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Boundary singularities of solutions to elliptic viscous Hamilton-Jacobi equations

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Abstract We study the boundary value problem with measures for (E1) $-\Delta u + g(|\nabla u|) = 0$ in a bounded domain Ω in \mathbb{R}^N , satisfying (E2) $u = \mu$ on $\partial\Omega$ and prove that if $g \in L^1(1, \infty; t^{-(2N+1)/N} dt)$ is nondecreasing (E1)-(E2) can be solved with any positive bounded measure. When $g(r) \geq r^q$ with $q > 1$ we prove that any positive function satisfying (E1) admits a boundary trace which is an outer regular Borel measure, not necessarily bounded. When $g(r) = r^q$ with $1 < q < q_c = \frac{N+1}{N}$ we prove the existence of a positive solution with a general outer regular Borel measure $\nu \neq \infty$ as boundary trace and characterize the boundary isolated singularities of positive solutions. When $g(r) = r^q$ with $q_c \leq q < 2$ we prove that

a necessary condition for solvability is that μ must be absolutely continuous with respect to the Bessel capacity $C_{\frac{2-q}{q}, q}$. We also characterize boundary removable sets for moderate and sigma-moderate solutions.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary and $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ a nondecreasing continuous function vanishing at 0. In this article we investigate several boundary data questions associated to nonnegative solutions of the following equation

$$-\Delta u + g(|\nabla u|) = 0 \quad \text{in } \Omega, \quad (1.1)$$

and we emphasize on the particular case of

$$-\Delta u + |\nabla u|^q = 0 \quad \text{in } \Omega. \quad (1.2)$$

where q is a real number mainly in the range $1 < q < 2$. We investigate first the generalized boundary value problem with measure associated to (1.1)

$$\begin{cases} -\Delta u + g(|\nabla u|) = 0 & \text{in } \Omega \\ u = \mu & \text{in } \partial\Omega \end{cases} \quad (1.3)$$

where μ is a measure on $\partial\Omega$. By a solution we mean an integrable function u such that $g(|\nabla u|) \in L^1_d(\Omega)$ where $d = d(x) := \text{dist}(x, \Omega)$ satisfying

$$\int_{\Omega} (-u\Delta\zeta + g(|\nabla u|)\zeta) dx = - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu \quad (1.4)$$

for all $\zeta \in X(\Omega) := \{\phi \in C_0^1(\overline{\Omega}) : \Delta\phi \in L^\infty(\Omega)\}$, where \mathbf{n} denotes the normal outward unit vector to $\partial\Omega$. The *integral subcriticality condition* for g is the following

$$\int_1^\infty g(s)s^{-\frac{2N+1}{N}} ds < \infty. \quad (1.5)$$

When $g(r) \leq r^q$, this condition is satisfied if $0 < q < q_c := \frac{N+1}{N}$. Our main existence result is the following.

Theorem 1.1 *Assume g satisfies (1.5). Then for any positive bounded Borel measure μ on $\partial\Omega$ there exists a maximal positive solution u_μ to problem (1.4). Furthermore the problem is closed for weak convergence of boundary data.*

Note that we do not know if problem (1.4) has a unique solution, *except if $g(r) = r^q$ with $0 < q < q_c$ and $\mu = c\delta_0$ in which case we prove that uniqueness holds.* A natural way for studying (1.1) is to introduce the notion of *boundary trace*. When $g(r) \geq r^q$ with $q > 1$ we prove in particular that the following result holds in which statement we denote $\Sigma_\delta = \{x \in \Omega : d(x) = \delta\}$ for $\delta > 0$:

Theorem 1.2 *Let u be any positive solution of (1.1). Then for any $x_0 \in \partial\Omega$ the following dichotomy occurs:*

(i) *Either there exists an open neighborhood U of x_0 such that*

$$\int_{\Omega \cap U} g(|\nabla u|) dx < \infty \quad (1.6)$$

and there exists a positive Borel measure μ_U on $\partial\Omega \cap U$ such that $u|_{\Sigma_\delta \cap U}$ converges to μ_U in the weak sense of measures when $\delta \rightarrow 0$.

(ii) *Or for any open neighborhood U of x_0 there holds*

$$\int_{\Omega \cap U} g(|\nabla u|) dx = \infty, \quad (1.7)$$

and

$$\lim_{\delta \rightarrow 0} \int_{\Sigma_\delta \cap U} u dS = \infty. \quad (1.8)$$

The set $\mathcal{S}(u)$ of boundary points x_0 with the property (ii) is closed and there exists a unique Borel measure μ on $\mathcal{R}(u) := \partial\Omega \setminus \mathcal{S}(u)$ such that $u|_{\Sigma_\delta}$ converges to μ in the weak sense of measures on $\mathcal{R}(u)$. The couple $(\mathcal{S}(u), \mu)$ is the boundary trace of u , denoted by $tr_{\partial\Omega}(u)$. The trace framework has also the advantage of pointing out some of the main questions which remain to be solved as it was done for the semilinear equation

$$-\Delta u + h(u) = 0 \quad \text{in } \Omega. \quad (1.9)$$

and the associated Dirichlet problem with measure

$$\begin{cases} -\Delta u + h(u) = 0 & \text{in } \Omega \\ u = \mu & \text{in } \partial\Omega, \end{cases} \quad (1.10)$$

where $h : \mathbb{R} \mapsto \mathbb{R}$ is a continuous nondecreasing function vanishing at 0. Much is known since the first paper of Gmira and Véron [15] and many developments are due to Marcus and Véron [27]–[30] in particular when (1.9) is replaced by

$$-\Delta u + |u|^{q-1} u = 0 \quad \text{in } \Omega. \quad (1.11)$$

with $q > 1$. We recall below some of the main aspects of the results dealing with (1.9)–(1.11), this will play the role of the breadcrumbs trail for our study.

- Problem (1.10) can be solved (in a unique way) for any bounded measure μ if h satisfies

$$\int_1^\infty (h(s) + |h(-s)|) s^{-\frac{2N}{N-1}} ds < \infty. \quad (1.12)$$

If $h(u) = |u|^{q-1} u$ the condition (1.12) is verified if and only if $1 < q < q_s$, the *subcritical range*; $q_s = \frac{N+1}{N-1}$ is a critical exponent for (1.11).

- When $1 < q < q_s$, boundary isolated singularities of positive solutions of (1.11) can be completely characterized i.e. if $u \in C(\overline{\Omega} \setminus \{0\})$ is a positive solution of (1.11) vanishing on

$\partial\Omega \setminus \{0\}$, then either it solves the associated Dirichlet problem with $\mu = c\delta_0$ for some $c \geq 0$ (*weak singularity*), or

$$u(x) \approx d(x)|x|^{-\frac{q+1}{q-1}} \quad \text{as } x \rightarrow 0. \quad (\text{strong singularity}) \quad (1.13)$$

- Always in the subcritical range it is proved that for any couple (\mathcal{S}, μ) where $\mathcal{S} \subset \partial\Omega$ is closed and μ is a positive Radon measure on $\mathcal{R} = \partial\Omega \setminus \mathcal{S}$ there exists a unique positive solution u of (1.11) with boundary trace (\mathcal{S}, μ) (in the sense defined in Theorem 1.2).

- When $q \geq q_s$, i.e. the *supercritical range*, any solution $u \in C(\overline{\Omega} \setminus \{0\})$ of (1.11) vanishing on $\partial\Omega \setminus \{0\}$ is identically 0, i.e. *isolated boundary singularities are removable*. This result due to Gmira-Véron has been extended, either by probabilistic tools by Le Gall [17], [18], Dynkin and Kuznetsov [12], [13], with the restriction $q_s \leq q \leq 2$, or by purely analytic methods by Marcus and Véron [27], [28] in the whole range $q_s \leq q$. The key tool for describing the problem is the Bessel capacity $C_{\frac{2}{q}, q'}$ in dimension $N - 1$. We list some of the most striking results. The associated Dirichlet problem can be solved with $\mu \in \mathfrak{M}^+(\partial\Omega)$ if and only if μ is absolutely continuous with respect to the $C_{\frac{2}{q}, q'}$ -capacity. If $K \subset \partial\Omega$ is compact and $u \in C(\overline{\Omega} \setminus K)$ is a solution of (1.11) vanishing on $\partial\Omega \setminus K$, then u is necessary zero if and only if $C_{\frac{2}{q}, q'}(K) = 0$. The complete characterization of positive solutions of (1.11) has been obtained by Mselati [26] when $q = 2$, Dynkin [11] when $q_s \leq q \leq 2$, and finally Marcus [25] when $q_s \leq q$; they proved in particular that any positive solution u is *sigma-moderate*, i.e. that there exists an increasing sequence of positive measures $\mu_n \in \mathfrak{M}^+(\partial\Omega)$ such that the sequence of the solutions $u = u_{\mu_n}$ of the associated Dirichlet problem with $\mu = \mu_n$ converges to u .

Concerning (1.2) we prove an existence result of solutions with a given trace belonging to the class of general outer regular Borel measures (not necessarily locally bounded).

Theorem 1.3 *Assume $1 < q < q_c$ and $\mathcal{S} \subsetneq \partial\Omega$ is closed and μ is a positive Radon measure on $\mathcal{R} := \partial\Omega \setminus \mathcal{S}$, then there exists a positive solution u of (1.2) such that $\text{tr}_{\partial\Omega}(u) = (\mathcal{S}, \mu)$.*

When $q_c \leq q < 2$ we prove a stronger result, using the characterization of singular solutions with strong singularities (see Theorem 1.6 below). When $q_c \leq q < 2$ we prove that Theorem 1.3 still holds with $\mu = 0$ if $\mathcal{S} = \overline{G}$ where $G \subsetneq \partial\Omega$ is relatively open, ∂G satisfies an interior sphere condition. Surprisingly the condition $\mathcal{S} \subsetneq \partial\Omega$ is necessary since there cannot exist any *large solution*, i.e. a solution which blows-up everywhere on $\partial\Omega$.

In order to characterize isolated singularities of positive solutions of (1.2) we introduce the following problem on the upper hemisphere S_+^{N-1} of the unit sphere in \mathbb{R}^N

$$\begin{cases} -\Delta' \omega + \left(\left(\frac{2-q}{q-1} \right)^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{q}{2}} - \frac{2-q}{q-1} \left(\frac{q}{q-1} - N \right) \omega = 0 & \text{in } S_+^{N-1} \\ \omega = 0 & \text{on } \partial S_+^{N-1}, \end{cases} \quad (1.14)$$

where ∇' and Δ' denote respectively the covariant gradient and the Laplace-Beltrami operator on S^{N-1} . To any solution ω of (1.14) we can associate a singular separable solution

u_s of (1.2) in $\mathbb{R}_+^N := \{x = (x_1, x_2, \dots, x_N) = (x', x_N) : x_N > 0\}$ vanishing on $\partial\mathbb{R}_+^N$ written in spherical coordinates $(r, \sigma) = (|x|, \frac{x}{|x|})$

$$u_s(x) = u_s(r, \sigma) = r^{-\frac{2-q}{q-1}} \omega(\sigma) \quad \forall x \in \overline{\mathbb{R}_+^N} \setminus \{0\}. \quad (1.15)$$

Theorem 1.4 *Problem (1.14) admits a positive solution if and only if $1 < q < q_c$. Furthermore this solution is unique and denoted by ω_s .*

This singular solution plays a fundamental role for describing isolated singularities.

Theorem 1.5 *Assume $1 < q < q_c$ and $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ is a positive solution of (1.2) which vanishes on $\partial\Omega \setminus \{0\}$. Then the following dichotomy occurs:*

(i) *Either there exists $c \geq 0$ such that $u = u_{c\delta_0}$ solves (1.3) with $g(r) = r^q$, $\mu = c\delta_0$ and*

$$u(x) = cP^\Omega(x, 0)(1 + o(1)) \quad \text{as } x \rightarrow 0 \quad (1.16)$$

where P^Ω is the Poisson kernel in Ω .

(ii) *Or $u = \lim_{c \rightarrow \infty} u_{c\delta_0}$ and*

$$\lim_{\substack{\Omega \ni x \rightarrow 0 \\ \frac{x}{|x|} \rightarrow \sigma \in S_+^{N-1}}} |x|^{\frac{2-q}{q-1}} u(x) = \omega_s\left(\frac{x}{|x|}\right). \quad (1.17)$$

We also give a sharp estimate from below for singular points of the trace

Theorem 1.6 *Assume $1 < q < q_c$ and u is a positive solution of (1.2) with boundary trace $(\mathcal{S}(u), \mu)$. Then for any $z \in \mathcal{S}(u)$ there holds*

$$u(x) \geq u_{\infty\delta_z}(x) := \lim_{c \rightarrow \infty} u_{c\delta_z}(x) \quad \forall x \in \Omega. \quad (1.18)$$

The description of $u_{\infty\delta_z}$ is provided by u_s defined in (1.15), up to a translation and a rotation.

The critical exponent q_c plays for removability of isolated boundary singularities (1.2) a similar role than q_s plays for (1.11) since we prove,

Theorem 1.7 *Assume $q \geq q_c$, then any nonnegative solution $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ of (1.2) vanishing on $\partial\Omega \setminus \{0\}$ is identically zero.*

The supercritical case for equation (1.2) can be understood using the Bessel capacity $C_{\frac{2-q}{q}, q'}$ in dimension $N - 1$, however we can only deal with moderate and sigma-moderate solutions. Following Dynkin [10] we define

Definition 1.8 *A positive solution u of (1.2) is moderate if there exists a bounded Borel measure $\mu \in \mathfrak{M}^+(\partial\Omega)$ such that u solves problem (1.3) with $g(r) = r^q$. It is sigma-moderate if there exists an increasing sequence $\{\mu_n\} \in \mathfrak{M}^+(\partial\Omega)$ such that the sequence of maximal solutions $\{u_{\mu_n}\}$ converges to u when $n \rightarrow \infty$, locally uniformly in Ω .*

Equivalently we shall prove that a positive solution u is moderate if and only if it is integrable in Ω and $|\nabla u| \in L_d^q(\Omega)$.

Theorem 1.9 *Assume $q \geq q_c$ and $K \subset \partial\Omega$ is compact and satisfies $C_{\frac{2-q}{q}, q'}(K) = 0$. Then any moderate solution $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{K\})$ of (1.2) vanishing on $\partial\Omega \setminus K$ is identically zero.*

As a corollary we prove that the above result remains true if u is a sigma-moderate solution of (1.2). The counterpart of this result is the following necessary condition for solving problem (1.3).

Theorem 1.10 *Assume $q \geq q_c$ and u is a moderate solution of (1.2) with boundary data $\mu \in \mathfrak{M}^+(\partial\Omega)$. Then μ is absolutely continuous with respect to the $C_{\frac{2-q}{q}, q'}$ -capacity.*

We end this article with some result concerning question of existence and removability of solutions of

$$-\Delta u + g(|\nabla u|) = \mu \quad \text{in } \Omega \quad (1.19)$$

where Ω is a bounded domain in \mathbb{R}^N and μ a positive bounded Radon measure on Ω . We prove that if g is a locally Lipschitz nondecreasing function vanishing at 0 and such that

$$\int_0^\infty g(s) s^{-\frac{2N-1}{N-1}} ds < \infty \quad (1.20)$$

then problem (1.19) admits a solution. In the power case

$$-\Delta u + |\nabla u|^q = \mu \quad \text{in } \Omega \quad (1.21)$$

with $1 < q < 2$, the critical exponent is $q^* = \frac{N}{N-1}$. We prove that a necessary condition for solving (1.21) with a positive Radon measure μ is that μ vanishes on Borel subsets E with $C_{1, q'}$ -capacity zero. The associated removability statement asserts that if K a compact subset of Ω such that $C_{1, q'}(K) = 0$, any positive solution of

$$-\Delta u + |\nabla u|^q = 0 \quad \text{in } \Omega \setminus K \quad (1.22)$$

can be extended as a solution in whole Ω .

2 The Dirichlet problem and the boundary trace

Throughout this article Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a C^2 boundary $\partial\Omega$ and c will denote a positive constant, independent of the data, the value of which may change from line to line. When needed the constant will be denoted by c_i for some indices $i = 1, 2, \dots$, or some dependence will be made explicit such as $c(a, b, \dots)$ for some data a, b, \dots

2.1 Boundary data bounded measures

We consider the following problem where μ belongs to the set $\mu \in \mathfrak{M}(\partial\Omega)$ of bounded Borel measures on $\partial\Omega$

$$\begin{cases} -\Delta u + g(|\nabla u|) = 0 & \text{in } \Omega \\ u = \mu & \text{in } \partial\Omega. \end{cases} \quad (2.1)$$

We assume that g belongs to the class \mathcal{G}_0 which means that $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a locally Lipschitz continuous nonnegative and nondecreasing function vanishing at 0. The *integral subcriticality condition* is the following

$$\int_1^\infty g(s) s^{-\frac{2N+1}{N}} ds < \infty. \quad (2.2)$$

If $g(r) = r^q$ the subcriticality condition is satisfied if $0 < q < q_c := \frac{N+1}{N}$.

Definition 2.1 A function $u \in L^1(\Omega)$ such that $g(|\nabla u|) \in L^1_d(\Omega)$ is a weak solution of (2.1) if

$$\int_\Omega (-u\Delta\zeta + g(|\nabla u|)\zeta) dx = - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu \quad (2.3)$$

for all $\zeta \in X(\Omega) := \{\phi \in C^1_0(\overline{\Omega}) : \Delta\phi \in L^\infty(\Omega)\}$.

If we denote respectively by G^Ω and P^Ω the Green kernel and the Poisson kernel in Ω , with corresponding operators \mathbb{G}^Ω and \mathbb{P}^Ω it is classical from linear theory that the above definition is equivalent to

$$u = \mathbb{P}^\Omega[\mu] - \mathbb{G}^\Omega[g(|\nabla u|)]. \quad (2.4)$$

We recall that $M^p_h(\Omega)$ denote the Marcinkiewicz space (or weak L^p space) of exponent $p \geq 1$ and weight $h > 0$ defined by

$$M^p_h(\Omega) = \left\{ v \in L^1_{loc}(\Omega) : \exists C \geq 0 \text{ s. t. } \int_E |v|h dx \leq C|E|_h^{1-\frac{1}{p}}, \forall E \subset \Omega, E \text{ Borel} \right\}, \quad (2.5)$$

where $|E|_h = \int \chi_E h dx$. The smallest constant C for which (2.5) holds is the Marcinkiewicz norm of v denoted by $\|v\|_{M^p_h}$ and the following inequality will be much useful:

$$|\{x : |v(x)| \geq \lambda\}|_h \leq \lambda^{-p} \|v\|_{M^p_h}^p \quad \forall \lambda > 0. \quad (2.6)$$

The main result of this section is the following existence and stability result for problem (2.1).

Theorem 2.2 Assume $g \in \mathcal{G}_0$ satisfies (2.2), then for any $\mu \in \mathfrak{M}^+(\partial\Omega)$ there exists a maximal solution $\bar{u} = \bar{u}_\mu$ to problem (2.1). Furthermore $u \in M^{\frac{N}{N-1}}(\Omega)$ and $|\nabla u| \in M^{\frac{N+1}{N}}_d(\Omega)$. Finally, if $\{\mu_n\}$ is a sequence of positive bounded measures on $\partial\Omega$ which converges to μ in the weak sense of measures and $\{u_{\mu_n}\}$ is a sequence of solutions of (2.1) with boundary data μ_n , then there exists a subsequence such that $u_{\mu_{n_k}}$ converges to a solution u_μ of (2.1) in $L^1(\Omega)$ and $g(|\nabla \bar{u}_{\mu_{n_k}}|)$ to $g(|\nabla \bar{u}_\mu|)$ in $L^1_d(\Omega)$.

We recall the following estimates [8], [15] and [39].

Proposition 2.3 *For any $\alpha \in [0, 1]$, there exist a positive constant c_1 depending on α , Ω and N such that*

$$\|\mathbb{G}^\Omega[\nu]\|_{L^1(\Omega)} + \|\mathbb{G}^\Omega[\nu]\|_{M_{d^\alpha}^{\frac{N+\alpha}{N+\alpha-2}}(\Omega)} \leq c_1 \|\nu\|_{\mathfrak{M}_{d^\alpha}(\Omega)}, \quad (2.7)$$

$$\|\nabla \mathbb{G}^\Omega[\nu]\|_{M_{d^\alpha}^{\frac{N+\alpha}{N+\alpha-1}}(\Omega)} \leq c_1 \|\nu\|_{\mathfrak{M}_{d^\alpha}(\Omega)}, \quad (2.8)$$

where

$$\|\nu\|_{\mathfrak{M}_{d^\alpha}(\Omega)} := \int_{\Omega} d^\alpha(x) d|\nu| \quad \forall \nu \in \mathfrak{M}(\Omega), \quad (2.9)$$

$$\|\mathbb{P}^\Omega[\mu]\|_{L^1(\Omega)} + \|\mathbb{P}^\Omega[f]\|_{M_d^{\frac{N}{N-1}}(\Omega)} + \|\mathbb{P}^\Omega[\mu]\|_{M_d^{\frac{N+1}{N}}(\Omega)} \leq c_1 \|\mu\|_{\mathfrak{M}(\partial\Omega)}, \quad (2.10)$$

$$\|\nabla \mathbb{P}^\Omega[\mu]\|_{M_d^{\frac{N+1}{N}}(\Omega)} \leq c_1 \|\mu\|_{\mathfrak{M}(\partial\Omega)}, \quad (2.11)$$

for any $\nu \in L_{d^\alpha}^1(\Omega)$ and any $\mu \in \mathfrak{M}(\partial\Omega)$.

Since $\partial\Omega$ is C^2 , there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$ and $x \in \Omega_\delta$ such that $d(x) < \delta$, there exists a unique $\sigma(x) \in \partial\Omega$ such that $|x - \sigma(x)| = d(x)$. We set $\sigma(x) = Proj_{\partial\Omega}(x)$. Furthermore, if $\mathbf{n} = \mathbf{n}_{\sigma(x)}$ is the normal outward unit vector to $\partial\Omega$ at $\sigma(x)$, we have $x = \sigma(x) - d(x)\mathbf{n}_{\sigma(x)}$. For $\delta \in (0, \delta_0]$, we set

$$\begin{aligned} \Omega_\delta &= \{x \in \Omega : d(x) \leq \delta\}, \\ \Omega'_\delta &= \{x \in \Omega : d(x) > \delta\}, \\ \Sigma_\delta &= \partial\Omega'_\delta = \{x \in \Omega : d(x) = \delta\}, \\ \Sigma &:= \Sigma_0 = \partial\Omega. \end{aligned}$$

For any $\delta \in (0, \delta_0]$, the mapping $x \mapsto (\delta(x), \sigma(x))$ defines a C^1 diffeomorphism from Ω_δ to $(0, \delta) \times \Sigma$. Therefore we can write $x = \sigma(x) - d(x)\mathbf{n}_{\sigma(x)}$ for every $x \in \Omega_\delta$. Any point $x \in \overline{\Omega}_{\delta_0}$ is represented by the couple $(\delta, \sigma) \in [0, \delta_0] \times \Sigma$ with formula $x = \sigma - \delta\mathbf{n}_\sigma$. This system of coordinates which will be made more precise in the boundary trace construction is called *flow coordinates*.

Proof of Theorem 2.2. Step 1: Construction of approximate solutions. Let $\{\mu_n\}$ be a sequence of $C^1(\partial\Omega)$ positive functions such that $\{\mu_n\}$ converges to μ in the weak sense of measures and $\|\mu_n\|_{L^1(\partial\Omega)} \leq c_2 \|\mu\|_{\mathfrak{M}(\partial\Omega)}$ for all n , where c_2 is a positive constant independent of n . We next consider the following problem

$$\begin{cases} -\Delta v + g(|\nabla(v + \mathbb{P}^\Omega[\mu_n])|) = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.12)$$

It is easy to see that 0 and $-\mathbb{P}^\Omega[\mu_n]$ are respectively supersolution and subsolution of (2.12). By [24, Theorem 6.5] there exists a solution $v_n \in W^{2,p}(\Omega)$ with $1 < p < \infty$ to problem (2.12) satisfying $-\mathbb{P}^\Omega[\mu_n] \leq v_n \leq 0$. Thus the function $u_n = v_n + \mathbb{P}^\Omega[\mu_n]$ is a solution of

$$\begin{cases} -\Delta u_n + g(|\nabla u_n|) = 0 & \text{in } \Omega \\ u_n = \mu_n & \text{on } \partial\Omega. \end{cases} \quad (2.13)$$

By the maximum principle, such solution is the unique solution of (2.13).

Step 2: We claim that $\{u_n\}$ and $\{|\nabla u_n|\}$ remain uniformly bounded respectively in $M^{\frac{N}{N-1}}(\Omega)$ and $M_d^{\frac{N+1}{N}}(\Omega)$. Let ξ be the solution to

$$\begin{cases} -\Delta \xi = 1 & \text{in } \Omega \\ \xi = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.14)$$

then there exists a constant $c_3 > 0$ such that

$$\frac{1}{c_3} < -\frac{\partial \xi}{\partial \mathbf{n}} < c_3 \text{ and } \frac{d(x)}{c_3} \leq \xi \leq c_3 d(x). \quad (2.15)$$

By multiplying the equation in (2.13) by ξ and taking integral on Ω , we obtain

$$\int_{\Omega} u_n dx + \int_{\Omega} g(|\nabla u_n|) \xi dx = - \int_{\partial\Omega} \mu_n \frac{\partial \xi}{\partial \mathbf{n}} dS,$$

which implies

$$\int_{\Omega} u_n dx + \int_{\Omega} d(x) g(|\nabla u_n|) dx \leq c_4 \|\mu\|_{\mathfrak{M}(\partial\Omega)} \quad (2.16)$$

where c_4 is a positive constant independent of n . By Proposition 2.3 and by noticing that $u_n \leq \mathbb{P}^{\Omega}[\mu_n]$, we get

$$\|u_n\|_{M^{\frac{N}{N-1}}(\Omega)} \leq \|\mathbb{P}^{\Omega}[\mu_n]\|_{M^{\frac{N}{N-1}}(\Omega)} \leq c_1 \|\mu_n\|_{L^1(\partial\Omega)} \leq c_1 c_2 \|\mu\|_{\mathfrak{M}(\partial\Omega)}. \quad (2.17)$$

Set $f_n = -g(|\nabla u_n|)$ then $f_n \in L_d^1(\Omega)$ and u_n satisfies

$$\int_{\Omega} (-u_n \Delta \zeta - f_n \zeta) dx = - \int_{\partial\Omega} \mu_n \frac{\partial \zeta}{\partial \mathbf{n}} dS \quad (2.18)$$

for any $\zeta \in X(\Omega)$. From (2.4) and Proposition 2.3, we derive that

$$\|\nabla u_n\|_{M_d^{\frac{N+1}{N}}(\Omega)} \leq c_1 \left(\|f_n\|_{L_d^1(\Omega, dx)} + \|\mu_n\|_{L^1(\partial\Omega)} \right), \quad (2.19)$$

which, along with (2.16), implies that

$$\|\nabla u_n\|_{M_d^{\frac{N+1}{N}}(\Omega)} \leq c_5 \|\mu\|_{\mathfrak{M}(\partial\Omega)} \quad (2.20)$$

where c_5 is a positive constant depending only on Ω and N . Thus the claim follows from (2.17) et (2.20).

Step 3: Existence of a solution. By standard results on elliptic equations and measure theory [9, Cor. IV 27], the sequences $\{u_n\}$ and $\{|\nabla u_n|\}$ are compact in $L_{loc}^1(\Omega)$. Therefore, there exist a subsequence, still denoted by $\{u_n\}$, and a function u such that $\{u_n\}$ converges to u in $L_{loc}^1(\Omega)$ and a.e. in Ω .

(i) The sequence u_n converges to u in $L^1(\Omega)$: let $E \subset \Omega$ be a Borel subset, then

$$\int_E u_n dx \leq |E|^{\frac{1}{N}} \|u_n\|_{M^{\frac{N}{N-1}}(\Omega)} \leq c_1 c_2 |E|^{\frac{1}{N}} \|\mu\|_{\mathfrak{M}(\partial\Omega)}. \quad (2.21)$$

The compactness follows by Vitali's theorem.

(ii) The sequence $g(|\nabla u_n|)$ converges to $g(|\nabla u|)$ in $L^1_d(\Omega)$: consider again a Borel set $E \subset \Omega$, $\lambda > 0$ and write

$$\int_E d(x)g(|\nabla u_n|)dx \leq \int_{E \cap \{x: |\nabla u_n(x)| \leq \lambda\}} d(x)g(|\nabla u_n|)dx + \int_{\{x: |\nabla u_n(x)| > \lambda\}} d(x)g(|\nabla u_n|)dx.$$

First

$$\int_{E \cap \{x: |\nabla u_n(x)| \leq \lambda\}} d(x)g(|\nabla u_n|)dx \leq g(\lambda)|E|_d. \quad (2.22)$$

Then

$$\int_{E \cap \{x: |\nabla u_n(x)| > \lambda\}} d(x)g(|\nabla u_n|)dx \leq -\int_{\lambda}^{\infty} g(s)d\omega_n(s)$$

where $\omega_n(s) = |\{x \in \Omega : |\nabla u_n(x)| > s\}|_d$. Using the fact that $g' \geq 0$ combined with (2.6) and (2.20), we get

$$\begin{aligned} -\int_{\lambda}^t g(s)d\omega_n(s) &= g(\lambda)\omega_n(\lambda) - g(t)\omega_n(t) + \int_{\lambda}^t \omega_n(s)g'(s)ds \\ &\leq g(\lambda)\omega_n(\lambda) - g(t)\omega_n(t) + c_6 \|\mu\|_{\mathfrak{M}(\partial\Omega)}^{\frac{N+1}{N}} \int_{\lambda}^t s^{-\frac{N+1}{N}} g'(s)ds \\ &\leq \left(\omega_n(\lambda) - c_6 \|\mu\|_{\mathfrak{M}(\partial\Omega)}^{\frac{N+1}{N}} \lambda^{-\frac{N+1}{N}} \right) g(\lambda) - \left(\omega_n(t) - c_6 \|\mu\|_{\mathfrak{M}(\partial\Omega)}^{\frac{N+1}{N}} t^{-\frac{N+1}{N}} \right) g(t) \\ &\quad + c_6 \frac{N+1}{N} \|\mu\|_{\mathfrak{M}(\partial\Omega)}^{\frac{N+1}{N}} \int_{\lambda}^t g(s) s^{-\frac{2N+1}{N}} ds. \end{aligned}$$

We have already used the fact that $\omega_n(\lambda) \leq c_6 \|\mu\|_{\mathfrak{M}(\partial\Omega)}^{\frac{N+1}{N}} \lambda^{-\frac{N+1}{N}}$, and since the condition (2.2) holds, $\liminf_{t \rightarrow \infty} t^{-\frac{N+1}{N}} g(t) = 0$. Letting $t \rightarrow \infty$ we derive

$$\int_{E \cap \{x: |\nabla u_n(x)| > \lambda\}} d(x)g(|\nabla u_n|)dx \leq c_6 \frac{N+1}{N} \|\mu\|_{\mathfrak{M}(\partial\Omega)}^{\frac{N+1}{N}} \int_{\lambda}^{\infty} g(s) s^{-\frac{2N+1}{N}} ds. \quad (2.23)$$

For $\epsilon > 0$ we fix λ in order the right-hand side of (2.23) be smaller than $\frac{\epsilon}{2}$. Thus, if $|E|_d \leq \frac{\epsilon}{2g(\lambda)+1}$, we obtain

$$\int_E d(x)g(|\nabla u_n|)dx \leq \epsilon. \quad (2.24)$$

The convergence follows again by Vitali's theorem. Next for any $\zeta \in X(\Omega)$, we have

$$\int_{\Omega} (-u_n \Delta \zeta + g(|\nabla u_n|)\zeta) dx = -\int_{\partial\Omega} \mu_n \frac{\partial \zeta}{\partial \mathbf{n}} dS \quad (2.25)$$

By taking into account the fact that $|\zeta| \leq cd$ in Ω , we can pass to the limit in each term in (2.25) and obtain (2.3); so u is a solution of (2.1). Clearly $u \in M^{\frac{N}{N-1}}(\Omega)$ and $|\nabla u| \in M_d^{\frac{N+1}{N}}(\Omega)$ from (2.4) and Proposition 2.3.

Step 4: Existence of a maximal solution. We first notice that any solution u of (2.1) is smaller than $\mathbb{P}^\Omega[\mu]$. Then $u \leq \mathbb{P}^\Omega[\mu]$ in Ω'_δ and by the maximum principle $u \leq u_\delta$ which satisfies

$$\begin{cases} -\Delta u_\delta + g(|\nabla u_\delta|) = 0 & \text{in } \Omega'_\delta \\ u_\delta = \mathbb{P}^\Omega[\mu] & \text{in } \Sigma_\delta. \end{cases} \quad (2.26)$$

As a consequence, $0 < \delta < \delta' \implies u_\delta \leq u_{\delta'}$ in $\Omega'_{\delta'}$ and $u_\delta \downarrow \bar{u}_\mu$ which is not zero if μ is so, since it is bounded from below by the already constructed solution u . We extend u_δ , $|\nabla u_\delta|$ and $g(|\nabla u_\delta|)$ by zero outside $\bar{\Omega}'_\delta$ and still denote them by the same expressions. Let $E \subset \Omega$ be a Borel set, then (2.21) becomes

$$\begin{aligned} \int_E u_\delta dx &\leq |E|^{\frac{1}{N}} \|u_\delta\|_{M^{\frac{N}{N-1}}(\Omega'_\delta)} \leq c_1 c_2 |E|^{\frac{1}{N}} \left\| \mathbb{P}^\Omega[\mu] \Big|_{\Sigma_\delta} \right\|_{\mathfrak{M}(\Sigma_\delta)} \\ &\leq c_1 c_2 c_7 |E|^{\frac{1}{N}} \|\mu\|_{\mathfrak{M}(\Sigma)}. \end{aligned} \quad (2.27)$$

Set $d_\delta(x) := \text{dist}(x, \Omega_\delta) = (d(x) - \delta)_+$ if $x \in \Omega_{\delta_0} := \Omega \setminus \Omega'_{\delta_0}$, we have, in denoting $E_\delta = E \cap \Omega'_\delta$

$$\int_{E_\delta \cap \{x: |\nabla u_\delta| > \lambda\}} d_\delta(x) g(|\nabla u_\delta|) dx \leq - \int_\lambda^\infty g(s) d\omega_\delta(s),$$

where $\omega_\delta(s) = |\{x \in \Omega : |\nabla u_\delta(x)| > s\}|_{d_\delta}$. Since $\left\| \mathbb{P}^\Omega[\mu] \Big|_{\Sigma_\delta} \right\|_{\mathfrak{M}(\Sigma_\delta)} \leq c_7 \|\mu\|_{\mathfrak{M}(\Sigma)}$, (2.22) and (2.23) become respectively

$$\int_{E_\delta \cap \{x: |\nabla u_\delta(x)| \leq \lambda\}} d_\delta(x) g(|\nabla u_\delta|) dx \leq g(\lambda) |E_\delta|_{d_\delta}. \quad (2.28)$$

and

$$\int_{E_\delta \cap \{x: |\nabla u_\delta(x)| > \lambda\}} d_\delta(x) g(|\nabla u_\delta|) dx \leq c_6 \frac{N+1}{N} \|\mu\|_{\mathfrak{M}^{\frac{N+1}{N}}} \int_\lambda^\infty g(s) s^{-\frac{2N+1}{N}} ds. \quad (2.29)$$

Combining (2.28) and (2.29) and noting that $|E_\delta|_{d_\delta} \leq |E|_d$, we obtain that for any $\epsilon > 0$ there exists $\lambda > 0$, independent of δ by (2.28), such that

$$\int_{E_\delta} d_\delta(x) g(|\nabla u_\delta|) dx \leq \epsilon. \quad (2.30)$$

provided $|E|_d \leq \frac{\epsilon}{2g(\lambda)+1}$.

Finally, if $\zeta \in X(\Omega)$ we denote by ζ_δ the solution of

$$\begin{cases} -\Delta \zeta_\delta = -\Delta \zeta & \text{in } \Omega'_\delta \\ \zeta_\delta = 0 & \text{in } \Sigma_\delta. \end{cases} \quad (2.31)$$

Then

$$\int_{\Omega'_\delta} (-u_\delta \Delta \zeta_\delta + g(|\nabla u_\delta|) \zeta_\delta) dx = - \int_{\Sigma_\delta} \frac{\partial \zeta_\delta}{\partial \mathbf{n}} \mathbb{P}^\Omega[\mu] dS \quad (2.32)$$

Clearly $|\zeta_\delta| \leq Cd_\delta$ and $\zeta_\delta \chi_{\Omega'_\delta} \rightarrow \zeta$ uniformly in Ω by standard elliptic estimates. Since the right-hand side of converges to $-\int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\mu$, it follows by Vitali's theorem that \bar{u}_μ satisfies (2.3).

Step 5: Stability. Consider a sequence of positive bounded measures $\{\mu_n\}$ which converges weakly to μ . By estimates (2.17) and (2.20), u_{μ_n} and $g(|\nabla u_{\mu_n}|)$ are relatively compact in $L^1_{loc}(\Omega)$ and respectively uniformly integrable in $L^1(\Omega)$ and $L^1_d(\Omega)$. Up to a subsequence, they converge a.e. to u and $g(|\nabla u|)$ for some function u . As in Step 3, u is a solution of (2.1). \square

A variant of the stability statement is the following result which will be much useful in the analysis of the boundary trace. The proof is similar as Step 4 in the proof of Theorem 2.2.

Corollary 2.4 . *Let g in \mathcal{G}_0 satisfy (2.2). Assume δ_n is a sequence decreasing to 0 and μ_n a sequence of positive bounded measures in $\Sigma_{\delta_n} = \partial\Omega'_{\delta_n}$ which converges to μ in the weak sense of measures and let u_{μ_n} be solutions of (2.1) with boundary data μ_n . Then there exists a subsequence $\{u_{\mu_{n_k}}\}$ of solutions of (2.1) with boundary data μ_{n_k} which converges to a solution u_μ with boundary data μ .*

2.2 Boundary trace

The construction of the boundary trace of positive solutions of (1.1) is a combination of tools developed in [27]–[29] with the help of a geometric construction from [3].

Definition 2.5 *Let $\mu_\delta \in \mathfrak{M}(\Sigma_\delta)$ for all $\delta \in (0, \delta_0)$ and $\mu \in \mathfrak{M}(\Sigma)$. We say that $\mu_\delta \rightarrow \mu$ as $\delta \rightarrow 0$ in the sense of weak convergence of measures if*

$$\lim_{\delta \rightarrow 0} \int_{\Sigma_\delta} \phi(\sigma(x)) d\mu_\delta = \int_{\Sigma} \phi d\mu \quad \forall \phi \in C_c(\Sigma). \quad (2.33)$$

A function $u \in C(\Omega)$ possesses a measure boundary trace $\mu \in \mathfrak{M}(\Sigma)$ if

$$\lim_{\delta \rightarrow 0} \int_{\Sigma_\delta} \phi(\sigma(x)) u(x) dS = \int_{\Sigma} \phi d\mu \quad \forall \phi \in C_c(\Sigma). \quad (2.34)$$

Similarly, if A is a relatively open subset of Σ , we say that u possesses a trace μ on A in the sense of weak convergence of measures if $\mu \in \mathfrak{M}(A)$ and (2.34) holds for every $\phi \in C_c(A)$.

We recall the following result [30, Cor 2.3], adapted here to (1.1),

Proposition 2.6 *Assume $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and let $u \in C^2(\Omega)$ be a positive solution of (1.1). Suppose that for some $z \in \partial\Omega$ there exists an open neighborhood U such that*

$$\int_{U \cap \Omega} g(|\nabla u|) d(x) dx < \infty. \quad (2.35)$$

Then $u \in L^1(K \cap \Omega)$ for every compact set $K \subset U$ and there exists a positive Radon measure ν on $\Sigma \cap U$ such that

$$\lim_{\delta \rightarrow 0} \int_{\Sigma_\delta \cap U} \phi(\sigma(x))u(x)dS = \int_{\Sigma \cap U} \phi d\nu \quad \forall \phi \in C_c(\Sigma \cap U). \quad (2.36)$$

Definition 2.7 Let $u \in C^2(\Omega)$ be a positive solution of (1.1). A point $z \in \partial\Omega$ is a regular boundary point of u if there exists an open neighborhood U of z such that (2.35) holds. The set of regular points is denoted by $\mathcal{R}(u)$. Its complement $\mathcal{S}(u) = \partial\Omega \setminus \mathcal{R}(u)$ is called the singular boundary set of u .

Clearly $\mathcal{R}(u)$ is relatively open and there exists a positive Radon measure μ on $\mathcal{R}(u)$ such that u admits $\mu := \mu(u)$ as a measure boundary trace on $\mathcal{R}(u)$ and $\mu(u)$ is uniquely determined. The couple $(\mathcal{S}(u), \mu)$ is called the boundary trace of u and denoted by $tr_{\partial\Omega}(u)$.

The main question is to determine the behaviour of u near $\mathcal{S}(u)$. The following result is proved in [30, Lemma 2.8].

Proposition 2.8 Assume $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and $u \in C^2(\Omega)$ be a positive solution of (1.1) with the singular boundary set $\mathcal{S}(u)$. If $z \in \mathcal{S}(u)$ is such that there exists an open neighborhood U' of z such that $u \in L^1(U' \cap \Omega)$, then for every neighborhood U of z there holds

$$\lim_{\delta \rightarrow 0} \int_{\Sigma_\delta \cap U} u(x)dS = \infty. \quad (2.37)$$

Corollary 2.9 Let $u \in C^2(\Omega)$ is a positive solution of (1.2) with $\frac{3}{2} < q \leq 2$. Then (2.37) holds for every $z \in \mathcal{S}(u)$.

Proof. This is a direct consequence of Lemma 3.2 since $\frac{q-2}{q-1} > -1$ implies $u \in L^1(\Omega)$. \square

We prove below that this result holds for any $1 < q \leq 2$.

Theorem 2.10 Assume $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous and satisfies

$$\liminf_{r \rightarrow \infty} \frac{g(r)}{r^q} > 0 \quad (2.38)$$

where $1 < q \leq 2$. If $u \in C^2(\Omega)$ is a positive solution of (1.1), then (2.37) holds for every $z \in \mathcal{S}(u)$.

Proof. Up to rescaling we can assume that $g(r) \geq r^q - \tau$ for some $\tau \geq 0$. We recall some results from [6] in the form exposed in [3, Sect 2]. There exist an open cover $\{\Sigma_j\}_{j=1}^k$ of Σ , an open set \mathcal{D} of \mathbb{R}^{N-1} and C^2 mappings T_j from \mathcal{D} to Σ_j with rank $N - 1$ such that for each $\sigma \in \Sigma_j$ there exists a unique $a \in \mathcal{D}$ with the property that $\sigma = T_j(a)$. The couples $\{\mathcal{D}, T_j^{-1}\}$ form a system of local charts of Σ . If we set $\Omega_j = \{x \in \Omega_{\delta_0} : \sigma(x) \in \Sigma_j\}$ and for any $j = 1, \dots, k$ the mapping

$$\Pi_j : (\delta, a) \mapsto x = T_j(a) - \delta \mathbf{n}$$

where \mathbf{n} is the outward unit normal vector to Σ at $T_j(a) = \sigma(x)$ is a C^2 diffeomorphism from $(0, \delta_0) \times \mathcal{D}$ to Ω_j . The Laplacian obtains the following expressions in terms of this system of flow coordinates provided the lines $\sigma_i = ct$ are the vector fields of the principal curvatures $\bar{\kappa}_i$ on Σ

$$\Delta = \Delta_\delta + \Delta_\sigma \quad (2.39)$$

where

$$\Delta_\delta = \frac{\partial^2}{\partial \delta^2} - (N-1)H \frac{\partial}{\partial \delta} \quad (2.40)$$

with $H = H(\delta, \cdot) = \frac{1}{N-1} \sum_{i=1}^{N-1} \frac{\bar{\kappa}_i}{1-\delta\bar{\kappa}_i}$ being the mean curvature of Σ_δ and

$$\Delta_\sigma = \frac{1}{\sqrt{|g|}} \sum_{i=1}^{N-1} \frac{\partial}{\partial \sigma_i} \left(\frac{\sqrt{|g|}}{\bar{g}_{ii}(1-\delta\bar{\kappa}_i + \kappa_{ii}\delta^2)} \frac{\partial}{\partial \sigma_i} \right). \quad (2.41)$$

In this expression, $\bar{g} = (\bar{g}_{ij})$ is the metric tensor on Σ and it is diagonal by the choice of coordinates and $|g| = \prod_{i=1}^{N-1} \bar{g}_{ii}(1-\delta\bar{\kappa}_i)^2$. In particular

$$|\nabla \xi|^2 = \sum_{i=1}^{N-1} \frac{\xi_{\sigma_i}^2}{\bar{g}_{ii}(1-\delta\bar{\kappa}_i + \kappa_{ii}\delta^2)} + \xi_\delta^2 \quad (2.42)$$

and

$$\nabla \xi \cdot \nabla \eta = \sum_{i=1}^{N-1} \frac{\xi_{\sigma_i} \eta_{\sigma_i}}{\bar{g}_{ii}(1-\delta\bar{\kappa}_i + \kappa_{ii}\delta^2)} + \xi_\delta \eta_\delta = \nabla_\sigma \xi \cdot \nabla_\sigma \eta + \xi_\delta \eta_\delta. \quad (2.43)$$

If $z \in \mathcal{S}(u)$ we can assume that $U_\Sigma := U \cap \Sigma$ is smooth and contained in a single chart Σ_j . Let ϕ be the first eigenfunction of Δ_σ in $W_0^{1,2}(U_\Sigma)$ and $\alpha > 1$ to be made precise later on. From $-\Delta_\delta u - \Delta_\sigma u + \frac{1}{2}(|\nabla u|^q - \tau) + \frac{1}{2}g(|\nabla u|) \leq 0$, we obtain by multiplying by ϕ^α and integrating over U_Σ

$$\begin{aligned} -\frac{d^2}{d\delta^2} \int_{U_\Sigma} u \phi^\alpha dS + (N-1) \int_{U_\Sigma} \frac{\partial u}{\partial \delta} \phi^\alpha H dS + \alpha \int_{U_\Sigma} \phi^{\alpha-1} \nabla_\sigma u \cdot \nabla_\sigma \phi dS \\ + \frac{1}{2} \int_{U_\Sigma} \phi^\alpha (|\nabla u|^q - \tau) dS + \frac{1}{2} \int_{U_\Sigma} \phi^\alpha g(|\nabla u|) dS \leq 0. \end{aligned} \quad (2.44)$$

Provided $\alpha > q' - 1$ we obtain by Hölder

$$\begin{aligned} \left| \int_{U_\Sigma} \phi^{\alpha-1} \nabla_\sigma u \cdot \nabla_\sigma \phi dS \right| &\leq \left(\int_{U_\Sigma} |\nabla u|^q \phi^\alpha dS \right)^{\frac{1}{q}} \left(\int_{U_\Sigma} |\nabla_\sigma \phi|^{q'} \phi^{\alpha-q'} dS \right)^{\frac{1}{q'}} \\ &\leq \frac{\epsilon}{q} \int_{U_\Sigma} |\nabla u|^q \phi^\alpha dS + \frac{\epsilon^{\frac{1}{1-q}}}{q'} \int_{U_\Sigma} |\nabla_\sigma \phi|^{q'} \phi^{\alpha-q'} dS, \end{aligned} \quad (2.45)$$

and

$$\left| \int_{U_\Sigma} \frac{\partial u}{\partial \delta} \phi^\alpha H dS \right| \leq \frac{\epsilon h}{q} \int_{U_\Sigma} |\nabla u|^q \phi^\alpha dS + \frac{\epsilon^{\frac{1}{1-q}}}{q'} \|H\|_{L^\infty} \int_{U_\Sigma} \phi^\alpha dS \quad (2.46)$$

with $\epsilon > 0$. We derive, with ϵ small enough,

$$\frac{d^2}{d\delta^2} \int_{U_\Sigma} u \phi^\alpha dS \geq \left(\frac{1}{2} - c_8 \epsilon\right) \int_{U_\Sigma} |\nabla u|^q \phi^\alpha dS + \frac{1}{2} \int_{U_\Sigma} \phi^\alpha g(|\nabla u|) dS - c'_8. \quad (2.47)$$

Integrating (2.47) twice yields to

$$\int_{U_\Sigma} u(\delta, \cdot) \phi^\alpha dS \geq \left(\frac{1}{2} - c_8 \epsilon\right) \int_\delta^{\delta_0} \int_{U_\Sigma} |\nabla u|^q \phi^\alpha dS (\tau - \delta) d\tau + \frac{1}{2} \int_{U_\Sigma} \phi^\alpha g(|\nabla u|) dS - c''_8. \quad (2.48)$$

Since $z \in \mathcal{S}(u)$, the right-hand side of (2.48) tends monotonically to ∞ as $\delta \rightarrow 0$, which implies that (2.37) holds. \square

Remark. It is often useful to consider the couple $(\mathcal{S}(u), \mu)$ defining the boundary trace of u as an outer regular Borel measure ν uniquely determined by

$$\nu(E) = \begin{cases} \mu(E) & \text{if } E \subset \mathcal{R}(u) \\ \infty & \text{if } E \cap \mathcal{S}(u) \neq \emptyset \end{cases} \quad (2.49)$$

for all Borel set $E \subset \partial\Omega$, and we will denote $tr_{\partial\Omega}(u) = \nu(u)$.

The integral blow-up estimate (2.37) without the growth estimate (2.38) provided (2.2) holds.

Theorem 2.11 *Assume $g \in \mathcal{G}_0$ satisfies (2.2). If $u \in C^2(\Omega)$ is a positive solution of (1.1), then (2.37) holds for every $z \in \mathcal{S}(u)$.*

Proof. By translation we assume $z = 0 \in \mathcal{S}(u)$ and (2.37) does not hold. We proceed by contradiction, assuming that there exists an open neighborhood U of z such that

$$\liminf_{\delta \rightarrow 0} \int_{\Sigma_\delta \cap U} u dS < \infty. \quad (2.50)$$

By Proposition 2.8, for any neighborhood U' of z there holds

$$\int_{\Omega \cap U'} u dx = \infty, \quad (2.51)$$

which implies

$$\limsup_{\delta \rightarrow 0} \int_{\Sigma_\delta \cap U'} u dS = \infty. \quad (2.52)$$

For $n \in \mathbb{N}_*$, we take $U' = B_{\frac{1}{n}}$; there exists a sequence $\{\delta_{n,k}\}_{k \in \mathbb{N}}$ satisfying $\lim_{k \rightarrow \infty} \delta_{n,k} = 0$ such that

$$\lim_{k \rightarrow \infty} \int_{\Sigma_{\delta_{n,k}} \cap B_{\frac{1}{n}}} u dS = \infty. \quad (2.53)$$

Then, for any $\ell > 0$, there exists $k_\ell := k_{n,\ell} \in \mathbb{N}$ such that

$$k \geq k_\ell \implies \int_{\Sigma_{\delta_{n,k}} \cap B_{\frac{1}{n}}} u dS \geq \ell \quad (2.54)$$

and $k_{n,\ell} \rightarrow \infty$ when $n \rightarrow \infty$. In particular there exists $m := m(\ell, n) > 0$ such that

$$\int_{\Sigma_{\delta_{n,k_\ell}} \cap B_{\frac{1}{n}}} \inf\{u, m\} dS = \ell. \quad (2.55)$$

By the maximum principle u is bounded from below in $\Omega'_{\delta_{n,k_\ell}}$ by the solution $v := v_{\delta_{n,k_\ell}}$ of

$$\begin{cases} -\Delta v + g(|\nabla v|) = 0 & \text{in } \Omega'_{\delta_{n,k_\ell}} \\ v = \inf\{u, m\} & \text{in } \Sigma_{\delta_{n,k_\ell}}. \end{cases} \quad (2.56)$$

When $n \rightarrow \infty$, $\int \inf\{u, m(\ell, n)\} dS$ converges in the weak sense of measures to $\ell \delta_0$. By Corollary 2.4 there exists a solution $u_{\ell\delta_0}$ such that $v_{\delta_{n,k_\ell}} \rightarrow u_{\ell\delta_0}$ when $n \rightarrow \infty$ and consequently $u \geq u_{\ell\delta_0}$ in Ω . Even if $u_{\ell\delta_0}$ may not be unique, this implies

$$\liminf_{\delta \rightarrow 0} \int_{\Sigma_\delta} u \zeta(x) dS \geq \lim_{\delta \rightarrow 0} \int_{\Sigma_\delta} u_{\ell\delta_0} \zeta(x) dS = \ell \quad (2.57)$$

for any nonnegative $\zeta \in C^\infty(\mathbb{R}^N)$ such that $\zeta = 1$ in a neighborhood of 0. Since ℓ is arbitrary we obtain

$$\liminf_{\delta \rightarrow 0} \int_{\Sigma_\delta} u \zeta(x) dS = \infty \quad (2.58)$$

which contradicts (2.50). \square

3 Boundary singularities

3.1 Boundary data unbounded measures

We recall that for any $q > 1$, any solution u of (1.2) bounded from below satisfies [19, Th A1] the following estimate: for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\sup_{d(x) \geq \epsilon} |\nabla u(x)| \leq C_\epsilon. \quad (3.1)$$

Later on Lions gave in [22, Th IV 1] a more precise estimate that we recall below.

Lemma 3.1 *Assume $q > 1$ and $u \in C^2(\Omega)$ is any solution of (1.2) in Ω . Then*

$$|\nabla u(x)| \leq C_1(N, q) (d(x))^{-\frac{1}{q-1}} \quad \forall x \in \Omega. \quad (3.2)$$

Similarly, the following result is proved in [22].

Lemma 3.2 *Assume $q > 1$ and $u \in C^2(\Omega)$ is a solution of (1.2) in Ω . Then*

$$|u(x)| \leq \frac{C_2(N, q)}{2-q} \left((d(x))^{\frac{q-2}{q-1}} - \delta_0^{\frac{q-2}{q-1}} \right) + \max\{|u(z)| : z \in \Sigma_{\delta_0}\} \quad \forall x \in \Omega \quad (3.3)$$

if $q \neq 2$, and

$$|u(x)| \leq C_3(N) (\ln \delta_0 - \ln d(x)) + \max\{|u(z)| : z \in \Sigma_{\delta_0}\} \quad \forall x \in \Omega \quad (3.4)$$

if $q = 2$, for some $C_2(N, q), C_3(N) > 0$.

Proof. Put $M_{\delta_0} := \max\{|u(z)| : z \in \Sigma_{\delta_0}\}$ and let $x \in \Omega_{\delta_0}$, $x = \sigma(x) - d(x)\mathbf{n}_{\sigma(x)}$, and $x_0 = \sigma(x) - \delta_0\mathbf{n}_{\sigma(x)}$. Then, using Lemma 3.1 and the fact that $\sigma(x) = \sigma(x_0)$,

$$\begin{aligned} |u(x)| &\leq M_{\delta_0} + \int_0^1 \left| \frac{d}{dt} u(tx + (1-t)x_0) \right| dt \\ &\leq M_{\delta_0} + c \int_0^1 (td(x) + (1-t)\delta_0)^{-\frac{1}{q-1}} (\delta_0 - d(x)) dt. \end{aligned} \quad (3.5)$$

Thus we obtain (3.3) or (3.4) according the value of q . \square

If $q = 2$ and u solves (1.2), $v = e^u$ is harmonic and positive while if $q > 2$, any solution remains bounded in Ω . Although this last case is interesting in itself, we will consider only the case $1 < q < 2$.

Lemma 3.3 *Assume $1 < q < 2$, $0 \in \partial\Omega$ and $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ is a solution of (1.2) in Ω which vanishes on $\partial\Omega \setminus \{0\}$. Then*

$$u(x) \leq C_4(q)|x|^{\frac{q-2}{q-1}} \quad \forall x \in \Omega. \quad (3.6)$$

Proof. For $\epsilon > 0$, we set

$$P_\epsilon(r) = \begin{cases} 0 & \text{if } r \leq \epsilon \\ \frac{-r^4}{2\epsilon^3} + \frac{3r^3}{\epsilon^2} - \frac{6r^2}{\epsilon} + 5r - \frac{3\epsilon}{2} & \text{if } \epsilon < r < 2\epsilon \\ r - \frac{3\epsilon}{2} & \text{if } r \geq 2\epsilon \end{cases}$$

and let u_ϵ be the extension of $P_\epsilon(u)$ by zero outside Ω . There exists R_0 such that $\Omega \subset B_{R_0}$. Since $0 \leq P'_\epsilon(r) \leq 1$ and P_ϵ is convex, $u_\epsilon \in C^2(\mathbb{R}^N)$ and it satisfies $-\Delta u_\epsilon + |\nabla u_\epsilon|^q \leq 0$. Furthermore u_ϵ vanishes in $B_{R_0}^c$. For $R \geq R_0$ we set

$$U_{\epsilon,R}(x) = C_4(q) \left((|x| - \epsilon)^{\frac{q-2}{q-1}} - (R - \epsilon)^{\frac{q-2}{q-1}} \right) \quad \forall x \in B_R \setminus B_\epsilon,$$

where $C_4(q) = (q-1)^{\frac{q-2}{q-1}}(2-q)^{-1}$, then $-\Delta U_{\epsilon,R} + |\nabla U_{\epsilon,R}|^q \geq 0$. Since u_ϵ vanishes on ∂B_{R_0} and is finite on ∂B_ϵ it follows $u_\epsilon \leq U_{\epsilon,R}$. Letting successively $\epsilon \rightarrow 0$ and $R \rightarrow \infty$ yields to (3.6). \square

Using regularity we can improve this estimate

Lemma 3.4 *Under the assumptions of Lemma 3.3 there holds*

$$|\nabla u(x)| \leq C_5(q, \Omega)|x|^{-\frac{1}{q-1}} \quad \forall x \in \Omega. \quad (3.7)$$

and

$$u(x) \leq C_6(q, \Omega)d(x)|x|^{-\frac{1}{q-1}} \quad \forall x \in \Omega. \quad (3.8)$$

Proof. For $\ell > 0$, we set

$$T_\ell[u](x) = \ell^{\frac{2-q}{q-1}} u(\ell x) \quad \forall x \in \Omega^\ell := \frac{1}{\ell}\Omega. \quad (3.9)$$

If $x \in \Omega$, we set $|x| = d$ and $u_d(y) = T_d[u](y) = d^{\frac{2-q}{q-1}}u(dy)$. Then u_d satisfies (1.2) in $\Omega^d = \frac{1}{d}\Omega$. Since $d \leq d^* := \text{diam}(\Omega)$, the curvature of $\partial\Omega^d$ is uniformly bounded and therefore standard a priori estimates imply that there exists c depending on the curvature of Ω^d and $\max\{|u_d(y)| : \frac{1}{2} \leq |y| \leq \frac{3}{2}\}$ such that

$$|\nabla u_d(z)| \leq c \quad \forall z \in \Omega^d, \frac{3}{4} \leq |z| \leq \frac{5}{4}. \quad (3.10)$$

By (3.6), c is uniformly bounded. Therefore $|\nabla u(dz)| \leq cd^{-\frac{1}{q-1}}$ which implies (3.7).

Next, if $x \in \Omega$ is such that $d(x) \geq \frac{1}{6}|x|$ then (3.10) follows from (3.6). If $x \in \Omega$ and $d(x) < \frac{1}{6}|x|$, let $P \in \partial\Omega \setminus \{0\}$ such that $|x - P| = d(x)$. By (3.7), we obtain

$$u(x) \leq c(q, \Omega)d(x) \int_0^1 |tx + (1-t)P|^{-\frac{1}{q-1}} dt, \quad (3.11)$$

which, combined with the following estimate

$$|tx + (1-t)P| \geq |x| - (1-t)d(x) \geq \frac{5}{6}|x|,$$

implies (3.8). \square

In the next statement we obtain a local estimate of positive solutions which vanish only on a part of the boundary.

Proposition 3.5 *Assume $1 < q < 2$. Then there exist $0 < r^* \leq \delta_0$ and $C_7 > 0$ depending on N, q and Ω such that for compact set $K \subset \partial\Omega$, $K \neq \partial\Omega$ and any positive solution $u \in C(\overline{\Omega} \setminus K) \cap C^2(\Omega)$ vanishing on $\partial\Omega \setminus K$ of (1.2), there holds*

$$u(x) \leq C_7(N, q, \Omega)d(x)(d_K(x))^{-\frac{1}{q-1}} \quad \forall x \in \Omega \quad \text{s.t.} \quad d(x) \leq r^*, \quad (3.12)$$

where $d_K(x) = \text{dist}(x, K)$.

Proof. The proof is based upon the construction of local barriers in spherical shells. We fix $x \in \Omega$ such that $d(x) \leq \delta_0$ and $\sigma(x) := \text{Proj}_{\partial\Omega}(x) \in \partial\Omega \setminus K$. Set $r = d_K(x)$ and consider $\frac{3}{4}r < r' < \frac{7}{8}r$, $\tau \leq 2^{-1}r'$ and $\omega_x = \sigma(x) + \tau\mathbf{n}_x$. Since $\partial\Omega$ is C^2 , there exists $r^* \leq \delta_0$, depending only on Ω such that $d_K(\omega_x) > \frac{7}{8}r$ provided $d(x) \leq r^*$. For $A, B > 0$ we define the functions $s \mapsto \tilde{v}(s) = A(r' - s)^{\frac{2-q}{q-1}} - B$ and $y \mapsto v(y) = \tilde{v}(|y - \omega_x|)$ respectively in $[0, r']$ and $B_{r'}(\omega_x)$. Then

$$\begin{aligned} -\tilde{v}''(s) - \frac{N-1}{s}\tilde{v}'(s) + |\tilde{v}'(s)|^q \\ = A \frac{2-q}{q-1} (r' - s)^{-\frac{q}{q-1}} \left(-\frac{1}{q-1} - \frac{(N-1)(r' - s)}{s} + \left(\frac{(2-q)A}{q-1} \right)^{q-1} \right). \end{aligned}$$

We choose A and $\tau > 0$ such that

$$\frac{1}{q-1} - 1 + N + \frac{(N-1)r'}{\tau} \leq \left(\frac{(2-q)A}{q-1} \right)^{q-1} \quad (3.13)$$

so that inequality $-\Delta v + |\nabla v|^q \geq 0$ holds in $B_r(\omega_x) \setminus B_\tau(\omega_x)$. We choose B so that $v(\sigma(x)) = \tilde{v}(\tau) = 0$, i.e. $B = A(r' - \tau)^{\frac{q-2}{q-1}}$. Since $\tau \leq \delta_0$, $B_\tau(\omega_x) \subset \Omega^c$ therefore $v \geq 0$ on $\partial\Omega$ and $v \geq u$ on $\Omega \cap \partial B_{r'}(\omega_x)$. By the maximum principle we obtain that $u \leq v$ in $\Omega \cap B_{r'}(\omega_x)$ and in particular $u(x) \leq v(x)$ i.e.

$$u(x) \leq A \left((r' - \tau - d(x))^{\frac{q-2}{q-1}} - (r' - \tau)^{\frac{q-2}{q-1}} \right) \leq \frac{A(2-q)}{q-1} (r' - \tau - d(x))^{-\frac{1}{q-1}} d(x). \quad (3.14)$$

If we take in particular $\tau = \frac{r'}{2}$ and $d(x) \leq \frac{r}{4}$, then $A = A(N, q)$ and

$$u(x) \leq c_9 r'^{-\frac{1}{q-1}} d(x). \quad (3.15)$$

where $c_9 = c_9(N, q)$. If we let $r' \rightarrow \frac{7}{8}r$ we derive (3.12). Next, if $x \in \Omega$ is such that $d(x) \leq \delta_0$ and $d(x) > \frac{1}{4}d_K(x)$, we combine (3.12) with Harnack inequality [37], and a standard connectedness argument we obtain that $u(x)$ remains locally bounded in Ω , and the bound on a compact subset G of Ω depends only on K, G, N and q . Since $d_K(x) \geq d(x) > \frac{1}{4}d_K(x)$ it follows from Lemma 3.2 that (3.12) holds. Finally (3.12) holds for every $x \in \Omega$ satisfying $d(x) \leq r^*$. \square

As a consequence we have existence of positive solutions of (1.2) in Ω with a locally unbounded boundary trace.

Corollary 3.6 *Assume $1 < q < q_c$. Then for any compact set $K \subsetneq \partial\Omega$, there exists a positive solution u of (1.2) in Ω such that $\text{tr}_{\partial\Omega}(u) = (\mathcal{S}(u), \mu(u)) = (K, 0)$.*

Proof. For any $0 < \epsilon$, we set $K_\epsilon = \{x \in \partial\Omega : d_K(x) < \epsilon\}$ and let ψ_ϵ be a sequence of smooth functions defined on $\partial\Omega$ such that $0 \leq \psi_\epsilon \leq 1$, $\psi_\epsilon = 1$ on K_ϵ , $\psi_\epsilon = 0$ on $\partial\Omega \setminus K_{2\epsilon}$ ($\epsilon < \epsilon_0$ so that $\partial\Omega \setminus K_{2\epsilon} \neq \emptyset$). Furthermore we assume that $\epsilon < \epsilon' < \epsilon_0$ implies $\psi_\epsilon \leq \psi_{\epsilon'}$. For $k \in \mathbb{N}^*$ let $u = u_{k,\epsilon}$ be the solution of

$$\begin{cases} -\Delta u + |\nabla u|^q = 0 & \text{in } \Omega \\ u = k\psi_\epsilon & \text{in } \partial\Omega. \end{cases} \quad (3.16)$$

By the maximum principle $(k, \epsilon) \mapsto u_{k,\epsilon}$ is increasing. Combining Proposition 3.5 with the same Harnack inequality argument as above we obtain that $u_{k,\epsilon}(x)$ remains locally bounded in Ω and satisfies (3.13), independently of k and ϵ . By regularity it remains locally compact in the C_{loc}^1 -topology of $\overline{\Omega} \setminus K$. If we set $u_{\infty,\epsilon} = \lim_{k \rightarrow \infty} u_{k,\epsilon}$, then it is a solution of (1.2) in Ω which satisfies

$$\lim_{x \rightarrow y \in K_\epsilon} u_{\infty,\epsilon}(x) = \infty \quad \forall y \in K_\epsilon,$$

locally uniformly in K_ϵ . Furthermore, if $y \in K_\epsilon$ is such that $\overline{B_\theta(y)} \cap \partial\Omega \subset K_\epsilon$, for any k large enough there exists $\theta_k < \theta$ such that

$$\int_{\partial\Omega} \chi_{\overline{B_{\theta_k}(y)} \cap \partial\Omega} dS = k^{-1}.$$

For any $\ell > 0$, $u_{k\ell, \epsilon}$ is minorized by $u := u_{k\ell, B_{\theta_k}(y) \cap \partial\Omega}$ which satisfies

$$\begin{cases} -\Delta u + |\nabla u|^q = 0 & \text{in } \Omega \\ u = k\ell \chi_{\overline{B_{\theta_k}(y) \cap \partial\Omega}} & \text{in } \partial\Omega. \end{cases} \quad (3.17)$$

When $k \rightarrow \infty$, $u_{k\ell, B_{\theta_k}(y)}$ converges to $u_{\ell\delta_y}$ by Theorem 2.2 for the stability and Theorem 3.17 for the uniqueness. It follows that $u_{\infty, \epsilon} \geq u_{\ell\delta_y}$. Letting $\epsilon \rightarrow 0$ and using the same local regularity-compactness argument we obtain that $u_K := u_{\infty, 0} = \lim_{\epsilon \rightarrow 0} u_{\infty, \epsilon}$ is a positive solution of (1.2) in Ω which vanishes on $\partial\Omega \setminus K$ and satisfies

$$u_K(x) \geq u_{\ell\delta_y} \implies \lim_{\delta \rightarrow 0} \int_{\Sigma_\delta \cap B_\tau(y)} u(x) dS \geq \ell,$$

for any $\tau > 0$. Since τ and ℓ are arbitrary, estimate (2.37) holds which implies that $y \in \mathcal{S}(u)$. Clearly $\mu(u) = 0$ on $\mathcal{R}(u) = \partial\Omega \setminus \mathcal{S}(u)$ which ends the proof. \square

In the supercritical case the above result cannot be always true since there exist removable boundary compact sets (see Section 4). The following result is proved by an easy adaptation of the ideas in the proof of Corollary 3.6.

Corollary 3.7 *Assume $q_c \leq q < 2$ and let $G \subset \partial\Omega$. We assume that the boundary $\partial_{\partial\Omega} G \subset \partial\Omega$ satisfies the interior boundary sphere condition relative to $\partial\Omega$ in the sense that for any $y \in \partial_{\partial\Omega} G$, there exists $\epsilon_y > 0$ and a sphere such that $B_{\epsilon_y} \cap \partial\Omega \subset G$ and $y \in \overline{B_{\epsilon_y}}$. If $\mathcal{S} := \overline{G} \neq \partial\Omega$ there exists a positive solution u of (1.2) with boundary trace $(\mathcal{S}, 0)$.*

Remark. It is noticeable that the condition for the singular set to be different from all the boundary is necessary as it is shown in a recent article by Alarcón-García-Melián and Quass [2]. When $q_c \leq q < 2$ and $\Theta \subset \partial\Omega$ it is always possible to construct a positive solution u_ϵ ($\epsilon > 0$) of (1.2) with boundary trace $(\Theta_\epsilon^c, 0)$, where $\Theta_\epsilon = \{x \in \partial\Omega : d_\Theta(x) < \epsilon\}$ and the complement is relative to $\partial\Omega$. Furthermore $\epsilon \mapsto u_\epsilon$ is decreasing. If Θ has an empty interior, Proposition 3.5 does not apply. We conjecture that the fact that $\lim_{\epsilon \rightarrow 0} u_\epsilon$ depends on some capacity estimates on Θ .

The condition that a solution vanishes outside a compact boundary set K can be weakened and replaced by a local integral estimate. The next result is fundamental for existence of a solution with a given general boundary trace.

Proposition 3.8 *Assume $1 < q < 2$, $U \subset \partial\Omega$ is relatively open and $\mu \in \mathfrak{M}(U)$ is a positive bounded Radon measure. Then for any compact set $\Theta \subset \Omega$ there exists a constant $C_8 = C_8(q, \Theta, \|\mu\|_{\mathfrak{M}(U)}) > 0$ such that any positive solution u of (1.2) in Ω with boundary trace (\mathcal{S}, μ') where \mathcal{S} is closed, $U \subset \partial\Omega \setminus \mathcal{S} := \mathcal{R}$ and μ' is a positive Radon measure on \mathcal{R} such that $\mu'|_U = \mu$, there holds*

$$u(x) \leq C_8 \quad \forall x \in \Theta. \quad (3.18)$$

Proof. We follow the notations of Theorem 2.10. Since the result is local, without loss of generality we can assume that U is smooth and contained in a single chart Σ_j . Estimates (2.44)-(2.48) are still valid under the form

$$\begin{aligned} & \int_U u(\delta, \cdot) \phi^\alpha dS - \int_U u(\delta_0, \cdot) \phi^\alpha dS \\ & \geq (1 - c_{10}\epsilon) \int_\delta^{\delta_0} \int_U |\nabla u|^q \phi^\alpha dS (\tau - \delta) d\tau - (\delta_0 - \delta) \int_U \frac{\partial u}{\partial \delta}(\delta_0, \cdot) \phi^\alpha dS - c'_{10}. \end{aligned} \quad (3.19)$$

Since the last term in the right-hand side of (3.19) is uniformly bounded by Lemma 3.1, it follows that we can let $\delta \rightarrow 0$ and derive, for some $c_{11} > 0$ depending on q and Ω

$$\int_U u(\delta_0, \cdot) \phi^\alpha dS + (1 - c_{10}\epsilon) \int_0^{\delta_0} \int_U |\nabla u|^q \tau dS d\tau \leq \int_U \phi^\alpha d\mu + c_3 \leq \|\mu\|_{\mathfrak{M}(U)} + c_{11}, \quad (3.20)$$

and c_1 and c_3 depend on the curvature H , N and q . This implies that there exist some ball $B_\alpha(a)$, $\alpha > 0$ and $a \in U$ such that $\overline{B_\alpha(a)} \cap \partial\Omega \subset U$ and

$$\int_{B_\alpha(a) \cap \Omega} |\nabla u|^q dx \leq \|\mu\|_{\mathfrak{M}(U)} + c_{11}, \quad (3.21)$$

Thus, if $B_\beta(b)$ is some ball such that $\overline{B_\beta(b)} \subset B_\alpha(a) \cap \Omega$, we have

$$\int_{B_\beta(b)} |\nabla u|^q dx \leq (d(b) - \beta)^{-1} \left(\|\mu\|_{\mathfrak{M}(U)} + c_{11} \right). \quad (3.22)$$

If in (3.19) we let $\delta \rightarrow 0$ and then replace δ_0 by $\delta \in (\delta_1, \delta_0]$ for $\delta_1 > 0$ we obtain

$$\int_{U_\Sigma} \phi^\alpha d\mu \geq \int_{U_\Sigma} u(\delta, \cdot) \phi^\alpha dS - (\delta_0 - \delta) \int_{U_\Sigma} \frac{\partial u}{\partial \delta}(\delta, \cdot) \phi^\alpha dS - c'_{11}. \quad (3.23)$$

By Lemma 3.1 the second term in the right-hand side remains bounded by a constant depending on δ_1 , H , N and q . Therefore $\int_{U_\Sigma} u(\delta, \cdot) \phi^\alpha dS$ remains bounded by a constant depending on the previous quantities and of $\|\mu\|_{\mathfrak{M}(U)}$ and consequently, assuming that $d(x) \geq \delta_1$ for all $x \in B_\beta(b)$ (i.e. $d(b) - \beta \geq \delta_1$)

$$u_{B_\beta(b)} := \frac{1}{|B_\beta(b)|} \int_{B_\beta(b)} u dx \leq c_{12} \quad (3.24)$$

and c_4 depends on δ_1 , H , N and q and $\|\mu\|_{\mathfrak{M}(U)}$. By Poincaré inequality

$$\left(\int_{B_\beta(b)} u^q dx \right)^{\frac{1}{q}} \leq c_5 \left(\int_{B_\beta(b)} |\nabla u|^q dx \right)^{\frac{1}{q}} + |B_\beta(b)|^{\frac{1}{q}} u_{B_\beta(b)}. \quad (3.25)$$

Combining (3.22) and (3.24) we derive that $\|u\|_{W^{1,q}(B_\beta(b))}$ remains bounded by a quantity depending only on δ_1 , H , N and q and $\|\mu\|_{\mathfrak{M}(U)}$. By the classical trace theorem in Sobolev

spaces, $\|u\|_{L^q(\partial B_{\frac{\beta}{2}}(b))}$ remains also uniformly bounded when the above quantities are so. By the maximum principle

$$u(x) \leq \mathbb{P}^{B_{\beta}(b)}[u|_{\partial B_{\beta}(b)}](x) \quad \forall x \in B_{\beta}(b), \quad (3.26)$$

where $\mathbb{P}^{B_{\beta}(b)}$ denotes the Poisson kernel in $B_{\beta}(b)$. Therefore, u remains uniformly bounded in $B_{\frac{\beta}{2}}(b)$ by some constant c_6 which also depends on $\|\mu\|_{\mathfrak{M}(U)}$, q , Ω , b and β , but not on u . We end the proof by Harnack inequality and a standard connectedness argument as it has already be used in Corollary 3.6. \square

The main result of this section is the following

Theorem 3.9 *Assume $1 < q < 2$, $K \subsetneq \partial\Omega$ is closed and μ is a positive Radon measure on $\mathcal{R} := \partial\Omega \setminus K$. Then there exists a solution of (1.2) such that $tr_{\partial\Omega}(u) = (K, \mu)$.*

Proof. For $\epsilon' > \epsilon > 0$ we set $\nu_{\epsilon, \epsilon'} = k\chi_{\overline{K}_{\epsilon'}} + \chi_{\overline{K}_{\epsilon}}\mu$ and denote by $u_{\epsilon, \epsilon', k, \mu}$ the maximal solution of

$$\begin{cases} -\Delta u + |\nabla u|^q = 0 & \text{in } \Omega \\ u = \nu_{\epsilon, \epsilon'} & \text{in } \partial\Omega. \end{cases} \quad (3.27)$$

We recall that $K_{\epsilon} := \{x \in \partial\Omega : d_K < \epsilon\}$, so that $\nu_{\epsilon, \epsilon'}$ is a positive bounded Radon measure. For $0 < \epsilon \leq \epsilon_0$ there exists $c \in \mathcal{R}$ and $\gamma > 0$ such that $\overline{B}_{\gamma}(c) \subset \overline{K}_{\epsilon_0}^c$. Since $\|\chi_{\overline{K}_{\epsilon}}\mu\|$ is uniformly bounded, it follows from Proposition 3.8 that $u_{\epsilon, \epsilon', k, \mu}$ remains locally bounded in Ω , uniformly with respect to k and ϵ . Furthermore $(k, \epsilon) \mapsto u_{\epsilon, \epsilon', k, \mu}$ is increasing with respect to k . If $u_{\epsilon, \epsilon', \infty, \mu} = \lim_{k \rightarrow \infty} u_{\epsilon, \epsilon', k, \mu}$, it is a solution of (1.2) in Ω . By the same argument as the one used in the proof of Corollary 3.6, any point $y \in K$ is such that $u_{\epsilon, \epsilon', \infty, \mu} \geq u_{\ell\delta_y}$ for any $\ell > 0$. Using maximum principle

$$(\epsilon_2 \leq \epsilon_1, \epsilon'_1 \leq \epsilon'_2, k_1 \leq k_2) \implies (u_{\epsilon_1, \epsilon'_1, k_1, \mu} \leq u_{\epsilon_2, \epsilon'_2, k_2, \mu}) \quad (3.28)$$

Since $u_{\epsilon, \epsilon', \infty, \mu}$ remains locally bounded in Ω independently of ϵ and ϵ' , we can set $u_{K, \mu} = \lim_{\epsilon' \rightarrow 0} \lim_{\epsilon \rightarrow 0} u_{\epsilon, \epsilon', \infty, \mu}$ then by the standard local regularity results $u_{K, \mu}$ is a positive solution of (1.2) in Ω . Furthermore $u_{K, \mu} > u_{\ell\delta_y}$, for any $y \in K$ and $\ell > 0$; thus the set of boundary singular points of $u_{K, \mu}$ contains K . In order to prove that $tr_{\partial\Omega}(u_{K, \infty}) = (K, \mu)$ consider a smooth relatively open set $U \subset \mathcal{R}$. Using the same function ϕ^{α} as in Proposition 3.8, we obtain from (3.20)

$$\int_{U_{\Sigma}} u_{K, \infty}(\delta_0, \cdot) \phi^{\alpha} dS + (1 - c_{10}\epsilon) \int_0^{\delta_0} \int_{U_{\Sigma}} |\nabla u_{K, \infty}|^q \tau dS d\tau \leq \int_U d\mu + c_{11}. \quad (3.29)$$

Therefore U is a subset of the set of boundary regular points of $u_{K, \infty}$, which implies $tr_{\partial\Omega}(u) = (K, \mu)$ by Proposition 2.6. \square

Remark. If $q_c \leq q < 2$, it is possible to solve (3.27) if μ is a smooth function defined in \mathcal{R} and to let successively $k \rightarrow \infty$; $\epsilon \rightarrow 0$ and $\epsilon' \rightarrow 0$ using monotonicity as before. The limit function u^* is a solution of (1.2) in Ω . If $tr_{\partial\Omega}(u^*) = (\mathcal{S}^*, \mu^*)$, then $\mathcal{S}^* \subset K$ and $\mu^*|_{\mathcal{R}} = \mu$. However interior points of K , if any, belong to \mathcal{S}^* (see Corollary 3.7).

3.2 Boundary Harnack inequality

We adapt below ideas from Bauman [5], Bidaut-Véron-Borghol-Véron [7] and Trudinger [36]-[37] in order to prove a *boundary Harnack inequality* which is one of the main tools for analyzing the behavior of positive solutions of (1.2) near an isolated boundary singularity. We assume that Ω is a bounded C^2 domain with $0 \in \partial\Omega$ and δ_0 has been defined for constructing the flow coordinates.

Theorem 3.10 *Assume $0 \in \partial\Omega$, $q > 1$. Then there exist $0 < r_0 \leq \delta_0$ and $C_9 > 0$ depending on q and Ω such that for any $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2r_0}(0))) \cap C^2(\Omega)$ positive solution of (1.2) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2r_0}(0)$ there holds*

$$\frac{u(y)}{C_9 d(y)} \leq \frac{u(x)}{d(x)} \leq \frac{C_9 u(y)}{d(y)} \quad (3.30)$$

for every $x, y \in B_{\frac{2r_0}{3}}(0) \cap \Omega$ satisfying $\frac{|y|}{2} \leq |x| \leq 2|y|$.

Since Ω is a bounded C^2 domain, it satisfies uniform sphere condition, i.e there exists $0r_0 >$ sufficiently small such that for any $x \in \partial\Omega$ the two balls $B_{r_0}(x - r_0\mathbf{n}_x)$ and $B_{r_0}(x + r_0\mathbf{n}_x)$ are subsets of Ω and $\overline{\Omega}^c$ respectively. We can choose $0 < r_0 < \min\{\delta_0, 3r^*\}$ where r^* is in Proposition 3.5.

We first recall the following chained property of the domain Ω [5].

Lemma 3.11 *Assume that $Q \in \partial\Omega$, $0 < r < r_0$ and $h > 1$ is an integer. There exists an integer N_0 depending only on r_0 such that for any points x and y in $\Omega \cap B_{\frac{3r}{2}}(Q)$ verifying $\min\{d(x), d(y)\} \geq r/2^h$, there exists a connected chain of balls B_1, \dots, B_j with $j \leq N_0 h$ such that*

$$\begin{aligned} x \in B_1, y \in B_j, \quad B_i \cap B_{i+1} \neq \emptyset \text{ for } 1 \leq i \leq j-1 \\ \text{and } 2B_i \subset B_{2r}(Q) \cap \Omega \text{ for } 1 \leq i \leq j. \end{aligned} \quad (3.31)$$

The next result is an internal Harnack inequality.

Lemma 3.12 *Assume $Q \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2r_0}{3}}(0)$ and $0 < r \leq |Q|/4$. Let $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2r_0}(0))) \cap C^2(\Omega)$ positive solution of (1.2) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2r_0}(0)$. Then there exists a positive constant $c_{12} > 1$ depending on N , q and r_0 such that*

$$u(x) \leq c_{12}^h u(y), \quad (3.32)$$

for every $x, y \in B_{\frac{3r}{2}}(Q) \cap \Omega$ such that $\min\{d(x), d(y)\} \geq r/2^h$ for some $h \in \mathbb{N}$.

Proof. We first notice that for any $\ell > 0$, $T_\ell[u]$ satisfies (1.2) in Ω^ℓ where T_ℓ is defined in (3.9). If we take in particular $\ell = |Q|$, we can assume $|Q| = 1$ and the curvature of the domain $\Omega^{|Q|}$ remains bounded. By Proposition 3.5

$$u(x) \leq 2^{\frac{2-q}{q-1}} C_7(N, q, \Omega) \quad \forall x \in B_{2r}(Q) \cap \Omega. \quad (3.33)$$

By Lemma 3.11 there exist an integer N_0 depending on r_0 and a connected chain of $j \leq N_0 h$ balls B_i with respectively radius r_i and centers x_i , satisfying (3.31). Hence due to [36,

Corollary 10] and [37, Theorem 1.1] there exists a positive constant c_2 depending on N, q and Ω such that for every $1 \leq i \leq j$,

$$\sup_{B_i} u \leq c'_{12} \inf_{B_i} u, \quad (3.34)$$

which yields to (3.32) with $c_{12} = c'_{12}{}^{N_0}$. \square

Lemma 3.13 *Assume the assumptions on Q and u of Lemma 3.12 are fulfilled. If $P \in \partial\Omega \cap B_r(Q)$ and $0 < s < r$, there exist two positive constants δ and c_{13} depending on N, q and Ω such that*

$$u(x) \leq c_{13} \frac{|x - P|^\delta}{s^\delta} M_{s,P}(u) \quad (3.35)$$

for every $x \in B_s(P) \cap \Omega$, where $M_{s,P}(u) = \max\{u(z) : z \in B_s(P) \cap \Omega\}$.

Proof. Notice that $B_s(P) \subset B_{2r}(Q)$. Up to the transformation $T|_Q$, we may assume $|Q| = 1$ and that u is bounded in $B_{2r}(Q) \cap \Omega$ as in (3.33). We fix $x \in B_s(P)$ and $s' \in (|x - P|, s)$. Set

$$\tilde{u} := \frac{u}{M_{s,P}(u)}$$

then $M_{s,P}(\tilde{u}) = 1$ and

$$-\Delta \tilde{u} + M_{s,P}^{q-1}(u) |\nabla \tilde{u}|^q = 0 \quad (3.36)$$

in $B_{2r}(Q) \cap \Omega$. It follows from the assumption and Young's inequality that

$$M_{s,P}^{q-1}(u) |\nabla \tilde{u}|^q \leq c'_{13} |\nabla \tilde{u}|^2 + c''_{13} |\nabla \tilde{u}|$$

where c, c' depend on q . By [37, Theorem 5.2] there exist $\delta = \delta(N, q, r_0) > 0$ and $c_{13} = c_{13}(N, q, r_0)$ such that

$$|\tilde{u}(z) - \tilde{u}(z')| \leq c_{13} \left(\frac{s'}{s}\right)^\delta \quad \forall z, z' \in B_{s'}(P) \cap \Omega,$$

which is equivalent to

$$|u(z) - u(z')| \leq c_{13} \left(\frac{s'}{s}\right)^\delta M_{s,P}(u) \quad \forall z, z' \in B_{s'}(P) \cap \Omega. \quad (3.37)$$

Now by taking $z = x$ and letting $z' \rightarrow P$, we obtain

$$u(x) \leq c_{13} \left(\frac{s'}{s}\right)^\delta M_{s,P}(u). \quad (3.38)$$

Since (3.38) holds true for every $s' > |x - P|$, (3.35) follows by rescaling. \square

Thanks to Lemma 3.12 and Lemma 3.13, we obtain the following result by proceeding as in [7, Lemma 5] and [5].

Corollary 3.14 Assume $Q \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2r_0}{3}}(0)$ and $0 < r \leq |Q|/8$. Let $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2r_0}(0))) \cap C^2(\Omega)$ positive solution of (1.2) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2r_0}(0)$. Then there exists a constant c_{14} depending only on N, q and Ω such that

$$u(x) \leq c_{14}u(Q - \frac{r}{2}\mathbf{n}_Q) \quad \forall x \in B_r(Q) \cap \Omega. \quad (3.39)$$

Lemma 3.15 Assume $Q \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2r_0}{3}}(0)$ and $0 < r \leq |Q|/8$. Let $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2r_0}(0))) \cap C^2(\Omega)$ positive solution of (1.2) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2r_0}(0)$. Then there exist $a \in (0, 1/2)$ and $c_{15} > 0$ depending on N, q and r_0 such that

$$\frac{1}{c_{15}} \frac{t}{r} \leq \frac{u(P - t\mathbf{n}_P)}{u(Q - \frac{r}{2}\mathbf{n}_Q)} \leq c_{15} \frac{t}{r} \quad (3.40)$$

for any $P \in B_r(Q) \cap \partial\Omega$ and $0 \leq t < \frac{a}{2}r$.

Proof. As above, we may assume $|Q| = 1$, thus estimate (3.33) holds.

Step 1: Lower estimate. Let $0 < \tau < \frac{a}{2}r < \frac{r}{4}$ be fixed. For $b > 0$ to be made precise later on, we define in $B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P)$

$$v(x) = V(s) := \frac{e^{-b(\frac{s}{r-\tau})^2} - e^{-\frac{b}{4}}}{e^{-\frac{b}{16}} - e^{-\frac{b}{4}}}, \quad (3.41)$$

where $s = |x - P + \frac{r}{2}\mathbf{n}_P|$. Since $-\Delta v + |\nabla v|^q = -V'' - \frac{N-1}{s}V' + |V'|^q$, this last expression is nonpositive if and only if

$$\begin{aligned} & 2b \left(\frac{s}{r-\tau} \right)^2 - N \\ & \geq (2b)^{q-1} \left(e^{-\frac{b}{16}} - e^{-\frac{b}{4}} \right)^{1-q} \left(\frac{s}{r-\tau} \right)^{2(q-1)} s^{2-q} e^{-b(q-1)(\frac{s}{r-\tau})^2}. \end{aligned} \quad (3.42)$$

Because $x \in B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P)$,

$$\frac{r-\tau}{4} \leq s \leq \frac{r-\tau}{2} < \frac{1}{4}.$$

Hence, if we choose $b = N + 6$ then (3.42) holds true, and thus

$$-\Delta v + |\nabla v|^q \leq 0 \quad (3.43)$$

in $B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P)$. Since $B_{r_0}(P - r_0\mathbf{n}_P) \subset \Omega$, it follows that $d(x) \geq \text{dist}(x, B_{r_0}(P - r_0\mathbf{n}_P)) \geq r/32$ for any $x \in B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap \partial B_{\frac{r}{4}}(P)$, which, along with Lemma 3.12, implies that

$$u(x) \geq c'_{15} u(Q - \frac{r}{2}\mathbf{n}_Q) \quad (3.44)$$

where

$$c'_{15} = c_{12}^5 \left(1 + 2^{\frac{2-q}{q-1}} C_7(N, q, \Omega) \right)$$

and $c(N, q, \Omega)$ is the constant in (3.12). Since $v \leq 1$ on $B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap \partial B_{\frac{r}{4}}(P)$,

$$u(x) \geq c'_{15}{}^{-1}u(Q - \frac{r}{2}\mathbf{n}_Q)v(x) \quad (3.45)$$

on $B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap \partial B_{\frac{r}{4}}(P)$. Moreover $c'_{15}{}^{-1}u(Q - \frac{r}{2}\mathbf{n}_Q) < c_{12}^{-5} < 1$, therefore $\tilde{v}(x) := c'_{15}{}^{-1}u(Q - \frac{r}{2}\mathbf{n}_Q)v(x)$ is a subsolution of (1.2) in $B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P)$. Consequently, by setting $w := u - \tilde{v}$, we get

$$-\Delta w + d \cdot \nabla w \geq 0$$

in $B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P)$ and $w \geq 0$ on $\partial(B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P))$, where $d = (d_1, \dots, d_N)$ and

$$d_i(x) = q \int_0^1 |\nabla(tu + (1-t)\tilde{v})|^{q-2} \partial_i(tu + (1-t)\tilde{v}) dt \quad \forall 1 \leq i \leq N.$$

Since $d \in L^\infty(B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P))^N$, by applying the maximum principle, we deduce that $u \geq \tilde{v}$ in $B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap \partial B_{\frac{r}{4}}(P)$. Finally, set $x_\tau = P - \tau\mathbf{n}_P$ then $x_\tau \in B_{\frac{r-\tau}{2}}(P - \frac{r}{2}\mathbf{n}_P) \cap B_{\frac{r}{4}}(P)$ and

$$v(x_\tau) \geq \frac{e^{-\frac{b}{4}}}{e^{-\frac{b}{16}} - e^{-\frac{b}{4}}} \left(1 - \left(1 - \frac{\tau}{r-\tau} \right)^2 \right) \geq c''_{15} \frac{\tau}{r} \quad (3.46)$$

where $c''_{15} = c''_{15}(N, q)$, which implies

$$\frac{u(x_\tau)}{u(Q - \frac{r}{2}\mathbf{n}_Q)} \geq c'_{15}{}^{-1} c''_{15} \frac{\tau}{r}. \quad (3.47)$$

Thus the left-hand side of (3.40) follows since τ is arbitrary in $(0, \frac{a}{2}r)$.

Step 2: Upper estimate. Let $a \in (0, 1/2)$ be a parameter to be determined later on. We can choose r_0 so that, $B_{3ar}(P + 3ar\mathbf{n}_P) \subset \bar{\Omega}^c$. Let ϕ_1 be the first eigenfunction of the Laplace operator $-\Delta$ in $B_3 \setminus \bar{B}_1$ with Dirichlet boundary condition and λ_1 is the corresponding eigenvalue. We normalize ϕ_1 by $\phi_1(x) = 1$ on $\{x : |x| = 2\}$ and set

$$\phi_{ar}(x) = \phi_1 \left(\frac{x - (P + ar\mathbf{n}_P)}{ar} \right),$$

thus

$$-\Delta \phi_{ar} = \frac{\lambda_1}{(ar)^2} \phi_{ar}(x) \geq 0$$

in $B_{3ar}(P + ar\mathbf{n}_P) \setminus \bar{B}_{ar}(P + ar\mathbf{n}_P)$ and vanishes on the boundary of this domain. We have $-\Delta \phi_{ar} \geq 0 \geq -\Delta u$ in $B_{2ar}(P + ar\mathbf{n}_P) \cap \Omega$. We can choose a small enough such that $B_{2ar}(P + ar\mathbf{n}_P) \subset B_r(Q)$. Then by Corollary 3.14,

$$u(x) \leq c_{14}u(Q - \frac{r}{2}\mathbf{n}_Q)$$

for $x \in \partial B_{2ar}(P + ar\mathbf{n}_P) \cap \Omega$. Set $\tilde{\phi}_{ar} := c_{14}u(Q - \frac{r}{2}\mathbf{n}_Q)\phi_{ar}$, then $-\Delta\tilde{\phi}_{ar} \geq 0 \geq -\Delta u$ in $B_{2ar}(P + ar\mathbf{n}_P) \cap \Omega$ and $\tilde{\phi}_{ar}$ dominates u on $\partial(B_{2ar}(P + ar\mathbf{n}_P) \cap \Omega)$. By the maximum principle, $u \leq \tilde{\phi}_{ar}$ in $B_{2ar}(P + ar\mathbf{n}_P) \cap \Omega$. In particular

$$u(P - t\mathbf{n}_P) \leq c_{14}\phi_1 \left(\frac{|P - t\mathbf{n}_P - (P + ar\mathbf{n}_P)|}{ar} \right) u(Q - \frac{r}{2}\mathbf{n}_Q).$$

Since $\phi_1(x) \leq c_{15}''d(x) = c_{15}''(|x| - 1)$ for every $1 \leq |x| \leq 2$, we obtain the right-hand side of (3.40). \square

Proof of Theorem 3.10. Assume $x \in B_{\frac{2r_0}{3}}(0) \cap \Omega$ and set $r = \frac{|x|}{8}$.

Step 1: Tangential estimate: we suppose $d(x) < \frac{a}{2}r$. Let $Q \in \partial\Omega \setminus \{0\}$ such that $|Q| = |x|$ and $x \in B_r(Q)$. By Lemma 3.15,

$$\frac{8}{c_{15}} \frac{u(Q - \frac{r}{2}\mathbf{n}_Q)}{|x|} \leq \frac{u(x)}{d(x)} \leq 8c_{15} \frac{u(Q - \frac{r}{2}\mathbf{n}_Q)}{|x|}. \quad (3.48)$$

We can connect $Q - \frac{r}{2}\mathbf{n}_Q$ with $-2r\mathbf{n}_0$ by m_1 (depending only on N) connected balls $B_i = B(x_i, \frac{r}{4})$ with $x_i \in \Omega$ and $d(x_i) \geq \frac{r}{2}$ for every $1 \leq i \leq m_1$. It follows from (3.34) that

$$c_{12}'^{-m_1} u(-2r\mathbf{n}_0) \leq u(Q - \frac{r}{2}\mathbf{n}_Q) \leq c_{12}'^{m_1} u(-2r\mathbf{n}_0),$$

which, together with (3.48) leads to

$$\frac{8}{c_{12}'^{m_1} c_{15}} \frac{u(-2r\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq 8c_{12}'^{m_1} c_{15} \frac{u(-2r\mathbf{n}_0)}{|x|}. \quad (3.49)$$

Step 2: Internal estimate: $d(x) \geq \frac{a}{2}r$. We can connect $-2r\mathbf{n}_0$ with x by m_2 (depending only on N) connected balls $B'_i = B(x'_i, \frac{a}{4}r)$ with $x'_i \in \Omega$ and $d(x'_i) \geq \frac{a}{2}r$ for every $1 \leq i \leq m_2$. By applying again (3.34) and keeping in mind the estimate $\frac{a}{4}|x| < d(x) \leq |x|$, we get

$$\frac{a}{4c_{12}'^{m_2}} \frac{u(-2r\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq \frac{4c_{12}'^{m_2}}{a} \frac{u(-2r\mathbf{n}_0)}{|x|}. \quad (3.50)$$

Step 3: End of proof. Take $\frac{|x|}{2} \leq s \leq 2|x|$, we can connect $-2r\mathbf{n}_0$ with $-s\mathbf{n}_0$ by m_3 (depending only on N) connected balls $B''_i = B(x''_i, \frac{r}{2})$ with $x''_i \in \Omega$ and $d(x''_i) \geq r$ for every $1 \leq i \leq m_3$. This fact, joint with (3.49) and (3.50), yields

$$\frac{1}{C'_9} \frac{u(-s\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq C'_9 \frac{u(-s\mathbf{n}_0)}{|x|} \quad (3.51)$$

where $C'_9 = C'_9(N, q, \Omega)$. Finally let $y \in B_{\frac{2r_0}{3}}(0) \cap \Omega$ satisfy $\frac{|x|}{2} \leq |y| \leq 2|x|$. By applying twice (3.51) we get (3.30) with $C_9 = C_9''$. \square

A direct consequence of Theorem 3.10 is the following useful form of boundary Harnack inequality.

Corollary 3.16 *Let $u_i \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2r_0}(0))) \cap C^2(\Omega)$ ($i = 1, 2$) be two nonnegative solutions of (1.2) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2r_0}(0)$. Then there exists a constant C_{10} depending on N, q and Ω such that for any $r \leq \frac{2r_0}{3}$*

$$\begin{aligned} & \sup \left(\frac{u_1(x)}{u_2(x)} : x \in \Omega \cap (B_r(0) \setminus B_{\frac{r}{2}}(0)) \right) \\ & \leq C_{10} \inf \left(\frac{u_1(x)}{u_2(x)} : x \in \Omega \cap (B_r(0) \setminus B_{\frac{r}{2}}(0)) \right). \end{aligned} \quad (3.52)$$

3.3 Isolated singularities

Theorem 2.2 assert the existence of a solution to (2.1) for any positive Radon measure μ if $g \in \mathcal{G}_0$ satisfies (2.1), and the question of uniqueness of this problem is still an open question, nevertheless when $\mu = \delta_a$ with $a \in \partial\Omega$, we have the following result

Theorem 3.17 *Assume $1 < q < q_c$, $a \in \partial\Omega$ and $c > 0$. Then there exists a unique solution $u := u_{c\delta_a}$ to*

$$\begin{cases} -\Delta u + |\nabla u|^q = 0 & \text{in } \Omega \\ u = c\delta_a & \text{on } \partial\Omega \end{cases} \quad (3.53)$$

Furthermore the mapping $c \mapsto u_{c\delta_a}$ is increasing.

Lemma 3.18 *Under the assumption of Theorem 3.17, there holds*

$$|\nabla u(x)| \leq C_{11}c|x-a|^{-N} \quad \forall x \in \Omega \quad (3.54)$$

with $C_{11} = C_{11}(N, q, \kappa) > 0$ where κ is the supremum of the curvature of $\partial\Omega$.

Proof. Up to a translation we may assume $a = 0$. By the maximum principle $0 < u(x) \leq cP^\Omega(x, 0)$ in Ω . For $0 < \ell \leq 1$, set $v_\ell = T_\ell[u]$ where T_ℓ is the scaling defined in (3.9), then v_ℓ satisfies

$$\begin{cases} -\Delta v_\ell + |\nabla v_\ell|^q = 0 & \text{in } \Omega^\ell \\ v_\ell = \ell^{\frac{2-q}{q-1}+1-N}c\delta_0 & \text{on } \partial\Omega^\ell \end{cases} \quad (3.55)$$

where $\Omega^\ell = \frac{1}{\ell}\Omega$ and by maximum principle

$$0 < v_\ell(x) \leq \ell^{\frac{2-q}{q-1}+1-N}cP^{\Omega^\ell}(x, 0) \quad \forall x \in \Omega^\ell.$$

Since the curvature of $\partial\Omega^\ell$ remains bounded when $0 < \ell \leq 1$, there holds (see [20])

$$\begin{aligned} & \sup\{|\nabla v_\ell(x)| : x \in \Omega^\ell \cap (B_2(0) \setminus B_{\frac{1}{2}}(0))\} \\ & \leq C'_{11} \sup\{v_\ell(x) : x \in \Omega^\ell \cap (B_3(0) \setminus B_{\frac{1}{3}}(0))\} \\ & \leq C'_{11} \ell^{\frac{2-q}{q-1}} \sup\{u(\ell x) : x \in \Omega^\ell \cap (B_3(0) \setminus B_{\frac{1}{3}}(0))\} \\ & \leq 3^{N-1} C'_{11} C''_{11} \ell^{\frac{2-q}{q-1}+1-N} \end{aligned} \quad (3.56)$$

with $C'_{11} = C'_{11}(N, q, \kappa)$ and $C''_{11} = C''_{11}(N)$. Consequently

$$\ell^{\frac{2-q}{q-1}+1} |\nabla u|(\ell x) \leq C_{11}(N, q, \Omega) c \ell^{\frac{2-q}{q-1}+1-N} \quad \forall x \in \Omega^\ell \cap (B_2(0) \setminus B_{\frac{1}{2}}(0)), \quad \forall \ell > 0$$

where $C_{11} = C'_{11}C''_{11}$. Set $\ell x = y$ and $|x| = 1$, then

$$|\nabla u|(y) \leq C_2(N, q, \Omega) |y|^{-N} \quad \forall y \in \Omega.$$

□

Lemma 3.19

$$\lim_{|x| \rightarrow 0} \frac{\mathbb{G}^\Omega[|x|^{-Nq}]}{P(x, 0)} = 0. \quad (3.57)$$

We recall the following estimates for the Green function ([7], [15], [38] and [39])

$$G^\Omega(x, y) \leq c_{16}d(x)|x - y|^{1-N} \quad \forall x, y \in \Omega, x \neq y$$

and

$$G^\Omega(x, y) \leq c_{16}d(x)d(y)|x - y|^{-N} \quad \forall x, y \in \Omega, x \neq y.$$

where $c_{16} = c_{16}(N, \Omega)$. Hence, for $\alpha \in (0, N + 1 - Nq)$, we obtain

$$\begin{aligned} G^\Omega(x, y) &\leq \left(c_{16}d(x)|x - y|^{1-N}\right)^\alpha \left(c_{16}d(x)d(y)|x - y|^{-N}\right)^{1-\alpha} \\ &= c_{16}d(x)d(y)^{1-\alpha}|x - y|^{\alpha-N} \quad \forall x, y \in \Omega, x \neq y, \end{aligned} \quad (3.58)$$

which follows that

$$\frac{\mathbb{G}^\Omega[|x|^{-Nq}]}{P^\Omega(x, 0)} \leq c_{16}|x|^N \int_{\mathbb{R}^N} |x - y|^{\alpha-N} |y|^{1-Nq-\alpha} dy \quad (3.59)$$

By the following identity (see [21, p. 124]),

$$\int_{\mathbb{R}^N} |x - y|^{\alpha-N} |y|^{1-Nq-\alpha} dy = c'_{16}|x|^{1-Nq} \quad (3.60)$$

where $c'_{16} = c'_{16}(N, \alpha)$, we obtain

$$\frac{\mathbb{G}^\Omega[|x|^{-Nq}]}{P^\Omega(x, 0)} \leq c_{16}c'_{16}|x|^{N+1-Nq}. \quad (3.61)$$

Since $N + 1 - Nq > 0$, (3.57) follows. □

Proof of Theorem 3.17. Since $u = c\mathbb{P}^\Omega[\delta_0] - \mathbb{G}^\Omega[|\nabla u|^q]$,

$$\lim_{|x| \rightarrow 0} \frac{u(x)}{P^\Omega(x, 0)} = c. \quad (3.62)$$

If $c = 0$ then problem admit a unique solution $u = 0$. If $c > 0$, let u and \tilde{u} be two solutions to (3.53). For any $\varepsilon > 0$, set $u_\varepsilon = (1 + \varepsilon)u$ then u_ε is a supersolution. By step 3,

$$\lim_{x \rightarrow 0} \frac{u_\varepsilon(x)}{P^\Omega(x, 0)} = (1 + \varepsilon)c.$$

By the maximum principle, $u_\varepsilon \geq \tilde{u}$ in $\Omega \setminus B_\varepsilon(0)$. Letting $\varepsilon \rightarrow 0$ yields to $u \geq \tilde{u}$ and the uniqueness follows. The monotonicity of $c \mapsto u_{c\delta_0}$ comes from (3.62). □

As a variant of the previous result we have its extension in some unbounded domains.

Theorem 3.20 *Assume $1 < q < q_c$, and either $\Omega = \mathbb{R}_+^N := \{x = (x', x_N) : x_N > 0\}$ or $\partial\Omega$ is compact with $0 \in \partial\Omega$. Then there exists one and only one solution to problem (3.53).*

Proof. The proof needs only minor modifications in order to take into account the decay of the solutions at ∞ . For $R > 0$ we set $\Omega_R = \Omega \cap B_R$ and denote by $u := u_{c\delta_0}^R$ the unique solution of

$$\begin{cases} -\Delta u + |\nabla u|^q = 0 & \text{in } \Omega_R \\ u = c\delta_0 & \text{on } \partial\Omega_R. \end{cases} \quad (3.63)$$

Then

$$u_{c\delta_0}^R(x) \leq cP^{\Omega_R}(x, 0) \quad \forall x \in \Omega_R. \quad (3.64)$$

Since $R \mapsto P^{\Omega_R}(\cdot, 0)$ is increasing, it follows from (3.62) that $R \mapsto u_{c\delta_0}^R$ is increasing too with limit u^* and there holds

$$u^*(x) \leq cP^\Omega(x, 0) \quad \forall x \in \Omega. \quad (3.65)$$

Estimate (3.54) is valid independently of R since the curvature of $\partial\Omega_R$ is bounded (or zero if $\Omega = \mathbb{R}_+^N$). By standard local regularity theory, $\nabla u_{c\delta_0}^R$ converges locally uniformly in $\bar{\Omega} \setminus B_\epsilon(0)$ for any $\epsilon > 0$ when $R \rightarrow \infty$, and thus $u^* \in C(\bar{\Omega} \setminus \{0\})$ is a positive solution of (1.2) in Ω which vanishes on $\partial\Omega \setminus \{0\}$. It admits therefore a boundary trace $tr_{\partial\Omega}(u^*)$. Estimate (3.65) implies that $\mathcal{S}(u^*) = \emptyset$ and $\mu(u^*)$ is a Dirac measure at 0, which is in fact $c\delta_0$ by combining estimates (3.62) for Ω_R , (3.64) and (3.65). Uniqueness follows from the same estimate. \square

We next consider the equation (1.2) in \mathbb{R}_+^N . We denote by $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$ are the spherical coordinates in \mathbb{R}^N and we recall the following representation

$$S_+^{N-1} = \left\{ (\sin \phi \sigma', \cos \phi) : \sigma' \in S^{N-2}, \phi \in [0, \frac{\pi}{2}) \right\},$$

$$\Delta v = v_{rr} + \frac{N-1}{r} v_r + \frac{1}{r^2} \Delta' v$$

where Δ' is the Laplace-Beltrami operator on S^{N-1} ,

$$\nabla v = v_r \mathbf{e} + \frac{1}{r} \nabla' v$$

where ∇' denotes the covariant derivative on S^{N-1} identified with the tangential derivative,

$$\Delta' v = \frac{1}{(\sin \phi)^{N-2}} ((\sin \phi)^{N-2} v_\phi)_\phi + \frac{1}{(\sin \phi)^2} \Delta'' v$$

where Δ'' is the Laplace-Beltrami operator on S^{N-2} . Notice that the function $\varphi_1(\sigma) = \cos \phi$ is the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$, with corresponding eigenvalue $\lambda_1 = N-1$ and we choose $\theta > 0$ such that $\tilde{\varphi}_1(\sigma) := \theta \cos \phi$ has mass 1 on S_+^{N-1} .

We look for a particular solution of

$$\begin{cases} -\Delta u + |\nabla u|^q = 0 & \text{in } \mathbb{R}_+^N \\ u = 0 & \text{on } \partial\mathbb{R}_+^N \setminus \{0\} = \mathbb{R}^{N-1} \setminus \{0\} \end{cases} \quad (3.66)$$

under the separable form

$$u(r, \sigma) = r^{-\beta} \omega(\sigma) \quad (r, \sigma) \in (0, \infty) \times S_+^{N-1}. \quad (3.67)$$

It follows from a straightforward computation that $\beta = \frac{2-q}{q-1}$ and ω satisfies

$$\begin{cases} \mathcal{L}\omega := -\Delta'\omega + \left(\left(\frac{2-q}{q-1} \right)^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{q}{2}} - \frac{2-q}{q-1} \left(\frac{q}{q-1} - N \right) \omega = 0 & \text{in } S_+^{N-1} \\ \omega = 0 & \text{on } \partial S_+^{N-1} \end{cases} \quad (3.68)$$

Multiplying (3.68) by φ_1 and integrating over S_+^{N-1} , we get

$$\begin{aligned} & \left[N - 1 - \frac{2-q}{q-1} \left(\frac{q}{q-1} - N \right) \right] \int_{S_+^{N-1}} \omega \varphi_1 dx \\ & + \int_{S_+^{N-1}} \left(\left(\frac{2-q}{q-1} \right)^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{q}{2}} \varphi_1 dx = 0. \end{aligned}$$

Therefore if $N - 1 \geq \frac{2-q}{q-1} \left(\frac{q}{q-1} - N \right)$ and in particular if $q \geq q_c$, there exists no nontrivial solution of (3.68).

In the next theorem we prove that if $N - 1 < \frac{2-q}{q-1} \left(\frac{q}{q-1} - N \right)$, or equivalently $q < \frac{N+1}{N}$, there exists a unique positive solution of (3.68).

Theorem 3.21 *Assume $1 < q < q_c$. There exists a unique positive solution $\omega_s := \omega \in W^{2,p}(S_+^{N-1})$ to (3.68) for all $p > 1$. Furthermore $\omega_s \in C^\infty(\overline{S_+^{N-1}})$.*

Proof. Step 1: Existence. We first claim that $\underline{\omega} := \gamma_1 \varphi_1^{\gamma_2}$ is a positive sub-solution of (3.68) where γ_i ($i = 1, 2$) will be determined later on. Indeed, we have

$$\begin{aligned} \mathcal{L}(\underline{\omega}) &= \gamma_1 \left[(N-1)\gamma_2 - \frac{2-q}{q-1} \left(\frac{q}{q-1} - N \right) \right] \varphi_1^{\gamma_2} - \gamma_1 \gamma_2 (\gamma_2 - 1) \varphi_1^{\gamma_2-2} |\nabla' \varphi_1|^2 \\ & \quad + \left[\left(\frac{2-q}{q-1} \right)^2 \gamma_1^2 \varphi_1^{2\gamma_2} + \gamma_1^2 \gamma_2^2 \varphi_1^{2(\gamma_2-1)} |\nabla' \varphi_1|^2 \right]^{\frac{q}{2}} \\ & \leq \gamma_1 \varphi_1^{\gamma_2} \left[(N-1)\gamma_2 - \frac{2-q}{q-1} \left(\frac{q}{q-1} - N \right) + 2 \left(\frac{2-q}{q-1} \right)^q \gamma_1^{q-1} \varphi_1^{(q-1)\gamma_2} \right] \\ & \quad - \gamma_1 \varphi_1^{\gamma_2-2} \left[\left(\frac{2-q}{q-1} \right)^q \gamma_1^{q-1} \varphi_1^{(q-1)\gamma_2+2} + \gamma_2 (\gamma_2 - 1) |\nabla' \varphi_1|^2 \right] \\ & \quad + \gamma_1^q \gamma_2^q \varphi_1^{q(\gamma_2-1)} |\nabla' \varphi_1|^q \\ & =: \gamma_1 \varphi_1^{\gamma_2} L_1 - \gamma_1 \varphi_1^{\gamma_2-2} L_2 + L_3. \end{aligned}$$

Since $q < q_c$, we can choose

$$1 < \gamma_2 < \frac{(N+q-Nq)(2-q)}{(N-1)(q-1)^2}.$$

Since $\varphi_1 \leq 1$, we can choose $\gamma_1 > 0$ small enough in order that $L_1 < 0$. Next, by Young's inequality, for any $\gamma_3 > 0$, we have

$$\varphi_1^{q(\gamma_2-1)-\gamma_2+2} |\nabla' \varphi_1|^q \leq \gamma_3 \varphi_1^{\frac{2(q(\gamma_2-1)-\gamma_2+2)}{2-q}} + \gamma_3^{\frac{q-2}{q}} |\nabla' \varphi_1|^2.$$

Since $q > 1$,

$$\frac{2(q(\gamma_2-1)-\gamma_2+2)}{2-q} > (q-1)\gamma_2+2,$$

hence

$$\varphi_1^{q(\gamma_2-1)-\gamma_2+2} |\nabla' \varphi_1|^q \leq \gamma_3 \varphi_1^{(q-1)\gamma_2+2} + \gamma_3^{\frac{q-2}{q}} |\nabla' \varphi_1|^2.$$

Therefore, if we choose γ_3 such that

$$(\gamma_1 \gamma_2)^{\frac{q(q-1)}{2-q}} (\gamma_2-1)^{-\frac{q}{2-q}} < \gamma_3 < \left(\frac{2-q}{q-1}\right)^q \gamma_2^{-q}$$

then $-\gamma_1 \varphi_1^{\gamma_2-2} L_2 + L_3 < 0$ and the claim follows.

Next, it is easy to see that $\bar{\omega} = \gamma_4$, with $\gamma_4 > 0$ large enough, is a supersolution of (3.68) and $\bar{\omega} > \underline{\omega}$ in \bar{S}_+^{N-1} . Therefore there exists a solution $\omega \in W^{2,p}(S_+^{N-1})$ to (3.68) such that $0 < \underline{\omega} \leq \omega \leq \bar{\omega}$ in S_+^{N-1} .

Step 2: Uniqueness. Suppose that ω_1 and ω_2 are two positive different solutions of (3.68) and by Hopf lemma $\nabla' \omega_i$ ($i = 1, 2$) does not vanish on S_+^{N-1} . Up to exchanging the role of ω_1 and ω_2 , we may assume $\max_{S_+^{N-1}} \omega_2 \geq \max_{S_+^{N-1}} \omega_1$ and

$$\lambda := \inf\{c > 1 : c\omega_1 > \omega_2 \text{ in } S_+^{N-1}\} > 1.$$

Set $\omega_{1,\lambda} := \lambda\omega_1$, then $\omega_{1,\lambda}$ is a positive supersolution to problem (3.68). Owing to the definition of $\omega_{1,\lambda}$, one of two following cases must occur.

Case 1: Either $\exists \sigma_0 \in S_+^{N-1}$ *such that* $\omega_{1,\lambda}(\sigma_0) = \omega_2(\sigma_0) > 0$ *and* $\nabla' \omega_{1,\lambda}(\sigma_0) = \nabla' \omega_2(\sigma_0)$.

Set $\omega_\lambda := \omega_{1,\lambda} - \omega_2$ then $\omega_\lambda \geq 0$ in \bar{S}_+^{N-1} , $\omega_\lambda(\sigma_0) = 0$, $\nabla' \omega_\lambda(\sigma_0) = 0$. Moreover,

$$-\Delta' \omega_\lambda + (H(\omega_{1,\lambda}, \nabla' \omega_{1,\lambda}) - H(\omega_2, \nabla' \omega_2)) - \frac{2-q}{q-1} \left(\frac{q}{q-1} - N \right) \omega_\lambda \geq 0. \quad (3.69)$$

where $H(s, \xi) = \left(\left(\frac{2-q}{q-1} \right)^2 s^2 + |\xi|^2 \right)^{\frac{q}{2}}$, $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. By the Mean Value theorem and (3.69),

$$-\Delta' \omega_\lambda + \frac{\partial H}{\partial \xi}(\bar{s}, \bar{\xi}) \nabla' \omega_\lambda + \left[\frac{\partial H}{\partial s}(\bar{s}, \bar{\xi}) - \frac{2-q}{q-1} \left(\frac{q}{q-1} - N \right) \right] \omega_\lambda \geq 0 \quad (3.70)$$

where \bar{s} and $\bar{\xi}_i$ are the functions with respect to $\sigma \in S_+^{N-1}$. Since $\omega_{1,\lambda}, \omega_2 \in C^1(\bar{S}_+^{N-1})$, we deduce that $\frac{\partial H}{\partial s}(\bar{s}, \bar{\xi}), \left| \frac{\partial H}{\partial \xi}(\bar{s}, \bar{\xi}) \right| \in L^\infty(\bar{S}_+^{N-1})$. So we may choose $\gamma_5 > 0$ large enough in other that

$$-\Delta' \omega_\lambda + \frac{\partial H}{\partial \xi}(\bar{s}, \bar{\xi}) \nabla' \omega_\lambda + \left[\gamma_5 - \frac{2-q}{q-1} \left(\frac{q}{q-1} - N \right) \right] \omega_\lambda \geq 0.$$

By maximum principle, ω_λ cannot achieve a non-positive minimum in S_+^{N-1} , which is a contradiction.

Case 2: or $\omega_{1,\lambda} > \omega_2$ in S_+^{N-1} and $\exists \sigma_0 \in \partial S_+^{N-1}$ such that

$$\frac{\partial \omega_{1,\lambda}}{\partial \mathbf{n}}(\sigma_0) = \frac{\partial \omega_2}{\partial \mathbf{n}}(\sigma_0). \quad (3.71)$$

Since $\omega_{1,\lambda}(\sigma_0) = 0$ and $\omega_{1,\lambda} \in C^1(\overline{S_+^{N-1}})$, there exists a relatively open subset $U \subset S_+^{N-1}$ such that $\sigma_0 \in \partial U$ and

$$\max_{\overline{U}} \omega_{1,\lambda} < q^{-\frac{1}{q-1}} \frac{q-1}{2-q} \left(\frac{q}{q-1} - N \right)^{\frac{1}{q-1}}. \quad (3.72)$$

We set $\omega_\lambda := \omega_{1,\lambda} - \omega_2$ as in case 1. It follows from (3.70) that

$$\begin{aligned} -\Delta' \omega_\lambda + \frac{\partial H}{\partial \xi}(\bar{s}, \bar{\xi}) \partial_{\sigma_i} \omega_\lambda &\geq \left[\frac{2-q}{q-1} \left(\frac{q}{q-1} - N \right) - \frac{\partial H}{\partial s}(\bar{s}, \bar{\omega}) \right] \omega_\lambda \\ &> \frac{2-q}{q-1} \left[\frac{q}{q-1} - N - q \left(\frac{2-q}{q-1} \right)^{q-1} \right] \omega_{1,\lambda}^{q-1} \omega_\lambda > 0 \end{aligned} \quad (3.73)$$

in U owing to (3.72). By Hopf's lemma $\frac{\partial \omega_\lambda}{\partial \mathbf{n}}(\sigma_0) < 0$, which contradicts (3.71). The regularity comes from the fact that $\omega^2 + |\nabla \omega|^2 > 0$ in $\overline{S_+^{N-1}}$. \square

When \mathbb{R}_+^N is replaced by a general C^2 bounded domain Ω , the role of ω_s is crucial for describing the boundary isolated singularities. In that case we assume that $0 \in \partial \Omega$ and the tangent plane to $\partial \Omega$ at 0 is $\partial \mathbb{R}_+^{N-1} := \{(x', 0) : x' \in \mathbb{R}^{N-1}\}$, with normal inward unit vector \mathbf{e}_N .

Proposition 3.22 *Assume $1 < q < q_c$ and $0 \in \partial \Omega$. Then*

$$\lim_{c \rightarrow \infty} u_{c\delta_0} = u_{\infty,0} \quad (3.74)$$

where $u_{\infty,0}$ is a positive solution of (3.53) with singularity at $a = 0$. Furthermore there holds

$$\lim_{\substack{\Omega \ni x \rightarrow 0 \\ \frac{x}{|x|} \rightarrow \sigma \in S_+^{N-1}}} |x|^{\frac{2-q}{q-1}} u_{\infty,0}(x) = \omega_s\left(\frac{x}{|x|}\right), \quad (3.75)$$

locally uniformly on S_+^{N-1} .

Proof. If u is the solution of a problem (3.53) in a domain Θ with boundary data $c\delta_a$, we denote it by $u_{c\delta_a}^\Theta$. Let B and B' be two open balls tangent to $\partial \Omega$ at 0 and such that $B \subset \Omega \subset B'^c$. Since $P^B(x, 0) \leq P^\Omega(x, 0) \leq P^{B'^c}(x, 0)$ it follows from Theorem 3.20 and (3.62) that

$$u_{c\delta_0}^B \leq u_{c\delta_0}^\Omega \leq u_{c\delta_0}^{B'^c}. \quad (3.76)$$

Because of uniqueness and whether Θ is B , Ω or B'^c , we have

$$T_\ell[u_{c\delta_0}^\Theta] = u_{c\ell^\theta\delta_0}^{\Theta^\ell} \quad \forall \ell > 0, \quad (3.77)$$

with $\Theta^\ell = \frac{1}{\ell}\Theta$ and $\theta := \frac{2-q}{q-1} + 1 - N$. Notice also that $c \mapsto u_{c\delta_0}^\Theta$ is increasing. Since $u_{c\delta_0}^\Theta(x) \leq c(q)|x|^{\frac{q-2}{q-1}}$ by (3.6), it follows that $u_{c\delta_0}^\Theta \uparrow u_{\infty,0}^\Theta$. As in the previous constructions, $u_{\infty,0}^\Theta$ is a positive solution of (1.2) in Θ , continuous in $\overline{\Theta} \setminus \{0\}$ and which vanishes on $\partial\Theta \setminus \{0\}$.

Step 1: $\Theta := \mathbb{R}_+^N$. Then $\Theta^\ell = \mathbb{R}^N$. Letting $c \rightarrow \infty$ in (3.77) yields to

$$T_\ell[u_{\infty,0}^{\mathbb{R}_+^N}] = u_{\infty,0}^{\mathbb{R}_+^N} \quad \forall \ell > 0. \quad (3.78)$$

Therefore $u_{\infty,0}^{\mathbb{R}_+^N}$ is self-similar and thus under the separable form (3.67). By Theorem 3.21,

$$u_{\infty,0}^{\mathbb{R}_+^N}(x) = |x|^{\frac{q-2}{q-1}} \omega_s\left(\frac{x}{|x|}\right). \quad (3.79)$$

Step 2: $\Theta := B$ or B'^c . In accordance with our previous notations, we set $B^\ell = \frac{1}{\ell}B$ and $B'^{c\ell} = \frac{1}{\ell}B'^c$ for any $\ell > 0$ and we have,

$$T_\ell[u_{\infty,0}^B] = u_{\infty,0}^{B^\ell} \quad \text{and} \quad T_\ell[u_{\infty,0}^{B'^c}] = u_{\infty,0}^{B'^{c\ell}} \quad (3.80)$$

and

$$u_{\infty,0}^{B^{\ell'}} \leq u_{\infty,0}^{B^\ell} \leq u_{\infty,0}^{\mathbb{R}_+^N} \leq u_{\infty,0}^{B'^{c\ell}} \leq u_{\infty,0}^{B'^{c\ell''}} \quad \forall 0 < \ell \leq \ell', \ell'' \leq 1. \quad (3.81)$$

When $\ell \rightarrow 0$ $u_{\infty,0}^{B^\ell} \uparrow \underline{u}_{\infty,0}^{\mathbb{R}_+^N}$ and $u_{\infty,0}^{B'^{c\ell}} \downarrow \overline{u}_{\infty,0}^{\mathbb{R}_+^N}$ where $\underline{u}_{\infty,0}^{\mathbb{R}_+^N}$ and $\overline{u}_{\infty,0}^{\mathbb{R}_+^N}$ are positive solutions of (1.2) in \mathbb{R}_+^N such that

$$u_{\infty,0}^{B^\ell} \leq \underline{u}_{\infty,0}^{\mathbb{R}_+^N} \leq u_{\infty,0}^{\mathbb{R}_+^N} \leq \overline{u}_{\infty,0}^{\mathbb{R}_+^N} \leq u_{\infty,0}^{B'^{c\ell}} \quad \forall 0 < \ell \leq 1. \quad (3.82)$$

This combined with the monotonicity of $u_{\infty,0}^{B^\ell}$ and $u_{\infty,0}^{B'^{c\ell}}$ implies that $\underline{u}_{\infty,0}^{\mathbb{R}_+^N}$ and $\overline{u}_{\infty,0}^{\mathbb{R}_+^N}$ vanishes on $\partial\mathbb{R}_+^N \setminus \{0\}$ and are continuous in $\overline{\mathbb{R}_+^N} \setminus \{0\}$. Furthermore there also holds for $\ell, \ell' > 0$,

$$T_{\ell'\ell}[u_{\infty,0}^B] = T_{\ell'}[T_\ell[u_{\infty,0}^B]] = u_{\infty,0}^{B^{\ell\ell'}} \quad \text{and} \quad T_{\ell'\ell}[u_{\infty,0}^{B'^c}] = T_{\ell'}[T_\ell[u_{\infty,0}^{B'^c}]] = u_{\infty,0}^{B'^{c\ell\ell'}}. \quad (3.83)$$

Letting $\ell \rightarrow 0$ and using (3.81) and the above convergence, we obtain

$$\underline{u}_{\infty,0}^{\mathbb{R}_+^N} = T_{\ell'}[\underline{u}_{\infty,0}^{\mathbb{R}_+^N}] \quad \text{and} \quad \overline{u}_{\infty,0}^{\mathbb{R}_+^N} = T_{\ell'}[\overline{u}_{\infty,0}^{\mathbb{R}_+^N}]. \quad (3.84)$$

Again this implies that $\underline{u}_{\infty,0}^{\mathbb{R}_+^N}$ and $\overline{u}_{\infty,0}^{\mathbb{R}_+^N}$ are separable solutions of (1.2) in \mathbb{R}_+^N vanishing on $\overline{\mathbb{R}_+^N} \setminus \{0\}$. Therefore they coincide with $u_{\infty,0}^{\mathbb{R}_+^N}(x)$.

Step 3: End of the proof. From (3.76) and (3.81) there holds

$$u_{\infty,0}^{B^\ell} \leq T_\ell[u_{\infty,0}^\Omega] \leq u_{\infty,0}^{B'^{c\ell}} \quad \forall 0 < \ell \leq 1. \quad (3.85)$$

Since the left-hand side and the right-hand side of (3.85) converge to the same function $u_{\infty,0}^{\mathbb{R}^N}(x)$, we obtain

$$\lim_{\ell \rightarrow 0} \ell^{\frac{2-q}{q-1}} u_{\infty,0}^{\Omega}(\ell x) = |x|^{\frac{q-2}{q-1}} \omega_s\left(\frac{x}{|x|}\right) \quad (3.86)$$

and this convergence holds in any compact subset of Ω . If we fix $|x| = 1$, we derive (3.75). \square

Remark. It is possible to improve the convergence in (3.75) by straightening $\partial\Omega$ near 0 (and thus to replace $u_{\infty,0}^{\Omega}$ by a function $\tilde{u}_{\infty,0}^{\Omega}$ defined in $B_{\epsilon} \cap \mathbb{R}_+^N$) and to obtain a convergence in $C^1(\overline{S_+^{N-1}})$.

Combining this result with Theorem 2.11 we derive

Corollary 3.23 *Assume $1 < q < q_c$ and $0 \in \partial\Omega$. If u is a positive solution of (1.2) with boundary trace $tr_{\partial\Omega}(u) = (\mathcal{S}(u), \mu(u)) = (\{0\}, 0)$ then $u \geq u_{\infty,0}^{\Omega}$.*

The next result asserts the existence of a maximal solution with boundary trace $(\{0\}, 0)$.

Proposition 3.24 *Assume $1 < q < q_c$ and $0 \in \partial\Omega$. Then there exists a maximal solution $U := U_{\infty,0}^{\Omega}$ of (1.2) with boundary trace $tr_{\partial\Omega}(U) = (\mathcal{S}(U), \mu(U)) = (\{0\}, 0)$. Furthermore*

$$\lim_{\substack{\Omega \ni x \rightarrow 0 \\ \frac{x}{|x|} \rightarrow \sigma \in S_+^{N-1}}} |x|^{\frac{2-q}{q-1}} U_{\infty,0}^{\Omega}(x) = \omega_s\left(\frac{x}{|x|}\right), \quad (3.87)$$

locally uniformly on S_+^{N-1} .

Proof. Step 1: Existence. Since $1 < q < q_c < \frac{N}{N-1}$, there exists a radial separable singular solution of (1.2) in $\mathbb{R}^N \setminus \{0\}$,

$$U_S(x) = \Lambda_{N,q} |x|^{\frac{q-2}{q-1}} \quad \text{with} \quad \Lambda_{N,q} = \left(\frac{q-1}{2-q}\right)^{q'} \left(\frac{(2-q)(N-(N-1)q)}{(q-1)^2}\right)^{\frac{1}{q-1}}. \quad (3.88)$$

By Lemma 3.3 there exists $c(q) > 0$ such that any positive solution u of (1.2) in Ω which vanishes on $\partial\Omega \setminus \{0\}$ satisfies $u(x) \leq c(q)|x|^{\frac{q-2}{q-1}}$ in Ω . Therefore, there $\Lambda^* := \Lambda^*(N, q) \geq \max\{\Lambda_N, q\}, c(q)$ such that $U^*(x) = \Lambda^* |x|^{\frac{q-2}{q-1}}$ is a supersolution of (1.2) in $\mathbb{R}^N \setminus \{0\}$ and dominates in Ω any solution u vanishing on $\partial\Omega \setminus \{0\}$. For $0 < \epsilon < \max\{|z| : z \in \Omega\}$, we denote by u_{ϵ} the solution of

$$\begin{cases} -\Delta u_{\epsilon} + |\nabla u_{\epsilon}|^q = 0 & \text{in } \Omega \setminus B_{\epsilon}(0) \\ u_{\epsilon} = 0 & \text{in } \partial\Omega \setminus B_{\epsilon}(0) \\ u_{\epsilon} = \Lambda^* \epsilon^{\frac{q-2}{q-1}} & \text{in } \Omega \cap \partial B_{\epsilon}(0). \end{cases} \quad (3.89)$$

If $\epsilon' < \epsilon$, $u_{\epsilon'}|_{\partial(\Omega \setminus B_{\epsilon'})} \leq u_{\epsilon}|_{\partial(\Omega \setminus B_{\epsilon})}$, therefore

$$u \leq u_{\epsilon'} \leq u_{\epsilon} \leq U^*(x) \quad \text{in } \Omega. \quad (3.90)$$

Letting ϵ to zero, u_{ϵ} decreases and converges to some $U_{\infty,0}^{\Omega}$ which vanishes on $\partial\Omega \setminus \{0\}$. By the regularity estimates already used in stability results, the convergence occurs in $C_{loc}^1(\overline{\Omega} \setminus \{0\})$.

$\{0\}$), $U_{\infty,0}^\Omega \in C(\overline{\Omega} \setminus \{0\})$ is a positive solution of (1.2) and it belongs to $C^2(\Omega)$; furthermore it has boundary trace $(\{0\}, 0)$ and for any positive solution u satisfying $tr_{\partial\Omega}(u) = (\{0\}, 0)$ there holds

$$u_{\infty,0}^\Omega \leq u \leq U_{\infty,0}^\Omega \leq U^*(x). \quad (3.91)$$

Therefore $U_{\infty,0}^\Omega$ is the maximal solution.

Step 2: $\Omega = \mathbb{R}_+^N$. Since

$$T_\ell[U^*]|_{|x|=\epsilon} = U^*|_{|x|=\epsilon} \quad \forall \ell > 0, \quad (3.92)$$

there holds

$$T_\ell[u_\epsilon] = u_{\frac{\epsilon}{\ell}} \quad (3.93)$$

Letting $\epsilon \rightarrow 0$ yields to $T_\ell[U_{\infty,0}^{\mathbb{R}_+^N}] = U_{\infty,0}^{\mathbb{R}_+^N}$. Therefore $U_{\infty,0}^{\mathbb{R}_+^N}$ is self-similar and coincide with $u_{\infty,0}^{\mathbb{R}_+^N}$.

Step 3: $\Omega = B$ or B^c . We first notice that the maximal solution is an increasing function of the domain. Since $T_\ell[u_\epsilon^\Theta] = u_{\frac{\epsilon}{\ell}}^\Theta$ where we denote by u_ϵ^Θ the solution of (3.89) in $\Theta \setminus B_\epsilon$ for any $\ell, \epsilon > 0$ and any domain Θ (with $0 \in \partial\Theta$), we derive as in Proposition 3.22-Step 2, using (3.93) and uniqueness,

$$T_\ell[U_{\infty,0}^B] = U_{\infty,0}^{B^\ell} \quad \text{and} \quad T_\ell[U_{\infty,0}^{B^c}] = U_{\infty,0}^{B'^c \ell} \quad (3.94)$$

and

$$U_{\infty,0}^{B^{\ell'}} \leq U_{\infty,0}^{B^\ell} \leq u_{\infty,0}^{\mathbb{R}_+^N} \leq U_{\infty,0}^{B'^c \ell} \leq U_{\infty,0}^{B'^c \ell''} \quad \forall 0 < \ell \leq \ell', \ell'' \leq 1. \quad (3.95)$$

As in Proposition 3.22, $U_{\infty,0}^{B^\ell} \uparrow \underline{U}_{\infty,0}^{\mathbb{R}_+^N} \leq U_{\infty,0}^{\mathbb{R}_+^N}$ and $U_{\infty,0}^{B'^c \ell} \downarrow \overline{U}_{\infty,0}^{\mathbb{R}_+^N} \geq U_{\infty,0}^{\mathbb{R}_+^N}$ where $\underline{U}_{\infty,0}^{\mathbb{R}_+^N}$ and $\overline{U}_{\infty,0}^{\mathbb{R}_+^N}$ are positive solutions of (1.2) in \mathbb{R}^N which vanishes on $\partial\mathbb{R}_+^N \setminus \{0\}$ and endow the same scaling invariance under T_ℓ . Therefore they coincide with $u_{\infty,0}^{\mathbb{R}_+^N}$.

Step 3: End of the proof. It is similar to the one of Proposition 3.22. \square

Combining Proposition 3.22 and Proposition 3.24 we can prove the final result

Theorem 3.25 *Assume $1 < q < q_c$ and $0 \in \partial\Omega$. Then $U_{\infty,0}^\Omega = u_{\infty,0}^\Omega$.*

Proof. We follow the method used in [15, Sec 4].

Step 1: Straightening the boundary. We represent $\partial\Omega$ near 0 as the graph of a C^2 function ϕ defined in $\mathbb{R}^{N-1} \cap B_R$ and such that $\phi(0) = 0, \nabla\phi(0) = 0$ and

$$\partial\Omega \cap B_R = \{x = (x', x_N) : x' \in \mathbb{R}^{N-1} \cap B_R, x_N = \phi(x')\}.$$

We introduce the new variable $y = \Phi(x)$ with $y' = x'$ and $y_N = x_N - \phi(x')$, with corresponding spherical coordinates in \mathbb{R}^N , $(r, \sigma) = (|y|, \frac{y}{|y|})$. If u is a positive solution of (1.2) in Ω vanishing on $\partial\Omega$, we set $\tilde{u}(y) = u(x)$, then a technical computation shows that \tilde{u} satisfies

with $\mathbf{n} = \frac{y}{|y|}$

$$\begin{aligned}
& r^2 \tilde{u}_{rr} \left(1 - 2\phi_r \langle \mathbf{n}, \mathbf{e}_N \rangle + |\nabla \phi|^2 \langle \mathbf{n}, \mathbf{e}_N \rangle^2 \right) \\
& + r \tilde{u}_r \left(N - 1 - r \langle \mathbf{n}, \mathbf{e}_N \rangle \Delta \phi - 2 \langle \nabla' \langle \mathbf{n}, \mathbf{e}_N \rangle, \nabla' \phi \rangle + r |\nabla \phi|^2 \langle \nabla' \langle \mathbf{n}, \mathbf{e}_N \rangle, \mathbf{e}_N \rangle \right) \\
& + \langle \nabla' \tilde{u}, \mathbf{e}_N \rangle \left(2\phi_r - |\nabla \phi|^2 \langle \mathbf{n}, \mathbf{e}_N \rangle - r \Delta \phi \right) \\
& + r \langle \nabla' \tilde{u}_r, \mathbf{e}_N \rangle \left(2 \langle \mathbf{n}, \mathbf{e}_N \rangle |\nabla \phi|^2 - 2\phi_r \right) - 2 \langle \nabla' \tilde{u}_r, \nabla' \phi \rangle \langle \mathbf{n}, \mathbf{e}_N \rangle \\
& + |\nabla \phi|^2 \langle \nabla' \langle \nabla' \tilde{u}, \mathbf{e}_N \rangle, \mathbf{e}_N \rangle - \frac{2}{r} \langle \nabla' \langle \nabla' \tilde{u}, \mathbf{e}_N \rangle, \nabla' \phi \rangle + \Delta' \tilde{u} \\
& + r^2 \left| \tilde{u}_r \mathbf{n} + \frac{1}{r} \nabla' \tilde{u} - (\phi_r \mathbf{n} + \frac{1}{r} \nabla' \phi) \langle \tilde{u}_r \mathbf{n} + \frac{1}{r} \nabla' \tilde{u}, \mathbf{e}_N \rangle \right|^q = 0.
\end{aligned} \tag{3.96}$$

Using the transformation $t = \ln r$ for $t \leq 0$ and $\tilde{u}(r, \sigma) = r^{\frac{q-2}{q-1}} v(t, \sigma)$, we obtain finally that v satisfies

$$\begin{aligned}
& (1 + \epsilon_1) v_{tt} + \left(N - \frac{2}{q-1} + \epsilon_2 \right) v_t + (\lambda_{N,q} + \epsilon_3) v + \Delta' v \\
& + \langle \nabla' v, \vec{\epsilon}_4 \rangle + \langle \nabla' v_t, \vec{\epsilon}_5 \rangle + \langle \nabla' \langle \nabla' v, \mathbf{e}_N \rangle, \vec{\epsilon}_6 \rangle \\
& - \left| \left(\frac{q-2}{q-1} v + v_t \right) \mathbf{n} + \nabla' \tilde{v} + \left\langle \left(\frac{q-2}{q-1} v + v_t \right) \mathbf{n} + \nabla' \tilde{v}, \mathbf{e}_N \right\rangle \vec{\epsilon}_7 \right|^q = 0,
\end{aligned} \tag{3.97}$$

on $(-\infty, \ln R] \times S_+^{N-1} := Q_R$ and vanishes on $(-\infty, \ln R] \times \partial S_+^{N-1}$, where

$$\lambda_{N,q} = \left(\frac{2-q}{q-1} \right) \left(\frac{q}{q-1} - N \right).$$

Furthermore the ϵ_j are uniformly continuous functions of t and $\sigma \in S^{N-1}$ for $j = 1, \dots, 7$, C^1 for $j = 1, 5, 6, 7$ and satisfy the following decay estimates

$$|\epsilon_j(t, \cdot)| \leq C e^t \quad \text{for } j = 1, \dots, 7 \quad \text{and} \quad |\epsilon_{jt}(t, \cdot)| + |\nabla' \epsilon_j| \leq c_{17} e^t \quad \text{for } j = 1, 5, 6, 7. \tag{3.98}$$

Since v , v_t and $\nabla' v$ are uniformly bounded and by standard regularity methods of elliptic equations [15, Lemma 4.4], there exist a constant $C > 0$ and $T < \ln R$ such that

$$\|v(t, \cdot)\|_{C^{2,\gamma}(\overline{S_+^{N-1}})} + \|v_t(t, \cdot)\|_{C^{1,\gamma}(\overline{S_+^{N-1}})} + \|v_{tt}(t, \cdot)\|_{C^{0,\gamma}(\overline{S_+^{N-1}})} \leq c'_{17} \tag{3.99}$$

for any $\gamma \in (0, 1)$ and $t \leq T - 1$. Consequently the set of functions $\{v(t, \cdot)\}_{t \leq 0}$ is relatively compact in the $C^2(\overline{S_+^{N-1}})$ topology and there exist η and a subsequence $\{t_n\}$ tending to $-\infty$ such that $v(t_n, \cdot) \rightarrow \eta$ when $n \rightarrow \infty$ in $C^2(\overline{S_+^{N-1}})$.

Step 2: End of the proof. Taking $u = u_{\infty,0}^\Omega$ or $u = U_{\infty,0}^\Omega$, with corresponding v , we already know that $v(t, \cdot)$ converges to ω_s , locally uniformly on S_+^{N-1} . Thus ω_s is the unique element in the limit set of $\{v(t, \cdot)\}_{t \leq 0}$ and $\lim_{t \rightarrow -\infty} v(t, \cdot) = \omega_s$ in $C^2(\overline{S_+^{N-1}})$. This implies in particular

$$\lim_{x \rightarrow 0} \frac{v_{\infty,0}^\Omega(x)}{U_{\infty,0}^\Omega(x)} = 1 \tag{3.100}$$

and uniqueness follows from maximum principle. \square

As a consequence we have a full characterization of positive solution with an isolated boundary singularity

Corollary 3.26 *Assume $1 < q < q_c$, $0 \in \partial\Omega$ and $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ is a positive solution of (1.2) vanishing on $\partial\Omega \setminus \{0\}$. Then either there exists $c \geq 0$ such that $u = u_{c\delta_0}$, or $u = u_{\infty,0}^\Omega = \lim_{c \rightarrow \infty} u_{c\delta_0}$.*

4 The supercritical case

In this section we consider the case $q \geq q_c$

4.1 Removable isolated singularities

Theorem 4.1 *Assume $2 > q \geq q_c$, then $u \equiv 0$.*

Proof. Step 1: Integral estimates. We consider a sequence of functions $\zeta_n \in C^\infty(\mathbb{R}^N)$ such that $\zeta_n(x) = 0$ if $|x| \leq \frac{1}{n}$, $\zeta_n(x) = 1$ if $|x| \geq \frac{2}{n}$, $0 \leq \zeta_n \leq 1$ and $|\nabla\zeta_n| \leq c_{18}n$, $|\Delta\zeta_n| \leq c_{18}n^2$ where c_{18} is independent of n . As a test function we take $\xi\zeta_n$ (where ξ is the solution to (2.14)) and we obtain

$$\int_{\Omega} (|\nabla u|^q \xi \zeta_n - u \zeta_n \Delta \xi) dx = \int_{\Omega} u (\xi \Delta \zeta_n + 2 \nabla \xi \cdot \nabla \zeta_n) dx = I + II. \quad (4.1)$$

Set $\Omega_n = \Omega \cap \{x : \frac{1}{n} < |x| \leq \frac{2}{n}\}$, then $|\Omega_n| \leq c'_{18}(N)n^{-N}$, thus

$$I \leq c_{18}C_4(q) \int_{\Omega_n} n^{\frac{2-q}{q-1}+2} \xi dx \leq c''_{18}n^{\frac{2-q}{q-1}+2-1-N} = c''_{18}n^{\frac{1}{q-1}-\frac{1}{q_c-1}}$$

since $\xi(x) \leq c_3\rho(x)$. Notice that $\frac{1}{q-1} - \frac{1}{q_c-1} \leq 0$.

$$II \leq c_{18}C_4(q) \int_{\Omega_n} n^{\frac{2-q}{q-1}+1} |\nabla \xi| dx \leq c_{19}n^{\frac{2-q}{q-1}+1-N} = c_{19}n^{\frac{1}{q-1}-\frac{1}{q_c-1}}.$$

Since the right-hand side of (4.1) remains uniformly bounded, it follows from monotone convergence theorem that

$$\int_{\Omega} (|\nabla u|^q \xi + u) dx < \infty. \quad (4.2)$$

More precisely, if $q > q_c$, $I + II$ goes to 0 as $n \rightarrow \infty$ which implies

$$\int_{\Omega} (|\nabla u|^q \xi + u) dx = 0.$$

Next we assume $q = q_c$. Since $|\nabla u| \in L^q_\rho(\Omega)$, $v := \mathbb{G}^\Omega[|\nabla u|^q] \in L^1(\Omega)$. Furthermore, $u + v$ is positive and harmonic in Ω . Its boundary trace is a Radon measure and since the boundary

trace $Tr(v)$ of v is zero, there exists $c \geq 0$ such that $Tr(u) = c\delta_0$. Equivalently, u solves the problem

$$\begin{cases} -\Delta u + |\nabla u|^{q_c} = 0 & \text{in } \Omega \\ u = c\delta_0 & \text{in } \partial\Omega. \end{cases} \quad (4.3)$$

Furthermore, since $u \in L^1(\Omega)$, $u(x) \leq cP(x, \cdot)$ in Ω . Therefore, if $c = 0$, so is u . Let us assume that $c > 0$.

Step 2: The flat case. Assume $\Omega = B_1^+ := B_1 \cap \mathbb{R}_+^N := \{x = (x', x_N) : x_N > 0\}$, where \cdot . We use the spherical coordinates $(r, \sigma) \in [0, \infty) \times S^{N-1}$ as above. Put

$$\bar{f} = \int_{S_+^{N-1}} f \tilde{\varphi}_1 dS$$

then

$$\bar{u}_{rr} + \frac{N-1}{r} \bar{u}_r - \frac{N-1}{r^2} \bar{u} = \overline{|\nabla u|^{q_c}} \quad (4.4)$$

Set $v(r) = r^{N-1} \bar{u}(r)$, then

$$v_{rr} + \frac{1-N}{r} v_r = r^{N-1} \overline{|\nabla u|^{q_c}}. \quad (4.5)$$

and

$$v_r(r) = r^{N-1} v_r(1) - r^{N-1} \int_r^1 \overline{|\nabla u|^{q_c}}(s) ds. \quad (4.6)$$

Since

$$\int_0^1 r^{N-1} \int_r^1 \overline{|\nabla u|^{q_c}}(s) ds = \frac{1}{N} \int_0^1 r^N \overline{|\nabla u|^{q_c}}(s) ds < \infty \quad (4.7)$$

it follows that there exists $\lim_{r \rightarrow 0} v(r) = \alpha \in [0, c]$. Let us assume that $0 < \alpha$. From (4.5),

$$(r^{1-N} v_r)_r = \overline{|\nabla u|^{q_c}} > 0$$

then

$$r_1^{1-N} v_r(r_1) = r_2^{1-N} v_r(r_2) + \int_{r_2}^{r_1} \overline{|\nabla u|^{q_c}} ds \quad \forall 0 < r_2 < r_1. \quad (4.8)$$

This implies that $v_r(r)$ keeps a constant sign on $(0, r_1)$ for some $r_1 > 0$. If $v_r < 0$, then

$$\bar{u}_r = ((1-N)v + rv_r) r^{-N} \implies \overline{|\nabla u|^{q_c}} \geq \left(\frac{(N-1)\alpha}{2} r^{-N} \right)^{q_c} \quad \forall 0 < r < r_2, \quad (4.9)$$

for some $0 < r_2 < r_1$. Which implies $\overline{|\nabla u|^{q_c}} \notin L^1_\rho(B_R^+)$, contradiction. Thus $v_r > 0$. By (4.6)

$$\int_0^1 \overline{|\nabla u|^{q_c}}(s) ds \leq v_r(1).$$

Using again (4.6) it implies $\lim_{r \rightarrow 0} v_r(r) = 0$. Thus (4.9) applies and we get the same contradiction. Therefore $\alpha = 0$, equivalently

$$\lim_{r \rightarrow 0} r^{N-1} \int_{S_+^{N-1}} u(r, \sigma) \tilde{\varphi}_1(\sigma) dS = 0. \quad (4.10)$$

Set $\Sigma := \{\sigma = (\sigma', \phi) \in S_+^{N-1} : 0 \leq \phi \leq \frac{\pi}{4}\}$, then

$$\lim_{r \rightarrow 0} r^{N-1} \int_{\Sigma} u(r, \sigma) dS = 0. \quad (4.11)$$

By Harnack inequality Theorem 3.10, and since $\tilde{\varphi} \leq \gamma$

$$\gamma^{-1} u(r, \tau) \leq \frac{u(r, \tau)}{\tilde{\varphi}_1(\tau)} \leq c_{20} u(r, \sigma) \quad \forall (\tau, \sigma) \in S_+^{N-1} \times \Sigma. \quad (4.12)$$

Integrating over Σ and using (4.11) it follows

$$\lim_{x \rightarrow 0} |x|^N \frac{u(x)}{d(x)} = 0. \quad (4.13)$$

By standard regularity methods, (4.12) can be improved in order to take into account that u vanishes on $\partial \mathbb{R}_+^N \setminus \{0\}$ and we get

$$\lim_{x \rightarrow 0} |x|^N \frac{u(x)}{d(x)} = 0 \iff \lim_{x \rightarrow 0} \frac{u(x)}{P_+^{\mathbb{R}_+^N}(x, 0)} = 0, \quad (4.14)$$

where $P_+^{\mathbb{R}_+^N}(x, 0)$ is the Poisson kernel in \mathbb{R}_+^N with singularity at 0. Since $P_+^{\mathbb{R}_+^N}(\cdot, 0)$ is a super solution and $u = o(P_+^{\mathbb{R}_+^N}(\cdot, 0))$, the maximum principle implies $u = 0$

Step 3: The general case. For $\ell > 0$, we set

$$v_\ell(x) = T_\ell[u](x) = \ell^{N-1} u(\ell x).$$

Then v_ℓ satisfies

$$\begin{cases} -\Delta v_\ell + |\nabla v_\ell|^{q_c} = 0 & \text{in } \Omega^\ell \\ v_\ell = 0 & \text{in } \partial \Omega^\ell \setminus \{0\} \end{cases} \quad (4.15)$$

Furthermore, $T_\ell[P^\Omega] = P^{\Omega^\ell}$ with $P^\Omega := P^{\Omega^1}$ and

$$u(x) \leq c P^\Omega(x, 0) \quad \forall x \in \Omega \implies v_\ell(x) \leq c P^{\Omega^\ell}(x, 0) \quad \forall x \in \Omega^\ell.$$

By standard a priori estimates [20], for any $R > 0$ there exists $M(N, q, R) > 0$ such that, if $\Gamma_R = B_{2R} \setminus B_R$,

$$\begin{aligned} & \sup \{ |v_\ell(x)| + |\nabla v_\ell(x)| : x \in \Gamma_R \cap \Omega^\ell \} \\ & + \sup \left\{ \frac{|\nabla v_\ell(x) - \nabla v_\ell(y)|}{|x - y|^\gamma} : (x, y) \in \Gamma_R \cap \Omega^\ell \right\} \leq M(N, q, R), \end{aligned} \quad (4.16)$$

where $\gamma \in (0, 1)$ is independent of $\ell \in (0, 1]$. Notice that these uniform estimates, up to the boundary, holds because the curvature of $\partial \Omega^\ell$ remains uniformly bounded when $\ell \in (0, 1]$. By compactness, there exist a sequence $\{\ell_n\}$ converging to 0 and function $v \in C^1(\overline{\mathbb{R}_+^N} \setminus \{0\})$ such that

$$\sup \{ |(v_{\ell_n} - v)(x)| + |\nabla(v_{\ell_n} - v)(x)| : x \in \Gamma_R \cap \Omega^{\ell_n} \} \rightarrow 0$$

Furthermore v satisfies

$$\begin{cases} -\Delta v + |\nabla v|^{q_c} = 0 & \text{in } \mathbb{R}_+^N \\ v = 0 & \text{in } \partial\mathbb{R}_+^N \setminus \{0\}. \end{cases} \quad (4.17)$$

From step 2, $v = 0$ and

$$\sup \{|u_\ell(x)| + |\nabla v_\ell(x)| : x \in \Gamma_R \cap \Omega^\ell\} \rightarrow 0;$$

therefore

$$\lim_{x \rightarrow 0} |x|^{N-1} u(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} |x|^N |\nabla u(x)| = 0. \quad (4.18)$$

Integrating from $\partial\Omega$, we obtain

$$\lim_{x \rightarrow 0} \frac{|x|^N}{\rho(x)} u(x) = 0. \quad (4.19)$$

Equivalently $u(x) = o(P^\Omega(x, 0))$ which implies $u = 0$ by the maximum principle. \square

4.2 Removable singularities

The next statement, valid for a positive solution of

$$-\Delta u = f \quad \text{in } \Omega, \quad (4.20)$$

is easy to prove:

Proposition 4.2 *Let u be a positive solution of (1.2). The following assertions are equivalent:*

- (i) u is moderate (Definition 1.8).
- (ii) $u \in L^1(\Omega)$, $|\nabla u| \in L_a^q(\Omega)$.
- (iii) The boundary trace of u is a positive bounded measure μ on $\partial\Omega$.

Let φ be the first eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$ normalized so that $\sup_\Omega \varphi = 1$ and λ be the corresponding eigenvalue. We start with the following simple result.

Lemma 4.3 *Let Ω be a bounded C^2 domain. Then for any $q \geq 1$, $0 \leq \alpha < 1$, $\gamma \in [0, \delta_0)$ and $u \in C^1(\Omega)$, there holds*

$$\begin{aligned} & \int_{\gamma < d(x) < \delta_0} (d(x) - \gamma)^{-\alpha} |u|^q dx \\ & \leq C_{12} \left(\max(|u(z)|^q : d(z) = \delta_0) + \int_{\gamma < d(x) < \delta_0} (d(x) - \gamma)^{q-\alpha} |\nabla u|^q dx \right) \end{aligned} \quad (4.21)$$

where $C_{12} = C_{12}(\alpha, q, \Omega)$. If $1 < q < 2$ and u is a solution of (1.2), we obtain, replacing d by φ ,

$$\int_\Omega \varphi^{1-q} |u|^q dx \leq C_{13} \left(1 + \int_\Omega \varphi |\nabla u|^q dx \right) \quad (4.22)$$

where $C_{13} = C_{13}(q, \Omega)$.

Proof. Without loss of generality, we can assume that u is nonnegative. By the system of flow coordinates introduced in section 2.2, for any $x \in \Omega_{\delta_0}$, we can write $u(x) = u(\delta, \sigma)$ where $\delta = d(x)$, $\sigma = \sigma(x)$ and $x = \sigma - \delta \mathbf{n}_\sigma$, thus

$$u(\delta, \sigma) - u(\delta_0, \sigma) = - \int_{\delta}^{\delta_0} \nabla u(\sigma - s \mathbf{n}_\sigma) \cdot \mathbf{n}_\sigma ds = - \int_{\delta}^{\delta_0} \frac{\partial u}{\partial s}(s, \sigma) ds$$

Set $m_{\delta_0} = \max(|u(z)| : d(x) = \delta_0)$, then

$$u(\delta, \sigma) \leq m_{\delta_0} - \int_{\delta}^{\delta_0} \frac{\partial u}{\partial s}(s, \sigma) ds.$$

Thus, multiplying both sides by $(\delta - \gamma)^{-\alpha}$ and integrating on (γ, δ_0) ,

$$\begin{aligned} & \int_{\gamma}^{\delta_0} (\delta - \gamma)^{-\alpha} u(\delta, \sigma) d\delta \\ & \leq \frac{(\delta_0 - \gamma)^{1-\alpha}}{1-\alpha} m_{\delta_0} + \int_{\gamma}^{\delta_0} (\delta - \gamma)^{-\alpha} \int_{\delta}^{\delta_0} |\nabla u(s, \sigma)| ds d\delta \\ & = \frac{(\delta_0 - \gamma)^{1-\alpha}}{1-\alpha} m_{\delta_0} + \frac{1}{1-\alpha} \int_{\gamma}^{\delta_0} (s - \gamma)^{1-\alpha} |\nabla u(s, \sigma)| ds. \end{aligned} \quad (4.23)$$

Integrating on Σ and using the fact that the mapping is a C^1 diffeomorphism, we get the claim when $q = 1$. If $q > 1$, we apply (4.23) to u^q instead of u and obtain

$$\begin{aligned} & \int_{\gamma}^{\delta_0} (\delta - \gamma)^{-\alpha} u^q(\delta, \sigma) d\delta \\ & \leq \frac{(\delta_0 - \gamma)^{1-\alpha}}{1-\alpha} m_{\delta_0}^q + \frac{q}{1-\alpha} \int_{\gamma}^{\delta_0} (s - \gamma)^{1-\alpha} u^{q-1} |\nabla u(s, \sigma)| ds \\ & \leq \frac{(\delta_0 - \gamma)^{1-\alpha}}{1-\alpha} m_{\delta_0}^q + \frac{q}{1-\alpha} \left(\int_{\gamma}^{\delta_0} (\delta - \gamma)^{-\alpha} u^q ds \right)^{\frac{1}{q}} \left(\int_{\gamma}^{\delta_0} (\delta - \gamma)^{q-\alpha} |\nabla u|^q ds \right)^{\frac{1}{q}}. \end{aligned} \quad (4.24)$$

Since the following implication is true

$$(A \geq 0, B \geq 0, M \geq 0, A^q \leq M^q + A^{q-1}B) \implies (A \leq M + B)$$

we obtain

$$\begin{aligned} & \left(\int_{\gamma}^{\delta_0} (\delta - \gamma)^{-\alpha} u^q(\delta, \sigma) d\delta \right)^{\frac{1}{q}} \\ & \leq \left(\frac{(\delta_0 - \gamma)^{1-\alpha}}{1-\alpha} \right)^{\frac{1}{q}} m_{\delta_0} + \frac{q}{1-\alpha} \left(\int_{\gamma}^{\delta_0} (\delta - \gamma)^{q-\alpha} |\nabla u|^q ds \right)^{\frac{1}{q}}. \end{aligned} \quad (4.25)$$

Inequality (4.21) follows as in the case $q = 1$. We obtain (4.22) with $\gamma = 0$ and using the fact that $c_{21}^{-1}d \leq \varphi \leq c_{21}d$ in Ω with $c_{21} = c_{21}(N)$. \square

Theorem 4.4 Assume $q_c \leq q < 2$. Let $K \subset \partial\Omega$ be compact such that $C_{\frac{2-q}{q}, q'}(K) = 0$. Then any positive moderate solution $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus K)$ of (1.2) such that $|\nabla u| \in L_d^q(\Omega)$ which vanishes on $\partial\Omega \setminus K$ is identically zero.

Proof. Let $\eta \in C^2(\Sigma)$ with value 1 in a neighborhood U_η of K and such that $0 \leq \eta \leq 1$, consider $\zeta = \varphi(\mathbb{P}^\Omega[1 - \eta])^{2q'}$. It is easy to check that ζ is an admissible test function since $\zeta(x) + |\nabla\zeta(x)| = O(d^{2q'+1}(x))$ in any neighborhood of $\{x \in \partial\Omega : \eta(x) = 1\}$. Then

$$\begin{aligned} \int_{\Omega} |\nabla u|^q \zeta dx &= \int_{\Omega} u \Delta \zeta dx \\ &= - \int_{\Omega} \nabla u \cdot \nabla \zeta dx. \end{aligned}$$

Next

$$\nabla \zeta = (\mathbb{P}^\Omega[1 - \eta])^{2q'} \nabla \varphi - 2q' (\mathbb{P}^\Omega[1 - \eta])^{2q'-1} \varphi \nabla \mathbb{P}^\Omega[\eta],$$

thus

$$\begin{aligned} \int_{\Omega} |\nabla u|^q \zeta dx &= - \int_{\Omega} (\mathbb{P}^\Omega[1 - \eta])^{2q'} \nabla \varphi \cdot \nabla u dx + 2q' \int_{\Omega} (\mathbb{P}^\Omega[1 - \eta])^{2q'-1} \varphi \nabla \mathbb{P}^\Omega[\eta] \cdot \nabla u \varphi dx \\ &= \int_{\Omega} u \nabla \cdot ((\mathbb{P}^\Omega[1 - \eta])^{2q'} \nabla \varphi) dx + 2q' \int_{\Omega} (\mathbb{P}^\Omega[1 - \eta])^{2q'-1} \varphi \nabla \mathbb{P}^\Omega[\eta] \cdot \nabla u \varphi dx. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\Omega} (\lambda u + |\nabla u|^q) \zeta dx &= -2q' \int_{\Omega} (\mathbb{P}^\Omega[1 - \eta])^{2q'-1} u \nabla \varphi \cdot \nabla \mathbb{P}^\Omega[\eta] dx + 2q' \int_{\Omega} (\mathbb{P}^\Omega[1 - \eta])^{2q'-1} \varphi \nabla u \cdot \nabla \mathbb{P}^\Omega[\eta] dx. \end{aligned} \quad (4.26)$$

Since $0 \leq \mathbb{P}^\Omega[1 - \eta] \leq 1$, $|\nabla \varphi| \leq c_{22}$ in Ω and by Hölder inequality,

$$\left| \int_{\Omega} (\mathbb{P}^\Omega[1 - \eta])^{2q'-1} u \nabla \varphi \cdot \nabla \mathbb{P}^\Omega[\eta] dx \right| \leq c_{22} \left(\int_{\Omega} \varphi^{1-q} u^q dx \right)^{\frac{1}{q}} \left(\int_{\Omega} \varphi |\nabla \mathbb{P}^\Omega[\eta]|^{q'} dx \right)^{\frac{1}{q'}}. \quad (4.27)$$

Using (4.22) and the fact that $|\nabla u| \in L_d^q(\Omega)$, we obtain

$$\left| \int_{\Omega} (\mathbb{P}^\Omega[1 - \eta])^{2q'-1} u \nabla \varphi \cdot \nabla \mathbb{P}^\Omega[\eta] dx \right| \leq c_{23} \left(1 + \|\nabla u\|_{L_d^q(\Omega)}^q \right)^{\frac{1}{q}} \left(\int_{\Omega} d |\nabla \mathbb{P}^\Omega[\eta]|^{q'} dx \right)^{\frac{1}{q'}}, \quad (4.28)$$

where $c_{23} = c_{23}(N, q, \Omega)$. Using again Hölder inequality, we can estimate the second term on the right-hand side of (4.26) as follows

$$\begin{aligned} \int_{\Omega} (\mathbb{P}^\Omega[1 - \eta])^{2q'-1} \varphi \nabla u \cdot \nabla \mathbb{P}^\Omega[\eta] dx &\leq \left(\int_{\Omega} |\nabla u|^q \varphi dx \right)^{\frac{1}{q}} \left(\int_{\Omega} \varphi |\nabla \mathbb{P}^\Omega[\eta]|^{q'} dx \right)^{\frac{1}{q'}} \\ &\leq c_{21} \|\nabla u\|_{L_d^q(\Omega)}^q \left(\int_{\Omega} d |\nabla \mathbb{P}^\Omega[\eta]|^{q'} dx \right)^{\frac{1}{q'}}. \end{aligned} \quad (4.29)$$

Combining (4.26), (4.28) and (4.29) we derive

$$\int_{\Omega} (|\nabla u|^q + \lambda u) \zeta dx \leq c'_{23} \left(1 + \|\nabla u\|_{L^q_d(\Omega)}^q\right)^{\frac{1}{q}} \left(\int_{\Omega} d|\nabla \mathbb{P}^{\Omega}[\eta]|^{q'} dx\right)^{\frac{1}{q'}}. \quad (4.30)$$

By [34, Proposition 7' and Lemma 4'],

$$\int_{\Omega} d|\nabla \mathbb{P}^{\Omega}[\eta]|^{q'} dx \leq c_{24} \|\eta\|_{W^{1-\frac{2}{q'}, q'}(\Sigma)}^{q'} = c \|\eta\|_{W^{\frac{2-q}{q}, q'}(\Sigma)}^{q'}, \quad (4.31)$$

which implies

$$\int_{\Omega} (|\nabla u|^q + \lambda u) \zeta dx \leq c_{25} \left(1 + \|\nabla u\|_{L^q_d(\Omega)}^q\right)^{\frac{1}{q}} \|\eta\|_{W^{\frac{2-q}{q}, q'}(\Sigma)} \quad (4.32)$$

where $c_{25} = c_{25}(N, q, \Omega)$. Since $C^{\frac{2-q}{q}, q'}(K) = 0$, there exists a sequence of functions $\{\eta_n\}$ in $C^2(\Sigma)$ such that for any n , $0 \leq \eta_n \leq 1$, $\eta_n \equiv 1$ on a neighborhood of K and $\|\eta_n\|_{W^{\frac{2-q}{q}, q'}(\Sigma)} \rightarrow 0$ and $\|\eta_n\|_{L^1(\Sigma)} \rightarrow 0$ as $n \rightarrow \infty$. By letting $n \rightarrow \infty$ in (4.32) with η replaced by η_n , we deduce that $\int_{\Omega} (|\nabla u|^q + \lambda u) \varphi dx = 0$ and the conclusion follows. \square

4.3 Admissible measures

Theorem 4.5 *Assume $q_c \leq q < 2$ and let u be a positive moderate solution of (1.2) with boundary data $\mu \in \mathfrak{M}^+(\partial\Omega)$. Then $\mu(K) = 0$ for any Borel subset $K \subset \partial\Omega$ such that $C^{\frac{2-q}{q}, q'}(K) = 0$.*

Proof. Without loss of generality, we can assume that K is compact. We consider test function η as in the proof of Theorem 4.4, put $\zeta = (\mathbb{P}^{\Omega}[\eta])^{2q'} \varphi$ and get

$$\int_{\Omega} (|\nabla u|^q \zeta - u \Delta \zeta) dx = - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\mu. \quad (4.33)$$

By Hopf lemma and since $\eta \equiv 1$ on K ,

$$- \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\mu \geq c\mu(K).$$

Since

$$-\Delta \zeta = \lambda \zeta + 4q'(\mathbb{P}^{\Omega}[1 - \eta])^{2q'-1} \nabla \varphi \cdot \nabla \mathbb{P}^{\Omega}[\eta] - 2q'(2q' - 1)(\mathbb{P}^{\Omega}[1 - \eta])^{2q'-2} \varphi |\nabla \mathbb{P}^{\Omega}[\eta]|^2,$$

we get

$$c_{26} \mu(K) \leq \int_{\Omega} \left((|\nabla u|^q + u\lambda) \zeta + 4q'(\mathbb{P}^{\Omega}[\eta])^{2q'-1} u \nabla \varphi \cdot \nabla \mathbb{P}^{\Omega}[\eta] \right) dx. \quad (4.34)$$

Using again the estimates (4.28) and (4.31), we obtain as in Theorem 4.4

$$\left| \int_{\Omega} (\mathbb{P}^{\Omega}[1 - \eta])^{2q'-1} u \nabla \mathbb{P}^{\Omega}[\eta] \cdot \nabla \varphi dx \right| \leq c'_{26} \left(1 + \|\nabla u\|_{L^q_d(\Omega)}^q\right)^{\frac{1}{q}} \|\eta\|_{W^{\frac{2-q}{q}, q'}(\Sigma)}. \quad (4.35)$$

Therefore

$$c_{26}\mu(K) \leq \int_{\Omega} (|\nabla u|^q + u\lambda)\zeta dx + c'_{26} \left(1 + \|\nabla u\|_{L^q_d(\Omega)}^q\right)^{\frac{1}{q}} \|\eta\|_{W^{\frac{2-q}{q},q'}(\Sigma)}. \quad (4.36)$$

As in Theorem 4.4, since $C_{\frac{2-q}{q},q'}(K) = 0$, there exists a sequence of functions $\{\eta_n\}$ in $C^2(\Sigma)$ such that for any n , $0 \leq \eta_n \leq 1$, $\eta_n \equiv 1$ on a neighborhood of K and $\|\eta_n\|_{W^{\frac{2-q}{q},q'}(\Sigma)} \rightarrow 0$ as $n \rightarrow \infty$. Thus $\|\eta_n\|_{L^1(\Sigma)} \rightarrow 0$ and $\zeta_n := (\mathbb{P}^\Omega[\eta_n])^{2q'} \varphi \rightarrow 0$ a.e. in Ω . Letting $n \rightarrow \infty$ in (4.36) with η and ζ replaced by η_n and ζ_n respectively and using the dominated convergence theorem, we deduce that $\mu(K) = 0$. \square

A Appendix: Removability in a domain

In the section we assume that Ω is a bounded open domain in \mathbb{R}^N with a C^2 boundary.

A.1 General nonlinearity

This appendix is devoted to the following equation

$$\begin{cases} -\Delta u + g(|\nabla u|) = \nu & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases} \quad (A.1)$$

where g is a continuous nondecreasing function vanishing at 0 and ν is a Radon measure. By a solution we mean a function $u \in L^1(\Omega)$ such that $g(|\nabla u|) \in L^1(\Omega)$ satisfying

$$\int_{\Omega} (-u\Delta\zeta + g(|\nabla u|)\zeta) dx = \int_{\Omega} \zeta d\nu \quad (A.2)$$

for all $\zeta \in X(\Omega)$. The integral subcriticality condition on g is the following

$$\int_1^\infty g(s)s^{-\frac{2N-1}{N-1}} ds < \infty \quad (A.3)$$

Theorem A.1 *Assume $g \in \mathcal{G}_0$ satisfies (A.3). Then for any positive bounded Borel measure ν in Ω there exists a maximal solution \bar{u}_ν of (A.1). Furthermore, if $\{\nu_n\}$ is a sequence of positive bounded measures in Ω which converges to a bounded measure ν in the weak sense of measures in Ω , then there exists a subsequence $\{\nu_{n_k}\}$ such that $\{u_{\nu_{n_k}}\}$ converges to a solution of (A.1).*

Proof. Since the proof follows the ideas of the one of Theorem 2.2, we just indicate the main modifications.

(i) Considering a sequence of functions $\nu_n \in C_0^\infty(\Omega)$ converging to ν , the approximate solutions are solutions of

$$\begin{cases} -\Delta w + g(|\nabla(w + \mathbb{G}^\Omega[\nu_n])|) = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (A.4)$$

(ii) The convergence is performed using

$$\|\mathbb{G}^\Omega[\nu]\|_{L^1(\Omega)} + \|\mathbb{G}^\Omega[\nu]\|_{M^{\frac{N}{N-2}}(\Omega)} + \|\nabla\mathbb{G}^\Omega[\nu]\|_{M^{\frac{N}{N-1}}(\Omega)} \leq c_1 \|\nu\|_{\mathfrak{M}(\Omega)} \quad (\text{A.5})$$

in Proposition 2.3.

(iii) For the construction of the maximal solution we consider u_δ solution of

$$\begin{cases} -\Delta u_\delta + g(|\nabla u_\delta|) = \nu & \text{in } \Omega'_\delta \\ u_\delta = \mathbb{G}^\Omega[\nu] & \text{in } \Sigma_\delta. \end{cases} \quad (\text{A.6})$$

Then consequently, $0 < \delta < \delta' \implies u_\delta \leq u_{\delta'}$ in $\Omega'_{\delta'}$ and $u_\delta \downarrow \bar{u}_\nu$. Using similar arguments as in the proof of Theorem 2.2 we deduce that \bar{u}_ν is the maximal solution of (A.1). \square

A.2 Power nonlinearity

We consider the following equation

$$-\Delta u + |\nabla u|^q = \nu \quad (\text{A.7})$$

where $1 < q < 2$. The study on the above equation also leads to a critical value $q^* = \frac{N}{N-1}$. In the subcritical case $1 < q < q^*$, if ν is a bounded Radon measure, then the problem

$$\begin{cases} -\Delta u + |\nabla u|^q = \nu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admit a unique solution $u \in L^1(\Omega)$ such that $|\nabla u|^q \in L^1(\Omega)$ (see [4] for solvability of a much more general class of equation). In the contrary, in the supercritical case, an internal singular set can be removable provided that its Bessel capacity is null. More precisely,

Theorem A.2 *Assume $q^* \leq q < 2$ and $K \subset \Omega$ is compact. If $C_{1,q'}(K) = 0$ then any positive solution $u \in C^2(\bar{\Omega} \setminus K)$ of*

$$-\Delta u + |\nabla u|^q = 0 \quad (\text{A.8})$$

in $\Omega \setminus K$ satisfying that $\int_\Omega \frac{\partial u}{\partial \mathbf{n}} dS$ is bounded remains bounded and can be extended as a solution of the same equation in Ω .

Proof. Let $\eta \in C_c^\infty(\Omega)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in a neighborhood of K . Put $\zeta = 1 - \eta$ and take $\zeta^{q'}$ for test function, then

$$-q' \int_\Omega \zeta^{q'-1} \nabla u \cdot \nabla \eta dx - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS + \int_\Omega \zeta^{q'} |\nabla u|^q dx = 0.$$

Since

$$\left| \int_\Omega \zeta^{q'-1} \nabla u \cdot \nabla \eta dx \right| \leq \left(\int_\Omega \zeta^{q'} |\nabla u|^q dx \right) \left(\int_\Omega |\nabla \eta|^{q'} dx \right)^{\frac{1}{q'}}.$$

Therefore

$$\int_\Omega \zeta^{q'} |\nabla u|^q dx \leq \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS + q' \left(\int_\Omega \zeta^{q'} |\nabla u|^q dx \right)^{\frac{1}{q}} \left(\int_\Omega |\nabla \eta|^{q'} dx \right)^{\frac{1}{q'}},$$

which implies

$$\int_{\Omega} \zeta^{q'} |\nabla u|^q dx \leq c_{27} \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS + c_{28} \int_{\Omega} |\nabla \eta|^{q'} dx. \quad (\text{A.9})$$

where $c_i = c_i(q)$ with $i = 27, 28$. Since $C_{1,q'}(K) = 0$, there exists a sequence $\{\eta_n\} \subset C_c^\infty(\Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood of K and $\|\nabla \eta_n\|_{L^{q'}(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$. Then the inequality (A.9) remains valid with η replaced by η_n and ζ replaced by $\zeta_n = 1 - \eta_n$. Thus, since $\zeta_n \rightarrow 1$ a.e. in Ω , we get

$$\int_{\Omega} |\nabla u|^q dx \leq c_{27} \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} dS.$$

Hence, from the hypothesis, we deduce that $|\nabla u| \in L^q(\Omega)$.

Next let $\eta \in C_0^\infty(\Omega)$ and η_n as above, then

$$\int_{\Omega} (1 - \eta_n) \nabla \eta \cdot \nabla u dx - \int_{\Omega} \eta \nabla \eta_n \cdot \nabla u dx + \int_{\Omega} (1 - \eta_n) \eta |\nabla u|^q dx = 0.$$

Since $|\nabla u| \in L^q(\Omega)$, we can let $n \rightarrow \infty$ and obtain by monotone and dominated convergence

$$\int_{\Omega} (\nabla \eta \cdot \nabla u + \eta |\nabla u|^q) dx = 0.$$

Regularity results imply that $u \in C^2(\Omega)$. □

Theorem A.3 *Assume $q^* \leq q < 2$ and $\nu \in \mathfrak{M}^+(\Omega)$. Let $u \in L^1(\Omega)$ with $|\nabla u| \in L^q(\Omega)$ is a solution of (A.7) in Ω . Then $\nu(E) = 0$ on Borel subsets $E \subset \Omega$ such that $C_{1,q'}(E) = 0$.*

Proof. Since ν is outer regular, it is sufficient to prove the result when E is compact. Let η_n be a sequence as in the previous theorem, then

$$\int_{\Omega} (\nabla u \cdot \nabla \eta_n + \eta_n |\nabla u|^q) dx = \int \eta_n d\nu \geq \nu(E). \quad (\text{A.10})$$

But the left-hand side of (A.10) is dominated by

$$\left(\int_{\Omega} |\nabla \eta_n|^{q'} dx \right)^{\frac{1}{q'}} \left(\int_{\Omega} \eta_n |\nabla u|^q dx \right)^{\frac{1}{q}} + \int_{\Omega} \eta_n |\nabla u|^q dx,$$

which goes to 0 when $n \rightarrow \infty$, both by the definition of the $C_{1,q'}$ -capacity and the fact that $\eta_n \rightarrow 0$ a.e. as $n \rightarrow \infty$ and is bounded by 1. Thus $\nu(E) = 0$. □

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