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ABSTRACT. We consider a nonlocal scalar conservation law proposed by Andrew C. Fowler to describe the dynamics of dunes, and we develop a numerical procedure based on splitting methods to approximate its solutions. We begin by proving the convergence of the well-known Lie formula, which is an approximation of the exact solution of order one in time. We next use the split-step Fourier method to approximate the continuous problem using the fast Fourier transform and the finite difference method. Our numerical experiments confirm the theoretical results.

1. INTRODUCTION

We consider the Fowler equation [11, 12]:

$$\begin{cases} \partial_t u(t,x) + \partial_x \left(\dfrac{u^2}{2}\right)(t,x) + \mathcal{I}[u(t,\cdot)](x) - \partial_x^2 u(t,x) = 0, \quad x \in \mathbf{R}, t > 0, \\ u(0,x) = u_0(x), \quad x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $u = u(t,x)$ represents the dune height and $\mathcal{I}$ is a nonlocal operator defined as follows: for any Schwartz function $\varphi \in \mathcal{S}(\mathbf{R})$ and any $x \in \mathbf{R}$,

$$\mathcal{I}[\varphi](x) := \int_0^{+\infty} |\xi|^{-\frac{1}{3}} \varphi''(x-\xi)\, d\xi. \tag{1.2}$$

We refer to [1, 2, 6] for theoretical results on this equation.

Remark 1.1. The nonlocal term $\mathcal{I}$ is anti-diffusive. Indeed, it has been proved in [1] that

$$\mathcal{F}\left(\mathcal{I}[\varphi]\right)(\xi) = -4\pi^2 \Gamma\left(\frac{2}{3}\right)\left(\frac{1}{2} - i\,\mathrm{sgn}(\xi)\frac{\sqrt{3}}{2}\right)|\xi|^{4/3}, \tag{1.3}$$

where $\mathcal{F}$ denotes the Fourier transform normalized in (1.11). Thus, $\mathcal{I}$ can been seen as a fractional power of order $2/3$ of the Laplacian, with the "bad" sign. It will be clear from the analysis below that our results can easily be extended to the case where $\mathcal{I}$ is replaced with a Fourier multiplier homogeneous of degree $\lambda \in]0,2[$, as in [4], and not only $\lambda = 4/3$.

We assume that the initial data u_0 belongs to $H^3(\mathbf{R})$, and thus (1.1) has a unique solution belonging to $C([0,t], H^3(\mathbf{R}))$ for all $t > 0$, from [1]. We will denote $u(t,\cdot)$ by $S^t u_0$; S^t maps $H^3(\mathbf{R})$ to itself. Duhamel's formula for the continuous problem (1.1) reads

$$u(t,\cdot) := S^t u_0 = K(t,\cdot) * u_0 - \frac{1}{2}\int_0^t \partial_x K(t-s,\cdot) * \left(S^s u_0\right)^2 \, ds, \tag{1.4}$$

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where $K(t,\cdot)=\mathcal{F}^{-1}\left(e^{-t\psi_{\mathcal{I}}}\right)$ is the kernel of the operator $\mathcal{I}-\partial_x^2$, and $\psi_{\mathcal{I}}$ is defined by

$$\psi_{\mathcal{I}}(\xi)=4\pi^2\xi^2-a_{\mathcal{I}}|\xi|^{4/3}+ib_{\mathcal{I}}\xi|\xi|^{1/3}, \tag{1.5}$$

where $a_{\mathcal{I}}, b_{\mathcal{I}}$ are positive constants.

Recently, to solve the Fowler equation some numerical experiments have been performed using mainly finite difference approximation schemes [3, 4]. However, these schemes are not effective because if we opt for an explicit scheme, numerical stability requires that the time step Δt is limited by $O(\Delta x^2)$. And, if we choose an implicit scheme, we have to solve a large system which is a computationally expensive operation. Thus, the splitting method becomes an interesting alternative to solve the Fowler model. To our knowledge, there is no convergence result in the literature for the splitting method associated to the Fowler equation. This method is more commonly used to split different physical terms, such as reaction and diffusion terms, see for instance [18]. Splitting methods have also been employed for solving a wide range of nonlinear wave equations. The basic idea of this method is to decompose the original problem into sub-problems and then to approximate the solution of the original problem by solving successively the sub-problems. Various versions of this method have been developed for the nonlinear Schrödinger, Korteweg-de-Vries and modified Korteweg-de-Vries equations, see for instance [13, 17, 19, 21].
For the Fowler model (1.1), we consider, separately, the linear Cauchy problem

$$\frac{\partial v}{\partial t}+\mathcal{I}[v(t,\cdot)]-\eta\,\partial_x^2 v=0;\quad v(0,x)=v_0(x), \tag{1.6}$$

and the nonlinear Cauchy problem

$$\frac{\partial w}{\partial t}+\partial_x\left(\frac{w^2}{2}\right)-\varepsilon\,\partial_x^2 w=0;\quad w(0,x)=w_0(x), \tag{1.7}$$

where ε,η are fixed *positive* parameters such that $\varepsilon+\eta=1$. Equation (1.7) is simply the viscous Burgers' equation. We denote by X^t and Y^t, respectively, the evolution operator associated with (1.6) and (1.7):

$$v(t,\cdot):=X^t v_0=D(t,\cdot)*v_0,$$

where $D(t,\cdot)=\mathcal{F}^{-1}\left(e^{-t\,\phi_{\mathcal{I}}}\right)$ with $\phi_{\mathcal{I}}(\xi)=4\pi^2\eta\xi^2-a_{\mathcal{I}}|\xi|^{4/3}+b_{\mathcal{I}}\xi|\xi|^{1/3}$, and

$$w(t,\cdot):=Y^t w_0=G(t,\cdot)*w_0-\frac{1}{2}\int_0^t \partial_x G(t-s,\cdot)*\left(Y^s w_0\right)^2\,ds, \tag{1.8}$$

where G is the heat kernel defined by

$$G(t,\cdot)=\mathcal{F}^{-1}(e^{-t(4\pi^2\varepsilon|.|^2)})=\frac{1}{\sqrt{4\pi\varepsilon t}}e^{-\frac{|.|^2}{4\varepsilon t}}.$$

Furthermore, the following L^2-estimate holds

$$\|Y^t w\|_{L^2(\mathbf{R})}\leqslant\|w\|_{L^2(\mathbf{R})}. \tag{1.9}$$

Let us explain the choice of this decomposition. First, we can remark that if we do not consider the nonlinear term in (1.1), the analytical solutions are available using the Fourier transform. Thus, the linear part may be computed efficiently using a fast Fourier transform (hereafter FFT) algorithms. Note also that the Laplacian and the fractional term $\mathcal{I}$ cannot be treated separately. Indeed, the equation $u_t+\mathcal{I}[u]=0$ is ill-posed. We next decide to handle the nonlinear term by adding a bit of viscosity in order to avoid shock problems in the standard Burgers' equation. Therefore, the splitting approach presented in this article differs from e.g. the one analyzed in [10], which corresponds to assuming $\varepsilon=0$ in the above definitions. The splitting operators associated to this approach when $\mathcal{I}=0$ (which

amount to considering alternatively the heat equation and the Burgers equation) have been studied in [15] (as well as other equations involving the Burgers nonlinearity, such as the KdV equation, see also [14]). The main difference with the result presented in [15] is that the operator $\mathcal{I}$ is not a differential operator, so its action on nonlinear terms is rather involved , while this point would be needed to compute the Lie commutator between the two operators

$$A(v) = \eta\partial_x^2 v - \mathcal{I}[v], \quad B(v) = -v\partial_x v + \varepsilon^2\partial_x^2 v,$$

as in e.g. [15, 17]. Note that since we consider the Lie splitting operator, we could in principle use the exact formula established in [7] for the local error. But then again, we face the problem to compute $[A, B]$, which is the only term that we cannot estimate in [7, Theorem 1]. We finally point out that we use smoothing effects associated to the viscous Burgers equation; see Corollaries 3.8 and 3.9.

We motivate this choice by the presence of artificial diffusion in classical numerical schemes used to solve the convection equations. An alternative to reduce this effect is to consider numerical schemes of high order which are usually computationally expensive and do not seem to be very useful for the Fowler model because of the diffusion term.

We consider the Lie formula defined by

$$Z_L^t = X^t Y^t. \tag{1.10}$$

The alternative definition $Z_L^t = Y^t X^t$ could be studied as well, leading to a similar result. Also, the following evolution operators

$$Z_S^t = X^{t/2} Y^t X^{t/2} \quad \text{or} \quad Z_S^t = Y^{t/2} X^t Y^{t/2},$$

corresponding to the Strang method [20] could be considered. Following the computations detailed in the present paper for the case (1.10), it would be possible to show that the other Lie formula generates a scheme of order one, and to prove that the Strang method is of order two (for smooth initial data), in the same fashion as in, e.g., [5, 17]. This fact is simply illustrated numerically in Section 6, to avoid a lengthy presentation. With Z_L^t given by (1.10), our main result is:

Theorem 1.2. *For all $u_0 \in H^3(\mathbf{R})$ and for all $T > 0$, there exist positive constants c_1, c_2 and Δt_0 such that for all $\Delta t \in]0, \Delta t_0]$ and for all $n \in \mathbf{N}$ such that $0 \leqslant n\Delta t \leqslant T$,*

$$\|(Z_L^{\Delta t})^n u_0 - S^{n\Delta t} u_0\|_{L^2(\mathbf{R})} \leqslant c_1\, \Delta t \quad \textit{and} \quad \|(Z_L^{\Delta t})^n u_0\|_{H^3(\mathbf{R})} \leqslant c_2.$$

Here, c_1, c_2 and Δt_0 depend only on T, $\rho = \max_{t\in[0,T]} \|S^t u_0\|_{H^2(\mathbf{R})}$, and $\|u_0\|_{H^3(\mathbf{R})}$.

Remark 1.3. It will follow from Lemma 3.11 that

$$\rho = \max_{t\in[0,T]} \|S^t u_0\|_{H^2(\mathbf{R})} \leqslant C_T(\|u_0\|_{H^1(\mathbf{R})})\|u_0\|_{H^2(\mathbf{R})},$$

for some nonlinear (increasing) function C_T depending on T.

In this paper, we begin by estimating the L^2-stability for error propagation. We next prove that the local error of the Lie formula is an approximation of order two in time. Finally we prove that this evolution operator represents a good approximation, of order one in time, of the evolution operator S^t in the sense of Theorem 1.2.
Furthermore, we apply Lie and Strang approximations in order to make some numerical simulations using the split-step Fourier experiments.

This paper is organised as follows. In the next section we give some properties related to the kernels G and K, and we prove two fractional Gronwall Lemmas. In Section 3, we

establish some estimates on X^t, Y^t, Z_L^t and S^t. In Section 4, we prove a local L^2 error estimate. Theorem 1.2 is proved in Section 5. We finally perform some numerical experiments which show that the Lie and Strang methods have a convergence rate in $\mathcal{O}\left(\Delta t\right)$ and $\mathcal{O}\left(\Delta t^2\right)$), respectively.

Notations.
- We denote by $\mathcal{F}$ the Fourier transform of f which is defined by: for all $\xi \in \mathbf{R}$,

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbf{R}} e^{-2i\pi x\xi} f(x)dx. \tag{1.11}$$

We denote by $\mathcal{F}^{-1}$ its inverse.
- We denote by $C_T(c_1, c_2, \cdots)$ a generic constant, strictly positive, which depends on parameters $c_1, c_2, \cdots$, and T. C is assumed to be a monotone increasing function of its arguments.

2. PRELIMINARIES

2.1. **Properties of the kernels.** We begin by recalling the properties of kernels involved in the present analysis.

Proposition 2.1 (Main properties of K, [1])**.** *The kernel K satisfies:*

(1) $\forall t > 0$, $K(t,\cdot) \in L^1(\mathbf{R})$ *and* $K \in C^\infty(]0,\infty[\times\mathbf{R})$.
(2) $\forall s,t > 0$, $K(s,\cdot) * K(t,\cdot) = K(s+t,\cdot)$.
(3) $\forall T > 0, \exists C_T > 0$ *such that for all* $t \in]0,T]$, $\|\partial_x K(t,\cdot)\|_{L^2(\mathbf{R})} \leqslant C_T\, t^{-3/4}$.
(4) $\forall T > 0, \exists C_T > 0$ *such that for all* $t \in]0,T]$, $\|\partial_x K(t,\cdot)\|_{L^1(\mathbf{R})} \leqslant C_T\, t^{-1/2}$.
(5) *For any* $u_0 \in L^2(\mathbf{R})$ *and* $t > 0$,

$$\|K(t,\cdot) * u_0\|_{L^2(\mathbf{R})} \leqslant e^{\alpha_0 t}\|u_0\|_{L^2(\mathbf{R})},$$

where $\alpha_0 = -\min \mathrm{Re}(\psi_{\mathcal{I}}) > 0$.

Proposition 2.2 (Main properties of G, [9])**.** *The kernel G satisfies:*

(1) $G \in C^\infty(]0,\infty[\times\mathbf{R})$.
(2) $\forall s,t > 0$, $G(s,\cdot) * G(t,\cdot) = G(s+t,\cdot)$.
(3) $\forall t > 0$, $\|G(t,\cdot)\|_{L^1(\mathbf{R})} = 1$.
(4) $\exists C_0 > 0$ *such that for all* $t > 0$, $\|\partial_x G(t,\cdot)\|_{L^2(\mathbf{R})} \leqslant C_0 t^{-3/4}$.
(5) $\exists C_1 > 0$ *such that for all* $t > 0$, $\|\partial_x G(t,\cdot)\|_{L^1(\mathbf{R})} \leqslant C_1 t^{-1/2}$.

Remark 2.3. The kernel D of $\mathcal{I} - \eta\,\partial_x^2$ has similar properties to the kernel K. Moreover, for all $t > 0$, we have

$$D(t,\cdot) * G(t,\cdot) = K(t,\cdot). \tag{2.1}$$

2.2. **Fractional Gronwall lemmas.**

Lemma 2.4 (Fractional Gronwall Lemma)**.** *Let $\phi : [0,T] \to \mathbf{R}_+$ be a bounded measurable function, and suppose that there are positive constants A, L and $\theta \in]0,1[$ such that for all $t \in [0,T]$,*

$$\phi(t) \leqslant A + L\frac{d^{-\theta}}{dt^{-\theta}}\phi(t), \tag{2.2}$$

where $\frac{d^{-\theta}}{dt^{-\theta}}$ is the Riemann–Liouville operator defined by

$$\frac{d^{-\theta}}{dt^{-\theta}}\phi(t) = \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\phi(s)\,ds.$$

Then there exists $C_T(\theta)$ such that

$$\phi(t) \leqslant e^{C_T(\theta)t}A, \quad \forall t \in [0,T].$$

Proof. The proof of this Lemma is well-known and is based on an iteration argument; see for instance [8, Lemma 4.3] or [22, Corollary 2]. We sketch the argument for the sake of completeness. Iterating inequality (2.2) once, we have

$$\begin{aligned}\phi(t) &\leqslant A + \frac{L}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\phi(s)\,ds \\ &\leqslant A + \frac{L}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\left(A + \frac{L}{\Gamma(\theta)}\int_0^s (s-r)^{\theta-1}\phi(r)\,dr\right)ds \\ &= A\left(1 + \frac{L}{\theta\Gamma(\theta)}T^\theta\right) + \frac{L^2}{\Gamma(\theta)^2}\int_0^t (t-s)^{\theta-1}\int_0^s (s-r)^{\theta-1}\phi(r)\,dr\,ds.\end{aligned}$$

From Fubini's Theorem, we get

$$\begin{aligned}\int_0^t (t-s)^{\theta-1}\int_0^s (s-r)^{\theta-1}\phi(r)\,dr\,ds &= \int_0^t \phi(r)\int_r^t (t-s)^{\theta-1}(s-r)^{\theta-1}ds\,dr \\ &= \int_0^t \phi(r)(t-r)^{2\theta-1}\left(\int_0^1 (1-\tau)^{\theta-1}\tau^{\theta-1}d\tau\right)dr \\ &= \beta(\theta,\theta)\int_0^t \phi(r)(t-r)^{2\theta-1}dr,\end{aligned}$$

where β is the beta function. Therefore, we have

$$\phi(t) \leqslant C_T(\theta)A + \frac{L^2}{\Gamma(\theta)^2}\beta(\theta,\theta)\int_0^t \phi(s)(t-s)^{2\theta-1}\,ds. \tag{2.3}$$

Iterating the estimate (2.3) n times, with $n\theta \geqslant 1$, we get the following estimate:

$$\phi(t) \leqslant \tilde{C}_T(\theta)A + \tilde{L}_T(\theta)\int_0^t \phi(s)(t-s)^{\alpha}ds,$$

with $\alpha \geqslant 0$, and where $\tilde{L}_T(\theta)$ is a positive constant which depends on T and θ. The lemma then follows from the classical Gronwall Lemma. □

Lemma 2.5 (Modified fractional Gronwall Lemma)**.** *Let $\phi : [0,T] \to \mathbf{R}_+$ be a bounded measurable function and P be a polynomial with positive coefficients and no constant term. We assume there exists two positive constants C and $\theta \in]0,1[$ such that for all $t \in [0,T]$,*

$$0 \leqslant \phi(t) \leqslant \phi(0) + P(t) + C\frac{d^{-\theta}}{dt^{-\theta}}\phi(t). \tag{2.4}$$

Then there exists $C_T(\theta)$ such that for all $t \in [0,T]$,

$$\phi(t) \leqslant C_T(\theta)\,\phi(0) + C_T(\theta)\,P(t).$$

Proof. Arguing as in the proof of Lemma 2.4, we iterate the previous inequality. After one iteration, we get

$$\begin{aligned}\phi(t) &\leqslant \tilde{C}\phi(0) + P(t) + \frac{C}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}P(s)ds \\ &\quad + \frac{C^2}{\Gamma(\theta)^2}\int_0^t (t-s)^{\theta-1}\left(\int_0^s (s-r)^{\theta-1}\phi(r)dr\right)ds \\ &\leqslant \tilde{C}\phi(0) + \tilde{C}P(t) + \frac{C^2}{\Gamma(\theta)^2}\beta(\theta,\theta)\int_0^t \phi(s)(t-s)^{2\theta-1}ds,\end{aligned}$$

where we have used the assumptions on P, and Fubini's Theorem again for the last term. Iterating sufficiently many times, we infer like in the proof of Lemma 2.4:

$$\phi(t) \leqslant c_0\phi(0) + c_0P(t) + \underline{C}\int_0^t \phi(s)(t-s)^{\alpha}ds, \tag{2.5}$$

with $\alpha > 0$. Set

$$\psi(t) = \left(c_0\phi(0) + c_0P(t) + \underline{C}\int_0^t \phi(s)(t-s)^{\alpha}ds\right)e^{-C_1t}.$$

Then

$$\begin{aligned}\psi'(t) = \Big(&c_0P'(t) + \underline{C}\alpha\int_0^t \phi(s)(t-s)^{\alpha-1}ds \\ &- C_1\Big(c_0\phi(0) + c_0P(t) + \underline{C}\int_0^t \phi(s)(t-s)^{\alpha}ds\Big)\Big)e^{-C_1t}.\end{aligned}$$

Using (2.5) to control the second term, and choosing C_1 sufficiently large, we infer:

$$\psi'(t) \leqslant c_0P'(t)e^{-C_1t}.$$

Since $P(0) = 0$, $\psi(0) = c_0\phi(0)$, for all $t \in [0,T]$,

$$\begin{aligned}\phi(t) \leqslant \psi(t)e^{C_1t} &\leqslant c_0\phi(0)e^{C_1t} + c_0\int_0^t P'(s)e^{C_1(t-s)}ds \\ &\leqslant c_0\phi(0)e^{C_1T} + c_0e^{C_1T}\int_0^t P'(s)\,ds \leqslant c_0e^{C_1T}\left(\phi(0) + P(t)\right).\end{aligned}$$

This completes the proof. □

3. Estimates on the various flows

3.1. **Estimates on linear flows.** In this paragraph, we collect several estimates concerning the convolutions with D, K and G, which will be useful in the estimates of the local error of the scheme.

Proposition 3.1. *Let $s \in \mathbf{R}$ and $\varphi \in H^s(\mathbf{R})$. Then $\mathcal{I}[\varphi] \in H^{s-4/3}(\mathbf{R})$ and we have*

$$\|\mathcal{I}[\varphi]\|_{H^{s-4/3}(\mathbf{R})} \leqslant 4\pi^2\Gamma\left(\frac{2}{3}\right)\|\varphi\|_{H^s(\mathbf{R})}. \tag{3.1}$$

Proof. For all $s \in \mathbf{R}$ and all $\varphi \in H^s(\mathbf{R})$, we have, using (1.3)

$$\begin{aligned}
\|\mathcal{I}[\varphi]\|_{H^{s-4/3}(\mathbf{R})} &= \left(\int_{\mathbf{R}} (1+|\xi|^2)^{s-4/3} |\mathcal{F}(\mathcal{I}[\varphi])(\xi)|^2 \, d\xi\right)^{1/2} \\
&= 4\pi^2 \Gamma\left(\frac{2}{3}\right) \left(\int_{\mathbf{R}} (1+|\xi|^2)^{s-4/3} \left|\frac{1}{2} - i \,\mathrm{sgn}(\xi) \frac{\sqrt{3}}{2}\right|^2 |\xi|^{8/3} |\mathcal{F}(\varphi)(\xi)|^2 \, d\xi\right)^{1/2} \\
&= 4\pi^2 \Gamma\left(\frac{2}{3}\right) \left(\int_{\mathbf{R}} \left(\frac{|\xi|^2}{1+|\xi|^2}\right)^{4/3} (1+|\xi|^2)^s |\mathcal{F}(\varphi)(\xi)|^2 \, d\xi\right)^{1/2} \\
&\leqslant 4\pi^2 \Gamma\left(\frac{2}{3}\right) \left[\int_{\mathbf{R}} (1+|\xi|^2)^s |\mathcal{F}(\varphi)(\xi)|^2 \, d\xi\right]^{1/2} = 4\pi^2 \Gamma\left(\frac{2}{3}\right) \|\varphi\|_{H^s(\mathbf{R})},
\end{aligned}$$

hence the result. □

Lemma 3.2. (1) *Let $n \in \mathbf{N}$. Then, for all $v \in H^n(\mathbf{R})$ and all $t > 0$,*

$$\|X^t v\|_{H^n(\mathbf{R})} \leqslant e^{\beta_0 t} \|v\|_{H^n(\mathbf{R})},$$

where $\beta_0 = -\min \mathrm{Re}(\phi_{\mathcal{I}}) > 0$.
(2) *Let $n \in \mathbf{N}$. There exists C such that for all $v \in H^2(\mathbf{R})$ and all $t > 0$,*

$$\|X^t v - v\|_{H^n(\mathbf{R})} \leqslant C\, t\, e^{\beta_0 t} \|v\|_{H^{n+2}(\mathbf{R})}. \tag{3.2}$$

Proof. Using Plancherel formula, we have

$$\begin{aligned}
\|X^t v\|^2_{L^2(\mathbf{R})} &= \|D(t,\cdot) * v\|^2_{L^2(\mathbf{R})} \\
&= \|\mathcal{F}\left(D(t,\cdot)\right) \mathcal{F} v\|^2_{L^2(\mathbf{R})} = \int_{\mathbf{R}} |\mathcal{F}\left(D(t,\cdot)\right)(\xi)|^2 |\mathcal{F} v(\xi)|^2 \, d\xi \\
&= \int_{\mathbf{R}} e^{-2t\phi_{\mathcal{I}}(\xi)} |\mathcal{F} v(\xi)|^2 \, d\xi \leqslant e^{2\beta_0 t} \|v\|^2_{L^2(\mathbf{R})}.
\end{aligned}$$

Moreover, since

$$\partial_x^n X^t v = D(t,\cdot) * \partial_x^n v$$

then, from again Plancherel formula, we have

$$\|\partial_x^n X^t v\|_{L^2(\mathbf{R})} \leqslant e^{\beta_0 t} \|\partial_x^n v\|_{L^2(\mathbf{R})},$$

hence the first point of the lemma.

Let $n \in \mathbf{N}$, $v \in H^{n+2}(\mathbf{R})$. We have

$$\|X^t v - v\|_{H^n(\mathbf{R})} = \left\| \int_0^t \dot{X^s}\, v \, ds \right\|_{H^n(\mathbf{R})}.$$

But from the definition of X^t, $\dot{X^s}$ is given by

$$\dot{X^s}\, v = \eta \partial_x^2 X^s v - \mathcal{I}[X^s v] = \eta X^s \partial_x^2 v - \mathcal{I}[X^s v],$$

since $X^s\partial_x^2 v = D(s,\cdot)*\partial_x^2 v = \partial_x^2\left(D(s,\cdot)*v\right)$. Thus, using Proposition 3.1 and the first point of this lemma, we get

$$\begin{aligned}\|X^t v - v\|_{H^n(\mathbf{R})} &\leqslant \eta\int_0^t \|X^s\partial_x^2 v\|_{H^n(\mathbf{R})}\,ds + \int_0^t \|\mathcal{I}[X^s v]\|_{H^n(\mathbf{R})}\,ds\\ &\leqslant \eta\, t\, e^{\beta_0 t}\|v\|_{H^{n+2}(\mathbf{R})} + \int_0^t \|\mathcal{I}[X^s v]\|_{H^n(\mathbf{R})}ds\\ &\leqslant \eta\, t\, e^{\beta_0 t}\|v\|_{H^{n+2}(\mathbf{R})} + 4\pi^2\Gamma\left(\frac{2}{3}\right)\int_0^t \|X^s v\|_{H^{n+4/3}(\mathbf{R})}ds\\ &\leqslant \eta\, t\, e^{\beta_0 t}\|v\|_{H^{n+2}(\mathbf{R})} + 4\pi^2\Gamma\left(\frac{2}{3}\right)\int_0^t \|X^s v\|_{H^{n+2}(\mathbf{R})}ds\\ &\leqslant \left(\eta + 4\pi^2\Gamma\left(\frac{2}{3}\right)\right) t\, e^{\beta_0 t}\|v\|_{H^{n+2}(\mathbf{R})},\end{aligned}$$

hence the result. □

Recalling that K corresponds to D in the case $\eta = 1$, we readily infer:

Corollary 3.3. *For all $w \in H^2(\mathbf{R})$ and all $t > 0$,*

$$\|K(t,\cdot)*w - w\|_{L^2(\mathbf{R})} \leqslant C\, t\, e^{\alpha_0 t}\|w\|_{H^2(\mathbf{R})}, \tag{3.3}$$

where C is a positive constant independent of t and w.

We conclude this paragraph with an analogous result on the heat kernel G:

Lemma 3.4. *For all $w \in H^2(\mathbf{R})$ and all $t > 0$,*

$$\|G(t,\cdot)*w - w\|_{L^2(\mathbf{R})} \leqslant \varepsilon\, t\, \|w\|_{H^2(\mathbf{R})}.$$

Proof. Proceeding as above, we have:

$$\begin{aligned}G(t,\cdot)*w - w &= \int_0^t \frac{\partial}{\partial t}\left(G(s,\cdot)*w\right)\,ds = \varepsilon\int_0^t \partial_x^2\left(G(s,\cdot)*w\right)\,ds\\ &= \varepsilon\int_0^t G(s,\cdot)*\partial_x^2 w\,ds.\end{aligned}$$

Taking the norm L^2 and using Proposition 2.2, Young's inequality yields

$$\|G(t,\cdot)*w - w\|_{L^2(\mathbf{R})} \leqslant \varepsilon\int_0^t \|G(s,\cdot)\|_{L^1(\mathbf{R})}\|\partial_x^2 w\|_{L^2(\mathbf{R})}\,ds \leqslant \varepsilon\, t\,\|w\|_{H^2(\mathbf{R})},$$

hence the result. □

3.2. **Estimates on Y^t.** We now turn to the viscous Burgers' equation (1.7):

$$\partial_t w - \varepsilon\partial_x^2 w + w\partial_x w = 0; \quad w_{|t=0} = w_0. \tag{3.4}$$

Remark 3.5 (Hopf–Cole transform). It is well-known that the change of unknown function

$$w = -2\varepsilon\frac{1}{\phi}\partial_x\phi = -2\varepsilon\partial_x\left(\ln\phi\right),$$

turns the viscous Burgers' equation into the heat equation:

$$\partial_t\phi - \varepsilon\partial_x^2\phi = 0.$$

We infer the explicit formula:

$$w(t,x) = -2\varepsilon\partial_x \ln\left(\frac{1}{\sqrt{4\pi\varepsilon t}}\int_{-\infty}^{+\infty}\exp\left(-\frac{(x-y)^2}{4\varepsilon t} - \frac{1}{2\varepsilon}\int_0^y w_0(z)dz\right)dy\right).$$

However, this formula does not seem very helpful in order to establish Proposition 3.6.

Proposition 3.6. *Let $w_0 \in H^1(\mathbf{R})$. Then* (3.4) *has a unique solution $w \in C(\mathbf{R}_+; H^1(\mathbf{R}))$. In addition, there exists $C = C(\varepsilon, \|w_0\|_{H^1(\mathbf{R})})$ such that for all $t \geqslant 0$,*

$$\|w(t)\|_{L^2(\mathbf{R})} \leqslant \|w_0\|_{L^2(\mathbf{R})}, \quad \|\partial_x w(t)\|_{L^2(\mathbf{R})} \leqslant \|w_0'\|_{L^2(\mathbf{R})} e^{C(t^{5/8}+t)}.$$

If in addition $w_0 \in H^n(\mathbf{R})$, for some $n \geqslant 2$, then $w \in C(\mathbf{R}_+; H^n(\mathbf{R}))$ and for all $T > 0$, there exists $M = M(\varepsilon, T, \|w_0\|_{H^2(\mathbf{R})})$ such that for all $t \in [0,T]$,

$$\|w(t)\|_{H^n(\mathbf{R})} \leqslant \|w_0\|_{H^n(\mathbf{R})} e^{Mt}.$$

Proof. The existence and uniqueness part being standard, we focus on the estimates. The L^2 estimate yields (formally, multiply (3.4) by w and integrate)

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 + \varepsilon\|\partial_x w(t)\|_{L^2}^2 = 0, \tag{3.5}$$

and the H^1 estimate (differentiate (3.4) with respect to x, multiply by $\partial_x w$ and integrate),

$$\frac{1}{2}\frac{d}{dt}\|\partial_x w(t)\|_{L^2}^2 + \varepsilon\|\partial_x^2 w(t)\|_{L^2}^2 = -\frac{1}{4}\int_{\mathbf{R}} \left(\partial_x w(t,x)\right)^3 dx. \tag{3.6}$$

The L^2 estimate (3.5) shows that the map $t \mapsto \|w(t)\|_{L^2}^2$ is non-increasing:

$$\|w(t)\|_{L^2} \leqslant \|w_0\|_{L^2}, \quad \forall t \geqslant 0.$$

An integration by parts and Cauchy–Schwarz inequality then yield

$$\|\partial_x w(t)\|_{L^2}^2 \leqslant \|w(t)\|_{L^2}\|\partial_x^2 w(t)\|_{L^2} \leqslant \|w_0\|_{L^2}\|\partial_x^2 w(t)\|_{L^2}. \tag{3.7}$$

In order to take advantage of the smoothing effect provided by the viscous part, integrate (3.6) in time and write

$$\varepsilon\int_0^t \|\partial_x^2 w(s)\|_{L^2}^2 ds \leqslant \frac{1}{2}\|w_0'\|_{L^2}^2 + \frac{1}{4}\int_0^t \|\partial_x w(s)\|_{L^3}^3 ds.$$

Gagliardo–Nirenberg inequality yields

$$\|\partial_x w\|_{L^3} \leqslant C\|\partial_x w\|_{L^2}^{5/6}\|\partial_x^2 w\|_{L^2}^{1/6},$$

so using (3.7), we infer:

$$\begin{aligned}\varepsilon\int_0^t \|\partial_x^2 w(s)\|_{L^2}^2 ds &\leqslant \frac{1}{2}\|w_0'\|_{L^2}^2 + C\int_0^t \|\partial_x w(s)\|_{L^2}^{5/2}\|\partial_x^2 w(s)\|_{L^2}^{1/2} ds \\ &\leqslant \frac{1}{2}\|w_0'\|_{L^2}^2 + C\|w_0\|_{L^2}^{5/4}\int_0^t \|\partial_x^2 w(s)\|_{L^2}^{7/4} ds.\end{aligned}$$

In view of Hölder inequality in the last integral in time,

$$\begin{aligned}\varepsilon\int_0^t \|\partial_x^2 w(s)\|_{L^2}^2 ds &\leqslant \frac{1}{2}\|w_0'\|_{L^2}^2 + C\|w_0\|_{L^2}^{5/4}\left(\int_0^t \|\partial_x^2 w(s)\|_{L^2}^2 ds\right)^{7/8} t^{1/8} \\ &\leqslant \frac{1}{2}\|w_0'\|_{L^2}^2 + \frac{\varepsilon}{2}\int_0^t \|\partial_x^2 w(s)\|_{L^2}^2 ds + C\left(\|w_0\|_{L^2}\right) t,\end{aligned}$$

where we have used Young inequality $ab \lesssim a^{8/7} + b^8$. We infer

$$\varepsilon \int_0^t \|\partial_x^2 w(s)\|_{L^2}^2 ds \leqslant \|w_0'\|_{L^2}^2 + C\left(\|w_0\|_{L^2}\right) t. \tag{3.8}$$

Gagliardo–Nirenberg inequality $\|f\|_{L^\infty} \leqslant \sqrt{2}\|f\|_{L^2}^{1/2}\|f'\|_{L^2}^{1/2}$ now yields

$$\begin{aligned}
\int_0^t \|\partial_x w(s)\|_{L^\infty} ds &\leqslant \sqrt{2}\int_0^t \|\partial_x w(s)\|_{L^2}^{1/2}\|\partial_x^2 w(s)\|_{L^2}^{1/2} ds \\
&\leqslant C\left(\varepsilon, \|w_0\|_{L^2}\right)\int_0^t \|\partial_x^2 w(s)\|_{L^2}^{3/4} ds \\
&\leqslant C\left(\varepsilon, \|w_0\|_{L^2}\right)\left(\int_0^t \|\partial_x^2 w(s)\|_{L^2}^2 ds\right)^{3/8} t^{5/8} \\
&\leqslant C\left(\varepsilon, \|w_0\|_{H^1}\right)(1+t)^{3/8} t^{5/8} \leqslant C\left(\varepsilon, \|w_0\|_{H^1}\right)\left(t^{5/8} + t\right),
\end{aligned}$$

where we have used (3.7), Hölder inequality and (3.8), successively.

Integrate the H^1 estimate (3.6) with respect to time, and now discard the viscous part whose contribution is non-negative:

$$\begin{aligned}
\|\partial_x w(t)\|_{L^2}^2 &\leqslant \|w_0'\|_{L^2}^2 + \frac{1}{2}\int_0^t \|\partial_x w(s)\|_{L^3}^3 ds \\
&\leqslant \|w_0'\|_{L^2}^2 + \frac{1}{2}\int_0^t \|\partial_x w(s)\|_{L^\infty}\|\partial_x w(s)\|_{L^2}^2 ds.
\end{aligned} \tag{3.9}$$

The first part of the proposition then follows from the Gronwall lemma.

To complete the proof of the proposition, we use the general H^n estimate, for $n \in \mathbf{N}$: set $\Lambda = (1-\partial_x^2)^{1/2}$. Applying Λ^n to (3.4) yields

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^n w(t)\|_{L^2} = -\int_{\mathbf{R}} \Lambda^n w(t,x)\Lambda^n\left(w\partial_x w\right)(t,x)dx + \varepsilon\int_{\mathbf{R}} \Lambda^n w(t,x)\Lambda^n\partial_x^2 w(t,x)dx.$$

Integrating by parts, the last term is non-positive, since

$$\int_{\mathbf{R}} \Lambda^n w(t,x)\Lambda^n\partial_x^2 w(t,x)dx = -\int_{\mathbf{R}} \left|\Lambda^n\partial_x w(t,x)\right|^2 dx.$$

Write

$$\begin{aligned}
\int_{\mathbf{R}} \Lambda^n w(t,x)\Lambda^n\left(w\partial_x w\right)(t,x)dx &= \int_{\mathbf{R}} \Lambda^n w(t,x)\left(w\partial_x\Lambda^n w\right)(t,x)dx \\
&\quad + \int_{\mathbf{R}} \Lambda^n w(t,x)\left(w\partial_x\Lambda^n w - \Lambda^n\left(w\partial_x w\right)\right)(t,x)dx.
\end{aligned}$$

Integrating by parts the first term yields

$$\begin{aligned}
\left|\int_{\mathbf{R}} \Lambda^n w(t,x)\left(w\partial_x\Lambda^n w\right)(t,x)dx\right| &= \frac{1}{2}\left|\int_{\mathbf{R}} \partial_x\left(\Lambda^n w(t,x)\right)^2 w(t,x)dx\right| \\
&= \frac{1}{2}\left|\int_{\mathbf{R}} \left(\Lambda^n w(t,x)\right)^2\partial_x w(t,x)dx\right| \\
&\leqslant \frac{\|\partial_x w(t)\|_{L^\infty}}{2}\|w(t)\|_{H^n}^2.
\end{aligned}$$

In view of Kato-Ponce estimate [16]

$$\|\Lambda^n(fg) - f\Lambda^n g\|_{L^2} \leqslant C\|\partial_x f\|_{L^\infty}\|g\|_{H^{n-1}} + \|f\|_{H^n}\|g\|_{L^\infty}, \tag{3.10}$$

we have (with $f = w$ and $g = \partial_x w$)

$$\|w\partial_x\Lambda^n w - \Lambda^n\left(w\partial_x w\right)\|_{L^2} \leqslant C\|\partial_x w\|_{L^\infty}\|w\|_{H^n}.$$

Leaving out the viscous term, Gronwall lemma yields the *a priori* estimate

$$\|w(t)\|_{H^n} \leqslant \|w_0\|_{H^n}\exp\left(C\int_0^t \|\partial_x w(s)\|_{L^\infty}ds\right), \tag{3.11}$$

where C depends only on $n \in \mathbf{N}$. In particular, for $n = 2$, Gronwall lemma implies $\|\partial_x^2 w(t)\|_{L^2} \leqslant \|w_0\|_{H^2}e^{C(t^{5/8}+t)}$, where $C = C(\varepsilon, \|w_0\|_{H^1})$. We bootstrap, thanks to Gagliardo–Nirenberg inequality again:

$$\|\partial_x w(t)\|_{L^\infty} \leqslant \sqrt{2}\|\partial_x w(t)\|_{L^2}^{1/2}\|\partial_x^2 w(t)\|_{L^2}^{1/2} \leqslant \sqrt{2}\|w_0\|_{H^2}e^{C(t^{5/8}+t)}.$$

Therefore, for $t \in [0,T]$,

$$\int_0^t \|\partial_x w(s)\|_{L^\infty}ds \leqslant \sqrt{2}\|w_0\|_{H^2} \times t \times e^{C(T^{5/8}+T)}.$$

The last estimates of the proposition then follow from (3.11). □

Lemma 3.7. *Let $T > 0$. For all $w \in H^1(\mathbf{R})$, there exists $C = C(T, \|w\|_{L^2(\mathbf{R})})$ such that*

$$\|Y^t w\|_{H^1(\mathbf{R})} \leqslant e^{Ct}\|w\|_{H^1(\mathbf{R})}, \quad \forall t \in [0,T]. \tag{3.12}$$

Proof. Differentiating the Duhamel formula (1.8) in space, we have

$$\partial_x Y^t w = G(t,\cdot) * \partial_x w - \int_0^t \partial_x G(t-s,\cdot) * (Y^s w)\partial_x(Y^s w)\,ds.$$

Using Young inequality and inequality (1.9), we infer:

$$\begin{aligned}\|\partial_x Y^t w\|_{L^2(\mathbf{R})} \leqslant{}& \|\partial_x w\|_{L^2(\mathbf{R})} \\ &+ \int_0^t \|\partial_x G(t-s,\cdot)\|_{L^2(\mathbf{R})}\|(Y^s w)\partial_x(Y^s w)\|_{L^1(\mathbf{R})}\,ds.\end{aligned}$$

In view of Proposition 2.2, this implies:

$$\begin{aligned}\|\partial_x Y^t w\|_{L^2(\mathbf{R})} \leqslant{}& \|\partial_x w\|_{L^2(\mathbf{R})} \\ &+ C_0\int_0^t (t-s)^{-3/4}\|(Y^s w)\partial_x(Y^s w)\|_{L^1(\mathbf{R})}\,ds.\end{aligned}$$

Writing

$$\|(Y^s w)\partial_x(Y^s w)\|_{L^1} \leqslant \|Y^s w\|_{L^2}\|\partial_x(Y^s w)\|_{L^2} \leqslant \|w\|_{L^2}\|\partial_x(Y^s w)\|_{L^2},$$

and invoking the fractional Gronwall Lemma 2.4 with $\theta = 1/4$, the lemma follows. □

Corollary 3.8. *Let $n \geqslant 1$ and $w \in H^n(\mathbf{R})$. Let $T, \alpha > 0$. If*

$$\|w\|_{L^2(\mathbf{R})} \leqslant \alpha,$$

then there exists c depending only on T and α such that

$$\|Y^t w\|_{H^n(\mathbf{R})} \leqslant e^{ct}\|w\|_{H^n(\mathbf{R})}, \quad \forall t \in [0,T].$$

Proof. Denote $w(t) = Y^t w$. From Lemma 3.7,

$$\|w(t)\|_{H^1(\mathbf{R})} \leqslant \|w_0\|_{H^1} e^{Ct},$$

where C depends only on α and T. From the proof of the first part of Proposition 3.6 (see the estimate after (3.8)), we infer

$$\int_0^t \|\partial_x w(s)\|_{L^\infty} ds \leqslant C(\alpha, T), \quad \forall t \in [0, T].$$

The corollary then stems from (3.11). □

3.3. **Estimates on the splitting operator Z_L^t.** Combining the estimates on X^t and Y^t established in the previous two sections, we infer:

Corollary 3.9. (1) *For all $u \in L^2(\mathbf{R})$ and all $t > 0$,*

$$\|Z_L^t\, u\|_{L^2(\mathbf{R})} \leqslant e^{\beta_0 t}\, \|u\|_{L^2(\mathbf{R})},$$

where $\beta_0 = -\min \mathrm{Re}(\phi_{\mathcal{I}}) > 0$.

(2) *Let $T > 0, n \in \mathbf{N}^*$ and $u \in H^n(\mathbf{R})$. There exists $C = C(T, \|u\|_{L^2(\mathbf{R})})$ such that for all $t \in [0, T]$,*

$$\|Z_L^t\, u\|_{H^n(\mathbf{R})} \leqslant e^{Ct} \|u\|_{H^n(\mathbf{R})}.$$

Proof. The first point is a direct consequence of the relation (1.9) and Lemma 3.2. The second point is readily established with Lemma 3.2 and Corollary 3.8. □

3.4. **Estimates on the exact flow S^t.**

Lemma 3.10 (L^2-a priori estimate)**.** *Let $u_0 \in L^2(\mathbf{R})$ and $T > 0$. Then, the unique mild solution $u \in C([0, T]; L^2(\mathbf{R})) \cap C(]0, T]; H^2(\mathbf{R}))$ of* (1.1) *satisfies, for all $t \in [0, T]$*

$$\|u(t, \cdot)\|_{L^2(\mathbf{R})} \leqslant e^{\alpha_0 t} \|u_0\|_{L^2(\mathbf{R})},$$

where $\alpha_0 = -\min \mathrm{Re}(\psi_{\mathcal{I}}) > 0$.

Proof. Multiplying (1.1) by u and integrating with respect to the space variable, we get:

$$\int_{\mathbf{R}} u_t u \, dx + \int_{\mathbf{R}} (\mathcal{I}[u] - u_{xx})\, u \, dx = 0$$

because the nonlinear term is zero. Using (1.3) and the fact that u and $\int_{\mathbf{R}} (\mathcal{I}[u] - \partial_{xx}^2 u) u \, dx$ are real, we get

$$\int_{\mathbf{R}} (\mathcal{I}[u] - \partial_{xx}^2 u) u \, dx = \int_{\mathbf{R}} \mathcal{F}^{-1}(\psi_{\mathcal{I}} \mathcal{F} u) u \, dx = \int_{\mathbf{R}} \psi_{\mathcal{I}} |\mathcal{F} u|^2 \, d\xi = \int_{\mathbf{R}} \mathrm{Re}(\psi_{\mathcal{I}}) |\mathcal{F} u|^2 \, d\xi.$$

We infer

$$\frac{1}{2} \frac{d}{dt} \|u(t, \cdot)\|_{L^2}^2 \quad \leqslant \quad \alpha_0 \|u(t)\|_{L^2}^2$$

where $\alpha_0 = -\min \mathrm{Re}(\psi_{\mathcal{I}}) > 0$. The result then follows from the Gronwall lemma. □

Lemma 3.11. *Let $n \in \mathbf{N}^*$, $u_0 \in H^n(\mathbf{R})$ and $T > 0$. There exists $C_T(\|u_0\|_{H^{n-1}(\mathbf{R})})$ such that the unique mild solution $u \in C([0, T]; H^n(\mathbf{R}))$ satisfies*

$$\|u(t, \cdot)\|_{H^n(\mathbf{R})} \leqslant C_T(\|u_0\|_{H^{n-1}(\mathbf{R})}) \|u_0\|_{H^n(\mathbf{R})}. \tag{3.13}$$

Proof. The proof is similar to the one given in Lemma 3.7. Differentiating the Duhamel formula (1.4) in space, we have

$$\partial_x u = K(t,\cdot) * \partial_x u_0 - \int_0^t \partial_x K(t-s,\cdot) * (u(s)\partial_x u(s))\, ds.$$

Using Young inequality and Proposition 2.1, we infer, for any integer $n \geqslant 1$:

$$\begin{aligned}\|\partial_x u(t)\|_{H^{n-1}(\mathbf{R})} &\leqslant e^{\alpha_0 t}\|\partial_x u_0\|_{H^{n-1}(\mathbf{R})} \\ &\quad + \int_0^t \|\partial_x K(t-s,\cdot)\|_{L^2(\mathbf{R})}\|u(s)\partial_x u(s)\|_{W^{n-1,1}(\mathbf{R})}\, ds.\end{aligned}$$

In view of Proposition 2.1, this implies:

$$\begin{aligned}\|\partial_x u(t)\|_{H^{n-1}(\mathbf{R})} &\leqslant e^{\alpha_0 t}\|\partial_x u_0\|_{H^{n-1}(\mathbf{R})} \\ &\quad + C_0\int_0^t (t-s)^{-3/4}\|u(s)\partial_x u(s)\|_{W^{n-1,1}(\mathbf{R})}\, ds.\end{aligned}$$

For $n = 1$, we use Lemma 3.10 to have

$$\|u(s)\partial_x u(s)\|_{L^1} \leqslant \|u(s)\|_{L^2}\|\partial_x u(s)\|_{L^2} \leqslant e^{\alpha_0 T}\|u_0\|_{L^2}\|\partial_x u(s)\|_{L^2}.$$

The fractional Gronwall Lemma 2.4 with $\theta = 1/4$ then yields

$$\|\partial_x u(t)\|_{L^2(\mathbf{R})} \leqslant e^{Ct+CT}\|\partial_x u_0\|_{L^2(\mathbf{R})},$$

where C depends only on T and $\|u_0\|_{L^2}$. From (1.9), this implies the lemma in the case $n = 1$. For $n \geqslant 1$, Leibniz rule and Cauchy–Schwarz inequality yield

$$\|u(s)\partial_x u(s)\|_{W^{n-1,1}(\mathbf{R})} \leqslant C(n)\|u(s)\|_{H^{n-1}(\mathbf{R})}\|\partial_x u(s)\|_{H^{n-1}(\mathbf{R})}.$$

The lemma then easily follows by induction on n. □

We will also need the fact that the flow map S^t is uniformly Lipschitzean on balls of $H^2(\mathbf{R})$.

Proposition 3.12. *Let $T, R > 0$. There exists $K = K(R,T) < \infty$ such that if*

$$\|u_0\|_{H^2(\mathbf{R})} \leqslant R,\ \|v_0\|_{H^2(\mathbf{R})} \leqslant R,$$

then

$$\|S^t u_0 - S^t v_0\|_{L^2(\mathbf{R})} \leqslant K\|u_0 - v_0\|_{L^2(\mathbf{R})}, \quad \forall t \in [0,T].$$

Proof. Set $u(t) = S^t u_0$, $v(t) = S^t v_0$ and $w = u - v$. It solves

$$\partial_t w + \mathcal{I}[w] - \partial_{xx}^2 w = v\partial_x v - u\partial_x u = -u\partial_x w - w\partial_x v. \tag{3.14}$$

The L^2 energy estimate yields:

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \int_{\mathbf{R}} w\,(u\partial_x w + w\partial_x v) \leqslant \alpha_0\|w(t)\|_{L^2}^2,$$

where the term $\int w\left(\mathcal{I}[w] - \partial_{xx}^2 w\right)$ has been estimated as in the proof of Lemma 3.10. We infer

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 &\leqslant \left(\alpha_0 + \frac{1}{2}\|\partial_x u(t)\|_{L^\infty} + \|\partial_x v(t)\|_{L^\infty}\right)\|w(t)\|_{L^2}^2 \\ &\leqslant C\left(1 + \|u(t)\|_{H^2} + \|v(t)\|_{H^2}\right)\|w(t)\|_{L^2}^2 \leqslant C(R,T)\|w(t)\|_{L^2}^2,\end{aligned}$$

where we have used Sobolev embedding and Lemma 3.11. Gronwall lemma yields the result. □

4. Local error estimate

As pointed out in the introduction, the formalism of Lie derivatives does not seem to be easy to use in the framework of this paper, since the action of the nonlocal operator $\mathcal{I}$ on nonlinearities is rather involved. As a consequence, we prove an L^2 error estimate by a rather pedestrian way.

Proposition 4.1 (L^2 local error estimate)**.** *Let $u_0 \in H^3(\mathbf{R})$. There exists $C\left(\|u_0\|_{L^2(\mathbf{R})}\right)$ such that for all $t \in [0,1]$,*

$$\|Z_L^t u_0 - S^t u_0\|_{L^2(\mathbf{R})} \leqslant C\left(\|u_0\|_{L^2(\mathbf{R})}\right) t^2 \|u_0\|^2_{H^3(\mathbf{R})}.$$

Proof. From the definition of Z_L^t and Remark 2.3, we have

$$\begin{aligned} Z_L^t u_0 = X^t Y^t u_0 &= X^t\left(G(t) * u_0 - \frac{1}{2}\int_0^t G(t-s) * \partial_x (Y^s u_0)^2\, ds\right) \\ &= D(t) * G(t) * u_0 - \frac{1}{2}\int_0^t D(t) * G(t-s) * \partial_x (Y^s u_0)^2\, ds \\ &= K(t) * u_0 - \frac{1}{2}\int_0^t D(t) * G(t-s) * \partial_x (Y^s u_0)^2\, ds. \end{aligned} \tag{4.1}$$

Thus, from Duhamel formula for the Fowler equation (1.4) and the Lie formula (4.1), we have:

$$\begin{aligned} Z_L^t u_0 - S^t u_0 &= \frac{1}{2}\int_0^t \partial_x K(t-s) * (S^s u_0)^2\, ds - \frac{1}{2}\int_0^t D(t) * \partial_x G(t-s) * (Y^s u_0)^2\, ds \\ &= \frac{1}{2}\int_0^t \partial_x K(t-s) * \left((S^s u_0)^2 - (Z_L^s u_0)^2\right)\, ds + R(t), \end{aligned} \tag{4.2}$$

where the remainder $R(t)$ is written as

$$R(t) = \frac{1}{2}\int_0^t R_1(s) ds, \quad \text{with } R_1(s) = \partial_x K(t-s) * (Z_L^s u_0)^2 - D(t) * \partial_x G(t-s,\cdot) * (Y^s u_0)^2.$$

Then, from Proposition 2.1, Corollary 3.9 and Lemma 3.10, we have, for $t \in [0,1]$:

$$\begin{aligned} \|Z_L^t u_0 - S^t u_0\|_{L^2(\mathbf{R})} &\leqslant \frac{1}{2}\int_0^t \|\partial_x K(t-s,\cdot)\|_{L^2(\mathbf{R})} \|(S^s u_0)^2 - (Z_L^s u_0)^2\|_{L^1(\mathbf{R})}\, ds \\ &\qquad + \|R(t)\|_{L^2(\mathbf{R})} \\ &\leqslant C\int_0^t (t-s)^{-3/4} \|S^s u_0 - Z_L^s u_0\|_{L^2(\mathbf{R})} \|S^s u_0 + Z_L^s u_0\|_{L^2(\mathbf{R})}\, ds + \|R(t)\|_{L^2(\mathbf{R})} \\ &\leqslant C(e^{\alpha_0 t} + e^{\beta_0 t})\|u_0\|_{L^2(\mathbf{R})} \int_0^t (t-s)^{-3/4} \|S^s u_0 - Z_L^s u_0\|_{L^2(\mathbf{R})}\, ds + \|R(t)\|_{L^2(\mathbf{R})}, \end{aligned}$$

where C is a positive constant. To estimate the remainder, we decompose it as follows

$$R_1(s) \quad = \quad T_1 + T_2 + T_3 + T_4,$$

where

$$\begin{aligned} T_1 &= K(t-s,\cdot) * \partial_x (Z_L^s u_0)^2 - \partial_x (Z_L^s u_0)^2, \\ T_2 &= \partial_x (Y^s u_0)^2 - G(t-s,\cdot) * \partial_x (Y^s u_0)^2, \\ T_3 &= G(t-s,\cdot) * \partial_x (Y^s u_0)^2 - D(t,\cdot) * G(t-s,\cdot) * \partial_x (Y^s u_0)^2, \\ T_4 &= \partial_x \left(Z_L^s u_0\right)^2 - \partial_x \left(Y^s u_0\right)^2. \end{aligned}$$

Let us first study the term T_1. From Corollaries 3.3 and 3.9, we have (recall that $t \in [0,1]$)

$$\begin{aligned}\|T_1\|_{L^2(\mathbf{R})} &= \|K(t-s,\cdot) * \partial_x (Z_L^s u_0)^2 - \partial_x (Z_L^s u_0)^2\|_{L^2(\mathbf{R})} \\ &\leqslant C e^{\alpha_0 (t-s)} (t-s) \|\partial_x (Z_L^s u_0)^2\|_{H^2(\mathbf{R})} \\ &\leqslant C e^{\alpha_0 (t-s)} (t-s) \|Z_L^s u_0\|^2_{H^3(\mathbf{R})} \\ &\leqslant C \left(\|u_0\|_{L^2(\mathbf{R})}\right) e^{\alpha_0 (t-s)} (t-s) \|u_0\|^2_{H^3(\mathbf{R})}.\end{aligned}$$

In the same way, from Lemma 3.4 and Corollary 3.8, we control the term T_2 as

$$\begin{aligned}\|T_2\|_{L^2(\mathbf{R})} &= \|\partial_x (Y^s u_0)^2 - G(t-s,\cdot) * \partial_x (Y^s u_0)^2\|_{L^2(\mathbf{R})} \\ &\leqslant \varepsilon\,(t-s) \|\partial_x (Y^s u_0)^2\|_{H^2(\mathbf{R})} \leqslant C \left(\|u_0\|_{L^2(\mathbf{R})}\right) (t-s) \|u_0\|^2_{H^3(\mathbf{R})}.\end{aligned}$$

From Lemma 3.2 and Corollary 3.8,

$$\begin{aligned}\|T_3\|_{L^2(\mathbf{R})} &= \|G(t-s,\cdot) * \partial_x (Y^s u_0)^2 - D(t,\cdot) * G(t-s,\cdot) * \partial_x (Y^s u_0)^2\|_{L^2(\mathbf{R})} \\ &\leqslant C\, e^{\beta_0 t}\, t\, \|G(t-s,\cdot) * \partial_x (Y^s u_0)^2\|_{H^2(\mathbf{R})} \\ &\leqslant C\, e^{\beta_0 t}\, t\, \|\partial_x (Y^s u_0)^2\|_{H^2(\mathbf{R})} \\ &\leqslant C\, \left(\|u_0\|_{L^2(\mathbf{R})}\right)\, e^{\beta_0 t}\, t\, \|u_0\|^2_{H^3(\mathbf{R})}.\end{aligned}$$

For the term T_4, write

$$\begin{aligned}\|T_4\|_{L^2(\mathbf{R})} &= \|\partial_x \left(Z_L^s u_0\right)^2 - \partial_x \left(Y^s u_0\right)^2 \|_{L^2(\mathbf{R})} \\ &= 2\|(Z_L^s u_0) \partial_x \left(Z_L^s u_0\right) - (Y^s u_0) \partial_x \left(Y^s u_0\right) \|_{L^2(\mathbf{R})}.\end{aligned}$$

By linearity of the evolution operator X^t, we have

$$\partial_x \left(Z_L^s u_0\right) = X^s \partial_x \left(Y^s u_0\right),$$

hence

$$\begin{aligned}\|T_4\|_{L^2(\mathbf{R})} &= 2\|(Z_L^s u_0) X^s \partial_x \left(Y^s u_0\right) - (Y^s u_0) \partial_x \left(Y^s u_0\right) \|_{L^2(\mathbf{R})} \\ &\leqslant 2\, \|X^s \partial_x \left(Y^s u_0\right) \left(X^s Y^s u_0 - Y^s u_0\right)\|_{L^2(\mathbf{R})} \\ &\quad + 2\, \|\left(Y^s u_0\right) \left(X^s \partial_x \left(Y^s u_0\right) - \partial_x \left(Y^s u_0\right)\right)\|_{L^2(\mathbf{R})}.\end{aligned}$$

Now from Sobolev embedding, Lemma 3.2 and Corollary 3.8, we get:

$$\begin{aligned}\|T_4\|_{L^2(\mathbf{R})} &\leqslant 2\|X^s \partial_x \left(Y^s u_0\right) \|_{L^\infty(\mathbf{R})} \|X^s Y^s u_0 - Y^s u_0\|_{L^2(\mathbf{R})} \\ &\quad + 2\|Y^s u_0\|_{L^\infty(\mathbf{R})} \|X^s \partial_x \left(Y^s u_0\right) - \partial_x \left(Y^s u_0\right) \|_{L^2(\mathbf{R})} \\ &\leqslant C \|X^s \partial_x \left(Y^s u_0\right) \|_{H^1(\mathbf{R})}\, e^{\beta_0 s}\, s\, \|Y^s u_0\|_{H^2(\mathbf{R})} \\ &\quad + C \|Y^s u_0\|_{H^1(\mathbf{R})}\, e^{\beta_0 s}\, s\, \|\partial_x \left(Y^s u_0\right) \|_{H^2(\mathbf{R})} \\ &\leqslant C\, e^{2\beta_0 s}\, s\, \|Y^s u_0\|^2_{H^2(\mathbf{R})} + C\, e^{\beta_0 s}\, s\, \|Y^s u_0\|^2_{H^3(\mathbf{R})} \\ &\leqslant C(\|u_0\|_{L^2(\mathbf{R})})\, e^{2\beta_0 s}\, s\, \|u_0\|^2_{H^3(\mathbf{R})}.\end{aligned}$$

Finally, since $R_1(s) = T_1 + T_2 + T_3 + T_4$ then for $0 \leqslant s \leqslant t \leqslant 1$,

$$\|R_1(s)\|_{L^2(\mathbf{R})} \leqslant C \left(\|u_0\|_{L^2(\mathbf{R})}\right)\, t\, \|u_0\|^2_{H^3(\mathbf{R})},$$

and by integration for $s \in [0,t]$,

$$\|R(t)\|_{L^2(\mathbf{R})} \leqslant C \left(\|u_0\|_{L^2(\mathbf{R})}\right)\, t^2\, \|u_0\|^2_{H^3(\mathbf{R})}.$$

We conclude by applying the modified fractional Gronwall Lemma 2.5. □

5. Proof of Theorem 1.2

The proof follows the same lines as in [15, Section 5]. Denote by $u_k = \left(Z_L^{\Delta t}\right)^k u_0$ the numerical solution, and

$$u_n^k = S^{(n-k)\Delta t} u_k,$$

which corresponds to the exact evolution of the numerical value u_k at time $t_k = k\Delta t$ up to time $t_n = n\Delta t$. From Lemma 3.10, there exists ρ such that

$$\|u(t)\|_{H^2(\mathbf{R})} \leqslant \rho, \quad \forall t \in [0,T].$$

We prove by induction that there exists $\gamma, \Delta t_0, c > 0$ such that if $0 < \Delta t \leqslant \Delta t_0$, for all $n \in \mathbf{N}$ with $n\Delta t \leqslant T$,

$$\begin{aligned}
&\|u_n\|_{L^2(\mathbf{R})} \leqslant 2\rho, \quad \|u_n - u(t_n)\|_{L^2(\mathbf{R})} \leqslant \gamma \Delta t,\\
&\|u_n\|_{H^3(\mathbf{R})} \leqslant e^{cn\Delta t}\|u_0\|_{H^3(\mathbf{R})} \leqslant C_0,
\end{aligned}$$

where $C_0 = e^{cT}\|u_0\|_{H^3}$. The above properties are satisfied for $n = 0$. Let $n \geqslant 1$, and suppose that the induction assumption is true for $0 \leqslant k \leqslant n-1$. Since $u_n = u_n^n$ and $u(t_n) = u_n^0$, we estimate

$$\begin{aligned}
\|u_n - u(t_n)\|_{L^2} &\leqslant \sum_{k=0}^{n-1} \|u_n^{k+1} - u_n^k\|_{L^2}\\
&\leqslant \sum_{k=0}^{n-1} \left\| S^{(n-k-1)\Delta t}\left(Z_L^{\Delta t} u_k\right) - S^{(n-k-1)\Delta t}\left(S^{\Delta t} u_k\right)\right\|_{L^2}.
\end{aligned}$$

For $k \leqslant n-2$, $Z_L^{\Delta t} u_k = u_{k+1}$ and Proposition 3.12 yields, along with the induction assumption,

$$\begin{aligned}
\|S^{\Delta t} u_k\|_{L^2} &\leqslant \|S^{\Delta t} u_k - S^{\Delta t} u(t_k)\|_{L^2} + \|S^{\Delta t} u(t_k)\|_{L^2}\\
&\leqslant K\|u_k - u(t_k)\|_{L^2} + \|u(t_{k+1})\|_{L^2} \leqslant K\gamma\Delta t + \rho,
\end{aligned}$$

which is bounded by 2ρ if $0 < \Delta t \leqslant \Delta t_0 \ll 1$. Up to replacing K with $\max(K,1)$, we obtain, for $k \leqslant n-1$ and $n\Delta t \leqslant T$,

$$\left\| S^{(n-k-1)\Delta t}\left(Z_L^{\Delta t} u_k\right) - S^{(n-k-1)\Delta t}\left(S^{\Delta t} u_k\right)\right\|_{L^2} \leqslant K\|Z_L^{\Delta t} u_k - S^{\Delta t} u_k\|_{L^2}.$$

Using Proposition 4.1, we infer

$$\left\| S^{(n-k-1)\Delta t}\left(Z_L^{\Delta t} u_k\right) - S^{(n-k-1)\Delta t}\left(S^{\Delta t} u_k\right)\right\|_{L^2} \leqslant CK(\Delta t)^2 e^{2ck\Delta t}\|u_0\|_{H^3}^2,$$

for some uniform constant C. Therefore,

$$\|u_n - u(t_n)\|_{L^2} \leqslant nCK(\Delta t)^2 e^{2cT}\|u_0\|_{H^3}^2 \leqslant CTKe^{2cT}\Delta t,$$

which yields the first two estimates of the induction, provided one takes $\gamma = CTKe^{2cT}$, which is uniform in n and Δt. Finally, the last estimate of the induction stems from Corollary 3.9.

6. NUMERICAL EXPERIMENTS

The aim of this section is to numerically verify the Lie method convergence rate in $\mathcal{O}(\Delta t)$ for the Fowler equation (1.1).
To solve the linear sub-equation (1.6), discrete Fourier transform is used and for the non-linear sub-equation (1.7), different numerical approximations can be used. Here, we use the finite difference method.
Since the discrete Fourier transform plays a key role in these schemes, we briefly review its definition, which can be found in most books. In some situation, when the mesh nodes number N is chosen to be $N = 2^p$ for some integer p, a fast Fourier transform (FFT) algorithm is used to further decrease the computation time. In this work we will use a subroutine implemented in Matlab. In this program, the interval $[0, 1]$ is discretized by N equidistant points, with spacing $\Delta x = 1/N$. The spatial grid points are then given by $x_j = j/N, j = 0, ..., N$. If $u_j(t)$ denotes the approximate solution to $u(t, x_j)$, the discrete Fourier transform of the sequence $\{u_j\}_{j=0}^{N-1}$ is defined by

$$\hat{u}(k) = \mathcal{F}_k^d(u_j) = \sum_{j=0}^{N-1} u_j e^{-2i\pi jk/N},$$

for $k = 0, \cdots, N-1$, and the inverse discrete Fourier transform is given by

$$u_j = \mathcal{F}_j^{-d}(\hat{u}_k) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{u}_k e^{2i\pi k x_j},$$

for $j = 0, \cdots, N-1$. Here $\mathcal{F}^d$ denotes the discrete Fourier transform and $\mathcal{F}^{-d}$ its inverse.

In what follows, the linear equation (1.6) is solved using the discrete Fourier transform and time marching is performed exactly according to

$$u_j^{n+1} = \mathcal{F}_j^{-d}\left(e^{-\phi_{\mathcal{I}}(k)\Delta t}\mathcal{F}_k^d(u_j^n)\right). \tag{6.1}$$

To approximate the viscous Burgers' equation (1.7), we use the following explicit centered scheme:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left[\left(\frac{u^2}{2}\right)_{j+1}^n - \left(\frac{u^2}{2}\right)_{j-1}^n\right] + \varepsilon\,\Delta t \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}, \tag{6.2}$$

which is stable under the CFL-Peclet condition

$$\Delta t = \min\left(\frac{\Delta x}{|v|}, \frac{\Delta x^2}{2\varepsilon}\right), \tag{6.3}$$

where v is an average value of u in the neighbourhood of (t^n, x_j).

Remark 6.1. In the case where the linear sub-equation (1.6) is solved using a finite difference scheme instead of a FFT computation, an additional stability condition is required, see [3]. Moreover, the computation time becomes very long because of the discretization of the nonlocal term which is approximated using a quadrature rule. Indeed, in [3], the Fowler equation has been discretized using finite difference method and the numerical analysis showed that this operation is computationally expensive. This observation has also motivated the use of splitting methods, in particular the implementation of split-step Fourier methods.

In order to avoid numerical reflections due to boundaries conditions and to justify the use of the FFT method, we consider initial data with compact support displayed in Figure 1 to perform numerical simulations.

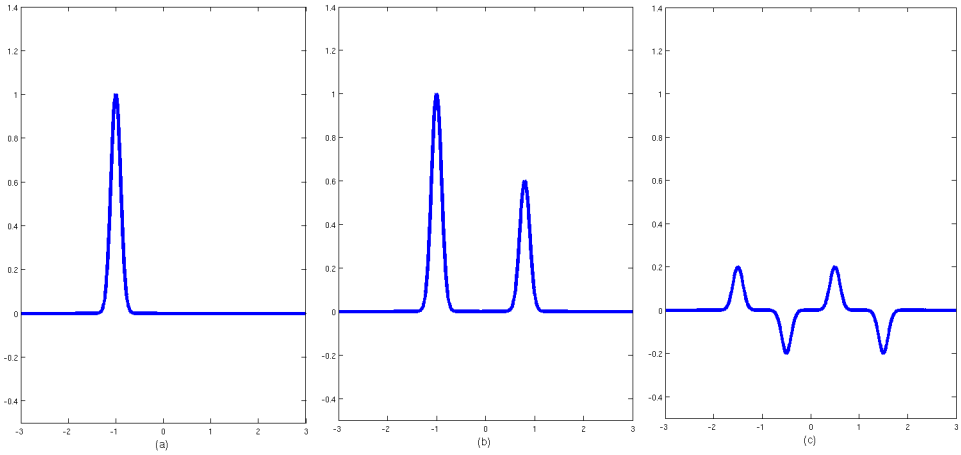

FIGURE 1. Initial data used for numerical experiments.

Since we do not know the exact solution of the Fowler equation, a classical numerical way to determine the convergence numerical order of schemes is to plot the logarithm of the error $||u_1(T) - u_2(T)||_{L^2}$ in function of the logarithm of the step time Δt, where u_1 and u_2 are computed for time steps $\Delta t/2$ and $\Delta t/4$, respectively, up to the final time T. Hence, the numerical order corresponds to the slope of the curve, see Figures 2, 3. For reference, a small line of slope one (resp. two) is added in Figure 2 (resp. 3). We see that the slopes for the three initial data match well and so we can conclude that numerical simulations are consistent with the theoretical results established above.

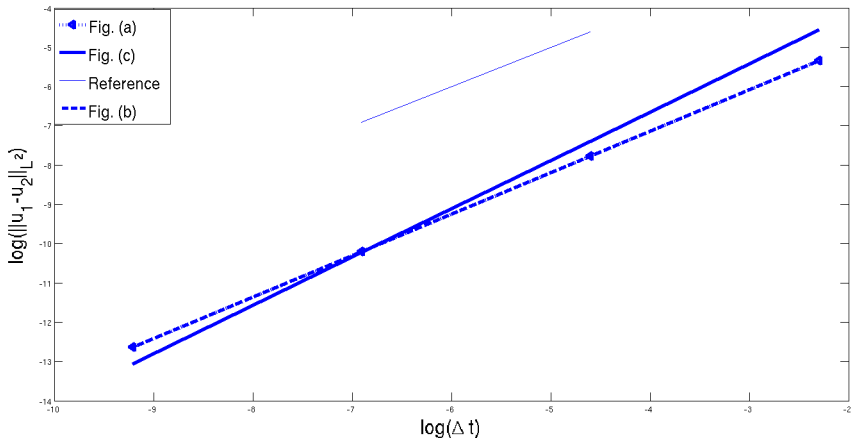

FIGURE 2. Lie method

We also study numerical convergence of Strang splittings using initial data displayed in Figure 1. Results are plotted in Figure 3. We can see that the Strang formulation is of order two in time for smooth initial data.

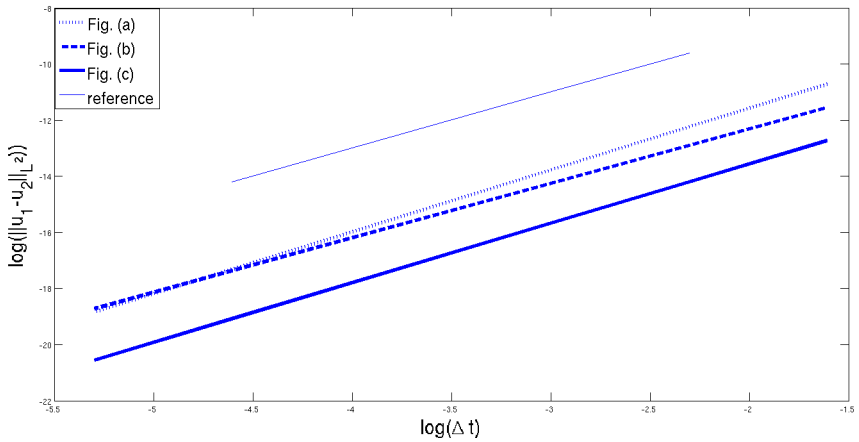

FIGURE 3. Strang method

From numerical simulations, we emphasize the fact that both formulas defining a Lie operator, as well as both formulas defining a Strang operator, lead to the same results.

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