N \int_0^T m_t \int_0^t m_s \int_0^s \sigma_n^2(u) e^{-\kappa_n(t+s-2u)} du ds dt \\
&\quad + \frac{\omega^2}{4} \sum_{n=1}^N (\rho_n^S)^2 \int_0^T m_t \int_0^t \sigma_n(s) \sqrt{m_s} e^{-\kappa_n(t-s)} \int_s^t \sigma_n(u) \sqrt{m_u} du ds \\
&\quad + \frac{\omega^2}{4} \int_0^T m_t \left( \sum_{n=1}^N \int_0^t \rho_n^S \sigma_n(s) \sqrt{m_s} e^{-\kappa_n(t-s)} ds \right)^2 dt. \tag{C.3}
\end{aligned}$$

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