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Stability behaviour of shells and imperfection sensitivity by the Asymptotic Numerical Method

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Abstract

This paper is concerned with stability behaviour and imperfection sensitivity of thin elastic shells. The aim is to determine the reduction of the critical buckling load as a function of the imperfection amplitude. For this purpose, the direct calculation of the so-called fold line connecting all the limit points of the equilibrium branches when the imperfection varies is performed. This fold line is the solution of an extended system demanding the criticality of the equilibrium. The Asymptotic Numerical Method is used as an alternative to Newton-like incremental-iterative procedures for solving this extended system. It results in a very robust and efficient path-following algorithm that takes the singularity of the tangent stiffness matrix into account. A modern and efficient finite shell element is considered for the discretization. The underlying shell theory uses the EAS concept. It allows finite rotations as well as thickness stretch. A numerical example involving a shape imperfection is presented.

1 Introduction

Since the pioneering work of Koiter [1], it is well established that thin shells, when submitted to geometrical instabilities, are often very sensitive to imperfections. It is all the more true for compressed axisymmetric structures or for optimized structures with geometrical imperfections (shape, thickness, ...). Usually this sensitivity is measured by the variation of the critical buckling load of the structure with respect to the imperfection amplitude. Since the pre-buckling response of the structure can be highly nonlinear, geometrical nonlinearities — as well as material nonlinearities if needed — must be taken into account during this type of analysis.

Several approaches have been proposed to perform such a sensitivity analysis. Reliability analyses rely on the calculation of the gradient of every variable with respect to the imperfection. They can be addressed in a deterministic [2] or probabilistic [3] way. These methods often provide a large amount of information, but they also require a lot of computations.

Another approach consists in using Koiter's asymptotic expansions in the neighbourhood of a bifurcation point. This method which was originally purely analytical has recently been coupled with the finite element method [4, 5]. With this method, a geometrical shape imperfection is usually defined as a linear compound of the linear buckling modes of the structure. An advantage is that the interaction between modes can be taken into account. Nevertheless, the range of validity of the solution is limited by the range of validity of the asymptotic expansions.

A third approach consists in evaluating the nonlinear response curve of the structure for an initial value of the imperfection and in isolating the limit point corresponding to the critical buckling load. Then, a continuation method is used to follow the evolution of the limit point when the imperfection amplitude varies, i.e. the so-called fold line connecting all the limit points is computed. For this purpose, an extended system which characterizes the limit points and in which the imperfection is an additional parameter has to be solved [6, 7]. This approach will be used in this paper. Its advantage is that the range of validity is not limited. With the continuation method, all the admissible values of the imperfection parameter can be studied. Nevertheless, one important drawback must be pointed out. This method can not be used in the case of plasticity because the loading history is not taken into account during the following of the fold line.

As for the classical post-buckling problem, the following of the fold line is usually performed by means of the Newton-Raphson method [8, 9]. In this paper, it is reconsidered by using the Asymptotic Numerical Method as an alternative to classical incremental-iterative methods. This method is inspired by the perturbation techniques which were developed by Thompson and Walker [10] for decomposing a nonlinear problem into a sequence of linear ones. These perturbation techniques have been revisited and efficiently solved by means of the finite element method by Cochelin, Damil and Potier-Ferry [11, 12, 13]. Since the solution obtained with the ANM is valid only in the vicinity of the starting point, a continuation method has been proposed by Cochelin [14], allowing a curve to be described in a step by step way.

The ANM has two major advantages. On one hand, its computational cost is reduced as compared to the incremental-iterative methods because the continuation procedure requires less tangent stiffness matrix decompositions. On the other hand, the continuation method is very robust and reliable. It permits to follow very complicated paths without jumping on other branches as it can happen with the Newton-Raphson method when the length of the predictor step is not carefully set. With the ANM, the step length is based on the equilibrium residual and can be computed a posteriori. By this way, it is guaranteed to be always optimal for each continuation step and no special procedure is needed for its resizing.

This method can be used for solving a large class of problems involving smooth or strong nonlinearities. For a non-exhaustive list, the interested reader should refer to [15, 16].

2 Augmented problem

This study will be restricted to geometrical nonlinearities (small strains, large displacements) and to a linear constitutive law. In this framework, the extended system that provides the fold line can be represented by the following set of discretized equations

$$\mathbf{H}(\mathbf{u}, \phi, \Lambda, \lambda) = \begin{pmatrix} \mathbf{F}(\mathbf{u}, \Lambda, \lambda) \\ \mathbf{F}_{,\mathbf{u}}(\mathbf{u}, \Lambda, \lambda) \cdot \phi \\ \|\phi\| - 1 \end{pmatrix} = \begin{pmatrix} \mathbf{f}(\mathbf{u}, \Lambda) - \lambda \mathbf{F}_e \\ \mathbf{K}_T(\mathbf{u}, \Lambda) \cdot \phi \\ \|\phi\| - 1 \end{pmatrix} = 0 \quad (1)$$

The first equation of (1) corresponds to the nonlinear equilibrium equations of the structure. In this equation, the amplitude of the imperfection is an additional variable. Therefore, the solution of this equilibrium equation is not a classical 2-D post-buckling *load-deflection* curve, but a 3-D *load-deflection-imperfection* surface that can be viewed as a continuous collection of load-deflection curves for all the possible values of the imperfection amplitude. Since we are only interested in the variation of the limit point that gives the critical buckling load, a constraint equation characterizing the singular points is added to the equilibrium equation. By this way, the surface is restricted to the fold line connecting all the limit points when the imperfection varies. In the extended system (1), this additional equation characterizes the existence of a null eigenvector ϕ associated with the null eigenvalue of the tangent operator \mathbf{K}_T at simple critical points. The last equation in (1) ensures the uniqueness of the solution and prevents ϕ from becoming excessively large or small, which could cause numerical problems. In the context of structural mechanics, this extended system has been studied or used by Moore and Spence [17], Wriggers *et al.* [18] and Wriggers and Simo [19], among others, for the precise calculation of limit and simple bifurcation points in the case of perfect structures, i.e. with Λ fixed. The mathematical study of (1) with Λ varying has been carried out by Jepson and Spence [20]. It has then been numerically investigated by Eriksson [6] and Deml and Wunderlich [7] for sensitivity analyses, in a finite element framework and using incremental-iterative strategies.

3 Asymptotic algorithm for the fold line following

3.1 Asymptotic expansions

We assume that the solution $(\mathbf{u}, \phi, \lambda, \Lambda)$ of the extended system (1) can be represented by a truncated power series expansion with respect to an additional parameter a

$$\begin{aligned} \mathbf{u}(a) &= \mathbf{u}_0 + a \mathbf{u}_1 + a^2 \mathbf{u}_2 + \dots + a^n \mathbf{u}_n \\ \phi(a) &= \phi_0 + a \phi_1 + a^2 \phi_2 + \dots + a^n \phi_n \\ \Lambda(a) &= \Lambda_0 + a \Lambda_1 + a^2 \Lambda_2 + \dots + a^n \Lambda_n \\ \lambda(a) &= \lambda_0 + a \lambda_1 + a^2 \lambda_2 + \dots + a^n \lambda_n \end{aligned} \quad (2)$$

where the point $(\mathbf{u}_0, \phi_0, \Lambda_0, \lambda_0)$ is a starting limit point that has been previously isolated. These series expansions depend on the path parameter a . Because of this new variable, an additional constraint

equation is needed. By analogy with the classical arc-length method, we choose the following definition for a

$$a = (\mathbf{u} - \mathbf{u}_0) \cdot \mathbf{u}_1 + (\Lambda - \Lambda_0)\Lambda_1 + (\lambda - \lambda_0)\lambda_1 \quad (3)$$

The next step consists in introducing the series expansions (2) into the nonlinear extended system (1) and in the definition of a (3). By identifying the terms with the same power of a , one obtains a succession of linear problems and the problem at order 1 reads

$$\begin{bmatrix} \mathbf{K}_T & 0 & \mathbf{F}_1 & -\mathbf{F}_e \\ \mathbf{K}_\phi & \mathbf{K}_T & \mathbf{F}_2 & 0 \\ 0 & \phi_0^T & 0 & 0 \\ \mathbf{u}_1^T & 0 & \Lambda_1 & \lambda_1 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \phi_1 \\ \Lambda_1 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \\ 1 \end{bmatrix} \quad \begin{array}{l} (N \text{ eq.}) \\ (N \text{ eq.}) \\ (1 \text{ eq.}) \\ (1 \text{ eq.}) \end{array} \quad (4)$$

The solution $(\mathbf{u}_1, \phi_1, \Lambda_1, \lambda_1)$ of this linear system is the tangential direction of the fold line at the starting point $(\mathbf{u}_0, \phi_0, \Lambda_0, \lambda_0)$. This system is exactly the same as for the predictor step of the Newton-Raphson method [6, 7]. Indeed, the matrix \mathbf{K}_ϕ and the two vectors \mathbf{F}_1 and \mathbf{F}_2 correspond to the differentials of the mapping \mathbf{F} evaluated at the starting point $(\mathbf{u}_0, \phi_0, \Lambda_0, \lambda_0)$

$$\mathbf{K}_T(\mathbf{u}_0, \Lambda_0) = \mathbf{f}_{,u}(\mathbf{u}_0, \Lambda_0) \quad (5)$$

$$\mathbf{K}_\phi(\mathbf{u}_0, \phi_0, \Lambda_0) = \mathbf{f}_{,uu}(\mathbf{u}_0, \Lambda_0) \cdot \phi_0 \quad (6)$$

$$\mathbf{F}_1(\mathbf{u}_0, \Lambda_0) = \mathbf{f}_{,\Lambda}(\mathbf{u}_0, \Lambda_0) \quad (7)$$

$$\mathbf{F}_2(\mathbf{u}_0, \phi_0, \Lambda_0) = \mathbf{f}_{,u\Lambda}(\mathbf{u}_0, \Lambda_0) \cdot \phi_0 \quad (8)$$

Because of the differentiation with respect to Λ , the vectors \mathbf{F}_1 and \mathbf{F}_2 depend on the type of imperfection (shape or thickness in the present paper) whereas the matrices \mathbf{K}_T and \mathbf{K}_ϕ do not. In the present case, with the use of the ANM, exact analytical forms of these differentials can be easily obtained. The derivation of the extended system (4) as well as the calculation of the directional derivatives \mathbf{K}_ϕ , \mathbf{F}_1 and \mathbf{F}_2 are fully described in [21]. Equating power-like terms at order p ($p \leq n$) yields the following system

$$\begin{bmatrix} \mathbf{K}_T & 0 & \mathbf{F}_1 & -\mathbf{F}_e \\ \mathbf{K}_\phi & \mathbf{K}_T & \mathbf{F}_2 & 0 \\ 0 & \phi_0^T & 0 & 0 \\ \mathbf{u}_1^T & 0 & \Lambda_1 & \lambda_1 \end{bmatrix} \begin{bmatrix} \mathbf{u}_p \\ \phi_p \\ \Lambda_p \\ \lambda_p \end{bmatrix} = \begin{bmatrix} \mathbf{F}_p^{nl} \\ \mathbf{G}_p^{nl} \\ h_p^{nl} \\ 0 \end{bmatrix} \quad \begin{array}{l} (N \text{ eq.}) \\ (N \text{ eq.}) \\ (1 \text{ eq.}) \\ (1 \text{ eq.}) \end{array} \quad (9)$$

The tangent augmented matrix is the same as for order 1. Only the r.h.s. terms are different. These terms contain the non-linearities of the problem. They are fully determined because they depend only on the solution at previous orders $(\mathbf{u}_r, \phi_r, \lambda_r, \Lambda_r)$ with $r < p$. As a result, recursively solving the succession of linear augmented systems will provide all the coefficients of the series (2). The r.h.s. terms depend on the considered type of imperfection and, as for the differentials (5)-(8), analytical expressions follow from the identification process. For lack of space, their expression will not be given here. The interested reader can refer to [21] for more details about their calculation.

3.2 Application to an EAS shell element

An eight noded finite shell element with finite rotations is used for the discretization. The deformation of this element is described by Reissner-Mindlin kinematics and a 6-parameter shell formulation, in which

an extensible director allows the thickness stretch of the shell, as presented by Büchter et al. [22]. The interest of this formulation lies in the use of difference displacements for the director update, instead of a classical rotational tensor, which is much more convenient for the series expansions. It is well-known that within this formulation, a significant locking phenomena connected to the linear thickness stretch distribution across the thickness occurs for cases dominated by bending. In order to avoid this problem, the Enhanced Assumed Strain method developed by Simo and Rifai [23] is applied to the transverse normal strain. Therefore, the *compatible* normal strain term is enhanced by an additional *incompatible* term. Since this additional term can be eliminated at the elementary level, the global shell formulation remains a 6-parameter formulation [22]. The adaptation of this formulation for use with the Asymptotic Numerical Method has been presented by Zahrouni et al. [24]. The extension to the present augmented problem is fully detailed in [21].

3.3 Solution procedure for the extended system

In practice, a deflated block elimination is used to solve the extended system (9). Thus, only subsystems of size N involving the matrix \mathbf{K}_T are considered. Such a block-elimination scheme can be found in Wriggers and Simo [19]. Its main interest relies on the fact that only the classical matrix \mathbf{K}_T needs to be decomposed, thus saving a large amount of calculation time. Besides this particular procedure, another numerical difficulty must be pointed out. Since all the solution points of the extended system \mathbf{H} are singular ones of \mathbf{F} , the matrix \mathbf{K}_T is singular all along the fold line connecting the computed solution points. That means that the classical matrix decomposition techniques cannot be used. A special procedure based on Lagrange multipliers is introduced to bypass this problem. A detailed description of the adaptation of the block-elimination and Lagrange multipliers procedures to the system (9) is given in [25].

3.4 Continuation procedure

Because of the limited radius of convergence of the series, only a fraction of the solution curve is obtained. Indeed, the accuracy of the solution deteriorates very quickly when the radius of convergence is reached. A criterium that gives the length of a step has been proposed by Cochelin [14]. Its justification is given in [21]. This criterium is based on the study of the residual of equilibrium equations (1)₁. In the case of our extended system, both residuals of the first and second equations of (1) must be monitored. However, numerical experiments have shown that the second equation is always more accurate than the first one. As a result, the classical criterium based on the equilibrium residual can be used. According to this criterium, for series truncated at order n , the maximal value of the path parameter a for which the solution satisfies a requested accuracy ε is given by

$$a_M = \left(\frac{\varepsilon}{\|\mathbf{F}_{n+1}^{nl}\|} \right)^{\frac{1}{n+1}} \quad (10)$$

Using this formula, the step length is set after all the coefficients of the series have been computed. By this way, the step length is guaranteed to be optimal. Once the step has been stopped, the starting point is updated and the global procedure is restarted. Thus, the solution curve is described in a step by step way, as it would be with the classical continuation algorithms. The strong point of this procedure is its robustness. Furthermore, it is completely automatic from the user's point of view. The only parameters that need to be chosen are the order n of the series and the accuracy ε . Setting n equal to 20 or 30 and $\varepsilon = 10^{-6}$ is often a good compromise.

4 Numerical example : compressed C-profile with a shape imperfection

The geometrical and material properties of this simply supported beam with a C-profile section, as well as the boundary conditions, are defined in Fig. 1. Each end of the beam is submitted to a compressive load $P=125000\lambda$. Using symmetry conditions, only one quarter of the C-profile was discretized. For this purpose, 140 Büchter EAS shell elements with 2850 dof were used.

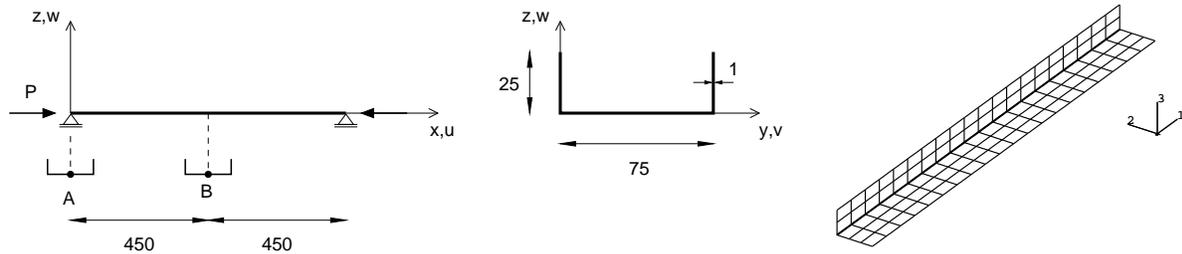


Figure 1: Geometrical properties and shell-element subdivision for one quarter of the C-profile. Material properties: $E=2100000$, $\nu=0.3$. Boundary conditions: $u(0, 450, 0)=0$, $v(0, 0, 0)=v(l, 0, 0)=0$, $w(0, y, 0)=w(l, y, 0)=0$

This example is particularly interesting because the global flexural mode (Euler mode) and the local modes are very close. As we will see, this modal interaction makes the beam very sensible to geometrical imperfections. This interaction between modes can be highlighted by studying the post-buckling behaviour of the beam. The equilibrium curve of the C-profile with an initial global flexural shape imperfection such that $w_B = -0.1$ (vertical displacement at point B) is plotted in Fig. 2. With such an initial imperfection, one might expect to trigger the corresponding flexural buckling mode. However, as can be seen on Fig. 3, the post-buckling shape combines not only the expected global flexural mode but also a local mode with 11 half-waves on the flanges of the beam. In fact, when the buckling occurs, the local 11-mode is also activated and the equilibrium curve has then a sharp turn (see Fig. 2b). This sharp turn does not correspond to a bifurcation but actually to a limit point. This particular post-buckling behaviour is due to the modal interaction. These results are in very good agreement with those obtained by Eriksson et al. [26].

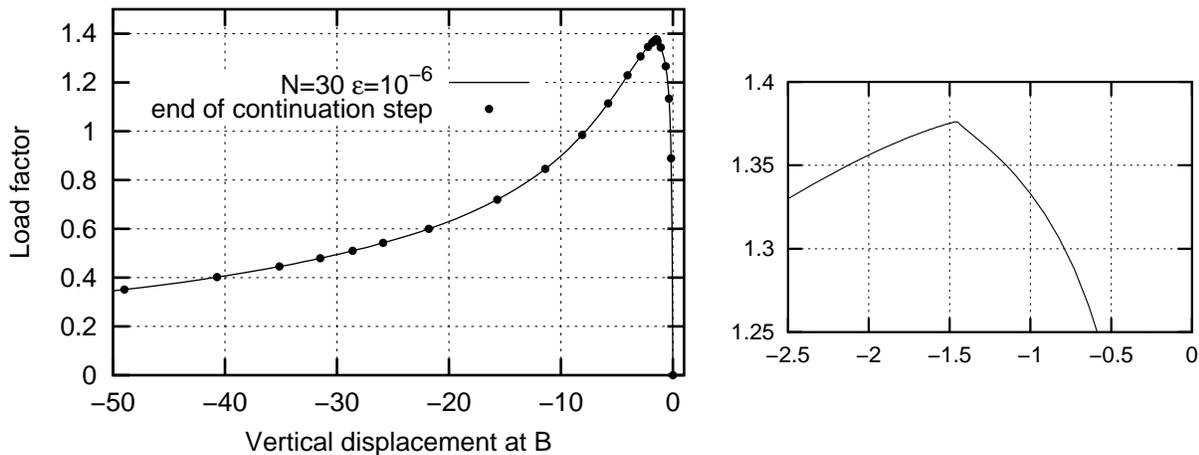


Figure 2: a) Load-displacement curve for the C-profile with an initial shape imperfection (global flexural mode) such that $w_B = -0.1$. b) Magnification of the area near the limit point

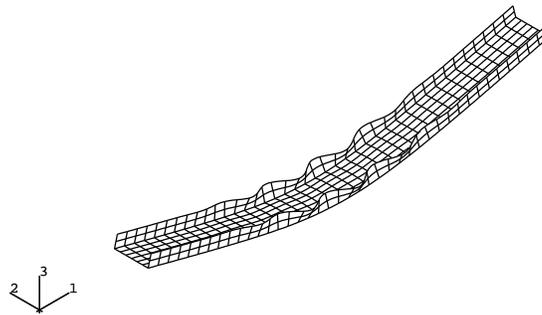


Figure 3: Deformed configuration of the complete C-profile along the post-buckling path.

For the imperfection sensitivity analysis, the same global flexural imperfection shape was used. A starting limit point was isolated on the equilibrium curve corresponding to the previous imperfection ($w_B = -0.1$). Then, the fold line following was initiated (see Fig. 4). The fold line was followed using the continuation procedure described in paragraph 3.4. For this purpose, series at order 30 and an accuracy $\varepsilon = 10^{-6}$ were used. To verify the validity of these results, the equilibrium curves for different values of the shape imperfection ($w_B = -0.1$, $w_B = -2$, $w_B = -10$, $w_B = -50$) are also plotted in Fig. 4. One can verify that the fold line passes through all the limit points.

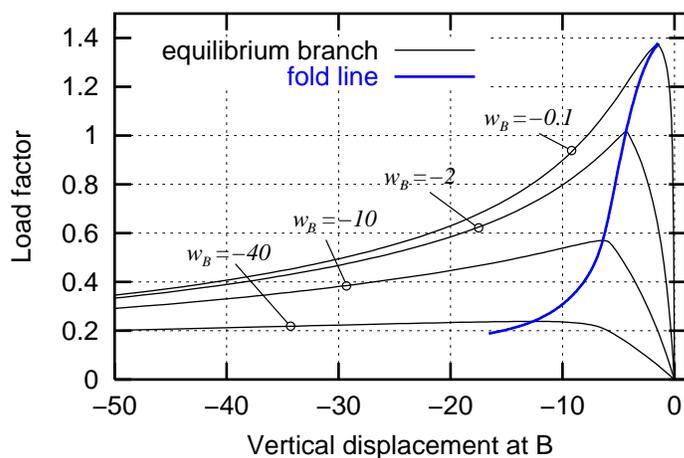


Figure 4: Fold line and equilibrium curves for different values of the imperfection amplitude.

In Fig. 5, the fold line is plotted in the *load-imperfection* plane. This representation is of prime interest for structural designers because one can directly read the value of the critical buckling load for a given imperfection amplitude. As mentioned before, the reduction of critical load is very important because of the modal interaction. For an imperfection amplitude such that $w_B = -50$, the reduction approaches 85%. It can also be noticed that the fold line stops for a imperfection value $w_B \simeq -55$. This is the extreme imperfection amplitude for which a limit point exists. Beyond this value, there is no limit point anymore on the equilibrium curve of the beam. On this curve, the ends of steps have also been plotted, so that one can see that the entire fold line was obtained with only 15 continuation steps, i.e. with only 15 decompositions of the tangent stiffness matrix K_T .

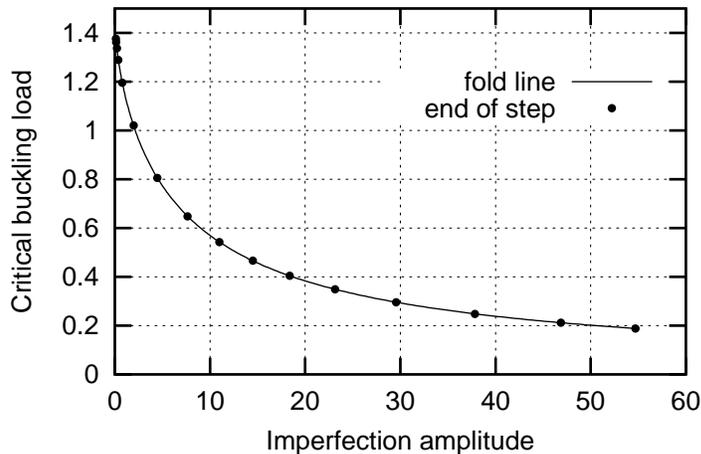


Figure 5: Reduction of the critical buckling load with respect to the amplitude of the shape imperfection.

5 Conclusions

This paper has described a general procedure for the imperfection sensitivity analysis of elastic structures. The developed calculation tool can handle global as well as local imperfections, in the case of thickness or geometrical shape defects.

The Asymptotic Numerical Method is used for the numerical treatment of the problem. As a result, the required calculation cost is significantly reduced as compared to the classical Newton-Raphson procedure. Moreover, the geometrical nonlinearities are treated without any approximation and exact analytical expressions are obtained for the directional derivatives of the tangent stiffness matrix. Thus, no approximation is introduced and the resulting overall algorithm is very accurate and the continuation procedure is very robust.

In order to deal with more complex structures, future developments will concern the introduction of material nonlinearities. Nevertheless, because of the dependence to the loading history, plasticity with unloading phenomenon can not be considered for the fold line following and the only possible nonlinear constitutive laws are restricted to nonlinear elasticity. In the case of plastic buckling, an alternative strategy to the fold line following must be used. Such a strategy can be found for example in [28].

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