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Convergence of a misanthrope process to the entropy solution of 1D problems

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Abstract

We prove the convergence, in some strong sense, of a Markov process called "a misanthrope process" to the entropy weak solution of a one-dimensional scalar nonlinear hyperbolic equation. Such a process may be used for the simulation of traffic flows. The convergence proof relies on the uniqueness of entropy Young measure solutions to the nonlinear hyperbolic equation, which holds for both the bounded and the unbounded cases. In the unbounded case, we also prove an error estimate. Finally, numerical results show how this convergence result may be understood in practical cases.

Keywords: misanthrope stochastic process, non linear scalar hyperbolic equation, entropy Young measure solution, traffic flow simulation, weak BV inequality

1 Introduction

Some recent models of traffic flows [8] are based on the notion of misanthrope process, introduced in [6, 1]. Such a process is a stochastic Markov jump process $(\eta_t)_{t\geq 0}$, usually defined on $\mathbb{N}^{\mathbb{Z}}$, which models the time evolution of occupation of discrete sites by a collection of identical objects. The random variable $\eta_t(n)$ represents the number of objects at site $n \in \mathbb{Z}$ at time t. Jumps of this process consist of jumps of an object from one site $n \in \mathbb{Z}$ to the next one $n+1$ with a rate denoted $b(n, n+1) \geq 0$. The aim of this paper is to prove that the values of a misanthrope process at time t may approximate the entropy weak solution $u(x,t)$ of a non linear hyperbolic equation, first on a bounded interval (A, B) of $\mathbb{R}$ (the bounded case), then on $\mathbb{R}$ (the unbounded case).
In the bounded case, the non linear equation is

$$\partial_t u(x,t) + \partial_x(f(u))(x,t) \quad = 0 \quad x \in (A,B), \;\; t \in \mathbb{R}^+, \tag{1}$$

for given reals $A < B$, where the partial derivatives of u with respect to time and space are respectively denoted by $\partial_t u$ and $\partial_x u$, with initial data

$$u(x,0) \quad = u^{\mathrm{ini}}(x), \quad x \in (A,B), \tag{2}$$

and some boundary conditions which are formally:

$$u(A,t) = \overline{u}(t), \; t \in \mathbb{R}^+, \tag{3}$$

and

$$u(B,t) = \overline{\overline{u}}(t), \; t \in \mathbb{R}^+. \tag{4}$$

In (1)-(4), we denote by u^{ini} a bounded measurable function defined for all $x \in (A,B)$, and by $\overline{u}, \overline{\overline{u}}$ bounded measurable functions defined for all $t \in \mathbb{R}^+$. We assume, without restricting the generality, that these three functions are a.e. valued in $[0, \overline{U}]$, for a given $\overline{U} \in \mathbb{R}^+$. Note that the weak sense of the boundary conditions (3)-(4), deduced from Otto's works [20], is precised below (in this paper, the regularity on the limit is not sufficient for using the strong sense for these boundary conditions given in [4]).

In the unbounded case the non linear equation is

$$\partial_t u(x,t) + \partial_x (f(u))(x,t) \quad = 0 \quad x \in \mathbb{R}, \;\; t \in \mathbb{R}^+, \tag{5}$$

with initial data

$$u(x,0) \quad = u^{\mathrm{ini}}(x), \quad x \in \mathbb{R}, \tag{6}$$

where u^{ini} denotes a bounded measurable function defined for all $x \in \mathbb{R}$, a.e. valued in $[0, \overline{U}]$.

Such studies of limit of stochastic process to the solution of hyperbolic equations are called "Euler hydrodynamics of attractive particle systems" in [2], [15], [3], [12]. Indeed, as described in [3], a misanthrope process $(\eta_t)_{t\geq 0}$ on $E = [\![0, K]\!]^{\mathbb{Z}}$ is said to have Euler hydrodynamic limit $u(x,t)$ if, for all continuous function Ψ from $\mathbb{R}$ to $\mathbb{R}^+$ with compact support, we have, for all $t \geq 0$

$$\lim_{N\to\infty} \mathbb{E}\left(\frac{1}{N} \sum_{y\in\mathbb{Z}} \eta_{Nt}(y) \; \Psi\left(\frac{y}{N}\right)\right) = \int_{\mathbb{R}} \Psi(x) \; u(x,t) \; \mathrm{d}x. \tag{7}$$

We may remark that, defining the real process $\nu_N(x,t)$ by $\nu_N(x,t) = \eta_{Nt}(y)$, for all $x \in [y/N, (y+1)/N)$, $y \in \mathbb{Z}$ and $t \in \mathbb{R}^+$, (7) is equivalent to the convergence for the weak-$\star$ topology of $L^\infty(\mathbb{R})$, and for all $t \in \mathbb{R}^+$, of the function $\mathbb{E}(\nu_N(\cdot,t))$ to $u(\cdot,t)$ as $N \to \infty$.
It is then proved that the Euler hydrodynamic limit $u(x,t)$ of a misanthrope process is the entropy weak solution of (5) such that the function f must check the condition

$$f(\rho) = \int b(\eta(0), \eta(1)) \; \mathrm{d}\mu_\rho(\eta), \; \forall \rho \in \mathcal{R},$$

where $\mathcal{R}$ is the largest subset of $[0, K]$ such that, for any $\rho \in \mathcal{R}$, the set of all measures μ on E, which are invariant for the process and shift-invariant with $\int \eta(0) \mathrm{d}\mu(\eta) = \rho$, has an extremal element denoted by μ_ρ (recall that a measure μ is invariant for the process if the process with initial distribution μ has the distribution μ for all $t > 0$); note that $\mathcal{R}$ is closed and necessarily contains 0 and K, but, as mentioned in [3], the relation $[0, K] = \mathcal{R}$ remains an open problem.
In this paper, although our starting point is also the notion of misanthrope process, we use a different way to approximate equations (1) and (5), with different definitions for both the limit sense and the function f. Let us first emphasize that we will handle the bounded case (A, B) as well as the unbounded one (in the first case, the study of the boundary terms has to be done). In the bounded case, we define, for $N \in \mathbb{N}^\star$, $h = (B - A)/N$ and we discretize the interval $[A, B]$ in constant size intervals $(A + (n-1)h, A + nh)$, $n \in \{1, \ldots, N\}$, each interval corresponding to a site for the stochastic process. In the unbounded case we discretize $\mathbb{R}$ in intervals $((n-1)h, nh)$, $n \in \mathbb{Z}$ of size h, each interval corresponding to a site for the stochastic process. In both cases, we discretize the continuous interval of possible values $[0, \bar{U}]$ of $u(x,t)$ in discrete values $(ki, \; i = 0, ..., K)$ with $K \in \mathbb{N}^\star$ and $k \in \mathbb{R}$ such that $k = \frac{\overline{U}}{K}$.
For a given function g such that

- g is Lipschitz continuous from $\mathbb{R}^2$ to $\mathbb{R}^+$, hence it is assumed to be nonnegative (we denote in this paper M the smallest Lipschitz constant of g)
- $g(x,y) = 0$ for all $(x,y) \in (]-\infty, 0] \times \mathbb{R}) \cup (\mathbb{R} \times [\overline{U}, +\infty[)$,
- $(x,y) \mapsto g(x,y)$, from $[0, \overline{U}]^2$ to $\mathbb{R}^+$, is nondecreasing with respect to x and nonincreasing with respect to y,
- the function f occurring in the nonlinear hyperbolic equation (1) is such that $f(x) = g(x,x)$, for all $x \in \mathbb{R}$,

we define the jump rate of the process by

$$b(x,y) = \frac{1}{h\,k} \; g(x,y), \; \forall x, y \in \mathbb{R}. \tag{8}$$

Thanks to the hypotheses on the function g, we get $b(0,.) = 0$ and $b(.,kK) = 0$.

In the bounded case, we define a non-homogeneous Markov jump process on $E = (k[\![0,K]\!])^{[\![1,N]\!]}$ with the following possible jumps and the associated rates defined at time t by:

- jump from η to $T^{0,1}(\eta)$ with $T^{0,1}(\eta)(1) = \eta(1) + k$ and $T^{0,1}(\eta)(j) = \eta(j)$ for all $j = 2,\dots,N$, and rate $b(\overline{u}(t), \eta(1))$,
- jump from η to $T^{n,n+1}(\eta)$ for $n = 1,\dots,N-1$, with $T^{n,n+1}(\eta)(n) = \eta(n)-k$, $T^{n,n+1}(\eta)(n+1) = \eta(n+1)+k$ and $T^{n,n+1}(\eta)(j) = \eta(j)$ for all $j = 1,\dots,N$ different from n and $n+1$, and rate $b(\eta(n), \eta(n+1))$,
- jump from η to $T^{N,N+1}(\eta)$ with $T^{N,N+1}(\eta)(N) = \eta(N) - k$ and $T^{N,N+1}(\eta)(j) = \eta(j)$ for all $j = 1,\dots,N-1$, and rate $b(\eta(N), \overline{\overline{u}}(t))$.

Note that the process is a homogeneous Markov jump process for almost everywhere constant boundary conditions. Using the following notation

$$\eta_t(0) = \overline{u}(t) \text{ and } \eta_t(N+1) = \overline{\overline{u}}(t),\ \forall t \in \mathbb{R}^+, \tag{9}$$

the corresponding Chapman-Kolmogorov equation reads:

$$\frac{\mathrm{d}\mathbb{E}\psi(\eta_t)}{\mathrm{d}t} = \mathbb{E}\left(\sum_{n=0}^{N} b(\eta_t(n), \eta_t(n+1))\left(\psi(T^{n,n+1}(\eta_t)) - \psi(\eta_t)\right)\right),\ \forall t \in \mathbb{R}^+,\ \forall \psi \in D, \tag{10}$$

where D is the set of all functions ψ from E to $\mathbb{R}^+$.
We associate to the process η_t the real process $\nu(x,t)$ defined from $(A,B) \times \mathbb{R}^+$ to $[0,\bar{U}]$ by

$$\nu(x,t) = \eta_t(n)\ \ \forall x \in [A + (n-1)h, A + nh),\ \forall n \in [\![1,N]\!],\ \forall t \in \mathbb{R}^+. \tag{11}$$

The final result for the bounded case (Theorem 2.2) is the convergence of

$$\int_0^T \int_A^B \mathbb{E}\left(|\nu(x,t) - u(x,t)|\right)\ \mathrm{d}x\mathrm{d}t = \int_0^T \sum_{1\le n\le N} \int_{[A+(n-1)h, A+nh)} \mathbb{E}\left(|\eta_t(n) - u(x,t)|\right)\ \mathrm{d}x\mathrm{d}t$$

to zero as h and k/h simultaneously tend to zero, where $u(x,t)$ is the entropy weak solution of equation (1), assuming that $\nu(x,0)$ converges in some sense to $u^{\mathrm{ini}}(x)$ as h and k/h simultaneously tend to zero. Note that this limit sense is stronger than that provided by (7), since it implies that $\mathbb{E}(\nu(x,t))$ converges in $L^1((A,B)\times(0,T))$ to u (although it would be possible, we do not consider here the framework $C^0\big(0,T;L^1((A,B))\big)$ for the sake of simplicity). The proof of these results are closely related to the methods involved in the convergence proofs for numerical schemes [5, 9, 19] (further works for providing an error estimate might also be done).

In the unbounded case, we define an homogeneous Markov jump process on $E = (k[\![0,K]\!])^{\mathbb{Z}}$ with the possible jumps from η to $T^{n,n+1}(\eta)$ for $n \in \mathbb{Z}$, with $T^{n,n+1}(\eta)(n) = \eta(n) - k$, $T^{n,n+1}(\eta)(n+1) = \eta(n+1) + k$ and $T^{n,n+1}(\eta)(j) = \eta(j)$ for all j different from n and $n+1$, and rate $b(\eta(n), \eta(n+1))$.
The corresponding Chapman-Kolmogorov equation reads:

$$\frac{\mathrm{d}\mathbb{E}\psi(\eta_t)}{\mathrm{d}t} = \mathbb{E}\left(\sum_{n\in\mathbb{Z}} b(\eta_t(n), \eta_t(n+1))\left(\psi(T^{n,n+1}(\eta_t)) - \psi(\eta_t)\right)\right),\ \forall t \in \mathbb{R}^+,\ \forall \psi \in D, \tag{12}$$

where D is the set of all bounded functions ψ from E to $\mathbb{R}^+$ depending on a finite number of coordinates.
We associate to the process η_t the real process $\nu(x,t)$ defined from $\mathbb{R} \times \mathbb{R}^+$ to $[0,\bar{U}]$ by

$$\nu(x,t) = \eta_t(n)\ \ \forall x \in [(n-1)h, nh),\ \forall n \in \mathbb{Z},\ \forall t \in \mathbb{R}^+. \tag{13}$$

In the unbounded case, we prove a similar convergence result to the bounded case, and moreover prove in Theorem 3.4 that, for given R and T, for a suitable initial value $\nu(x,0)$ close from $u^{\text{ini}}(x)$ (assumed to have bounded variations) and for $k = h^2$, if $u(x,t)$ is the entropy weak solution of equation (5), we have

$$\int_0^T \int_{-R}^R \mathbb{E}\left(|\nu(x,t) - u(x,t)|\right) \, \mathrm{d}x\mathrm{d}t \leq C_e h^{\frac{1}{4}}, \tag{14}$$

where C_e only depends on R, T, g and u^{ini}. Inequality (14) can be equivalently written

$$\int_0^T \sum_{\{n\in\mathbb{Z}/|nh|\leq R, |(n-1)h|\leq R\}} \int_{[(n-1)h,nh)} \mathbb{E}\left(|\eta_t(n) - u(x,t)|\right) \, \mathrm{d}x\mathrm{d}t \leq C_e h^{\frac{1}{4}} \tag{15}$$

which brings into play values $\eta_t(n)$ of the process at a finite number of sites n.

So we obtain an approximation of the entropy solution of (1) and (5) with the help of a misanthrope process in a different way than Euler hydrodynamic. The sense of the limit is different and the relation between the function f in the hyperbolic equation and the misanthrope process is explicitly given by $f(x) = g(x,x)$. Note that different functions $g(.,.)$ could match a given function $f(.)$ with respecting the above hypotheses (see also [13], [17] and references therein):

- the Godunov numerical flux: introduced in [14], it may be summarized by the following expression.

$$g(a,b) = \begin{cases} \min\{f(\xi), \xi \in [a,b]\} \text{ if } a \leq b, \\ \max\{f(\xi), \xi \in [b,a]\} \text{ if } b \leq a. \end{cases} \tag{16}$$

- the splitting numerical flux: assume $f = f_1 + f_2$, with f_1, $f_2 \in C^1(\mathbb{R},\mathbb{R})$, $f_1'(s) \geq 0$ and $f_2'(s) \leq 0$ for all $s \in [0, \overline{U}]$ (such a decomposition for f is always possible, see the modified Lax-Friedrichs numerical flux below), and take

$$g(a,b) = f_1(a) + f_2(b).$$

  Note that if $f' \geq 0$, taking $f_1 = f$ and $f_2 = 0$, the splitting numerical flux boils down to the upwind numerical flux, i.e. $g(a,b) = f(a)$.

- the modified Lax-Friedrichs numerical flux, also called the Rusanov numerical flux in different frameworks: take

$$g(a,b) = \frac{f(a) + f(b)}{2} + D(a-b), \tag{17}$$

  with $D \in \mathbb{R}$ such that $2D \geq \max\{|f'(s)|, s \in [U_m, \overline{U}]\}$.

We remark that in the last two examples the functions defined above can be negative. We could easily extend the definition of the misanthrope process by assuming that, if $b(n,n+1) < 0$, then $-b(n,n+1)$ denotes the rate of jump of such an object from the site $n+1$ to the site n. Then all the results of this paper would hold as well. But, for the sake of simplicity, we only consider here positive jump rates.

This paper is organized as follows: the second section is devoted to the bounded case, while the third section deals with the unbounded case; numerical computations on the Riemann problem are presented in section 4. Some tracks for further research are finally shown in a short conclusion.

2 The bounded case

In this section, our aim is to prove that the misanthrope process, defined by the Chapman-Kolmogorov equation (10) and a given initial probability measure on D, converges to an entropy weak solution in some sense. Let us recall this notion in the bounded case as given in [20].
Let us denote $a\top b$ the maximum of a and b, $a\perp b$ the minimum of a and b, for all real values a, b, and $C^1_c([A,B]\times\mathbb{R}^+,\mathbb{R}^+)$ the set of the restriction to $[A,B]\times\mathbb{R}^+$ of the non-negative C^1 functions with compact support from $\mathbb{R}^2$ to $\mathbb{R}^+$.

Definition 2.1 (Entropy weak solution) *Let $f\in C^1(\mathbb{R},\mathbb{R})$ (or $f:\mathbb{R}\to\mathbb{R}$ Lipschitz continuous) be given, let $u^{\rm ini}\in L^\infty((A,B))$, and $\overline{u},\overline{\overline{u}}\in L^\infty(\mathbb{R}^+)$ be given functions. We say that u is a weak entropy solution of problem* (1)-(4) *if:*

- $u\in L^\infty((A,B)\times(0,\infty))$,
- *there exists $M>0$ such that, for all functions $\zeta^\top(s)=s\top\kappa-\kappa$ and $F^\top(s)=f(s\top\kappa)-f(\kappa)$ with $\kappa\in\mathbb{R}$, we have for all $\varphi\in C^1_c([A,B]\times\mathbb{R}^+,\mathbb{R}^+)$*

$$\begin{aligned}&\int_0^\infty\int_A^B(\zeta^\top(u)\partial_t\varphi(x,t)+F^\top(u)\,\partial_x\varphi(x,t))\,{\rm d}x{\rm d}t\\&+M\int_0^\infty(\zeta^\top(\overline{u}(t))\varphi(A,t)+\zeta^\top(\overline{\overline{u}}(t))\varphi(B,t))\,{\rm d}t+\int_A^B\zeta^\top(u^{\rm ini})\varphi(x,0)\,{\rm d}x\geq 0,\end{aligned}\tag{18}$$

- *the same inequality as* (18) *holds replacing $\zeta^\top(s)$ by $\zeta^\perp(s)=\kappa-s\perp\kappa$ and $F^\top(s)$ by $F^\perp(s)=f(\kappa)-f(s\perp\kappa)$.*

We have the following fundamental theorem [20].

Theorem 2.1 *Let $f\in C^0(\mathbb{R})$ be Lipschitz continuous , $u^{\rm ini}\in L^\infty((A,B))$, $\overline{u},\overline{\overline{u}}\in L^\infty(\mathbb{R}^+)$, then there exists a unique entropy weak solution in the sense of Definition* (2.1) *to Problem (1).*

Remark 2.1 *The solution of* (18) *does not depends on the choice of M (the value M is chosen in this paper as the Lipschitz constant of g). A uniqueness result on a larger class of objects (Young measures instead of measurable functions) is used in this paper for the proof of convergence (this result is proved in [22] for the purpose of the convergence study of a numerical scheme [10]). It is interesting to remark that if one replaces in* (18)*, the set of function $\zeta^\top$ or $\zeta^\perp$ by the set of all entropies $|u-\kappa|$ (as done in the unbounded case, see Definition 3.1), one has an existence result (since $|u-\kappa|=s\top\kappa-\kappa+\kappa-s\perp\kappa$) but no uniqueness result, see [22] for a counter-example to uniqueness.*

2.1 Estimates

The first step is to obtain a discrete entropy inequality for the misanthrope process.

Lemma 2.1 (Discrete entropy inequalities) *Let $\zeta\in C^2(\mathbb{R})$ be a convex function (i.e. $\zeta''(\kappa)\geq 0$ for all $\kappa\in\mathbb{R}$). Let η_t be the misanthrope process defined by the Chapman-Kolmogorov equation* (10) *and a given initial probability measure on D. Then the following inequalities hold:*

$$\begin{aligned}&h\frac{{\rm d}\mathbb{E}\left(\zeta(\eta_t(n))\right)}{{\rm d}t}+\mathbb{E}\left(G_\zeta(\eta_t(n),\eta_t(n+1))-G_\zeta(\eta_t(n-1),\eta_t(n))\right)\leq k\,g(\overline{U},0)\max_{s\in[0,\overline{U}]}\zeta''(s),\\&\forall n\in\{1,\dots,N\},\end{aligned}\tag{19}$$

denoting $G_\zeta=G_\zeta^\top$ or $G_\zeta^\perp$, for given $X<0<\overline{U}<Y$, where

$$\begin{aligned}G_\zeta^\top(x,y)&=\textstyle\int_X^Y\zeta''(s)(g(x\top s,y\top s)-f(s))\,{\rm d}s+g(x,y)\zeta'(X),\quad\forall x,y\in[X,Y],\\G_\zeta^\perp(x,y)&=\textstyle\int_X^Y\zeta''(s)(f(s)-g(x\perp s,y\perp s))\,{\rm d}s+g(x,y)\zeta'(Y),\quad\forall x,y\in[X,Y].\end{aligned}\tag{20}$$

Proof of Lemma 2.1

Since we have $\zeta \in C^2(\mathbb{R})$, we get for all $\eta \in E$ and for a given $n \in \{1, \dots, N\}$, that there exists $s_n^-(\eta) \in (\eta(n) - k, \eta(n))$ such that

$$\frac{1}{k}\big(\zeta(\eta(n) - k) - \zeta(\eta(n))\big) = -\zeta'(\eta(n)) + \frac{k}{2}\zeta''(s_n^-(\eta)),$$

and that there exists $s_n^+(\eta) \in (\eta(n), \eta(n) + k)$ such that

$$\frac{1}{k}\big(\zeta(\eta(n) + k) - \zeta(\eta(n))\big) = \zeta'(\eta(n)) + \frac{k}{2}\zeta''(s_n^+(\eta)).$$

Hence, from Chapman-Kolmogorov equation (10), we obtain, choosing, for $n \in \{1, \dots, N\}$, $\psi \in D$ such that $\psi(\eta) = \zeta(\eta(n))$ for all $\eta \in E$,

$$\begin{aligned} &h\frac{\mathrm{d}\mathbb{E}\zeta(\eta_t(n))}{\mathrm{d}t} + \mathbb{E}\left(\zeta'(\eta_t(n))(g(\eta_t(n), \eta_t(n+1)) - g(\eta_t(n-1), \eta_t(n)))\right) \\ &= \frac{k}{2}\mathbb{E}\big(\zeta''(s_n^-(\eta_t))g(\eta_t(n), \eta_t(n+1)) + \zeta''(s_n^+(\eta_t))g(\eta_t(n-1), \eta_t(n))\big),\ \forall n \in \{1, \dots, N\}. \end{aligned}$$

Thanks to the monotonicity properties of g, we have the properties, for $x, y, z, s \in \mathbb{R}$ (defining $\mathrm{sign}^+(s) = 1$ if $s > 0$, 0 otherwise):

$$\mathrm{sign}^+(y - s)(g(y, z) - g(s, s)) \geq g(y\top s, z\top s) - g(s, s),$$

and

$$\mathrm{sign}^+(y - s)(g(s, s) - g(x, y)) \geq g(s, s) - g(x\top s, y\top s),$$

and therefore

$$\mathrm{sign}^+(y - s)(g(y, z) - g(x, y)) \geq g(y\top s, z\top s) - g(s, s) + g(s, s) - g(x\top s, y\top s),$$

easily checked by considering all cases. Since we may write,

$$\zeta'(y) = \int_X^Y \zeta''(s)\mathrm{sign}^+(y - s)\,\mathrm{d}s + \zeta'(X),$$

we get, using (20) and $\zeta''(s) \geq 0$ for all $s \in \mathbb{R}$,

$$\zeta'(y)(g(y, z) - g(x, y)) \geq G_\zeta^\top(y, z) - G_\zeta^\top(x, y),$$

thanks to Definition (20) of $G_\zeta^\top$, which concludes the proof of (19) in this case. Turning to the case $G_\zeta = G_\zeta^\perp$, we write, for $x, y, z, s \in \mathbb{R}$ (defining $\mathrm{sign}^-(s) = -1$ if $s < 0$, 0 otherwise):

$$\mathrm{sign}^-(y - s)(g(y, z) - g(s, s)) \geq g(s, s) - g(y\perp s, z\perp s),$$

and

$$\mathrm{sign}^-(y - s)(g(s, s) - g(x, y)) \geq g(x\perp s, y\perp s) - g(s, s),$$

and therefore

$$\mathrm{sign}^-(y - s)(g(y, z) - g(x, y)) \geq g(s, s) - g(y\perp s, z\perp s) + g(x\perp s, y\perp s) - g(s, s),$$

again verified by considering all cases. Since we may write

$$\zeta'(y) = \int_X^Y \zeta''(s)\mathrm{sign}^-(y - s)\,\mathrm{d}s + \zeta'(Y),$$

we get, using (20) and $\zeta''(s) \geq 0$ for all $s \in \mathbb{R}$,

$$\zeta'(y)(g(y,z) - g(x,y)) \geq G_\zeta^\perp(y,z) - G_\zeta^\perp(x,y),$$

thanks to Definition (20) of $G_\zeta^\perp$, which concludes the proof of (19) in this case. ■

Let us write the following lemma, which provides an inequality used in the convergence proof playing the same role as the so-called "weak BV inequalities" in the case of deterministic numerical schemes (see [5, 9, 11] for the use of such inequalities).

Lemma 2.2 *Let $T > 0$. Let η_t be the misanthrope process defined by the Chapman-Kolmogorov equation* (10) *and a given initial probability measure on D. Then there exists $C \in \mathbb{R}$, only depending on T, $\overline{U}$, g such that, assuming $\frac{k}{h} \leq 1$,*

$$\int_0^T \sum_{n=0}^N \mathbb{E}\left(H(\eta_t(n), \eta_t(n+1))\right) \mathrm{d}t \leq \frac{C}{\sqrt{h}}, \tag{21}$$

where H is defined by

$$H(a,b) = \max_{(c,d)\in\mathcal{C}(a,b)} |g(c,d) - f(c)| + \max_{(c,d)\in\mathcal{C}(a,b)} |g(c,d) - f(d)|, \ \forall a,b \in \mathbb{R}, \tag{22}$$

denoting, for all $a, b \in \mathbb{R}$, by $\mathcal{C}(a,b) = \{(c,d) \in [a\perp b, a\top b]^2;\ (d-c)(b-a) \geq 0\}$.

Proof of Lemma 2.2

In this proof, we shall denote by C_i ($i \in \mathbb{N}$) various quantities only depending on $g, \overline{U}, T$.
Applying the Cauchy-Schwarz inequality to the left hand side of (21) yields

$$\left(\int_0^T \sum_{n=0}^N \mathbb{E}\left(H(\eta_t(n), \eta_t(n+1))\right)\ \mathrm{d}t\right)^2 \leq 2T(N+1)\int_0^T \sum_{n=0}^N \mathbb{E}\left(H_2(\eta_t(n), \eta_t(n+1))\right)\ \mathrm{d}t$$

with

$$H_2(a,b) = \max_{(c,d)\in\mathcal{C}(a,b)} (g(c,d) - f(c))^2 + \max_{(c,d)\in\mathcal{C}(a,b)} (g(c,d) - f(d))^2, \ \forall a,b \in \mathbb{R}.$$

Thanks to the monotonicity properties of g (and using the fact that $g(s,s) = f(s)$), the following inequality holds, for any $(c,d) \in \mathcal{C}(a,b)$:

$$\int_a^b (f(s) - g(a,b))\ \mathrm{d}s \geq \int_c^d (f(s) - g(a,b))\ \mathrm{d}s \geq \int_c^d (f(s) - g(c,d))\ \mathrm{d}s \geq 0. \tag{23}$$

Let us remark that the following property

$$|\int_c^d (\theta(s) - \theta(c))\ \mathrm{d}s| \geq \frac{1}{2G}(\theta(d) - \theta(c))^2, \ \ \forall c,d \in \mathbb{R}. \tag{24}$$

holds for all monotone, Lipschitz continuous function $\theta : \mathbb{R} \to \mathbb{R}$, with a Lipschitz constant $G > 0$. Indeed, let us assume, for instance, that θ is nondecreasing and $c < d$ (the other cases are similar). Then, one has $\theta(s) \geq \varphi(s)$, for all $s \in [c,d]$, where $\varphi(s) = \theta(c)$ for $s \in [c, d-l]$ and $\varphi(s) = \theta(c) + (s-d+l)G$ for $s \in [d-l, d]$, with $lG = \theta(d) - \theta(c)$, and therefore:

$$\int_c^d (\theta(s) - \theta(c))\ \mathrm{d}s \geq \int_c^d (\varphi(s) - \theta(c))\ \mathrm{d}s = \frac{l}{2}(\theta(d) - \theta(c)) = \frac{1}{2G}(\theta(d) - \theta(c))^2.$$

Applying (24), we can notice that

$$\int_c^d (f(s) - g(c,d))\ \mathrm{d}s \geq \int_c^d (g(c,s) - g(c,d))\mathrm{d}s \geq \frac{1}{2M}(f(c) - g(c,d))^2, \tag{25}$$

and

$$\int_c^d (f(s) - g(c,d))\,\mathrm{d}s \geq \int_c^d (g(s,d) - g(c,d))\mathrm{d}s \geq \frac{1}{2M}(f(d) - g(c,d))^2. \tag{26}$$

Multiplying (25) and (26) by $1/2$, taking the maximum for $(c,d) \in \mathcal{C}(a,b)$, and adding the two equations yields, with (23),

$$\int_a^b (f(s) - g(a,b))\,\mathrm{d}s \geq \frac{1}{4M} H_2(a,b) \tag{27}$$

So we have to find a bound for

$$T_1 = \int_0^T \sum_{n=0}^N \mathbb{E}\left(\int_{\eta_t(n)}^{\eta_t(n+1)} (f(s) - g(\eta_t(n), \eta_t(n+1)))\,\mathrm{d}s\right)\,\mathrm{d}t$$

If Φ is a primitive of the function $(\cdot)f'(\cdot)$, an integration by parts yields, for all $(a,b) \in \mathbb{R}^2$,

$$\begin{aligned}
\Phi(\eta_t(n+1)) - \Phi(\eta_t(n)) &= \int_{\eta_t(n)}^{\eta_t(n+1)} s f'(s)\,\mathrm{d}s \\
&= \eta_t(n+1)(f(\eta_t(n+1)) - g(\eta_t(n), \eta_t(n+1)) - \eta_t(n)(f(\eta_t(n)) - g(\eta_t(n), \eta_t(n+1)) \\
&\quad - \int_{\eta_t(n)}^{\eta_t(n+1)} (f(s) - g(\eta_t(n), \eta_t(n+1))\,\mathrm{d}s
\end{aligned} \tag{28}$$

and we can write $T_1 = T_3 + T_2$ with

$$T_3 = \int_0^T \sum_{n=0}^N \mathbb{E}\left(\begin{array}{l} \eta_t(n)(g(\eta_t(n), \eta_t(n+1)) - f(\eta_t(n))) \\ +\eta_t(n+1)(f(\eta_t(n+1)) - g(\eta_t(n), \eta_t(n+1))) \end{array}\right)\,\mathrm{d}t,$$

and

$$T_2 = \int_0^T \sum_{n=0}^N \mathbb{E}\Big(\Phi(\eta_t(n)) - \Phi(\eta_t(n+1))\Big)\,\mathrm{d}t = \int_0^T \mathbb{E}\Big(\Phi(\overline{u}(t)) - \Phi(\overline{\overline{u}}(t))\Big)\,\mathrm{d}t.$$

It is clear that $T_2 \leq C_2$.
Using Chapman-Kolmogorov formula (10) with $\psi(\eta) = \eta(n)^2$ for $n \in \{1, \ldots, N\}$, we have

$$\begin{aligned}
&h\sum_{n=1}^N \mathbb{E}\left(\eta_t(n)^2\right) - h\sum_{n=1}^N \mathbb{E}\left(\eta_0(n)^2\right) \\
&= \int_0^T \sum_{n=1}^N \mathbb{E}\left((k - 2\eta_t(n))g(\eta_t(n), \eta_t(n+1)) + (k + 2\eta_t(n))g(\eta_t(n-1), \eta_t(n))\right)\mathrm{d}t.
\end{aligned}$$

Then we get

$$\begin{aligned}
&h\sum_{n=1}^N \mathbb{E}\left(\eta_t(n)^2\right) - h\sum_{n=1}^N \mathbb{E}\left(\eta_0(n)^2\right) \\
&= 2\int_0^T k\sum_{n=0}^N \mathbb{E}\left(g(\eta_t(n), \eta_t(n+1))\right)\,\mathrm{d}t \\
&- \int_0^T \mathbb{E}\left((2\eta_t(N+1) + k)g(\eta_t(N), \eta_t(N+1)) + (k - 2\eta_t(0))g(\eta_t(0), \eta_t(1))\right)\mathrm{d}t \\
&+2\int_0^T (\eta_t(N+1)f(\eta_t(N+1)) - \eta_t(0)f(\eta_t(0)))\,\mathrm{d}t - 2T_3.
\end{aligned}$$

(recall that we denote by $\eta_t(0) = \overline{u}(t)$ and $\eta_t(N+1) = \overline{\overline{u}}(t)$). We have

$$\int_0^T k \sum_{n=0}^{N} \mathbb{E}\left(g(\eta_t(n), \eta_t(n+1))\right) \, \mathrm{d}t \leq C_5 k(N+1) \leq \frac{C_4 k}{h}.$$

This gives, using simple bounds for $\eta(n)$, $\overline{u}$, $\overline{\overline{u}}$ that

$$T_3 \leq C_1 + \tfrac{C_4 k}{h}.$$

We can then deduce :

$$\frac{1}{4M} \int_0^T \sum_{n=0}^{N} \mathbb{E}\left(H_2(\eta_t(n), \eta_t(n+1))\right) \, \mathrm{d}t \leq C_1 + \frac{C_4 k}{h} + C_2 \leq C_3. \tag{29}$$

This completes the proof of Lemma 2.2.

■

Let us recall that $\nu(x,t)$ is a function from $\mathbb{R} \times \mathbb{R}^+$ to $[0, \bar{U}]$ associated to η_t by $\nu(x,t) = \eta_t(n)$ if $x \in [nh, (n+1)h)$. The following lemma proves an entropy inequality associated to $\nu(x,t)$.

Lemma 2.3 *Let η_t be the misanthrope process defined by the Chapman-Kolmogorov equation* (10) *and a given initial probability measure on D. Let ν be the real process defined by* (11)*. Let $\kappa \in [0, \overline{U}]$ be given and $\zeta^\top(s) = s\top\kappa - \kappa$ and $F^\top(s) = f(s\top\kappa) - f(\kappa)$. Then the following inequality holds:*

$$\begin{aligned}
&\int_0^{+\infty} \int_A^B \mathbb{E}\left(\zeta^\top(\nu(x,t))\right) \, \partial_t \varphi(x,t) \, \mathrm{d}x\mathrm{d}t + \int_A^B \zeta^\top(u^{\mathrm{ini}}(x)) \varphi(x,0) \, \mathrm{d}x \\
&+ M \int_0^{+\infty} \zeta^\top(\overline{u}(t)) \varphi(A,t) \, \mathrm{d}t + M \int_0^{+\infty} \zeta^\top(\overline{\overline{u}}(t)) \varphi(B,t) \, \mathrm{d}t \\
&+ \int_0^{+\infty} \int_A^B \mathbb{E}(F^\top(\nu(x,t))) \, \partial_x \varphi(x,t) \, \mathrm{d}x\mathrm{d}t \geq - \int_0^{+\infty} \varphi(A,t) \mathrm{d}\overline{\mu}(t) - \int_0^{+\infty} \varphi(B,t) \mathrm{d}\overline{\overline{\mu}}(t) \\
&- \int_A^B \varphi(x,0) \mathrm{d}\mu^{\mathrm{ini}}(x) - \int_{(A,B)\times\mathbb{R}^+} \varphi(x,t) \mathrm{d}\mu_0(x,t) - \int_{(A,B)\times\mathbb{R}^+} (|\partial_t \varphi(x,t)| + |\partial_x \varphi(x,t)|) \mathrm{d}\mu_1(x,t),
\end{aligned} \tag{30}$$

for all $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R}^+)$ (recall that M is a Lipschitz constant for g).
The same inequality holds replacing $\zeta^\top(s)$ by $\zeta^\perp(s) = \kappa - s\perp\kappa$ and $F^\top(s)$ by $F^\perp(s) = f(\kappa) - f(s\perp\kappa)$.
In (30)*, the measures μ_0, μ_1, $\overline{\mu}$, $\overline{\overline{\mu}}$ and μ^{ini} verify the following properties:*

1. *For all $T > 0$, there exists C_0 depending only on T, g and $\overline{U}$ such that*

$$\mu_0([A,B] \times [0,T]) + \overline{\mu}([0,T]) + \overline{\overline{\mu}}([0,T]) \leq C_0(k + \sqrt{\frac{k}{h}}). \tag{31}$$

2. *For all $T > 0$, there exists C_1 depending only on g, u^{ini}, $\overline{U}$ and T such that, for $h < R$,*

$$\mu_1([A,B] \times [0,T]) \leq C_1(\sqrt{h} + \sqrt{\frac{k}{h}}). \tag{32}$$

3. *The measure μ^{ini} is the measure of density $\mathbb{E}|u^{\mathrm{ini}}(.) - \nu(.,0)| + C^{\mathrm{ini}}\sqrt{\frac{k}{h}}$ with respect to the Lebesgue measure, where C^{ini} only depends on T, g and $\overline{U}$.*

Proof of Lemma 2.3

We remark that we cannot directly apply Lemma 2.1 to $\zeta = \zeta^\top$ since Lemma 2.1 involves $C^2(\mathbb{R})$ convex functions. Therefore, we approximate the function $\zeta^\top$, for a given $\kappa \in [0, \overline{U}]$, by a regular function $\zeta_\varepsilon^\top$, defined, for a value $\varepsilon \in (0,1)$ which will be chosen later, and for a mollifier ρ (defined as a nonnegative element of $C_c^\infty(\mathbb{R})$ with support included in $[-1,1]$ and integral equal to 1), by

$$\zeta_\varepsilon^\top(x) = \int_{\mathbb{R}} (y\top\kappa - \kappa)\frac{1}{\varepsilon}\rho(\frac{x-y}{\varepsilon})\,\mathrm{d}y.$$

We then get that, for given $X < -1 < \overline{U} + 1 < Y$, we have $(\zeta_\varepsilon^\top)'(X) = 0$ and $(\zeta_\varepsilon^\top)'(Y) = 1$, and

$$|\zeta^\top(x) - \zeta_\varepsilon^\top(x)| \leq \int_{\mathbb{R}} |y-x|\frac{1}{\varepsilon}\rho(\frac{x-y}{\varepsilon})\,\mathrm{d}y \leq \varepsilon,\ \forall x \in \mathbb{R}. \tag{33}$$

Let us denote $G_\varepsilon^\top$ the function $G_{\zeta_\varepsilon^\top}^\top$ as defined by (20). We have, for $x \in [X,Y]$, $(\zeta_\varepsilon^\top)''(x) = \frac{1}{\varepsilon}\rho(\frac{x-\kappa}{\varepsilon})$, which leads to

$$G_\varepsilon^\top(x,y) = \int_X^Y \frac{1}{\varepsilon}\rho(\frac{s-\kappa}{\varepsilon})(g(x\top s, y\top s) - f(s))\,\mathrm{d}s,\ \forall x,y \in [X,Y].$$

We define $G^\top(x,y)$ by

$$G^\top(x,y) = g(x\top\kappa, y\top\kappa) - f(\kappa),\ \forall x,y \in [X,Y],$$

and we get, since M is a Lipschitz constant for g,

$$|G^\top(x,y) - G_\varepsilon^\top(x,y)| \leq 2M\int_X^Y |s-\kappa|\frac{1}{\varepsilon}\rho(\frac{s-\kappa}{\varepsilon})\,\mathrm{d}s \leq 2M\varepsilon,\ \forall x,y \in [X,Y]. \tag{34}$$

We have, for $x, x' \in [X,Y]$, that

$$\zeta_\varepsilon^\top(x) - \zeta_\varepsilon^\top(x') = \int_{\mathbb{R}} (y\top\kappa - \kappa)\frac{1}{\varepsilon}\rho(\frac{x-y}{\varepsilon})\,\mathrm{d}y - \int_{\mathbb{R}} (y\top\kappa - \kappa)\frac{1}{\varepsilon}\rho(\frac{x'-y}{\varepsilon})\,\mathrm{d}y,$$

Hence, changing y in the second integral in $y - x + x'$,

$$\begin{aligned} |\zeta_\varepsilon^\top(x) - \zeta_\varepsilon^\top(x')| &\leq \int_{\mathbb{R}} \big|(y\top\kappa - \kappa) - ((y-x+x')\top\kappa - \kappa)\big|\frac{1}{\varepsilon}\rho(\frac{x-y}{\varepsilon})\,\mathrm{d}y \\ &\leq \int_{\mathbb{R}} |x'-x|\frac{1}{\varepsilon}\rho(\frac{x-y}{\varepsilon})\,\mathrm{d}y = |x'-x|. \end{aligned} \tag{35}$$

It leads to

$$\int_X^Y \frac{1}{\varepsilon}\rho(\frac{y-\kappa}{\varepsilon})(f(x\top y) - f(x\top\kappa))\,\mathrm{d}y \leq \int_X^Y \frac{1}{\varepsilon}\rho(\frac{y-\kappa}{\varepsilon})M|y-\kappa|\mathrm{d}y \leq M\varepsilon, \tag{36}$$

(recall that M is also a Lipschitz constant for f).

Let $\varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}, \mathbb{R}^+)$ be given. Applying Lemma 2.1 to $\zeta = \zeta_\varepsilon^\top$, we multiply Inequality (19) by $\frac{1}{h}\int_{(n-1)h}^{nh} \varphi(A+x,t)\,\mathrm{d}x$, sum over $n = 1,\ldots,N$ and integrate the resulting equation with respect to t. This gives

$$T_1 + T_2 \leq T_3, \tag{37}$$

with

$$T_1 = \int_0^{+\infty} \sum_{n=1}^N h\frac{\mathrm{d}\mathbb{E}\left(\zeta_\varepsilon^\top(\eta_t(n))\right)}{\mathrm{d}t}\frac{1}{h}\int_{(n-1)h}^{nh} \varphi(A+x,t)\,\mathrm{d}x\mathrm{d}t,$$

$$T_2 = \int_0^{+\infty} \sum_{n=1}^N \mathbb{E}\left(G_\varepsilon^\top(\eta_t(n), \eta_t(n+1)) - G_\varepsilon^\top(\eta_t(n-1), \eta_t(n))\right)\frac{1}{h}\int_{(n-1)h}^{nh} \varphi(A+x,t)\,\mathrm{d}x\mathrm{d}t,$$

and

$$T_3 = C\sum_{n=1}^{N}\int_0^{+\infty}\max_{s\in[X,Y]}(\zeta_\varepsilon^\top)''(s)kg(\overline{U},0)\frac{1}{h}\int_{(n-1)h}^{nh}\varphi(A+x,t)\,\mathrm{d}x\mathrm{d}t.$$

We may write

$$T_1 = \int_0^{+\infty}\int_A^B \frac{\mathrm{d}\mathbb{E}\left(\zeta_\varepsilon^\top(\nu(x,t))\right)}{\mathrm{d}t}\varphi(x,t)\,\mathrm{d}x\mathrm{d}t,$$

which provides, thanks to an integrate by parts with respect to time,

$$T_1 = -\int_0^{+\infty}\int_A^B \mathbb{E}\left(\zeta_\varepsilon^\top(\nu(x,t))\right)\,\partial_t\varphi(x,t)\,\mathrm{d}x\mathrm{d}t - \int_A^B \mathbb{E}\left(\zeta_\varepsilon^\top(\nu(x,0))\right)\varphi(x,0)\,\mathrm{d}x. \tag{38}$$

We have

$$T_1 = -\int_0^{+\infty}\int_A^B \mathbb{E}(\zeta^\top(\nu(x,t)))\partial_t\varphi(x,t)\,\mathrm{d}x\mathrm{d}t - \int_A^B \zeta^\top(u^{\mathrm{ini}}(x))\varphi(x,0)\,\mathrm{d}x + T_4 + T_5 + T_6,$$

with T_4, T_5 and T_6 such that

$$T_4 = \int_0^{+\infty}\int_A^B \mathbb{E}\left(\zeta^\top(\nu(x,t)) - \zeta_\varepsilon^\top(\nu(x,t))\right)\,\partial_t\varphi(x,t)\,\mathrm{d}x\mathrm{d}t.$$

$$T_5 = \int_A^B \mathbb{E}\left(\zeta_\varepsilon^\top(u^{\mathrm{ini}}(x)) - \zeta_\varepsilon^\top(\nu(x,0))\right)\varphi(x,0)\,\mathrm{d}x.$$

$$T_6 = \int_A^B (\zeta^\top(u^{\mathrm{ini}}(x)) - \zeta_\varepsilon^\top(u^{\mathrm{ini}}(x))\varphi(x,0)\,\mathrm{d}x.$$

We get from (33) that

$$|T_4| \le \varepsilon\int_0^{+\infty}\int_A^B |\partial_t\varphi(x,t)|\mathrm{d}x\mathrm{d}t,$$

and

$$|T_6| \le \varepsilon\int_A^B \varphi(x,0)\,\mathrm{d}x,$$

and, from (35), we may write

$$|T_5| \le \int_A^B \mathbb{E}|u^{\mathrm{ini}}(x) - \nu(x,0)|\,\varphi(x,0)\,\mathrm{d}x.$$

Turning to the study of T_2, we can write

$$T_2 = T_7 - T_8 + (T_2 - T_9) + (T_9 - T_{10} + T_8) + (T_{10} - T_7),$$

with

$$T_7 = -\int_0^{+\infty}\int_A^B \mathbb{E}(f(\nu(x,t)\top\kappa) - f(\kappa))\partial_x\varphi(x,t)\,\mathrm{d}x\mathrm{d}t,$$

$$T_8 = \int_0^{+\infty}(\mathbb{E}(G^\top(\eta_t(0),\eta_t(1))\varphi(A,t) - G^\top(\eta_t(N),\eta_t(N+1))\varphi(B,t))\,\mathrm{d}t,$$

$$\begin{aligned} T_9 \quad &= \int_0^{+\infty}\sum_{n=1}^{N}\mathbb{E}\left(G_\varepsilon^\top(\eta_t(n),\eta_t(n+1)) - F_\varepsilon^\top(\eta_t(n))\right)\varphi(A+nh,t)\,\mathrm{d}t \\ &+ \int_0^{+\infty}\sum_{n=1}^{N}\mathbb{E}\left(F_\varepsilon^\top(\eta_t(n)) - G_\varepsilon^\top(\eta_t(n-1),\eta_t(n))\right)\varphi(A+(n-1)h,t)\,\mathrm{d}t, \end{aligned}$$

and

$$T_{10} = -\int_0^{+\infty}\int_A^B \mathbb{E}\left(F_\varepsilon^\top(\nu(x,t))\right)\partial_x\varphi(x,t)\,\mathrm{d}x\mathrm{d}t,$$

where $F_\varepsilon^\top$ is the function defined by

$$F_\varepsilon^\top(x) = \int_X^Y \frac{1}{\varepsilon}\rho(\frac{y-\kappa}{\varepsilon})(f(x\top y)-f(y))\mathrm{d}y,\ \forall x\in\mathbb{R}. \tag{39}$$

In order to compare T_2 with T_9 we write

$$\begin{aligned} T_2 &= \int_0^{+\infty}\sum_{n=1}^N \mathbb{E}\left(G_\varepsilon^\top(\eta_t(n),\eta_t(n+1)) - F_\varepsilon^\top(\eta_t(n))\right)\frac{1}{h}\int_{(n-1)h}^{nh}\varphi(A+x,t)\,\mathrm{d}x\mathrm{d}t \\ &+ \int_0^{+\infty}\sum_{n=1}^N \mathbb{E}\left(F_\varepsilon^\top(\eta_t(n)) - G_\varepsilon^\top(\eta_t(n-1),\eta_t(n))\right)\frac{1}{h}\int_{(n-1)h}^{nh}\varphi(A+x,t)\,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Using the inequalities

$$|\varphi(A+nh,t) - \frac{1}{h}\int_{(n-1)h}^{nh}\varphi(A+x,t)\,\mathrm{d}x| \le \int_{(n-1)h}^{nh}|\partial_x\varphi(A+x,t)|\mathrm{d}x,$$

$$|\varphi(A+(n-1)h,t) - \frac{1}{h}\int_{(n-1)h}^{nh}\varphi(A+x,t)\,\mathrm{d}x| \le \int_{(n-1)h}^{nh}|\partial_x\varphi(A+x,t)|\mathrm{d}x,$$

$$|G_\varepsilon^\top(x,y) - F_\varepsilon^\top(y)| \le 2H(x,y),\ |G_\varepsilon^\top(x,y) - F_\varepsilon^\top(x)| \le 2H(x,y) \qquad \forall x,y\in[X,Y],$$

we get

$$|T_9 - T_2| \le 2\int_0^{+\infty}\sum_{n=1}^N \mathbb{E}\left(H(\eta_t(n),\eta_t(n+1)) + H(\eta_t(n-1),\eta_t(n))\right)\int_{(n-1)h}^{nh}|\partial_x\varphi(A+x,t)|\mathrm{d}x\mathrm{d}t.$$

We define the measure μ_1 by

$$\begin{aligned} &\int_{\mathbb{R}\times\mathbb{R}^+}\psi(x,t)\,\mathrm{d}\mu_1(x,t) \\ &= C\int_0^{+\infty}\sum_{n=1}^N \mathbb{E}\left(H(\eta_t(n),\eta_t(n+1)) + H(\eta_t(n-1),\eta_t(n))\right)\int_{(n-1)h}^{nh}|\partial_x\psi(x,t)|\mathrm{d}x\mathrm{d}t, \end{aligned}$$

which is, thanks to Lemma 2.2, such that (32) holds.
We have

$$T_{10} - T_9 = \int_0^{+\infty}(\mathbb{E}(G_\varepsilon^\top(\eta_t(0),\eta_t(1))\varphi(A,t) - G_\varepsilon^\top(\eta_t(N),\eta_t(N+1))\varphi(B,t))\,\mathrm{d}t.$$

then using (34), we get

$$|T_9 - T_{10} + T_8| \le 2M\varepsilon\int_0^{+\infty}(\varphi(A,t)+\varphi(B,t))\,\mathrm{d}t.$$

We now remark that

$$G^\top(a,b) = g(a\top\kappa, b\top\kappa) - f(\kappa) \le g(a\top\kappa,\kappa) - f(\kappa) \le M(a\top\kappa - \kappa),$$

and that

$$-G^\top(a,b) = f(\kappa) - g(a\top\kappa, b\top\kappa) \le f(\kappa) - g(\kappa, b\top\kappa) \le M(b\top\kappa - \kappa).$$

This leads to

$$T_8 \leq M \int_0^{+\infty} \left(\zeta^\top(\overline{u}(t))\varphi(A,t) + \zeta^\top(\overline{\overline{u}}(t))\varphi(B,t)\right) \, \mathrm{d}t.$$

Besides, we have, using (36),

$$|T_7 - T_{10}| \leq 2M\varepsilon \int_0^{+\infty} \int_A^B |\partial_x \varphi(x,t)| \mathrm{d}x\mathrm{d}t.$$

Turning to the study of T_3, we finally write that

$$|T_3| \leq \frac{Ckg(\overline{U},0)}{\varepsilon h} \int_0^{+\infty} \int_A^B \varphi(x,t) \, \mathrm{d}x\mathrm{d}t.$$

We then define the measure μ_0 by

$$\int_{\mathbb{R}\times\mathbb{R}^+} \psi(x,t) \, \mathrm{d}\mu_0(x,t) = \left(\varepsilon + \frac{Ckg(\overline{U},0)}{\varepsilon h}\right) \int_0^{+\infty} \int_A^B \psi(x,t) \, \mathrm{d}x\mathrm{d}t,$$

and we choose $\varepsilon = \sqrt{\frac{k}{h}}$.
Gathering all the above results, and doing similarly for the "bottom" case, this completes the proof of Lemma 2.3. ∎

2.2 Convergence study

We now state and prove a convergence result.

Theorem 2.2 *Let us consider a sequence $(h_i, k_i)_{i\in\mathbb{N}}$ with $h_i \to 0$ and $k_i/h_i \to 0$ as $i \to \infty$. Let us denote by $\nu_i(x,t)$ the process $\nu(x,t)$ associated by* (11) *to the misanthrope process, defined by the Chapman-Kolmogorov equation* (10) *and a given initial probability measure on D, with parameters h_i, k_i. If we assume that*

$$\lim_{i\to\infty} \mathbb{E}\left(\int_A^B \left|u^{\mathrm{ini}}(x) - \nu_i(x,0)\right| \, \mathrm{d}x\right) = 0,$$

the process $\nu_i(x,t)$ converges to the unique entropy weak solution $u(x,t)$ of the equation (1), in the sense that for all $T > 0$

$$\lim_{i\to\infty} \int_{[A,B]\times[0,T]} \mathbb{E}\left(|\nu_i(x,t) - u(x,t)|\right) \, \mathrm{d}x\mathrm{d}t = 0$$

Proof of Theorem 2.2
Let us define the Young measure μ_i, for any $i \in \mathbb{N}$, by

$$\int_{\mathbb{R}} \zeta(s) \, \mathrm{d}\mu_i(x,t)(s) = \mathbb{E}\left(\zeta(\nu_i(x,t))\right), \; t \in \mathbb{R}^+, \; x \in \mathbb{R}, \; \forall \zeta \in C^0(\mathbb{R}).$$

We first remark that there exists a subsequence, again denoted μ_i, and a Young measure μ limit for the nonlinear weak-$\star$ topology, that is

$$\begin{aligned}
&\lim_{i\to\infty} \int_{[A,B]\times\mathbb{R}^+} \varphi(x,t) \int_{\mathbb{R}} \zeta(s) \, \mathrm{d}\mu_i(x,t)(s) \, \mathrm{d}x\mathrm{d}t \\
&= \int_{[A,B]\times\mathbb{R}^+} \varphi(x,t) \int_{\mathbb{R}} \zeta(s) \, \mathrm{d}\mu(x,t)(s) \, \mathrm{d}x\mathrm{d}t, \; \forall \varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}, \mathbb{R}^+), \; \forall \zeta \in C^0(\mathbb{R}).
\end{aligned}$$

In order to justify this, let us develop in the framework of this paper an argument which is classical in the L^∞ framework [7]. There exists a sequence $(\zeta_j)_{j\in\mathbb{N}}$ of elements of $C^0([0,\bar{U}])$, dense in $C^0([0,\bar{U}])$ for the uniform

convergence topology. Then the sequence $\left(\int_{\mathbb{R}} \zeta_1(s)\ \mathrm{d}\mu_i(x,t)(s)\right)_{i\in\mathbb{N}}$ is bounded in $L^\infty((A,B)\times\mathbb{R}^+)$. Then there exists a subsequence and $g_1 \in L^\infty((A,B)\times\mathbb{R}^+)$ such that $\int_{\mathbb{R}} \zeta_1(s)\ \mathrm{d}\mu_i(x,t)(s)$ converges to g_1 for the weak-$\star$ topology of $L^\infty((A,B)\times\mathbb{R}^+)$. By a diagonal process, we may extract a subsequence again denoted $(\mu_i)_{i\in\mathbb{N}}$ such that for all $j\in\mathbb{N}$ the sequence of functions $(x,t)\to\int_{\mathbb{R}} \zeta_j(s)\ \mathrm{d}\mu_i(x,t)(s)$ tends to g_j for the weak-$\star$ topology of $L^\infty((A,B)\times\mathbb{R}^+)$. By density, for all $\zeta\in C^0([0,\bar{U}])$, there exists a function $g_\zeta\in L^\infty((A,B)\times\mathbb{R}^+)$ such that $\int_{\mathbb{R}} \zeta(s)\ \mathrm{d}\mu_i(x,t)(s)$ converges to $g_\zeta(x,t)$, as $i\to\infty$, for the weak-$\star$ topology of $L^\infty((A,B)\times\mathbb{R}^+)$. Now, by considering the Lebesgue points of all functions $(g_j)_{j\in\mathbb{N}}$, we build a subset of $(A,B)\times\mathbb{R}^+$ whose complementary in $(A,B)\times\mathbb{R}^+$ has zero Lebesgue measure. For (x,t) in this subset, we may consider the application $\zeta\to g_\zeta(x,t)$, checking that (x,t) is a Lebesgue point as well for any function $\zeta\in C^0([0,\bar{U}])$. This application defines a Young measure on $(A,B)\times\mathbb{R}^+$ since it is continuous.
For a given $\kappa\in\mathbb{R}$, we pass to the limit $i\to\infty$ in (30). Hence we get that μ is such that

$$\begin{aligned}
&\int_0^{+\infty}\int_A^B\int_{\mathbb{R}} (s\top\kappa-\kappa)\ \mathrm{d}\mu(x,t)(s)\ \partial_t\varphi(x,t)\ \mathrm{d}x\mathrm{d}t + \int_A^B (u^{\mathrm{ini}}(x)\top\kappa-\kappa)\varphi(x,0)\ \mathrm{d}x\\
&+M\int_0^{+\infty}\varphi(A,t)(\overline{u}(t)\top\kappa-\kappa)\ \mathrm{d}t + M\int_0^{+\infty}\varphi(B,t)(\overline{\overline{u}}(t)\top\kappa-\kappa)\ \mathrm{d}t\\
&+\int_0^{+\infty}\int_A^B\int_{\mathbb{R}} (f(s\top\kappa)-f(\kappa))\mathrm{d}\mu(x,t)(s)\ \partial_x\varphi(x,t)\ \mathrm{d}x\mathrm{d}t \geq 0.
\end{aligned}$$

We have also the same inequality with functions $\kappa - s\perp\kappa$. Then μ is an entropy Young measure solution (also called "process" solution in [22]) of the problem. Thanks to the uniqueness result, given as Theorem 2 in [22], we know that this entropy Young measure solution resumes to the entropy weak solution $u(x,t)$. We then have

$$\lim_{i\to\infty}\int_{[A,B]\times\mathbb{R}^+}\varphi(x,t)\int_{\mathbb{R}} s^2\mathrm{d}\mu_i(x,t)(s)\ \mathrm{d}x\mathrm{d}t = \int_{[A,B]\times\mathbb{R}^+}\varphi(x,t)u(x,t)^2\mathrm{d}x\mathrm{d}t,$$

and

$$\lim_{i\to\infty}\int_{[A,B]\times\mathbb{R}^+}\varphi(x,t)u(x,t)\int_{\mathbb{R}} s\mathrm{d}\mu_i(x,t)(s)\ \mathrm{d}x\mathrm{d}t = \int_{[A,B]\times\mathbb{R}^+}\varphi(x,t)u(x,t)\mathrm{d}x\mathrm{d}t,$$

which shows that

$$\lim_{i\to\infty}\int_{[A,B]\times\mathbb{R}^+}\varphi(x,t)\int_{\mathbb{R}} (s-u(x,t))^2\mathrm{d}\mu_i(x,t)(s)\ \mathrm{d}x\mathrm{d}t = 0.$$

Hence we conclude the proof of the theorem taking $\varphi=1$ on $((A,B)\times[0,T])$ and using the Cauchy-Schwarz inequality. ∎

3 The unbounded case

In this section, we now aim to prove that the misanthrope process, defined by the Chapman-Kolmogorov equation (12) and a given initial probability measure, converges to some entropy weak solution that we have to define in the unbounded case (this is simpler than in the bounded case).

Definition 3.1 (Entropy weak solution) Let $f\in C^1(\mathbb{R},\mathbb{R})$ and $u^{\mathrm{ini}}\in L^\infty(\mathbb{R})$. The entropy weak solution to Problem (5) is a function u such that

- $u\in L^\infty(\mathbb{R}\times\mathbb{R}^+)$,
- the following inequality holds

$$\begin{aligned}
&\int |u(x,t)-\kappa|\partial_t\varphi(x,t)\ \mathrm{d}t\mathrm{d}x + \int\Big(f(u(x,t)\top\kappa)-f(u(x,t)\perp\kappa)\Big)\ \partial_x\varphi(x,t)\ \mathrm{d}t\mathrm{d}x\\
&+\int_{\mathbb{R}}|u^{\mathrm{ini}}(x)-\kappa|\varphi(x,0)\ \mathrm{d}x\ \geq\ 0, \qquad \forall\varphi\in C_c^1(\mathbb{R}\times\mathbb{R}^+,\mathbb{R}^+), \forall\kappa\in\mathbb{R}.
\end{aligned} \tag{40}$$

We have the following fundamental theorem [16] (let us observe that the unbounded case has been solved a long time before the bounded one).

Theorem 3.1 *Let $f \in C^0(\mathbb{R})$ be Lipschitz continuous , $u^{\rm ini} \in L^\infty(\mathbb{R})$, then there exists a unique entropy weak solution to Problem (5).*

3.1 Estimates

Estimates in the unbounded case are obtained in a similar way as in the bounded case. The next lemma is similar to Lemma 2.1 and is proved using $G_\zeta = \frac{1}{2}(G_\zeta^\top + G_\zeta^\perp)$.

Lemma 3.1 (Discrete entropy inequalities) *Let $\zeta \in C^2(\mathbb{R})$ be a convex function. Let η_t be the misanthrope process defined by the Chapman-Kolmogorov equation* (12) *and a given initial probability measure on D.*
Then the following inequality holds:

$$h\frac{\mathrm{d}\mathbb{E}\left(\zeta(\eta_t(n))\right)}{\mathrm{d}t} + \mathbb{E}\left(G_\zeta(\eta_t(n),\eta_t(n+1)) - G_\zeta(\eta_t(n-1),\eta_t(n))\right) \leq k\, g(\overline{U},0) \max_{\kappa\in[0,\overline{U}]} \zeta''(\kappa),\ \forall n \in \mathbb{Z}, \tag{41}$$

denoting, for given $X < 0 < \overline{U} < Y$,

$$G_\zeta(x,y) = \frac{1}{2}\int_X^Y \zeta''(\kappa)(g(x\top\kappa, y\top\kappa) - g(x\perp\kappa, y\perp\kappa))\mathrm{d}\kappa + g(x,y)\ \frac{\zeta'(X)+\zeta'(Y)}{2},\ \forall x,y \in [X,Y]. \tag{42}$$

The next lemma is similar to Lemma 2.2. The differences come from the fact that, in the unbounded case, one needs in introduce arbitrary bounds, related to the support of test functions in the entropy formulation.

Lemma 3.2 *Let η_t be the misanthrope process defined by the Chapman-Kolmogorov equation* (12) *and a given initial probability measure on D. Let $T > 0$, $R > 0$, $A, B \in \mathbb{N}$ such that $R < Bh < R+1$ and $-R > Ah > -R-1$. Then there exists $C \in \mathbb{R}$, only depending on T, R, $\overline{U}$, g such that, for $h < R$, assuming $\frac{k}{h} \leq 1$,*

$$\int_0^T \sum_{n=A}^{B-1} \mathbb{E}\left(H(\eta_t(n),\eta_t(n+1))\right)\mathrm{d}t \leq \ \frac{C}{\sqrt{h}}, \tag{43}$$

with H defined by (22).

One more time, the next lemma is similar to Lemma 2.3.

Lemma 3.3 *Let η_t be the misanthrope process defined by the Chapman-Kolmogorov equation* (12) *and a given initial probability measure on D. Let ν be the real process defined by* (13)*. Let $\kappa \in \mathbb{R}$ be given. Then the following inequality holds:*

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}} \mathbb{E}\left(|\nu(x,t)-\kappa|\right)\ \partial_t\varphi(x,t)\ \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} |u^{\rm ini}(x)-\kappa|\varphi(x,0)\ \mathrm{d}x \\
&+ \int_0^{+\infty}\int_{\mathbb{R}} \mathbb{E}(f(\nu(x,t)\top\kappa) - f(\nu(x,t)\perp\kappa))\ \partial_x\varphi(x,t)\ \mathrm{d}x\mathrm{d}t\ \geq \\
&- \int_{\mathbb{R}} \varphi(x,0)\mathrm{d}\mu^{\rm ini}(x) - \int_{\mathbb{R}\times\mathbb{R}^+} \varphi(x,t)\mathrm{d}\mu_0(x,t) - \int_{\mathbb{R}\times\mathbb{R}^+} (|\partial_t\varphi(x,t)| + |\partial_x\varphi(x,t)|)\mathrm{d}\mu_1(x,t),
\end{aligned} \tag{44}$$

for all $\varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R},\mathbb{R}^+)$, and where the measures $\mu_0,\ \mu_1$ and $\mu^{\rm ini}$ verify the following properties:

1. *For all $R > 0$ and $T > 0$, there exists C_0 depending only on R and T, g and $\overline{U}$ such that*

$$\mu_0([-R,R]\times[0,T]) \leq C_0\sqrt{\frac{k}{h}}. \tag{45}$$

2. *For all $R > 0$ and $T > 0$, there exists C_1 depending only on g, $u^{\rm ini}$, $\overline{U}$, R and T such that, for $h < R$,*

$$\mu_1([-R,R]\times[0,T]) \le C_1(\sqrt{h}+\sqrt{\frac{k}{h}}). \tag{46}$$

3. *The measure $\mu^{\rm ini}$ is the measure of density $\mathbb{E}|u^{\rm ini}(.)-\nu(.,0)|+\dfrac{C^{\rm ini}}{\sqrt{Kh}}$ with respect to the Lebesgue measure, where $C^{\rm ini}$ only depends on R and T, g and $\overline{U}$.*

3.2 Convergence study

The following convergence result may be proved as in the bounded case, its proof relies on the preceding lemma and on the uniqueness theorem given in [7].

Theorem 3.2 *Let us consider a sequence h_i, k_i with $h_i \to 0$ and $k_i/h_i \to 0$. Let us denote by $\nu_i(x,t)$ the process $\nu(x,t)$ associated by* (13) *to the misanthrope process defined by the Chapman-Kolmogorov equation* (12) *and a given initial probability measure on D with parameters h_i, k_i. If we assume that for all $A < B$*

$$\lim_{i\to\infty}\mathbb{E}\left(\int_A^B \left|u^{\rm ini}(x)-\nu_i(x,0)\right|\,{\rm d}x\right)=0,$$

the process $\nu_i(x,t)$ converges to the unique entropy weak solution $u(x,t)$ of the equation (5), in the sense that for all $T > 0$ and all $A < B$

$$\lim_{i\to\infty}\int_{[A,B]\times[0,T]}\mathbb{E}\left(|\nu_i(x,t)-u(x,t)|\right)\,{\rm d}x{\rm d}t=0$$

3.3 Error estimate

Let us finally provide an error estimate in the unbounded case, the proof of which is inspired by that of error estimates for deterministic numerical schemes [9]. To this purpose, we assume that $u^{\rm ini}$ has locally bounded variations (that we denote by $u^{\rm ini}\in BV_{\rm loc}(\mathbb{R})$), which simply means that its derivative in the distribution sense is a measure which is not necessarily finite (for example, $u^{\rm ini}=0$ on $[2k,2k+1)$ and $u^{\rm ini}=1$ on $[2k+1,2k+2)$, for all $k\in\mathbb{Z}$).

Lemma 3.4 *Let $u^{\rm ini}\in BV_{\rm loc}(\mathbb{R})\cap L^\infty(\mathbb{R})$. Let ν be a stochastic process on $\mathbb{R}\times\mathbb{R}^+$, valued in in a bounded subset of $\mathbb{R}$, such that there exists measures $\mu_0,\ \mu_1$ on $\mathbb{R}\times\mathbb{R}^+$ and $\mu^{\rm ini}$ on $\mathbb{R}$ such that*

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}}\mathbb{E}\Big(|\nu(x,t)-\kappa|\partial_t\varphi(x,t)+(f(\nu(x,t)\top\kappa)-f(\nu(x,t)\perp\kappa))\partial_x\varphi(x,t)\Big){\rm d}x{\rm d}t\\&+\int_{\mathbb{R}}|u^{\rm ini}(x)-\kappa|\varphi(x,0){\rm d}x\\&\ge-\int_{\mathbb{R}\times\mathbb{R}^+}\Big(|\partial_t\varphi(x,t)|+|\partial_x\varphi(x,t)|\Big){\rm d}\mu_1(x,t)-\int_{\mathbb{R}\times\mathbb{R}^+}\varphi(x,t){\rm d}\mu_0(x,t)-\int_{\mathbb{R}}\varphi(x,0){\rm d}\mu^{\rm ini}(x),\\&\forall\kappa\in\mathbb{R},\ \ \forall\varphi\in C_c^\infty(\mathbb{R}\times\mathbb{R}^+,\mathbb{R}^+).\end{aligned}\tag{47}$$

Let u be the unique entropy weak solution of (5) *in the sense of Definition 3.1.*
Let $\psi\in C_c^\infty(\mathbb{R}\times\mathbb{R}^+,\mathbb{R}^+)$ be given, and let $S=\{\psi\neq 0\}=\{(x,t)\in\mathbb{R}\times\mathbb{R}^+;\ \psi(x,t)\neq 0\}$ and $S_0=\{\psi(\cdot,0)\neq 0\}=\{x\in\mathbb{R};\ \psi(x,0)\neq 0\}$. Then there exists C only depending on $\|\psi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}$, $\|\partial_t\psi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}$, $\|\partial_x\psi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}$, f, S, S_0 and $u^{\rm ini}$, such that

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}}\mathbb{E}\Big[|\nu(x)-u(x,t)|\partial_t\psi(x,t)+\Big(f(\nu(x)\top u(x,t))-f(\nu(x)\perp u(x,t))\Big)(\partial_x\psi(x,t))\Big]\,{\rm d}x{\rm d}t\\&\ge -C(\mu^{\rm ini}(S_0)+(\mu_1(S))^{\frac12}+(\mu_1+\mu_0)(S)).\end{aligned}\tag{48}$$

Proof of Lemma 3.4

The main tool of the proof (the so-called Krushkov's "double variable technique") is to take $\kappa = u(y,s)$ in (47), $\kappa = \nu(x,t)$ in (40) and to introduce mollifiers in order to have y close to x and s close to t. Let $\rho \in C_c^\infty(\mathbb{R},\mathbb{R})$ satisfying the following properties:

$$\text{supp}(\rho) = \{x \in \mathbb{R};\ \rho(x) \neq 0\} \subset [-1,0], \tag{49}$$

$$\rho(x) \geq 0,\ \forall x \in \mathbb{R},$$

$$\int_{\mathbb{R}} \rho(x)\mathrm{d}x = 1.$$

For example, one might define $\widetilde{\rho}$ by $\widetilde{\rho}(x) = \exp(-1/(1-(2x+1)^2))$ for $x \in (-1,0)$ and 0 elsewhere, and $\rho(x) = \widetilde{\rho}(x)/\int_{-1}^0 \widetilde{\rho}(y)\mathrm{d}y$. For $r \in \mathbb{R}$, $r \in [1,+\infty)$, one defines $\rho_r(x) = r\rho(rx)$, for all $x \in \mathbb{R}$.

Let $\psi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+,\mathbb{R}^+)$, and let $\varphi : (\mathbb{R}\times\mathbb{R}^+)^2 \to \mathbb{R}^+$ be defined by:

$$\varphi(x,t,y,s) = \psi(x,t)\rho_r(x-y)\rho_r(t-s).$$

Note that, for any $(y,s) \in \mathbb{R}\times\mathbb{R}^+$, one has $\varphi(\cdot,\cdot,y,s) \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+,\mathbb{R}^+)$ and, for any $(x,t) \in \mathbb{R}\times\mathbb{R}^+$, one has $\varphi(x,t,\cdot,\cdot) \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+,\mathbb{R}^+)$. Let us consider, in (40) (in which we denote the space and time variables by (y,s)), $\varphi(x,t,\cdot,\cdot)$ as test function, and $\kappa = \nu(x,t)$. We then take the mean value and integrate for $(x,t) \in \mathbb{R}\times\mathbb{R}^+$. We thus get, remarking that, thanks to (49), $\varphi(x,t,y,0) = 0$ for all $t \geq 0$,

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} \mathbb{E}\Big[& |u(y,s)-\nu(x,t)|\partial_s\varphi(x,t,y,s) \\ & +\Big(f(u(y,s)\top\nu(x,t)) - f(u(y,s)\bot\nu(x,t))\Big)\partial_y\varphi(x,t,y,s)\Big]\mathrm{d}y\mathrm{d}s\mathrm{d}x\mathrm{d}t \geq 0, \\ & \forall\kappa \in \mathbb{R},\ \ \forall\varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+,\mathbb{R}^+).\end{aligned} \tag{50}$$

We now consider, in (47), $\varphi(\cdot,\cdot,y,s)$ as test function and $\kappa = u(y,s)$. We then integrate (47) for $(y,s) \in \mathbb{R}\times\mathbb{R}^+$. We then get

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} \mathbb{E}\Big(&& |\nu(x,t)-u(y,s)|\partial_t\varphi(x,t,y,s) \\ & && +(f(\nu(x,t)\top u(y,s)) - f(\nu(x,t)\bot u(y,s)))\partial_x\varphi(x,t,y,s)\Big)\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \\ + & \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} && |u^{\text{ini}}(x)-u(y,s)|\varphi(x,0,y,s)\mathrm{d}x\mathrm{d}y\mathrm{d}s \quad \geq \\ - & \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}\times\mathbb{R}^+} && \Big(|\partial_t\varphi(x,t,y,s)| + |\partial_x\varphi(x,t,y,s)|\Big)\mathrm{d}\mu_1(x,t)\mathrm{d}y\mathrm{d}s \\ - & \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}\times\mathbb{R}^+} && \varphi(x,t,y,s)\mathrm{d}\mu_0(x,t)\mathrm{d}y\mathrm{d}s \\ - & \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} && |\varphi(x,0,y,s)|\mathrm{d}\mu^{\text{ini}}(x)\mathrm{d}y\mathrm{d}s, \end{aligned} \tag{51}$$

Adding the two inequalities (50) and (51) and using $\partial_s\varphi(x,t,y,s) + \partial_t\varphi(x,t,y,s) = \partial_t\psi(x,t)\rho_r(x-y)\rho_r(t-s)$ and $\partial_x\varphi(x,t,y,s) + \partial_y\varphi(x,t,y,s) = \partial_x\psi(x,t)\rho_r(x-y)\rho_r(t-s)$, yield

$$E_{11} + E_{12} + E_{14} \geq -E_2, \tag{52}$$

where

$$E_{11} = \int_0^\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} \mathbb{E}\Big[|\nu(x,t) - u(y,s)|\partial_t\psi(x,t)\rho_r(x-y)\rho_r(t-s)\Big]\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s,$$

$$E_{12} = \int_0^\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} \mathbb{E}\Big[\Big(f(\nu(x,t)\top u(y,s)) - f(\nu(x,t)\bot u(y,s))\Big) \partial_x\psi(x,t)\rho_r(x-y)\rho_r(t-s)\Big]\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s,$$

$$E_{14} = \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} |u^{\mathrm{ini}}(x) - u(y,s)|\psi(x,0)\rho_r(x-y)\rho_r(-s)\mathrm{d}y\mathrm{d}s\mathrm{d}x$$

and

$$\begin{aligned} E_2 = \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}\times\mathbb{R}^+} \Big(&|\rho_r(x-y)(\partial_t\psi(x,t)\rho_r(t-s) + \psi(x,t)\rho'_{1,r}(t-s))| \\ &+|\rho_r(t-s)(\partial_x\psi(x,t)\rho_r(x-y) + \psi(x,t)\rho'_r(x-y))|\Big) d\mu_1(x,t)\mathrm{d}y\mathrm{d}s \\ &+\int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}\times\mathbb{R}^+} \rho_r(x-y)\psi(x,t)\rho_r(t-s)d\mu_0(x,t)\mathrm{d}y\mathrm{d}s \\ &+\int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} |\psi(x,0)\rho_r(x-y)\rho_r(-s)|d\mu^{\mathrm{ini}}(x)\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{53}$$

Let us now study the four terms of (52). In the following, the notation C_i ($i \in \mathbb{N}$) will refer to various real quantities only depending on $\|\psi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}$, $\|\partial_t\psi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}$, $\|\partial_x\psi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}$, f, S, S_0 and u^{ini}.
Equality (53) leads, thanks to $\int_{\mathbb{R}} |\rho'_r(s)|\mathrm{d}s = r\int_{\mathbb{R}} |\rho'(s)|\mathrm{d}s$, to

$$E_2 \le (r+1)C_1\mu_1(S) + C_2\mu^{\mathrm{ini}}(S_0) + C_2\mu_0(S). \tag{54}$$

Let us handle the term E_{11}. For all $x \in \mathbb{R}$ and for all $t \in \mathbb{R}^+$, one has, using (49),

$$\int_{\mathbb{R}} \int_0^\infty \rho_r(x-y)\rho_{1,r}(t-s)\mathrm{d}s\mathrm{d}y = 1.$$

Then,

$$\begin{aligned} &|E_{11} - \int_0^{+\infty} \int_{\mathbb{R}} \mathbb{E}\Big[|\nu(x,t) - u(x,t)|\partial_t\psi(x,t)\Big]\,\mathrm{d}x\mathrm{d}t| \le \\ &\int_0^\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} \Big[|u(x,t) - u(y,s)||\partial_t\psi(x,t)|\rho_r(x-y)\rho_r(t-s)\Big]\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \le \|\partial_t\psi\|_\infty \varepsilon(r,S), \end{aligned}$$

with

$$\varepsilon(r,S) = \sup\{\|u - u(\cdot+\eta, \cdot+\tau)\|_{L^1(S)},\ |\eta| \le \frac{1}{r},\ 0 \le \tau \le \frac{1}{r}\}. \tag{55}$$

Since $u^{\mathrm{ini}} \in BV(\mathbb{R})$, the function u belongs to $BV(\mathbb{R}\times(-T,T))$, for all $T > 0$, setting, for instance, $u(.,t) = u^{\mathrm{ini}}$ for $t < 0$ (see [16] where this result is proved passing to the limit on numerical schemes).
Then,

$$\varepsilon(r,S) \le \frac{C_3}{r}. \tag{56}$$

Hence,

$$|E_{11} - \int_0^{+\infty} \int_{\mathbb{R}} \mathbb{E}\Big[|\nu(x,t) - u(x,t)|\partial_t\psi(x,t)\Big]\,\mathrm{d}x\mathrm{d}t| \le \frac{C_4}{r}. \tag{57}$$

In the same way, using $|f(a\top b) - f(a\top c)| \le M|b-c|$ and $|f(a\bot b) - f(a\bot c)| \le M|b-c|$ for all a, b, $c \in [0,\overline{U}]$ where M is the Lipschitz constant of f in $[0,\overline{U}]$,

$$|E_{12} - \int_0^{+\infty} \int_{\mathbb{R}} \mathbb{E}\Big(f(\nu(x,t)\top u(x,t)) - f(\nu(x,t)\bot u(x,t))\Big) \\ \partial_x \psi(x,t) \mathrm{d}x\mathrm{d}t| \leq C_5 \varepsilon(r,S) \leq \tfrac{C_6}{r}. \tag{58}$$

In order to estimate E_{14}, let us take in (40) (in which we denote the space and time variables by (y,s)), for $x \in \mathbb{R}$ fixed, $\varphi = \varphi(x,\cdot,\cdot)$, with

$$\varphi(x,y,s) = \psi(x,0)\rho_r(x-y)\int_s^{\infty} \rho_r(-\tau)\mathrm{d}\tau,$$

and $\kappa = u^{\mathrm{ini}}(x)$. Note that $\varphi(x,\cdot,\cdot) \in C_c^{\infty}(\mathbb{R}\times\mathbb{R}^+,\mathbb{R}^+)$. We then integrate the resulting inequality with respect to $x \in \mathbb{R}$. We get

$$-E_{14} + E_{15} + E_{16} \geq 0,$$

with

$$E_{15} = -\int_0^{\infty}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_s^{\infty} (f(u(y,s)\top u^{\mathrm{ini}}(x)) - f(u(y,s)\bot u^{\mathrm{ini}}(x))) \\ \psi(x,0)\rho_r'(x-y)\rho_r(-\tau)\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}s,$$

$$E_{16} = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{\infty} \psi(x,0)\rho_r(x-y)\rho_r(-\tau)|u^{\mathrm{ini}}(x) - u^{\mathrm{ini}}(y)|\mathrm{d}\tau\mathrm{d}y\mathrm{d}x.$$

To bound E_{15}, one introduces E_{15b} defined as

$$E_{15b} = -\int_0^{\infty}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_s^{\infty} (f(u(y,s)\top u^{\mathrm{ini}}(y)) - f(u(y,s)\bot u^{\mathrm{ini}}(y))) \\ \psi(x,0)\rho_r'(x-y)\rho_r(-\tau)\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}s.$$

Integrating by parts for the x variable yields

$$E_{15b} = \int_0^{\infty}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_s^{\infty} (f(u(y,s)\top u^{\mathrm{ini}}(y)) - f(u(y,s)\bot u^{\mathrm{ini}}(y))) \\ \partial_x\psi(x,0)\rho_r(x-y)\rho_r(-\tau)\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}s.$$

Then, noting that the time support of this integration is reduced to $s \in [0,1/r]$, one has

$$E_{15b} \leq \frac{C_{11}}{r}. \tag{59}$$

Furthermore, one has

$$|E_{15} - E_{15b}| \leq C_{12}\int_0^{\infty}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_s^{\infty} |u^{\mathrm{ini}}(x) - u^{\mathrm{ini}}(y)||\rho_r'(x-y)|\psi(x,0)\rho_r(-\tau)\mathrm{d}\tau\mathrm{d}y\mathrm{d}x\mathrm{d}s,$$

which is bounded by $C_{13}\varepsilon_0(r,S_0)$, since the time support of the integration is reduced to $s \in [0,1/r]$, where $\varepsilon_0(r,S_0)$ is defined by

$$\varepsilon_0(r,S_0) = \sup\{\int_{S_0} |u^{\mathrm{ini}}(x) - u^{\mathrm{ini}}(x+\eta)|\mathrm{d}x;\ |\eta| \leq \frac{1}{r}\}. \tag{60}$$

Since $u^{\mathrm{ini}} \in BV(\mathbb{R})$, one has $\varepsilon_0(r,S_0) \leq C_{14}/r$ and therefore, with (59), $E_{15} \leq C_{15}/r$, and one also has that the term E_{16} is bounded by C_{16}/r. Hence, since $E_{14} \leq E_{15} + E_{16}$,

$$E_{14} \leq \frac{C_{17}}{r}. \tag{61}$$

Using (52), (54), (57), (58), (61), one obtains

$$\int_0^{+\infty}\int_{\mathbb{R}}\mathbb{E}\Big[|\nu(x,t)-u(x,t)|\partial_t\psi(x,t)+\Big(f(\nu(x,t)\top u(x,t))-f(\nu(x,t)\bot u(x,t))\Big)\partial_x\psi(x,t)\Big]\mathrm{d}x\mathrm{d}t$$
$$\geq -C_1(r+1)\mu_1(S)-C_2(\mu^{\mathrm{ini}}(S_0)+\mu_0(S))-\frac{C_{18}}{r},$$

which, taking $r=1/\sqrt{\mu_1(S)}$ if $0<\mu_1(S)\leq 1$ ($r\to\infty$ if $\mu_1(S)=0$ and $r=1$ if $\mu_1(S)>1$), gives (48).
This concludes the proof of Lemma 3.4. ■

Theorem 3.3 *Let $u^{\mathrm{ini}}\in BV_{\mathrm{loc}}(\mathbb{R})\cap L^\infty(\mathbb{R})$, let u be the unique entropy weak solution of the problem. Let us assume the same hypotheses as in Lemma 3.4,*
Then, for all $R>0$ and all $T>0$ there exists C_e and $\bar{R}$, only depending on R, T, f and u^{ini}, such that the following inequality holds:

$$\int_0^T\int_{-R}^R\mathbb{E}|\nu(x,t)-u(x,t)|\,\mathrm{d}x\mathrm{d}t\leq C_e(\mu^{\mathrm{ini}}([-\bar{R},\bar{R}])+[\mu_1([-\bar{R},\bar{R}]\times[0,T])]^{\frac{1}{2}}$$
$$+(\mu_1+\mu_0)([-\bar{R},\bar{R}]\times[0,T])).$$

Proof of Theorem 3.3
Let $R>0$ and $T>0$. Let M be a Lipschitz constant of f in $[0,\overline{U}]$ (indeed, since $f\in C^1(\mathbb{R},\mathbb{R})$, one has $M=\sup\{|f'(s)|;\,s\in[0,\overline{U}]\}$).
Let $\rho\in C_c^1(\mathbb{R}^+,[0,1])$ be a function such that $\rho(r)=1$ if $r\in[0,R+MT]$, $\rho(r)=0$ if $r\in[R+MT+1,\infty)$ and $\rho'(r)\leq 0$, for all $r\in\mathbb{R}^+$ (ρ only depends on R, T, f and u^{ini}).
One takes, in (48), ψ defined by

$$\psi(x,t)=\rho(|x|+Mt)\tfrac{T-t}{T},\ \text{for } x\in\mathbb{R} \text{ and } t\in[0,T],$$
$$\psi(x,t)=0,\ \text{for } x\in\mathbb{R} \text{ and } t\geq T.$$

Note that $\rho(|x|+Mt)=1$, if $(x,t)\in[-R,R]\times[0,T]$.
The function ψ is not in $C_c^\infty(\mathbb{R}\times\mathbb{R}^+,\mathbb{R}^+)$, but, using a usual regularization technique, it may be proved that such a function can be considered in (48), in which case Inequality (48) writes, with $\bar{R}=R+MT+1$,

$$\int_0^T\int_{\mathbb{R}}\mathbb{E}\Big[|\nu(x,t)-u(x,t)|\Big(\frac{T-t}{T}M\rho'(|x|+Mt)-\frac{1}{T}\rho(|x|+Mt)\Big)+$$
$$\Big(f(\nu(x,t)\top u(x,t))-f(\nu(x,t)\bot u(x,t))\Big)\tfrac{T-t}{T}\rho'(|x|+Mt)\tfrac{x}{|x|})\Big]\,\mathrm{d}x\mathrm{d}t\ \geq$$
$$-C(\mu^{\mathrm{ini}}([-\bar{R},\bar{R}])+(\mu_1([-\bar{R},\bar{R}]\times[0,T]))^{\frac{1}{2}}+(\mu_1+\mu_0)([-\bar{R},\bar{R}]\times[0,T])),$$

where C only depends on R, T, f and u^{ini}.
Since $\rho'\leq 0$, one has

$$\Big(f(\nu(x,t)\top u(x,t))-f(\nu(x,t)\bot u(x,t))\Big)\,\frac{T-t}{T}\rho'(|x|+Mt)(\frac{x}{|x|})\Big)\leq$$
$$|\nu(x,t)-u(x,t)|\tfrac{T-t}{T}M(-\rho'(|x|+Mt)),$$

and therefore, since $\rho(|x|+Mt)=1$, if $(x,t)\in[-R,R]\times[0,T]$,

$$\int_0^T\int_{[-R,R]}\mathbb{E}|\nu(x,t)-u(x,t)|\mathrm{d}x\mathrm{d}t\leq CT(\mu^{\mathrm{ini}}([-\bar{R},\bar{R}])+(\mu_1([-\bar{R},\bar{R}]\times[0,T]))^{\frac{1}{2}}$$
$$+(\mu_1+\mu_0)([-\bar{R},\bar{R}]\times[0,T])).$$

This completes the proof of Theorem 3.3. ■

We then deduce the following theorem.

Theorem 3.4 *Let η_t be the misanthrope process defined by the Chapman-Kolmogorov equation* (12) *and the initial probability measure given by the Dirac measure on the function defined, for all $n \in \mathbb{Z}$, by the closest element of $k\mathbb{Z}$ to $\frac{1}{h}\int_{(n-1)h}^{nh} u^{\mathrm{ini}}(x)\,\mathrm{d}x$. Let ν be the real process defined by* (13)*. For all $R > 0$ and $T > 0$, assuming $u^{\mathrm{ini}} \in BV_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, taking $k = h^2$, then there exists C_e, only depending on R, T, g, u^{ini}, such that the following inequality holds:*

$$\int_0^T \int_{-R}^R \mathbb{E}|\nu(x,t) - u(x,t)|\,\mathrm{d}x\mathrm{d}t \leq C_e h^{\frac{1}{4}}. \tag{62}$$

4 Numerical computations on the Riemann problem

For a given $\overline{U} > 0$, one considers Problem (1)-(4) with $(A, B) = (0, 1)$, assuming that u^{ini} is defined, for given real values $u_l, u_r \in [0, \overline{U}]$ and $x_0 \in (0, 1)$, by

$$u^{\mathrm{ini}}(x) = \begin{cases} u_l & \text{if } x < x_0 \\ u_r & \text{otherwise} \end{cases} \tag{63}$$

and that the functions $\overline{u}$ and $\overline{\overline{u}}$ are respectively defined by $\overline{u}(t) = u_l$ and $\overline{\overline{u}}(t) = u_r$ for all $t \in \mathbb{R}^+$ (this problem is the generalization of the Riemann problem to the bounded setting since this boundary condition allows to reproduce the unbounded solution at least for a finite time which is not precisely given in the examples below).

In the linear case $f(u) = v\,u$, $v \in \mathbb{R}$, the entropy weak solution to Problem (1)-(4) (which is also in this case the unique weak solution) is then obtained by translation of the initial condition:

$$u(x,t) = u^{\mathrm{ini}}(x - tv) = \begin{cases} u_l & \text{if} \quad x - tv < x_0 \\ u_r & \text{if} \quad x - tv > x_0 \end{cases}$$

If the flux function f is non-linear, the entropy weak solution shows shock or rarefaction waves: shock waves are discontinuity lines between two density levels, that propagate over the time. Rarefaction waves are regular transitions between density levels. The entropy condition allows to specify the unique physical solution. When f is strictly convex or concave, the solution is deduced from the sign of $u_l - u_r$.

Assuming f strictly convex (resp. strictly concave), $u_l > u_r$ (resp. $u_l < u_r$) and denoting

$$\sigma = \frac{f(u_r) - f(u_l)}{u_r - u_l},$$

then the entropy weak solution u of Problem (1)-(4) is the shock wave starting at x_0 and propagating at the constant speed σ

$$u(x,t) = \begin{cases} u_l & \text{if} \quad (x - x_0)/t < \sigma, \\ u_r & \text{if} \quad (x - x_0)/t > \sigma. \end{cases}$$

The solution is a shock wave starting at x_0 and propagating at the constant speed σ.

If one assumes f strictly convex (resp. strictly concave) and $u_l < u_r$ (resp. $u_l > u_r$) then the entropy weak solution of Problem (1)-(4) is the rarefaction wave given by

$$u(x,t) = \begin{cases} u_l & \text{if} \quad (x - x_0)/t < f'(u_l), \\ G((x - x_0)/t) & \text{if} \quad f'(u_l) \leq (x - x_0)/t \leq f'(u_r), \\ u_r & \text{if} \quad (x - x_0)/t > f'(u_r), \end{cases}$$

with G is the reciprocal function to f', which means that, for a given value $u_0 \in [u_l \perp u_r, u_l \top u_r]$, then $u(x_0 + tf'(u_0), t) = u_0$.

In the application of this model to traffic flow [18, 21], the flux function f is assumed to be concave. The nondecreasing part of the function corresponds to a free traffic state into which traffic characteristics propagate down-stream. The nonincreasing part describes an interactive or congested traffic state. For this traffic state, the characteristics propagate up-stream. These aspects are observed on real traffic data [8].

With the aim to model traffic flow, one considers henceforth the two following positive and concave flux functions:

$$f_1(u) = \begin{cases} u & 0 \leq u \leq 1/2 \\ 1-u & 1/2 < u \leq 1 \end{cases} \tag{64}$$

and

$$f_2(u) = u(1-u). \tag{65}$$

Hence $f_1 \in C^0(\mathbb{R})$ is a triangular and piecewise linear flux function while $f_2 \in C^\infty(\mathbb{R})$ is a regular polynomial one.

One proposes to numerically approximate the solution of Problem (1)-(4) by using the misanthrope process η_t defined on $E = (h^2[\![0, N^2]\!])^{[\![1,N]\!]}$ with $N \in \mathbb{N}^\star$ and $h = 1/N$ (one assumes with the previous notation that $k = h^2$, $\overline{U} = 1$ and therefore $K = \overline{U}/k = N^2$). The process is characterized by the Chapman-Kolmogorov equation (10).

The jump rate of the process, given by (8), depends on the numerical flux function. One uses and compares two numerical fluxes in the numerical experiments: the Godunov numerical flux (16) and a modified positive Rusanov (or Lax-Friedrichs) numerical flux given by

$$g(a,b) = \frac{1}{2}\left(f(a) + f(b) + a - b\right)_+,$$

with $(x)_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$ (see figure 1 and 4).

One uses an event-driven evolution scheme to simulate the stochastic process η. Each site occupied by at least one particle has an exponential clock giving the jump time of a particle towards the next site, or creation (resp. deletion) of a particle for the first (resp. last) site. The exponential times are calibrated by the jump rate function b. The process is simulated as follows. At the initial time, one calculates the jump time of each site according to their jump rate. Then, at each step (*i.e.* at each jump), one has to:

1. Select the site with the minimal time;
2. Transfer a particle from this site to the next one, or create or delete a particle;
3. Update the global time of the system with the jump time of the section selected;
4. Calculated the new jump times of the sections whose jump rate has been modified.

In the simulation experiments, one considers two initial conditions:

Exp. 1 leading to a rarefaction wave,

$$u^{\text{ini}}(x) = \begin{cases} u_l = 0.8 & \text{if } x < x_0 = 0.5 \\ u_r = 0.2 & \text{otherwise,} \end{cases}$$

(see Figures 2 and 5);

Exp. 2 leading to a shock wave,

$$u^{\text{ini}}(x) = \begin{cases} u_l = 0.4 & \text{if } x < x_0 = 0.8 \\ u_r = 0.7 & \text{otherwise,} \end{cases}$$

(see Figures 3 and 6).

One compares the numerical approximations respectively obtained using the Godunov and Rusanov numerical fluxes. The simulation are done for different values of the parameter h (equal to 0.05, 0.02 and 0.01).

One observes that the precision of the approximations is similar for the two numerical fluxes when the flux function is triangular since the jump rate functions are close. The approximation solutions are more accurate when using the Godunov numerical flux than the Rusanov one if the flux function is the polynomial $f : u \mapsto u(1-u)$. As expected, in all the cases, the approximations are more and more accurate as h tends towards 0.

5 Conclusion

We show the convergence of a misanthrope process to the entropy solution of a hyperbolic equation, both on a bounded and a non-bounded space. The proof of the convergence is based on the Chapman-Kolmogorov equation and on the uniqueness of the entropy Young measure solution. In the non-bounded case, we provide an error estimate through a method close to those involved in the convergence proofs for deterministic numerical schemes. The use of this framework enables a strong limit sense of the stochastic process to the entropy weak solution of the problem (note that, even in the linear case, one should use the nonlinear framework for proving such a convergence).

From a numerical point of view, the stochastic continuous-time misanthrope process might be compared to first order deterministic discrete-time numerical schemes, in particular from the points of view of the computing time and the accuracy. It seems clear that the error estimate provided in this paper is not sharp, in a similar way to that obtained for deterministic numerical schemes. The error estimate including boundary terms remains to be studied.

Finally, the results given in this paper show the close relation between modeling traffic flows using misanthrope processes and using the limit nonlinear scalar hyperbolic equation. Large work remains to be done for improving the accuracy of these models.

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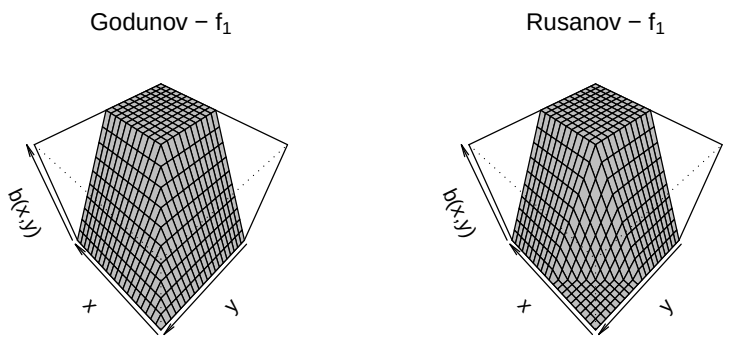

Figure 1: Jump rate function for the triangular flux (64). Left, case of the Godunov numerical flux and, right, case of the modified Rusanov one.

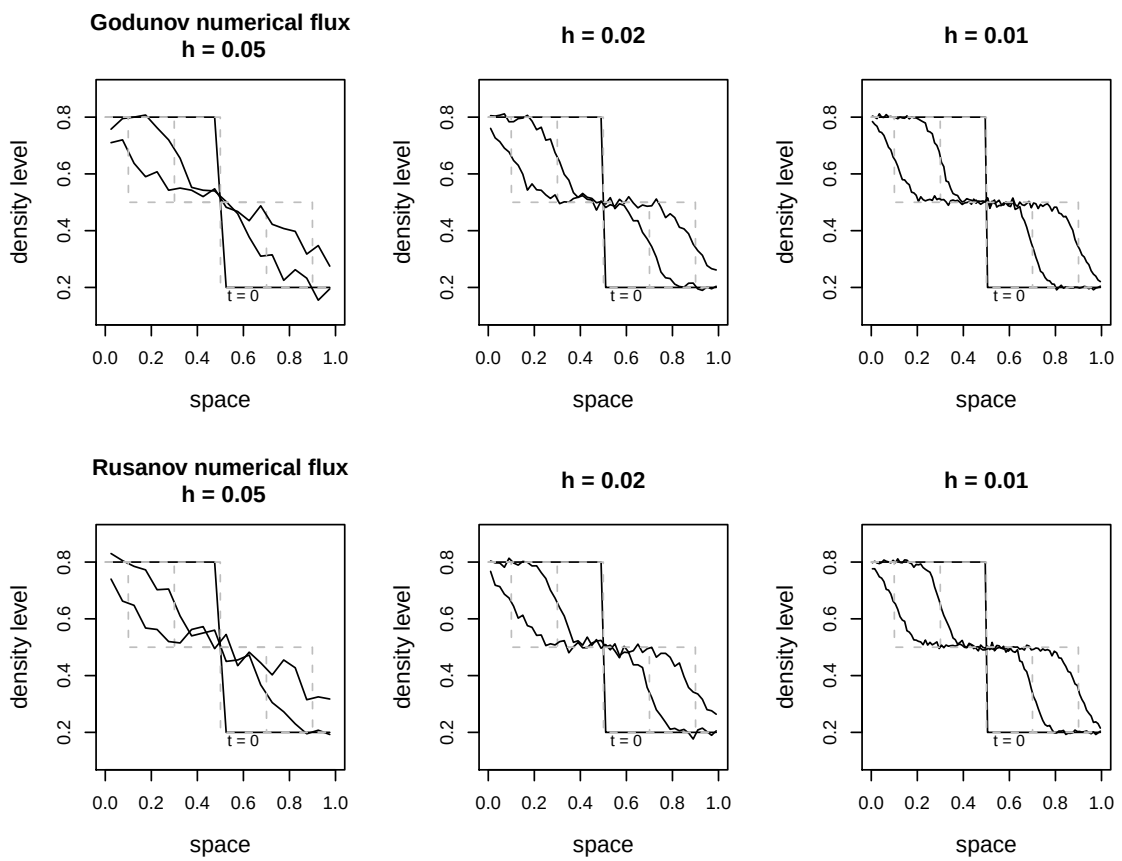

Figure 2: Exp. 1 – case of the triangular flux (64). Solution (in gray dotted lines) and the misanthrope process plot at time $t = \{0, 0.2, 0.4\}$ for $h = \{0.05, 0.02, 0.01\}$. Top, one uses the Godunov numerical flux and bottom the modified Rusanov one.

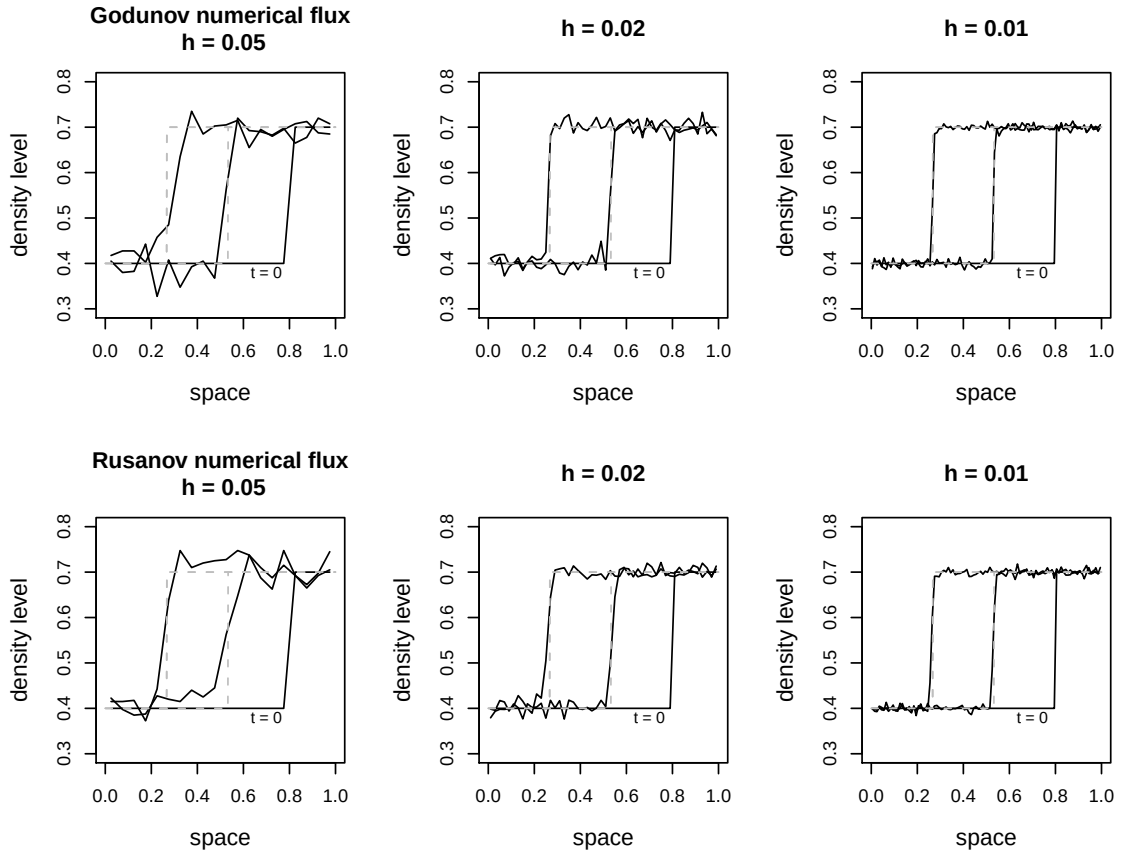

Figure 3: Exp. 2 – case of the triangular flux (64). Solution (in gray dotted lines) and the misanthrope process plot at time $t = \{0, 0.8, 1.6\}$ for $h = \{0.05, 0.02, 0.01\}$. Top, one uses the Godunov numerical flux and bottom the modified Rusanov one.

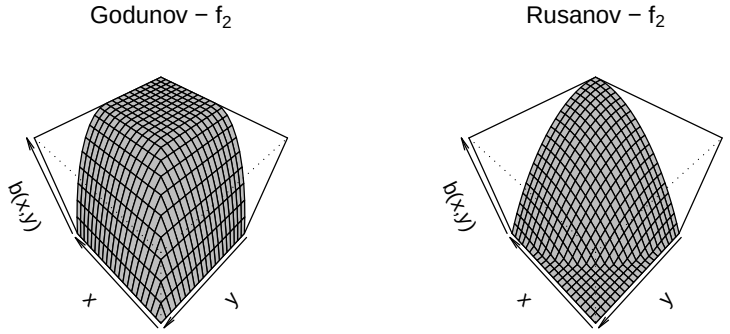

Figure 4: Jump rate function for the polynomial flux (65). Left, case of the Godunov numerical flux and, right, case of the modified Rusanov one.

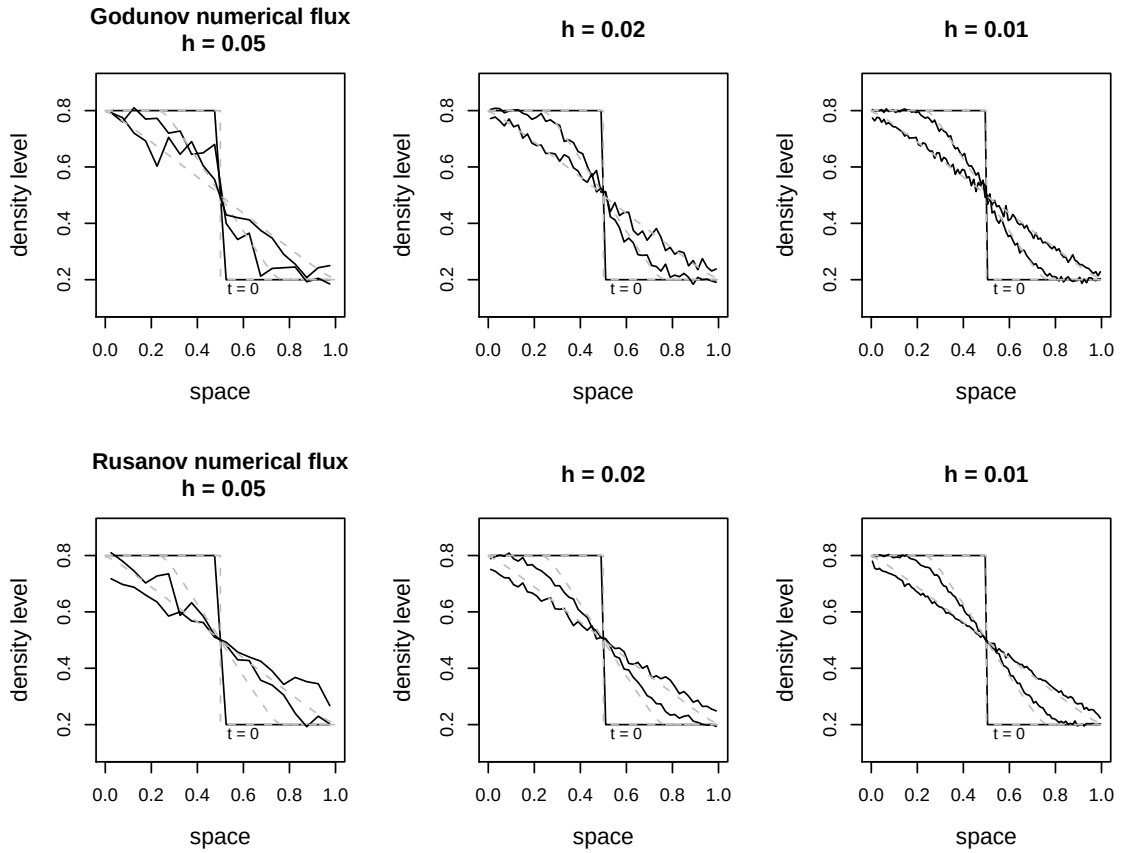

Figure 5: Exp. 1 – case of the polynomial flux (65). Solution (in gray dotted lines) and the misanthrope process plot at time $t = \{0, 0.4, 0.8\}$ for $h = \{0.05, 0.02, 0.01\}$. Top, one uses the Godunov numerical flux and bottom the modified Rusanov one.

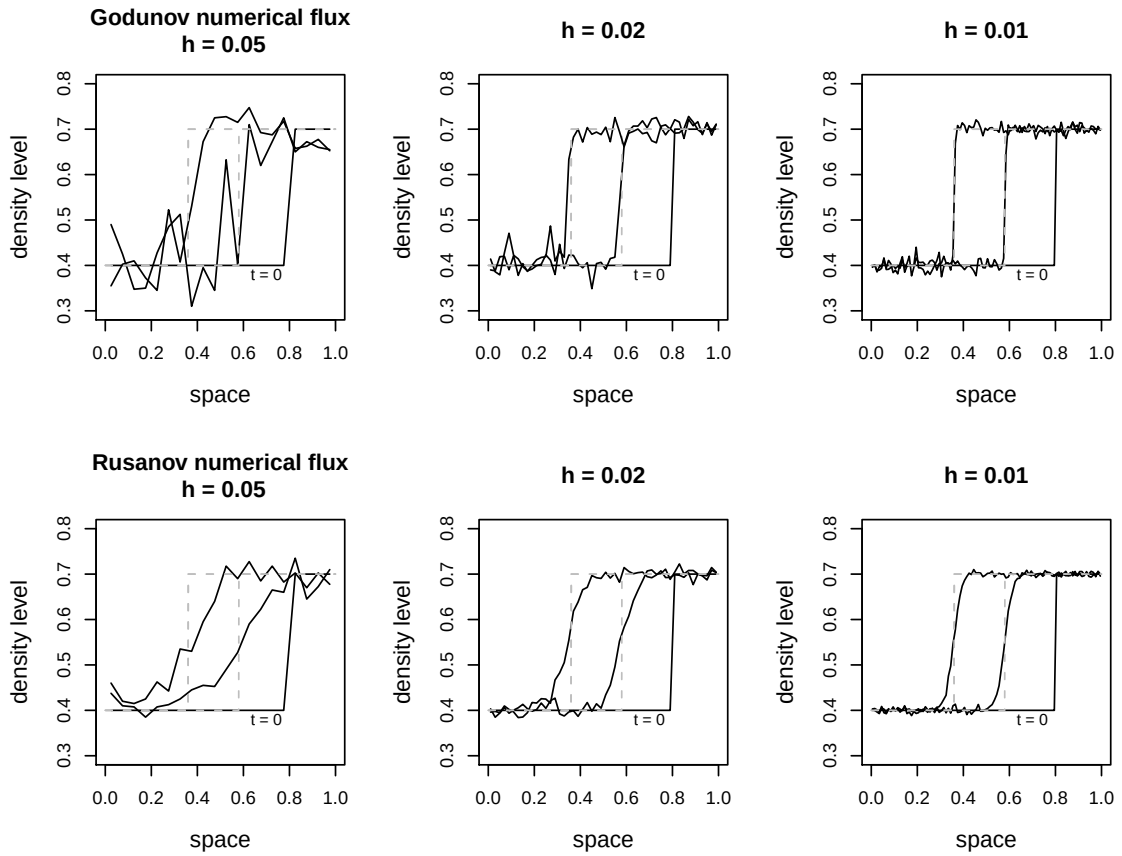

Figure 6: Exp. 2 – case of the polynomial flux (65). Solution (in gray dotted lines) and the misanthrope process plot at time $t = \{0, 2.2, 4.4\}$ for $h = \{0.05, 0.02, 0.01\}$. Top, one uses the Godunov numerical flux and bottom the modified Rusanov one.