A new class of $(\mathcal{H}^k, 1)$-rectifiable subsets of metric spaces

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Abstract

The main motivation of this paper arises from the study of Carnot–Carathéodory spaces, where the class of 1-rectifiable sets does not contain smooth non-horizontal curves; therefore a new definition of rectifiable sets including non-horizontal curves is needed. This is why we introduce in any metric space a new class of curves, called continuously metric differentiable of degree k, which are Hölder but not Lipschitz continuous when $k > 1$. Replacing Lipschitz curves by this kind of curves we define $(\mathcal{H}^k, 1)$-rectifiable sets and show a density result generalizing the corresponding one in Euclidean geometry. This theorem is a consequence of computations of Hausdorff measures along curves, for which we give an integral formula. In particular, we show that both spherical and usual Hausdorff measures along curves coincide with a class of dimensioned lengths and are related to an interpolation complexity, for which estimates have already been obtained in Carnot–Carathéodory spaces.

Contents

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1 Introduction

The main motivation of this paper arises from the study of Carnot–Carathéodory spaces. Recall that such a metric space (M, d) is defined by a sub-Riemannian manifold $(M, \mathcal{D}, g)$, where M is a smooth manifold, $\mathcal{D}$ a subbundle of TM and g a Riemannian metric on $\mathcal{D}$. The absolutely continuous paths which are almost everywhere tangent to $\mathcal{D}$ are called horizontal and their length is obtained as in Riemannian geometry integrating the norm of their tangent vectors. The distance d is defined as the infimum of length of horizontal paths between two given points.

By construction, only horizontal paths may have finite length and may be Lipschitz with respect to the distance. In contrast to the Euclidean case, both properties are independent on the regularity: all smooth non-horizontal paths have infinite length and are not Lipschitz. This gives rise to two kind of questions.

The first query concerns the measure of non-horizontal paths: what kind of notion is the best suited? One of our motivation is that, from an intrinsic point of view, computing measures of paths should allow to determine metric invariants of curves and thus recover metrically the structure of the manifold [14]. Since non-horizontal paths have a metric dimension greater than one (see [17]), Hausdorff measures are the most natural candidates. However they pose two problems: first they can hardly be computed (except for specific cases [1]), second they do not appear as integrals along the path, which is what we expect for a measure generalizing the notion of length.

The second question comes from geometric measure theory. A typical problem in this field is whether it is possible to characterize the geometric structure of a set using only measures. This gave rise to the notion of rectifiable sets, which is based on Lipschitz functions, and to density results in Euclidean (see [6, 9, 24] and [21] for a complete presentation) and general metric spaces [18]. In the context of Carnot–Carathéodory spaces rectifiable sets have been studied in Heisenberg groups (see [11, 22]) and a different notion of rectifiability was proposed in [19]. However, in these spaces the class of Lipschitz paths is quite poor and does not include non-horizontal smooth curves which consequently are not rectifiable in the usual sense. To take into account the latter curves we need to define rectifiability through a larger class of paths, intrinsically characterized by the distance.

In this paper we address these issues in any metric space, not only in Carnot–Carathéodory ones, by defining a class of curves in the spirit of ([2, 18]). Namely, we introduce curves on a metric space (M, d) that are continuously metric differentiable of degree k (m-$\mathcal{C}^1_k$ for short) as continuous curves $\gamma : [a, b] \to M$ such that the map

$$t \mapsto \mathrm{meas}^k_t(\gamma) = \left(\lim_{s \to 0} \frac{d(\gamma(t+s), \gamma(t))}{|s|^{1/k}} \right)^k$$

is well-defined and continuous (see Definition 1). In an Euclidean space, this definition is useless since the class of m-$\mathcal{C}^1_k$ curves with non-zero measure is empty when $k > 1$ (see Proposition 2). However, in the sub-Riemannian context, for integer values of k this class of curves contains some smooth non-horizontal paths (see Proposition 1).

For m-$\mathcal{C}^1_k$ curves we can compute different kind of measures. First, we examine the Hausdorff measures: the usual ones $\mathcal{H}^k$ and the spherical ones $\mathcal{S}^k$. Second, we study the k-dimensional length

of a curve $\gamma : [a, b] \to M$ introduced in [8] and defined by

$$\mathrm{Length}_k(\gamma([a,b])) = \int_a^b \mathrm{meas}_t^k(\gamma)dt.$$

Third, we consider a measure based on approximations by finite sets called interpolation complexity (see [13, 16]). The first result of the paper (Theorem 1) states that for an injective m-$\mathcal{C}^1_k$ curve $\gamma : [a, b] \to M$ we have

$$\mathcal{H}^k(C) = \mathcal{S}^k(C) = \mathrm{Length}_k(C),$$

where $C = \gamma([a, b])$. It also provides a relation between $\mathcal{H}^k(C)$ and the interpolation complexity. On the one hand, Theorem 1 gives an integral formula for the Hausdorff measure. On the other hand, it essentially implies that the considered measures are equivalent. Another interesting property of injective m-$\mathcal{C}^1_k$ curves with non-zero k-dimensional measure is that the k-dimensional density of $\mathcal{H}^k\lfloor_C$ exists and is constant along the curve (see Proposition 3).

We define $(\mathcal{H}^k, 1)$-rectifiable sets as sets that are covered, up to $\mathcal{H}^k$-null sets, by countable unions of m-$\mathcal{C}^1_k$ curves (see Definition 3). This notion is modeled on the definition of $(\mathcal{H}^k, k)$-rectifiable sets in $\mathbb{R}^n$, which are sets that are covered, up to $\mathcal{H}^k$-null sets, by countable unions of image of C^1 maps from $\mathbb{R}^k$ to $\mathbb{R}^n$. Thanks to the properties of m-$\mathcal{C}^1_k$ curves, we show a density result for sets that are rectifiable according to our definition. Namely, the second main theorem of the paper (Theorem 2) states that if a set S is $\mathcal{H}^k$-measurable and satisfies $\mathcal{H}^k(S) < +\infty$, then being $(\mathcal{H}^k, 1)$-rectifiable implies that the upper and lower densities of $\mathcal{H}^k\lfloor_S$ are bounded by positive constants.

Theorem 2 is inspired by the result proved in Federer [10, Th. 3.2.19], which states that for a $\mathcal{H}^k$ measurable subset E of the Euclidean n-space $(\mathcal{H}^k, k)$-rectifiability implies that the measure $\mathcal{H}^k\lfloor_E$ has k-dimensional density equal to 1 at $\mathcal{H}^k$-almost every point of E. The converse of this fact was proved for $k = 1$ and for a general measure μ in [23]. Much later, Preiss showed not only that the converse holds true for any k, but also a stronger result: there exists a constant $c > 1$ (depending only on n and k) such that if

$$0 < \limsup_{r\to 0^+} \frac{\mu(E \cap B(x, r))}{r^k} \le c \liminf_{r\to 0^+} \frac{\mu(E \cap B(x, r))}{r^k} < +\infty, \quad \text{for a.e. } x \in E,$$

then E is (μ, k)-rectifiable. Our Theorem 2 implies that an estimate of the type above is satisfied by $(\mathcal{H}^k, 1)$-rectifiable sets. An open question is whether an analogous of Preiss' result still holds in non-Euclidean metric spaces with our definition of $(\mathcal{H}^k, 1)$-rectifiability.

Another open problem is to show a Marstrand's type result (see [20, Th. 1]) for $(\mathcal{H}^k, 1)$-rectifiable subsets, at least in Carnot–Carathéodory spaces. In Section 2.2 we construct m-$\mathcal{C}^1_k$ curves in sub-Riemannian manifolds having nonzero k-dimensional measure for integer values of $k \ge 1$. When the curve is absolutely continuous, it is easy to see that being m-$\mathcal{C}^1_k$ with non-vanishing k-dimensional measure implies that k is an integer (see Corollary 1). The question is whether such result holds true without assuming absolute continuity.

The structure of the paper is the following. In Section 2 we give the definition of m-$\mathcal{C}^1_k$ curves in metric spaces and construct them in Carnot–Carathéodory spaces. We then study measures along curves. In Section 3.1 we recall different notions of measures. In Section 3.2 we show an auxiliary result for m-$\mathcal{C}^1_k$ injective curves with nonzero k-dimensional measure. In Section 3.3 we analyse m-$\mathcal{C}^1_k$ curves in (the Euclidean space or a) Riemannian manifold. The main theorem concerning injective m-$\mathcal{C}^1_k$ curves is proved in Section 3.4. Some possible generalizations to non m-$\mathcal{C}^1_k$ curves

are discussed in Section 3.5. Finally, in Section 4 we define $(\mathcal{H}^k, 1)$-rectifiable sets and prove the density result.

2 m-$\mathcal{C}^1_k$ curves

Throughout the paper (M, d) denotes a metric space.

2.1 Definitions

Let $\gamma : [a, b] \to M$ be a continuous curve, where $a, b \in \mathbb{R}$, and let $k \geq 1$ be a real number.

Definition 1. We say that γ is *m-differentiable of degree k at $t \in [a, b]$* if the limit

$$\lim_{\substack{s\to 0\\ t+s\in[a,b]}} \frac{d(\gamma(t+s), \gamma(t))}{|s|^{1/k}} \tag{1}$$

exists and is finite. In this case, we call this limit the *metric derivative of degree k of γ at t* and we define moreover the *k-dimensional infinitesimal measure of γ at t* as

$$\mathrm{meas}^k_t(\gamma) = \left(\lim_{\substack{s\to 0\\ t+s\in[a,b]}} \frac{d(\gamma(t+s), \gamma(t))}{|s|^{1/k}} \right)^k.$$

When γ is not m-differentiable of degree k at t we set $\mathrm{meas}^k_t(\gamma) = +\infty$.

For the case $k = 1$, the notion of metric derivative is classical, see [3, Def. 4.1.2]. The k-dimensional infinitesimal measures of curves were introduced in the context of sub-Riemannian geometry in [8].

Note that if γ is m-differentiable of degree k at t then, for any k',

$$\lim_{\substack{s\to 0\\ t+s\in[a,b]}} \frac{d(\gamma(t+s), \gamma(t))}{|s|^{1/k'}} = \lim_{\substack{s\to 0\\ t+s\in[a,b]}} \frac{1}{|s|^{1/k'-1/k}} \frac{d(\gamma(t+s), \gamma(t))}{|s|^{1/k}}.$$

Therefore, for any $k' > k$, $\mathrm{meas}^{k'}_t(\gamma) = 0$. If moreover $\mathrm{meas}^k_t(\gamma) > 0$, then for any $k' < k$ $\mathrm{meas}^{k'}_t(\gamma) = +\infty$.

Definition 2. Given $k \geq 1$, we say that γ is *differentiable of class m-$\mathcal{C}^1_k$ on $[a, b]$* (m-$\mathcal{C}^1_k$ for short) if for every $t \in [a, b]$ the curve is m-differentiable of degree k at t and the map $t \mapsto \mathrm{meas}^k_t(\gamma)$ is continuous.

Clearly, γ is m-$\mathcal{C}^1_k$ if and only if the limit in (1) exists and depends continuously on t.

We shall see in the next section that when a smooth structure on M exists, m-$\mathcal{C}^1_k$ curves need not be differentiable in the usual sense. The following lemma states that they are Hölder continuous of exponent $1/k$ as functions from an interval to the metric space (M, d).

Lemma 1. *Let $\gamma : [a, b] \to M$ be m-$\mathcal{C}^1_k$ on $[a, b]$, $k \geq 1$. For any t and $t + s$ in $[a, b]$,*

$$d(\gamma(t), \gamma(t+s)) = |s|^{1/k}(\mathrm{meas}^k_t(\gamma)^{1/k} + \epsilon_t(s)), \tag{2}$$

where $\epsilon_t(s)$ tends to zero as s tends to zero uniformly with respect to t.

This is a direct consequence of the continuity of $t \mapsto \mathrm{meas}^k_t(\gamma)$ on the compact interval $[a, b]$.

2.2 Construction of m-$\mathcal{C}^1_k$ curves

In this section we consider a class of metric spaces which are also smooth manifolds and construct smooth m-$\mathcal{C}^1_k$ curves on them with non-vanishing metric derivative of degree k for some integer values of k. The analysis of this class of spaces is the main motivation of this paper.

Let (M, d) be a metric space defined by a sub-Riemannian manifold $(M, \mathcal{D}, g)$, i.e., M is a smooth manifold, $\mathcal{D}$ a subbundle of TM, g a Riemannian metric on $\mathcal{D}$, and d is the associated sub-Riemannian distance. We assume that Chow's condition is satisfied: let $\mathcal{D}^s$ denote the $\mathbb{R}$-linear span of brackets of degree $< s$ of vector fields tangent to $\mathcal{D}^1 = \mathcal{D}$; then, at every $p \in M$, there exists an integer $r = r(p)$ such that $\mathcal{D}^{r(p)}(p) = T_pM$, that is,

$$\{0\} \subset \mathcal{D}^1(p) \subset \mathcal{D}^2(p) \subset \cdots \subset \mathcal{D}^{r(p)}(p) = T_pM. \tag{3}$$

Let $A \subset M$. A point $p \in A$ is said *A-regular* if the sequence of dimensions $n_i(q) = \dim \mathcal{D}^i(q)$, $i = 1, \dots r(q)$ remains constant for $q \in A$ near p, and *A-singular* otherwise. The set A is said *equiregular* if every point of A is A-regular. A curve $\gamma : [a, b] \to M$ is *equiregular* if $\gamma([a, b])$ is equiregular.

Proposition 1. *Let $\gamma : [a, b] \to M$ be an equiregular curve of class $\mathcal{C}^1$ and $k \in \mathbb{N}$ such that $\dot\gamma(t) \in \mathcal{D}^k(\gamma(t))$ for every $t \in [a, b]$. Then γ is m-$\mathcal{C}^1_k$ on $[a, b]$.*

If moreover $\dot\gamma(t) \notin \mathcal{D}^{k-1}(\gamma(t))$ for a given $t \in [a, b]$ then $\mathrm{meas}^k_t(\gamma) \neq 0$.

The proof of this proposition is based on the notions of nilpotent approximation and privileged coordinates (see [5]) and some results in [8]. We do not give the complete argument, but only the underlying ideas. All the facts that here are simply claimed are already established and complete proofs can be found in the cited literature.

Sketch of the proof. Since $\gamma([a, b])$ is equiregular, the integers w_i defined by

$$w_i = j, \quad \text{if } n_{j-1}(\gamma(t)) < i \leq n_j(\gamma(t)), \quad i = 1, \dots, n,$$

do not depend on t. We define for $s \geq 0$ the dilation $\delta_s : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\delta_s z = (s^{w_1} z_1, \dots, s^{w_n} z_n).$$

Moreover, locally there exist n vector fields $Y_1, \dots, Y_n$ whose values at each $\gamma(t)$ form a basis of $T_{\gamma(t)}M$ adapted to the filtration (3) at $\gamma(t)$, in the sense that, for every integer $i \geq 1$, $Y_1(\gamma(t)), \dots, Y_{n_i}(\gamma(t))$ is a basis of $\mathcal{D}^i(\gamma(t))$. The local diffeomorphism

$$x \in \mathbb{R}^n \mapsto \exp(x_n Y_n) \circ \cdots \circ \exp(x_1 Y_1)(\gamma(t))$$

defines a system of coordinates $\phi^t : q \mapsto x = (x_1, \dots, x_n)$ on a neighborhood of $\gamma(t)$, satisfying $\phi^t(\gamma(t)) = 0$. Following [5, Sec. 5.3], there exists a sub-Riemannian distance $\widehat{d}_t$ on $\mathbb{R}^n$ such that

- $\widehat{d}_t$ is homogeneous under the dilation δ_s, i.e., $\widehat{d}_t(\delta_s x, \delta_s x') = s\widehat{d}_t(x, x')$ for all $s \geq 0, x, x' \in \mathbb{R}^n$;
- when defined, the mapping $t \mapsto \widehat{d}_t(\phi^t(q), \phi^t(q'))$ is continuous;
- for q in a neighborhood of $\gamma(t)$, $d(\gamma(t), q) = \widehat{d}_t(0, \phi^t(q))(1 + \epsilon_t(\widehat{d}_t(0, \phi^t(q))))$, where $\epsilon_t(s)$ tends to zero as s tends to zero uniformly with respect to t.

The coordinates ϕ^t are privileged at $\gamma(t)$ and the distance $\widehat{d}_t$ is the sub-Riemannian distance associated with a nilpotent approximation at $\gamma(t)$.

Set $\phi^t(\gamma(t)) = (\gamma_1(t), \dots, \gamma_n(t))$. By the construction in the proof of [8, Le. 12], the limit

$$\lim_{\substack{s\to 0\\ t+s\in[a,b]}} \delta_{|s|^{-1/k}}\phi^t(\gamma(t+s))$$

exists at every t and is equal to $x(t) = (x_1(t), \dots, x_n(t))$, where

$$x_j(t) = \begin{cases} 0, & w_j \neq k \\ \dot\gamma_j(t), & w_j = k. \end{cases}$$

Using the properties of $\widehat{d}_t$, we have

$$\lim_{s\to 0} \frac{d(\gamma(t+s),\gamma(t))}{|s|^{1/k}} = \lim_{s\to 0} \frac{\widehat{d}_t(\phi^t(\gamma(t+s)),0)}{|s|^{1/k}} = \lim_{s\to 0} \widehat{d}_t(\delta_{|s|^{-1/k}}\phi^t(\gamma(t+s)),0) = \widehat{d}_t(x(t),0).$$

As a consequence, $\mathrm{meas}^k_t(\gamma)$ exists and is equal to $\widehat{d}_t(x(t),0)^k$. Since the components of $x(t)$ are continuous and the distance $\widehat{d}_t$ depends continuously on t, γ is m-$\mathcal{C}^1_k$. If moreover $\dot\gamma(t) \notin \mathcal{D}^{k-1}(\gamma(t))$ for a given $t \in [a,b]$ then $x(t) \neq 0$, whence $\mathrm{meas}^k_t(\gamma) \neq 0$. ■

Let us explain the construction in Proposition 1 through an example.

Example 1. Consider the Heisenberg group, that is, the sub-Riemannian manifold $(\mathbb{R}^3, \mathcal{D}, g)$ where $\mathcal{D}$ is the linear span of the vector fields

$$X_1(x,y,z) = (1,0,-y/2), \quad X_2(x,y,z) = (0,1,x/2),$$

and $g = dx^2 + dy^2$. Denote by d the Carnot–Carathéodory distance associated with the Heisenberg group. Recall that d is homogeneous with respect to the dilation

$$\delta_\lambda(x,y,z) = (\lambda x, \lambda y, \lambda^2 z), \quad \lambda \geq 0.$$

and it is invariant with respect to the group law

$$(x,y,z) * (x',y',z') = \left(x+x', y+y', z+z' + \frac{1}{2}(xy' - x'y)\right).$$

Moreover, for each point $(x,y,z) \in \mathbb{R}^3$, $\mathcal{D}^2(x,y,z) = \mathbb{R}^3$ as $[X_1, X_2](x,y,z) = (0,0,1)$.

Let $\gamma(t) = (0,0,t)$, for $t \in \mathbb{R}$. Then γ is of class m-$\mathcal{C}^1_2$ and $\mathrm{meas}^2_t(\gamma)$ is a positive constant. This is a consequence of Proposition 1 as γ is smooth and, for all $t \in \mathbb{R}$, $\dot\gamma(t) \in \mathcal{D}^2(\gamma(t))$. Let us compute explicitly $\mathrm{meas}^2_t(\gamma)$. Notice first that $d(\gamma(t+s),\gamma(t)) = d((0,0,s),(0,0,0))$, since d is invariant with respect to the group law. Hence, using the homogeneity of d and the fact $d((0,0,1),0) = d((0,0,-1),0)$,

$$\lim_{s\to 0} \frac{d(\gamma(t+s),\gamma(t))}{|s|^{1/2}} = \lim_{s\to 0} \frac{d((0,0,s),0)}{|s|^{1/2}} = d((0,0,1),0) = 2\sqrt{\pi},$$

the value of the distance resulting from an isoperimetric problem. Note that such a computation can be generalized to any contact sub-Riemannian manifold, see [8, Th. 22].

Remark 1. Note that the equiregularity assumption is essential to obtain the continuity of $\widehat{d}_t$ and ϕ^t with respect to t. In particular the proof of [8, Le. 12] is not valid without this hypothesis[1]. This assumption has also an intrinsic meaning. Indeed it is shown in [8] that the k-dimensional measure $\mathrm{meas}_t^k(\gamma)$ can actually be defined through the distance on the metric tangent space to (M,d) at $\gamma(t)$. Since the metric tangent space does not vary continuously with respect to t around C-singular points, where $C=\gamma([a,b])$, in general non equiregular curves may not be m-$\mathcal{C}^1_k$.

Note that for every integer $k\in\{1,\dots,r(p)\}$, where p is regular, there exist $\mathcal{C}^1$ equiregular curves with tangent vector belonging to $\mathcal{D}^k\setminus\mathcal{D}^{k-1}$. As a consequence, for such integers k the class of m-$\mathcal{C}^1_k$ curves with non-vanishing metric derivative of degree k is not empty. For instance, this is the case in the Heisenberg group for $k=2$, and in the Engel group (see below) for $k=2,3$. On the contrary, the next proposition states that in the Riemannian case, i.e., when $\mathcal{D}=TM$, the class of m-$\mathcal{C}^1_k$ curves with non-vanishing derivative is empty except for $k=1$ (the proof of Proposition 2 is postponed to Section 3.3).

Proposition 2. *Let (M,g) be a Riemannian manifold. Let $k\geq 1$ and assume that $\gamma:[a,b]\to M$ is a m-$\mathcal{C}^1_k$ curve such that $t\mapsto \mathrm{meas}_t^k(\gamma)$ does not vanish identically. Then $k=1$.*

Let $\gamma:[a,b]\to M$ be of class m-$\mathcal{C}^1_1$ and such that $\mathrm{meas}_t^1(\gamma)\neq 0$ for every $t\in[a,b]$. Then γ is horizontal, i.e., it is absolutely continuous and $\dot\gamma(t)\in\mathcal{D}^1(\gamma(t))$ almost everywhere on $[a,b]$. To see this, remark that by construction, γ is Lipschitz with respect to the sub-Riemannian distance. The metric g defined on $\mathcal{D}$ can be extended (at least in a tubular neighbourhood of $\gamma([a,b])$) to a Riemannian metric $\tilde g$ on TM. In this way we obtain a Riemannian distance on M which is not greater than the sub-Riemannian distance. Hence γ is Lipschitz with respect to the chosen Riemannian distance which in turn implies that γ is absolutely continuous. Therefore, by [7, Pr. 5] γ is horizontal, i.e., $\dot\gamma(t)\in\mathcal{D}^1(\gamma(t))$ almost everywhere on $[a,b]$.

Using Proposition 1, this fact can be partially generalized to the case $k>1$ under the following form.

Corollary 1. *Let $k\geq 1$ and let $\gamma:[a,b]\to M$ be equiregular and of class m-$\mathcal{C}^1_k$, with $\mathrm{meas}_t^k(\gamma)\not\equiv 0$. If γ is absolutely continuous, then k is the smallest integer m such that $\dot\gamma(t)\in\mathcal{D}^m(\gamma(t))$ almost everywhere.*

In particular Corollary 1 states that if $\mathrm{meas}_t^k(\gamma)\not\equiv 0$ then k is an integer, provided that γ is absolutely continuous. An open question is whether the latter condition is necessary. If this were not the case then we would obtain a Marstrand's type Theorem [20, Th. 1] for m-$\mathcal{C}^1_k$ curves: indeed we shall see in Proposition 3 that along injective m-$\mathcal{C}^1_k$ curves with non-vanishing k-dimensional measure the density of $\mathcal{H}^k$ exists and is constant.

Nevertheless, a m-$\mathcal{C}^1_k$ curve need not be $\mathcal{C}^1$ in the usual sense as it is shown below.

Example 2. Consider the Engel group, that is, the sub-Riemannian manifold $(\mathbb{R}^4,\mathcal{D},g)$ where $\mathcal{D}$ is the linear span of the vector fields

$$X_1(x,y,z,w)=(1,0,0,0),\quad X_2(x,y,z,w)=(0,1,x,x^2/2),$$

and $g=dx^2+dy^2$. Let $\gamma(t)=(0,0,W(t),\varphi(t))$, where $\varphi\in\mathcal{C}^1$ and W is the Weierstrass function

$$W(t)=\sum_{n=0}^{\infty}\alpha^n(\cos(\beta^n\pi t)-1),\quad t\in\mathbb{R},$$

[1]The statement of Lemma 12 in [8] is incorrect. Indeed without equiregularity formula (3) therein does not hold.

where $0 < \alpha < 1$, $\beta > 1$, and $\alpha\beta > 1$ see [25]. It was proved in [15] that $W(t)$ is continuous, nowhere differentiable on the real line, and satisfies

$$W(t+h) - W(t) = O(|h|^\xi), \quad \xi = \frac{\log(1/\alpha)}{\log \beta} < 1, \tag{4}$$

uniformly with respect to $t \in \mathbb{R}$. Then, choosing α, β such that $\xi > 2/3$, γ is continuous and m-$\mathcal{C}^1_3$, but nowhere differentiable. Indeed, it is not hard to verify that the sub-Riemannian distance d satisfies the following homogeneity property

$$\lambda d((0,0,\bar z,\bar w),(0,0,z,w)) = d(0,(0,0,\lambda^2(z-\bar z),\lambda^3(w-\bar w))),$$

for every $\lambda \geq 0$. Then we have

$$\lim_{s\to 0} \frac{1}{|s|^{1/3}} d((0,0,W(t),\varphi(t)),(0,0,W(t+s),\varphi(t+s))) = \lim_{s\to 0} d\left(0,\left(0,0,\frac{O(|s|^\xi)}{|s|^{2/3}},\varphi'(t)\right)\right).$$

Since $\xi > 2/3$, γ is m-differentiable of degree 3 at each t and $\mathrm{meas}^3_t(\gamma) = d(0,(0,0,0,\varphi'(t))^3$, which is non-zero for a suitable choice of φ. Therefore, γ is m-$\mathcal{C}^1_3$ and by the properties of $W(t)$, γ is nowhere differentiable.

Notice that if γ is m-$\mathcal{C}^1_k$ and $k' \geq k$, then γ is m-$\mathcal{C}^1_{k'}$. Define $k_\gamma \geq 1$ as the infimum of $k \geq 1$ such that γ is m-$\mathcal{C}^1_k$. Then k_γ need not be an integer as it is shown in the next example. Moreover, γ is not necessarily m-$\mathcal{C}^1_{k_\gamma}$.

Example 3. Consider the sub-Riemannian structure of Example 2 and the curve $\gamma(t) = (0,0,W(t),0)$. Then $k_\gamma = 2/\xi$ may be any real number greater than 2 (see (4)), but γ is not m-$\mathcal{C}^1_{k_\gamma}$.

3 Measures along curves

This section is devoted to compute Hausdorff (and spherical Hausdorff) measures of continuous curves and to establish a relation with the k-dimensional length and with the complexity.

3.1 Different notions of measures

Denote by $\operatorname{diam} S$ the diameter of a set $S \subset M$. Let $k \geq 0$ be a real number. For every set $A \subset M$, we define the *k-dimensional Hausdorff measure* $\mathcal{H}^k$ of A as $\mathcal{H}^k(A) = \lim_{\epsilon\to 0^+} \mathcal{H}^k_\epsilon(A)$, where

$$\mathcal{H}^k_\epsilon(A) = \inf\left\{\sum_{i=1}^\infty (\operatorname{diam} S_i)^k \,:\, A \subset \bigcup_{i=1}^\infty S_i,\ \operatorname{diam} S_i \leq \epsilon,\ S_i \text{ closed set}\right\},$$

and the *k-dimensional spherical Hausdorff measure* $\mathcal{S}^k$ of A as $\mathcal{S}^k(A) = \lim_{\epsilon\to 0^+} \mathcal{S}^k_\epsilon(A)$, where

$$\mathcal{S}^k_\epsilon(A) = \inf\left\{\sum_{i=1}^\infty (\operatorname{diam} S_i)^k \,:\, A \subset \bigcup_{i=1}^\infty S_i,\ S_i \text{ is a ball},\ \operatorname{diam} S_i \leq \epsilon\right\}.$$

In the Euclidean space $\mathbb{R}^n$, k-dimensional Hausdorff measures are often defined as $2^{-k}\alpha(k)\mathcal{H}^k$ and $2^{-k}\alpha(k)\mathcal{S}^k$, where $\alpha(k)$ is defined from the usual gamma function as $\alpha(k) = \Gamma(\frac{1}{2})^k/\Gamma(\frac{k}{2}+1)$.

This normalization factor is necessary for the n-dimensional Hausdorff measure and the Lebesgue measure coincide on $\mathbb{R}^n$.

For a given set $A \subset M$, $\mathcal{H}^k(A)$ is a decreasing function of k, infinite when k is smaller than a certain value, and zero when k is greater than this value. We call *Hausdorff dimension* of A the real number

$$\dim_{\mathcal{H}} A = \sup\{k \,:\, \mathcal{H}^k(A) = \infty\} = \inf\{k \,:\, \mathcal{H}^k(A) = 0\}.$$

Note that $\mathcal{H}^k \leq \mathcal{S}^k \leq 2^k \mathcal{H}^k$, so the Hausdorff dimension can be defined equally from Hausdorff or spherical Hausdorff measures.

When the set A is a curve, another kind of dimensioned measures can be obtained from the integration of k-dimensional infinitesimal measures. Let $\gamma : [a, b] \to M$ be a continuous curve and $C = \gamma([a, b])$. For $k \geq 1$, we define the *k-dimensional length* of C as

$$\text{Length}_k(C) = \int_a^b \text{meas}_t^k(\gamma)\, dt. \tag{5}$$

where $\text{meas}_t^k(\gamma)$ is as in Definition 1 (these lengths were introduced in [8] in the sub-Riemannian context). Thanks to the properties of $\text{meas}_t^k(\gamma)$, $\text{Length}_k(\gamma)$ is a decreasing function of k, infinite when k is smaller than a certain value, and zero when k is greater than this value. We call this value the *length dimension of C*.

Another way to measure the set C is to study its approximations by finite sets (see [16] and [14, p. 278]). Here we only consider approximations by ϵ-chains of C, i.e., sets of points $q_1 = \gamma(a)$, ..., $q_N = \gamma(b)$ in C such that $d(q_i, q_{i+1}) \leq \epsilon$. The *interpolation complexity* $\sigma_{\text{int}}(C, \epsilon)$ is the minimal number of points in an ϵ-chain of C. This complexity has been computed in several cases in [12].

Remark 2. Notice that for any injective m-$\mathcal{C}^1_1$ curve the equality $\mathcal{H}^1(C) = \text{Length}_1(C)$ holds (see [3, Th. 4.1.6, 4.4.2]).

3.2 m-$\mathcal{C}^1_k$ curves with non-vanishing k-dimensional measure

In this section we prove the following proposition about m-$\mathcal{C}^1_k$ curves with non-vanishing k-dimensional measure. This result is the first step to prove Theorem 1.

Proposition 3. *Let $\gamma : [a, b] \to M$ be an injective m-$\mathcal{C}^1_k$ curve and $C = \gamma([a, b])$. Assume $\text{meas}_t^k(\gamma) \neq 0$ for every t. Then*

$$\mathcal{H}^k(C) = \mathcal{S}^k(C) = \text{Length}_k(C) \tag{6}$$

$$\lim_{\epsilon \to 0^+} \epsilon^k \sigma_{\text{int}}(C, \epsilon) = \text{Length}_k(C), \tag{7}$$

and for every $q \in C$

$$\lim_{r \to 0^+} \frac{\mathcal{H}^k(C \cap B(q, r))}{2r^k} = 1. \tag{8}$$

Remark 3. Equations (6), (8) hold when we replace $[a, b]$ by the open interval (a, b). Also, they hold for unbounded intervals. Therefore, thanks to the regularity of $\mathcal{L}^1$ and $\mathcal{H}^k$ measures, equations (6), (8) are still verified when we replace C by $\gamma(A)$, for any measurable set $A \subset [a, b]$.

If we drop the injectivity assumption we obtain the following weaker result.

Corollary 2. *Let $\gamma : [a,b] \to M$ be a m-$\mathcal{C}^1_k$ curve and $C = \gamma([a,b])$. Assume* $\mathrm{meas}^k_t(\gamma) \neq 0$ *for every t. Then*

$$\mathcal{H}^k(C) = \mathcal{S}^k(C) \leq \mathrm{Length}_k(C). \tag{9}$$

Proof of Corollary 2. Since $\mathrm{meas}^k_t(\gamma) \neq 0$ for every $t \in [a,b]$, γ is locally injective. Hence $[a,b]$ is the disjoint union of a finite family of intervals I_i such that $\gamma|_{I_i}$ is injective. For each i there exists a measurable subset $A_i \subset I_i$ such that $C = \cup\gamma(A_i)$ and the sets $\gamma(A_i)$ are pairwise disjoint. Using Remark 3, formula (6) applies to each $\gamma(A_i)$. Since $\mathcal{H}^k(C) = \sum_i \mathcal{H}^k(\gamma(A_i))$ and $\mathcal{S}^k(C) = \sum_i \mathcal{S}^k(\gamma(A_i))$, we obtain (9).

■

The proof of Proposition 3 is based on the following result for bi-Hölder continuous curves.

Lemma 2. *Let $\gamma : [0,T] \to M$ be an injective curve and $C = \gamma([0,T])$. Assume that there exist positive constants δ_-, δ_+, and η such that*

$$\delta_- |s|^{1/k} \leq d\big(\gamma(t), \gamma(t+s)\big) \leq \delta_+ |s|^{1/k}, \tag{10}$$

for every $t, t+s \in [0,T]$ with $|s| < \eta$. Then

$$\delta_-^k T \leq \mathcal{H}^k(C) \leq \delta_+^k T, \tag{11}$$

$$\delta_-^k T \leq \liminf_{\epsilon \to 0^+} \epsilon^k \sigma_{\mathrm{int}}(C,\epsilon) \leq \limsup_{\epsilon \to 0^+} \epsilon^k \sigma_{\mathrm{int}}(C,\epsilon) \leq \delta_+^k T, \tag{12}$$

$$\mathcal{S}^k(C) \geq \mathcal{H}^k(C) \geq \left(\frac{\delta_-}{\delta_+}\right)^{2k} \mathcal{S}^k(C), \tag{13}$$

and, for every $t \in [0,T]$ and $r > 0$ small enough,

$$\left(\frac{\delta_-}{\delta_+}\right)^k \leq \frac{\mathcal{H}^k(C \cap B(\gamma(t), r))}{2r^k} \leq \left(\frac{\delta_+}{\delta_-}\right)^k. \tag{14}$$

Proof. Let $\epsilon > 0$ be smaller than $\delta_+ \eta^{1/k}$. We denote by N the smallest integer such that $T \leq N(\frac{\epsilon}{\delta_+})^k$ and define $t_0, \ldots, t_N$ by

$$t_i = i\big(\frac{\epsilon}{\delta_+}\big)^k \quad \text{for } i = 0, \ldots, N-1, \qquad t_N = T.$$

Set $S_i = \gamma([t_{i-1}, t_i])$, $i = 1, \ldots, N$. For t, t' in S_i, one has $|t - t'| \leq \epsilon^k/\delta_+^k$; it follows from (10) that

$$d(\gamma(t), \gamma(t')) \leq \delta_+ |t - t'|^{1/k} \leq \epsilon, \tag{15}$$

which in turn implies $\mathrm{diam}\, S_i \leq \epsilon$. Thus $\mathcal{H}^k_\epsilon(C) \leq \sum_i (\mathrm{diam}\, S_i)^k \leq N\epsilon^k$. Using $(N-1)\epsilon^k < T\delta_+^k$, we obtain

$$\mathcal{H}^k_\epsilon(C) \leq \delta_+^k T + \epsilon^k.$$

It also results from inequality (15) that $\gamma(t_0), \ldots, \gamma(t_N)$ is an ϵ-chain of C which implies

$$\epsilon^k \sigma_{\mathrm{int}}(C,\epsilon) \leq \delta_+^k T + \epsilon^k,$$

Taking the limit as $\epsilon \to 0$ in the preceding inequalities, we find

$$\mathcal{H}^k(C) \leq \delta_+^k T \quad \text{and} \quad \limsup_{\epsilon \to 0^+} \epsilon^k \sigma_{\mathrm{int}}(C,\epsilon) \leq \delta_+^k T.$$

We now prove converse inequalities for $\mathcal{H}^k$ and $\sigma_{\rm int}$. Fix $\epsilon > 0$ and consider a countable family $S_1, S_2, \dots$ of closed subsets of M such that $C \subset \bigcup_i S_i$ and $\operatorname{diam} S_i \leq \epsilon$. For every $i \in \mathbb{N}$, we set $I_i = \gamma^{-1}(S_i \cap C)$. As γ is injective, if ϵ is small enough then it results from (10) that for any t, t' in I_i there holds

$$\operatorname{diam} S_i \geq d(\gamma(t), \gamma(t')) \geq \delta_- |t - t'|^{1/k},$$

which implies $\mathcal{L}^1(I_i) \leq (\operatorname{diam} S_i)^k / \delta_-^k$. Note that $T \leq \sum_i \mathcal{L}^1(I_i)$ since the sets I_i cover $[0, T]$. It follows that $\mathcal{H}^k_\epsilon(C) \geq T\delta_-^k$, that is,

$$\mathcal{H}^k_\epsilon(C) \geq \delta_-^k T. \tag{16}$$

In the same way, an ϵ-chain $\gamma(t_0) = \gamma(0), \dots, \gamma(t_N) = \gamma(T)$ of C satisfies $N\epsilon^k \geq T\delta_-^k$ since the injectivity of γ assures that

$$\epsilon \geq d(\gamma(t_{i-1}), \gamma(t_i)) \geq \delta_- |t_i - t_{i-1}|^{1/k}.$$

It follows that $\epsilon^k \sigma_{\rm int}(C, \epsilon) \geq \delta_-^k T$. Taking the limit as $\epsilon \to 0$ in this inequality and in (16), we find $\mathcal{H}^k(C) \geq \delta_-^k T$ and $\liminf_{\epsilon \to 0^+} \epsilon^k \sigma_{\rm int}(C, \epsilon) \geq \delta_-^k T$, which completes the proof of (11) and (12).

The first inequality in (13) always holds. Before proving the second one, let us recall a standard result in geometric measure theory (see for instance [10, 2.10.18, (1)]). *Let X be a metric space and μ be a regular measure on X such that the closed balls in X are μ-measurable. If*

$$\limsup_{\substack{y \in B(x,r) \\ r \to 0^+}} \frac{\mu(B(y,r))}{(\operatorname{diam} B(y,r))^k} \geq \lambda,$$

for every point $x \in X$, then $\mu(X) \geq \lambda \mathcal{S}^k(X)$. We will apply this result to the metric space $(C, d|_C)$ and to the measure $\mu = \mathcal{H}^k \lfloor_C$.

If $t \in [0, T]$, then for $r > 0$ small enough, there holds

$$\gamma([t - \tfrac{r^k}{\delta_+^k}, t + \tfrac{r^k}{\delta_+^k}]) \subset C \cap B(\gamma(t), r) \subset \gamma([t - \tfrac{r^k}{\delta_-^k}, t + \tfrac{r^k}{\delta_-^k}]). \tag{17}$$

The diameter of this set then satisfies

$$\operatorname{diam}(C \cap B(\gamma(t), r)) \leq d\left(\gamma\left(t - \frac{r^k}{\delta_+^k}\right), \gamma\left(t + \frac{r^k}{\delta_+^k}\right)\right) \leq \delta_+ \frac{2^{\frac{1}{k}} r}{\delta_-}.$$

Moreover, applying (11) to the curve γ restricted to $[t - \frac{r^k}{\delta_+^k}, t + \frac{r^k}{\delta_+^k}]$, we obtain

$$\mathcal{H}^k(C \cap B(\gamma(t), r)) \geq \delta_-^k \frac{2r^k}{\delta_+^k}.$$

Thus we have, for every point $\gamma(t') \in C$,

$$\limsup_{\substack{r \to 0^+ \\ \gamma(t) \in B(\gamma(t'), r)}} \frac{\mathcal{H}^k \lfloor_C (B(\gamma(t), r))}{(\operatorname{diam}(C \cap B(\gamma(t), r)))^k} \geq \left(\frac{\delta_-}{\delta_+}\right)^{2k},$$

which implies $\mathcal{H}^k(C) \geq \left(\frac{\delta_-}{\delta_+}\right)^{2k} \mathcal{S}^k(C)$.

Finally, formula (14) results from (11) applied to the restrictions of γ in (17).

■

Proof of Proposition 3. By definition,

$$\mathrm{Length}_k(C) = \int_a^b \mathrm{meas}_t^k(\gamma)\, dt.$$

Note that the k-dimensional length does not depend on the parameterization [8, Le. 16]. Thus, up to a reparameterization by the k-length, we assume that γ is defined on the interval $[0, T]$, with $T = \mathrm{Length}_k(C)$, and that $\mathrm{meas}_t^k(\gamma) \equiv 1$.

Fix $\delta > 0$. Then, by Lemma 1, there exists $\eta > 0$ so that the hypothesis of Lemma 2 is satisfied with $\delta_- = 1-\delta$ and $\delta_+ = 1+\delta$. We let δ tends to zero in inequalities (11)–(14) and the proposition follows. ■

Remark 4. Another way to measure C using approximations by finite sets is to consider ϵ-nets, i.e., sets of points $q_1, \ldots, q_n \in M$ such that the union of closed balls $B(q_i, \epsilon)$ covers C, and the metric entropy $e(C, \epsilon)$ which is the minimal number of points in an ϵ-net of C. Under the assumptions of Proposition 3, the following estimates can be deduced for a m-$\mathcal{C}_k^1$ curve:

$$\frac{\mathcal{S}^k(C)}{2^k} \leq \liminf_{\epsilon\to 0^+} \epsilon^k e(C,\epsilon) \leq \limsup_{\epsilon\to 0^+} \epsilon^k e(C,\epsilon) \leq \frac{\mathcal{S}^k(C)}{2}.$$

3.3 The Riemannian case

Let us come back to the case where (M, d) is a Carnot–Carathéodory space associated with a sub-Riemannian manifold $(M, \mathcal{D}, g)$. A consequence of Proposition 3 is that if the structure is Riemannian, i.e., $\mathcal{D} = TM$, then the class of m-$\mathcal{C}_k^1$ curves having non-zero metric derivative of degree k is empty if $k > 1$, as stated in Proposition 2.

Proof of Proposition 2. Let $s \in [a, b]$ such that $\mathrm{meas}_s^k(\gamma) \neq 0$. Thus, restricted to a small enough neighbourhood $I = [s-\delta, s+\delta]$ of s, the curve γ is injective and $\mathrm{meas}_t^k(\gamma) \neq 0$ on I. Up to reparameterizing $\gamma|_I$, we may assume moreover $\mathrm{meas}_t^k(\gamma) \equiv 1$ on I. Also, it is sufficient to consider the case $M = \mathbb{R}^n$ and d is the Euclidean distance on $\mathbb{R}^n$. Denote by C the set $\gamma(I)$. By Proposition 3, we have $0 < \mathcal{H}^k(C) < +\infty$ and for every $t \in I$

$$\lim_{r\to 0} \frac{\mathcal{H}^k(C \cap B(\gamma(t), r))}{2r^k} = 1.$$

Moreover, by Lemma 1, there exist $0 < \rho < 1$ such that

$$(1-\rho)|t-t'|^{1/k} \leq ||\gamma(t) - \gamma(t')|| \leq (1+\rho)|t-t'|^{1/k}, \quad \forall\, t, t' \in I.$$

The proposition then results from the lemma below. ■

Lemma 3. *Let $k \geq 1$ and let $\gamma : [a, b] \to \mathbb{R}^n$ be a bi-Hölder curve of exponent $1/k$, i.e.,*

$$\delta_-|t-t'|^{1/k} \leq ||\gamma(t) - \gamma(t')|| \leq \delta_+|t-t'|^{1/k}, \quad \forall\, t, t' \in [a, b], \tag{18}$$

with $\delta_-, \delta_+ \neq 0$. Set $C = \gamma([a, b])$. If $0 < \mathcal{H}^k(C) < +\infty$ and if there exits a positive constant c such that for every $t \in [a, b]$

$$\lim_{r\to 0} \frac{\mathcal{H}^k(C \cap B(\gamma(t), r))}{r^k} = c,$$

then $k = 1$.

Proof. Under the assumptions of the lemma, Marstrand's Theorem [20, Th. 1] assures that $k \in \mathbb{N}$ and $k \in \{1, 2, \dots, n\}$. Applying Preiss' result [24], there exists a countable family of k-dimensional submanifolds $N_i \subset \mathbb{R}^n$ such that $\mathcal{H}^k(C \setminus \cup_i N_i) = 0$. Since $\mathcal{H}^k(C) > 0$, there exists i such that $\mathcal{H}^k(C \cap N_i) > 0$. Let us rename N_i by N. Then $\mathcal{L}^k_N(C \cap N) = \mathcal{H}^k \lfloor_N (C) > 0$, where $\mathcal{L}^k_N$ is the k-dimensional Lebesgue measure on N. Hence there exists a density point $\gamma(t_0) \in C \cap N$, that is, a point such that

$$\lim_{r \to 0} \frac{\mathcal{L}^k_N(C \cap N \cap B_N(\gamma(t_0), r))}{\mathcal{L}^k_N(B_N(\gamma(t_0), r))} = 1,$$

where $B_N(\gamma(t_0), r)$ is the open ball in N centered at $\gamma(t_0)$ of radius r.

Let us identify N with $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_{k+1} = \cdots = x_n = 0\}$ by choosing local coordinates around $\gamma(t_0)$. Using the inequalities (18) and the density point t_0 it is not hard to prove that there exists a bi-Lipschitz homeomorphism from $B_N(\gamma(t_0), \delta_-)$ endowed with the Euclidean distance to $(-1, 1)$ endowed with the distance $|\cdot|^{1/k}$ (see for instance the argument in the proof of [4, Pr. 4.12]). Since the topological dimension of $B_N(\gamma(t_0), \delta_-)$ is k, then k must be equal to 1.

■

3.4 Comparison of measures for m-$\mathcal{C}^1_k$ curves

Next theorem generalizes the first part of Proposition 3 to the case when the metric derivative may vanish. Namely, it compares the $\mathcal{H}^k$ measure and the $\mathcal{S}^k$ measure of sets that are images of m-$\mathcal{C}^1_k$ curves. Also, it provides a relation among such measures, the k-length and the behaviour of the complexity of the curve.

Theorem 1. *Let $\gamma : [a, b] \to M$ be an injective m-$\mathcal{C}^1_k$ curve and $C = \gamma([a, b])$. Then*

$$\mathcal{H}^k(C) = \mathcal{S}^k(C) = \mathrm{Length}_k(C) = \lim_{\epsilon \to 0} \epsilon^k \sigma_{\mathrm{int}}(C, \epsilon).$$

If moreover $\mathcal{H}^k(C) > 0$ or $\mathrm{Length}_k(C) > 0$, then for every $k' \geq 1$

$$\mathcal{H}^{k'}(C) = \mathcal{S}^{k'}(C) = \mathrm{Length}_{k'}(C) = \lim_{\epsilon \to 0} \epsilon^{k'} \sigma_{\mathrm{int}}(C, \epsilon).$$

Remark 5. When M is a sub-Riemannian manifold, in many cases Gauthier and coauthors (see [12] and references therein) computed the interpolation complexity of curves as integral of some geometric invariants. Jointly with Theorem 1, these results provide a way of computing Hausdorff measures of curves as well as a geometric interpretation of Hausdorff and infinitesimal measures.

Corollary 3. *Let $\gamma : [a, b] \to M$ be a m-$\mathcal{C}^1_k$ curve. Then, for every measurable set $A \subset [a, b]$,*

$$\mathcal{S}^k(\gamma(A)) = \mathcal{H}^k(\gamma(A)) \quad \text{and} \quad \mathcal{H}^k(\gamma(A)) \leq \mathrm{Length}_k(\gamma(A)).$$

If moreover γ is injective then

$$\mathcal{H}^k(\gamma(A)) = \mathrm{Length}_k(\gamma(A)).$$

Remark 6. Let us consider the case where γ is an injective m-$\mathcal{C}^1_k$ curve. Recalling the definition of Length_k, we have

$$\mathcal{H}^k(C) = \int_a^b \mathrm{meas}^k_t(\gamma)\, dt,$$

that is, we have an integral formula for the k-dimensional Hausdorff measure. Moreover, Theorem 1 implies that the Hausdorff dimension $k_{\mathcal{H}}$ of C coincides with the length dimension of C. If in addition $\mathcal{H}^{k_{\mathcal{H}}}(C)$ (or $\mathrm{Length}_{k_{\mathcal{H}}}(C)$) is finite, then Corollary 3 implies that $\mathcal{H}^{k_{\mathcal{H}}}\lfloor_C$ is absolutely continuous with respect to the push-forward measure[2] $\gamma_*\mathcal{L}^1$ and that its Radon–Nikodym derivative is $\mathrm{meas}_t^k(\gamma)$.

Proof of Theorem 1. Clearly it suffices to prove the first statement of the theorem. Consider the (possibly empty) open subset of $[a,b]$

$$I = \{t \in [a,b] \ : \ \mathrm{meas}_t^k(\gamma) \neq 0\}, \tag{19}$$

and its complementary $I^c = [a,b] \setminus I$. The set I is the union of a disjointed countable family of open subintervals I_i of $[a,b]$. Note that, since $\mathrm{meas}_t^k(\gamma) = 0$ for all $t \in I^c$, one has

$$\mathrm{Length}_k(C) = \int_I \mathrm{meas}_t^k(\gamma)dt = \sum_i \int_{I_i} \mathrm{meas}_t^k(\gamma)dt.$$

By Remark 3, we have the equality

$$\mathcal{H}^k(\gamma(I_i)) = \int_{I_i} \mathrm{meas}_t^k(\gamma)dt, \quad \forall i. \tag{20}$$

Since γ is injective, we have

$$\mathcal{H}^k(C) \geq \sum_i \mathcal{H}^k(\gamma(I_i)),$$

whence we obtain $\mathcal{H}^k(C) \geq \mathrm{Length}_k(C)$.

The next step is to prove the converse inequality. Let $\delta > 0$. Since the function $t \mapsto \mathrm{meas}_t^k(\gamma)^{1/k}$ is uniformly continuous on $[a,b]$, there exists $\eta > 0$ such that, if $t, t' \in [a,b]$ and $|t - t'| < \eta$, then $|\mathrm{meas}_t^k(\gamma)^{1/k} - \mathrm{meas}_{t'}^k(\gamma)^{1/k}| < \delta$. In the covering $I = \bigcup_i I_i$, only a finite number N_δ of subintervals I_i may have a Lebesgue measure greater than η. Up to reordering, we assume $\mathcal{L}^1(I_i) < \eta$ if $i > N_\delta$. Set $J = I^c \cup \bigcup_{i>N_\delta} I_i$. Since the restriction of $\mathrm{meas}_t^k(\gamma)$ to I^c is identically zero, there holds $\mathrm{meas}_t^k(\gamma)^{1/k} < \delta$ for every $t \in J$.

The k-dimensional Hausdorff measure of C satisfies

$$\mathcal{H}^k(C) \leq \sum_{i \leq N_\delta} \mathcal{H}^k(\gamma(I_i)) + \mathcal{H}^k(\gamma(J)) = \sum_{i \leq N_\delta} \int_{I_i} \mathrm{meas}_t^k(\gamma)dt + \mathcal{H}^k(\gamma(J)), \tag{21}$$

in view of (20).

It remains to compute $\mathcal{H}^k(\gamma(J))$. Being the complementary of $\bigcup_{i \leq N_\delta} I_i$ in $[a,b]$, J is the disjointed union of $N_\delta + 1$ closed subintervals $J_i = [a_i, b_i]$ of $[a,b]$. For each one of these intervals we will proceed as in the proof of Proposition 3.

[2]Given a Borel set $E \subset M$ the push-forward measure $\gamma^*\mathcal{L}^1$ is defined by

$$\gamma_*\mathcal{L}^1(E) = \mathcal{L}^1(\gamma^{-1}(E \cap C)).$$

Let $\epsilon > 0$ and $i \in \{1, \dots, N_\delta + 1\}$. We denote by N' the smallest integer such that $b_i - a_i \leq N'(\frac{\epsilon}{2\delta})^k$ and define $t_0, \dots, t_{N'}$ by

$$t_j = a_i + j\big(\frac{\epsilon}{2\delta}\big)^k \quad \text{for } j = 0, \dots, N' - 1, \qquad t_{N'} = b_i.$$

We then set $S_j = \gamma([t_j, t_{j-1}])$. Applying Lemma 1, we get, for any $t, t' \in [t_j, t_{j-1}]$,

$$d\big(\gamma(t), \gamma(t')\big) = |t - t'|^{1/k}(\mathrm{meas}_t^k(\gamma)^{1/k} + \epsilon_t(t - t')).$$

Note that $\mathrm{meas}_t^k(\gamma)^{1/k} < \delta$ since $t \in J$. Note also that, if ϵ is small enough, then $\epsilon_t(|t - t'|)$ is smaller than δ. Therefore $d\big(\gamma(t), \gamma(t')\big) < 2\delta|t - t'|^{1/k} \leq \epsilon$ and $\operatorname{diam} S_j \leq \epsilon$. As a consequence

$$\mathcal{H}_\epsilon^k(\gamma(J_i)) \leq N'\epsilon^k \leq (2\delta)^k(b_i - a_i) + \epsilon^k,$$

and $\mathcal{H}^k(\gamma(J_i)) \leq (2\delta)^k(b_i - a_i)$. It follows that

$$\mathcal{H}^k(\gamma(J)) \leq \sum_{i \leq N_\delta + 1} (2\delta)^k(b_i - a_i) \leq (2\delta)^k(b - a).$$

Finally, formula (21) yields

$$\mathcal{H}^k(C) \leq \sum_{i \leq N_\delta} \int_{I_i} \mathrm{meas}_t^k(\gamma)dt + (2\delta)^k(b - a).$$

Letting $\delta \to 0$, we get $\mathcal{H}^k(C) \leq \int_I \mathrm{meas}_t^k(\gamma)dt = \mathrm{Length}_k(C)$, and thus $\mathcal{H}^k(C) = \mathrm{Length}_k(C)$. Similarly we can show that $\mathcal{S}^k(C)$ and the limit of $\epsilon^k\sigma_{\mathrm{int}}(C, \epsilon)$ are equal to $\mathrm{Length}_k(C)$. ■

Proof of Corollary 3. When γ is injective, the conclusions follow from Theorem 1 and from the regularity of $\mathcal{L}^1$ and $\mathcal{H}^k$ measures (see Remark 3).

Assume now that γ is not injective. We slightly modify the second part of the proof of Theorem 1 replacing the equality in (21) by

$$\mathcal{H}^k(C) \leq \sum_{i \leq N_\delta} \mathcal{H}^k(\gamma(I_i)) + \mathcal{H}^k(\gamma(J)) \leq \sum_{i \leq N_\delta} \int_{I_i} \mathrm{meas}_t^k(\gamma)dt + \mathcal{H}^k(\gamma(J)),$$

which is a consequence of Corollary 2. This shows that $\mathcal{H}^k(C) \leq \mathrm{Length}_k(C)$ and therefore $\mathcal{H}^k(\gamma(A)) \leq \mathrm{Length}_k(\gamma(A))$. Moreover, we have $\mathcal{H}^k(\gamma(I^c)) = 0$, where I^c is the complementary of the set I defined in (19), which in turn implies $\mathcal{S}^k(\gamma(I^c)) = 0$. Thus

$$\mathcal{S}^k(C) = \mathcal{S}^k(\gamma(I)) = \mathcal{H}^k(\gamma(I)) = \mathcal{H}^k(C),$$

where the second equality results from Corollary 2. ■

3.5 Generalization to non m-$\mathcal{C}_k^1$ curves

In this section we present some possible generalizations of the preceding results (in particular Theorem 1) to non m-$\mathcal{C}_k^1$ curves.

Consider first the case of a continuous curve $\gamma : [a, b] \to M$, $C = \gamma([a, b])$. For $k \geq 1$, we define I^k to be the set of points $t \in [a, b]$ such that $\mathrm{meas}_t^k(\gamma)$ is not continuous at t (that is, such that

γ is not m-$\mathcal{C}^1_k$ at t). A standard argument of measure theory allows to show the following fact. Assume that $[a,b]\setminus I^k$ is an open subset of $[a,b]$ of full $\mathcal{L}^1$ measure and that $\mathcal{H}^k(\gamma(I^k))=0$. Then the conclusions of Corollary 3 still hold. Moreover, if γ is injective, then the equalities between $\mathcal{H}^k(C)$, $\mathcal{S}^k(C)$, and $\mathrm{Length}_k(C)$ as in Theorem 1 hold true. The result on the complexity is not valid anymore, since $\lim_{\epsilon\to 0}\epsilon^k\sigma_{\mathrm{int}}(C,\epsilon)$ is not a measure. A curve C satisfying the properties above actually appears as a particular case of $(\mathcal{H}^k,1)$-rectifiable set, which will be studied in the next section.

It is however worth to mention a consequence of the result claimed above (and of Proposition 1) in the context of Carnot–Carathéodory spaces. Let $(M,\mathcal{D},g)$ be a sub-Riemannian manifold, $\gamma:[a,b]\to M$ be an absolutely continuous injective curve, and $C=\gamma([a,b])$. Let $m_C\geq 1$ be the smallest integer such that $\dot\gamma(t)\in\mathcal{D}^{m_C}(\gamma(t))$ almost everywhere. We denote by I_C the set of points $t\in[a,b]$ such that either γ is not $\mathcal{C}^1$ at t or $\gamma(t)$ is C-singular.

Corollary 4. *Assume that $[a,b]\setminus I_C$ is an open subset of $[a,b]$ of full $\mathcal{L}^1$ measure and $\mathcal{H}^{m_C}(\gamma(I_C))=0$. Then, for any $k\geq 1$,*

$$\mathcal{H}^k(C)=\mathcal{S}^k(C)=\mathrm{Length}_k(C),\qquad \textit{and}\qquad \dim_{\mathcal{H}} C=m_C.$$

When the sub-Riemannian manifold is equiregular, it is already known [14, p. 104] that the Hausdorff dimension of a one-dimensional submanifold C is the smallest integer k such that $T_qC\subset\mathcal{D}^k(q)$ for every $q\in C$. Corollary 4 generalizes this fact.

Any injective m-$\mathcal{C}^1_k$ curve being bi-Hölder (see Lemma 1), it is also natural to generalize our results to such curves. Consider then a bi-Hölder curve of exponent $1/k$, i.e. a curve $\gamma:[a,b]\to M$, i.e., there exist positive constants δ_- and δ_+ such that, for every $t,t+s\in[a,b]$,

$$\delta_-|s|^{1/k}\leq d\big(\gamma(t),\gamma(t+s)\big)\leq\delta_+|s|^{1/k}.$$

For such a curve $C=\gamma([a,b])$, the k-dimensional length does not always exists but Lemma 2 gives estimates of $\mathcal{H}^k(C)$ and $\mathcal{S}^k(C)$ in function of $T=b-a$, and the following result for upper and lower density:

$$\left(\frac{\delta_-}{\delta_+}\right)^k\leq\liminf_{r\to 0^+}\frac{\mathcal{H}^k(C\cap B(q,r))}{2r^k}\leq\limsup_{r\to 0^+}\frac{\mathcal{H}^k(C\cap B(q,r))}{2r^k}\leq\left(\frac{\delta_+}{\delta_-}\right)^k. \tag{22}$$

Let us remark that there is no hope to obtain a density result such as (8) for bi-Hölder curves. Indeed Assouad proved in [4] that, for any $k<n$, there exist bi-Hölder curves of exponent $1/k$ from $(-1,1)$ to $\mathbb{R}^n$ (both endowed with a Euclidean metric). When $k>1$ the density of these curves cannot be constant, for otherwise Lemma 3 would yield a contradiction. This strongly hints that there is not a Rademacher's-type result in this context, that is, being bi-Hölder of exponent $1/k$ does not imply being m-$\mathcal{C}^1_k$ almost everywhere.

In what follows, in particular in the definition of $(\mathcal{H}^k,1)$-rectifiability, we will work with m-$\mathcal{C}^1_k$ curves and not with bi-Hölder curves. The drawback is that our definitions will not be invariant under bi-Lipschitz equivalence of metric spaces since the m-$\mathcal{C}^1_k$ property is not invariant under such equivalence, contrarily to the bi-Hölder property. However we think that rectifiable sets should be defined as sets which admit almost everywhere a metric derivative. As noticed above, the use of bi-Hölder curves would not guarantee such a property.

4 $(\mathcal{H}^k, 1)$-rectifiable sets and a density result

In this section we use m-$\mathcal{C}^1_k$ curves to define a new class of $(\mathcal{H}^k, 1)$-rectifiable subsets of metric spaces.

Consider the Euclidean space $\mathbb{R}^n$. Recall that, given a positive measure μ on Borel subsets of $\mathbb{R}^n$, a subset $S \subset \mathbb{R}^n$ is (μ, k)-*rectifiable* if there exists a countable family of Lipschitz functions $\gamma_i : V_i \to \mathbb{R}^n$, $i \in \mathbb{N}$, where V_i is a bounded subset of $\mathbb{R}^k$, such that $\mu(S \setminus \cup_{i\in\mathbb{N}}\gamma_i(V_i)) = 0$ (see [10, 3.2.14]). Considering $\mu = \mathcal{H}^{k'}$ on $\mathbb{R}^n$ (with the Euclidean structure), one has that if $k' > k$ then there are no $(\mathcal{H}^{k'}, k)$-rectifiable sets of positive $\mathcal{H}^{k'}$ measure. This follows from the requirement of γ_i being Lipschitz. On the other hand, if one consider images under Hölder continuous functions γ_i then the case $k' > k$ becomes of interest. This suggests the next definition.

Consider a metric space (M, d).

Definition 3. A subset $S \subset M$ is $(\mathcal{H}^k, 1)$-*rectifiable* if there exists a countable family of m-$\mathcal{C}^1_k$ curves $\gamma_i : I_i \to M$, $i \in \mathbb{N}$, I_i closed interval in $\mathbb{R}$ such that

$$\mathcal{H}^k(S \setminus \cup_{i\in\mathbb{N}}\gamma_i(I_i)) = 0.$$

Remark 7. If M is a manifold and d is the distance associated with a Riemannian structure on M, the class of $(\mathcal{H}^k, 1)$-rectifiable sets with positive and finite $\mathcal{H}^k$ measure is empty unless $k = 1$ (see Proposition 2). Since m-$\mathcal{C}^1_1$ curves are Lipschitz, in this case Definition 3 coincides with the usual definition of $(\mathcal{H}^1, 1)$-rectifiable sets. Conversely, when (M, d) is the Carnot–Carathéodory space associated with a genuine sub-Riemannian manifold, there exist $(\mathcal{H}^k, 1)$-rectifiable sets (of positive and finite $\mathcal{H}^k$ measure) for some integers $k > 1$ (see Section 2.2).

When a subset is $\mathcal{H}^k$-measurable and has finite $\mathcal{H}^k$ measure, being $(\mathcal{H}^k, 1)$-rectifiable implies boundedness for the lower and upper densities of the measure $\mathcal{H}^k\lfloor_S$.

Theorem 2. *Assume $S \subset M$ is a $(\mathcal{H}^k, 1)$-rectifiable and $\mathcal{H}^k$-measurable set such that $\mathcal{H}^k(S) < +\infty$. Then for $\mathcal{H}^k$-almost every $q \in S$*

$$2 \le \liminf_{r\to 0^+} \frac{\mathcal{H}^k(S \cap B(q, r))}{r^k} \le \limsup_{r\to 0^+} \frac{\mathcal{H}^k(S \cap B(q, r))}{r^k} \le 2^k. \tag{23}$$

Recall that in [24, Co. 5.5] it was proved that, in the Euclidean case, there exists a constant $c > 0$ such that if a μ-measurable subset $E \subset \mathbb{R}^n$ with finite μ measure satisfies

$$0 < \limsup_{r\to 0^+} \frac{\mu(E \cap B(x, r))}{r^k} \le c \liminf_{r\to 0^+} \frac{\mu(E \cap B(x, r))}{r^k} < +\infty, \tag{24}$$

for μ-almost every $x \in E$, then E is (μ, k)-rectifiable. This result provides a characterization of rectifiable sets as the converse is also true (see [10, Th. 3.2.19]). Theorem 2 implies that if $S \subset M$ is $(\mathcal{H}^k, 1)$-rectifiable in the sense of Definition 3 then

$$0 < \limsup_{r\to 0^+} \frac{\mathcal{H}^k(S \cap B(q, r))}{r^k} \le 2^{k-1} \liminf_{r\to 0^+} \frac{\mathcal{H}^k(S \cap B(q, r))}{r^k} < +\infty, \tag{25}$$

for $\mathcal{H}^k$-almost every $q \in S$. The last estimate is, mutatis mutandis, the assumption (24) in the result by Preiss. An open question is whether the same conclusion of [24, Cor. 5.5] holds with

our notion of $(\mathcal{H}^k, 1)$-rectifiable sets. Namely, is condition (25) for $\mathcal{H}^k$-almost every $q \in S \subset M$ sufficient to show that a $\mathcal{H}^k$-measurable set S of finite $\mathcal{H}^k$ measure is $(\mathcal{H}^k, 1)$-rectifiable in the sense of Definition 3?

Proof of Theorem 2. By assumption, there exists a countable family of m-$\mathcal{C}^1_k$ curves $\gamma_i : I_i \to M$ such that I_i is a closed interval and $\mathcal{H}^k(S \setminus \cup_i \gamma_i(I_i)) = 0$. Since by Corollary 3, for every i, $\mathcal{H}^k(\gamma_i(\{t \mid \mathrm{meas}^k_t(\gamma_i) = 0\})) = 0$, we may assume $\mathrm{meas}^k_t(\gamma_i) \neq 0$ for every $t \in I_i$ and then, by a reparameterization, $\mathrm{meas}^k_t(\gamma_i) \equiv 1$. This implies that every γ_i is locally injective. Hence, without loss of generality, we may assume that every γ_i is injective and moreover that the sets $\gamma_i(I_i)$ are pairwise disjoint.

Since $\mathcal{H}^k(S) < +\infty$, to prove the upper bound

$$\limsup_{r \to 0^+} \frac{\mathcal{H}^k(S \cap B(q,r))}{r^k} \leq 2^k,$$

for $\mathcal{H}^k$-almost every $q \in M$, it suffices to use [10, 2.10.19 (5)].

Let us show the lower bound in (23), namely, that

$$\liminf_{r \to 0^+} \frac{\mathcal{H}^k(S \cap B(q,r))}{r^k} \geq 2, \tag{26}$$

for $\mathcal{H}^k$-almost every $q \in S$. Let $\tilde{I}_i = \gamma_i^{-1}(\gamma_i(I_i) \cap S)$. Then $\cup_{i \in N} \gamma_i(\tilde{I}_i) \subset S$ and $\mathcal{H}^k(S \setminus \cup_{i \in N} \gamma_i(\tilde{I}_i)) = 0$. We may assume $\mathcal{H}^k(\gamma_i(\tilde{I}_i)) > 0$ for each i. Then, by Corollary 3, since $\mathrm{meas}^k_t(\gamma_i) \equiv 1$, $\mathcal{L}^1(\tilde{I}_i) = \mathcal{H}^k(\gamma_i(\tilde{I}_i)) > 0$. Therefore almost every $t \in \tilde{I}_i$ is a density point for the Lebesgue measure on $\tilde{I}_i$, i.e.,

$$\lim_{r \to 0} \frac{\mathcal{L}^1(\tilde{I}_i \cap B(t,r))}{2r} = 1,$$

where $B(t,r) = (t-r, t+r)$. Hence, for $\mathcal{H}^k$-almost every $q \in S$ there exist a unique i and a unique $t \in \tilde{I}_i$ such that $q = \gamma_i(t)$ and t is a density point for $\mathcal{L}^1 \lfloor_{\tilde{I}_i}$. Since $\gamma_i(\tilde{I}_i) \subset S$, we deduce

$$\frac{\mathcal{H}^k(S \cap B(q,r))}{r^k} \geq \frac{\mathcal{H}^k(\gamma_i(\tilde{I}_i) \cap B(q,r))}{r^k} = \frac{\mathcal{L}^1(\tilde{I}_i \cap \gamma_i^{-1}(B(q,r)))}{r^k},$$

the last equality following by Corollary 3. Now, for any $\delta > 0$, from Lemma 1, for $|t-s| \leq \frac{r^k}{(1+\delta)^k}$ we have

$$d\big(\gamma(t), \gamma(s)\big) \leq |t-s|^{1/k}(1+\delta) \leq r.$$

This implies $B(t, r^k/(1+\delta)^k) \subset \gamma_i^{-1}(B(q,r))$. Therefore

$$\frac{\mathcal{L}^1(\tilde{I}_i \cap \gamma_i^{-1}(B(q,r)))}{r^k} \geq \frac{\mathcal{L}^1(\tilde{I}_i \cap B(t, r^k/(1+\delta)^k)))}{r^k}.$$

The right-hand side of the inequality above tends to $2/(1+\delta)^k$, as r goes to 0, since t is a density point for $\mathcal{L}^1 \lfloor_{\tilde{I}_i}$. Letting δ go to 0, we conclude

$$\liminf_{r \to 0} \frac{\mathcal{L}^1(\tilde{I}_i \cap \gamma_i^{-1}(B(q,r)))}{r^k} \geq 2,$$

which shows (26). ∎

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