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# KERNEL DENSITY ESTIMATION FOR STATIONARY RANDOM FIELDS

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## Abstract

In this paper, under natural and easily verifiable conditions, we prove the  $\mathbb{L}^1$ -convergence and the asymptotic normality of the Parzen-Rosenblatt density estimator for stationary random fields of the form  $X_k = g(\varepsilon_{k-s}, s \in \mathbb{Z}^d)$ ,  $k \in \mathbb{Z}^d$ , where  $(\varepsilon_i)_{i \in \mathbb{Z}^d}$  are i.i.d real random variables and  $g$  is a measurable function defined on  $\mathbb{R}^{\mathbb{Z}^d}$ . Such kind of processes provides a general framework for stationary ergodic random fields. A Berry-Esseen's type central limit theorem is also given for the considered estimator.

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*Short title*: Kernel density estimation for random fields.

## 1 Introduction and main results

Let  $(X_i)_{i \in \mathbb{Z}}$  be a stationary sequence of real random variables defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with an unknown marginal density  $f$ . The kernel density estimator  $f_n$  of  $f$  introduced by Rosenblatt [20] and Parzen [19] is defined for any positive integer  $n$  and any  $x$  in  $\mathbb{R}$  by

$$f_n(x) = \frac{1}{nb_n} \sum_{i=1}^n K\left(\frac{x - X_i}{b_n}\right)$$

where  $K$  is a probability kernel and the bandwidth  $b_n$  is a parameter which converges slowly to zero such that  $nb_n$  goes to infinity. The literature dealing with the asymptotic properties of  $f_n$  when the observations  $(X_i)_{i \in \mathbb{Z}}$  are independent is very extensive (see Silverman [22]). Parzen [19] proved that when  $(X_i)_{i \in \mathbb{Z}}$  is i.i.d. and the bandwidth  $b_n$  goes to zero such that  $nb_n$  goes to infinity then  $(nb_n)^{1/2}(f_n(x_0) - \mathbb{E}f_n(x_0))$  converges in distribution to the normal law with zero mean and variance  $f(x_0) \int_{\mathbb{R}} K^2(t) dt$ . Under the same conditions on the bandwidth, this result was extended by Wu and Mielniczuk

[28] for causal linear processes with i.i.d. innovations and by Dedecker and Merlevède [10] for strongly mixing sequences. In this paper, we are interested by the kernel density estimation problem in the setting of dependent random fields indexed by  $\mathbb{Z}^d$  where  $d$  is a positive integer. The question is not trivial since  $\mathbb{Z}^d$  does not have a natural ordering for  $d \geq 2$ . In recent years, there is a growing interest in asymptotic properties of kernel density estimators for random fields. One can refer for example to Carbon et al. ([3], [4]), Cheng et al. [8], El Machkouri [13], Hallin et al. [16], Tran [24] and Wang and Woodroffe [25]. In [24], the asymptotic normality of the kernel density estimator for strongly mixing random fields was obtained using the Bernstein's blocking technique and coupling arguments. Using the same method, the case of linear random fields with i.i.d. innovations was handled in [16]. In [13], the central limit theorem for the Parzen-Rosenblatt estimator given in [24] was improved using the Lindeberg's method (see [18]). In particular, a simple criterion on the strong mixing coefficients is provided and the only condition imposed on the bandwidth is  $n^d b_n \rightarrow \infty$  which is similar to the usual condition imposed in the independent case (see Parzen [19]). In [13], the regions where the random field is observed are reduced to squares but a careful reading of the proof allows us to state that the main result in [13] still holds for very general regions  $\Lambda_n$ , namely those which the cardinality  $|\Lambda_n|$  goes to infinity such that  $|\Lambda_n| b_n$  goes to zero as  $n$  goes to infinity (see Assumption **(A2)** below). In [8], Cheng et al. investigated the asymptotic normality of the kernel density estimator for linear random fields with i.i.d. innovations using a martingale approximation method (initiated by Cheng and Ho [7]) but it seems that there is a mistake in their proof (see Remark 6 in [25]). Since the mixing property is often unverifiable and might be too restrictive, it is important to provide limit theorems for nonmixing and possibly nonlinear random fields. If  $d$  is a positive integer, we consider in this work a field  $(X_i)_{i \in \mathbb{Z}^d}$  of identically distributed real random variables with a marginal density  $f$  such that

$$X_i = g(\varepsilon_{i-s}; s \in \mathbb{Z}^d), \quad i \in \mathbb{Z}^d, \quad (1)$$

where  $(\varepsilon_j)_{j \in \mathbb{Z}^d}$  are i.i.d. random variables and  $g$  is a measurable function defined on  $\mathbb{R}^{\mathbb{Z}^d}$ . In the one-dimensional case ( $d = 1$ ), the class (1) includes linear as well as many widely used nonlinear time series models as special cases. More importantly, it provides a very general framework for asymptotic theory for statistics of stationary time series (see [26] and the review paper [27]).

Let  $(\varepsilon'_j)_{j \in \mathbb{Z}^d}$  be an i.i.d. copy of  $(\varepsilon_j)_{j \in \mathbb{Z}^d}$  and consider for any positive integer  $n$  the coupled version  $X_i^*$  of  $X_i$  defined by  $X_i^* = g(\varepsilon_{i-s}^*; s \in \mathbb{Z}^d)$  where  $\varepsilon_j^* = \varepsilon_j \mathbb{1}_{\{j \neq 0\}} +$

$\varepsilon'_0 \mathbb{1}_{\{j=0\}}$  for any  $j$  in  $\mathbb{Z}^d$ . In other words, we obtain  $X_i^*$  from  $X_i$  by just replacing  $\varepsilon_0$  by its copy  $\varepsilon'_0$ . Following Wu [26], we introduce appropriate dependence measures: let  $i$  in  $\mathbb{Z}^d$  and  $p > 0$  be fixed. If  $X_i$  belongs to  $\mathbb{L}_p$  (that is,  $\mathbb{E}|X_i|^p$  is finite), we define the physical dependence measure  $\delta_{i,p} = \|X_i - X_i^*\|_p$  where  $\|\cdot\|_p$  is the usual  $\mathbb{L}^p$ -norm and we say that the random field  $(X_i)_{i \in \mathbb{Z}^d}$  is  $p$ -stable if  $\sum_{i \in \mathbb{Z}^d} \delta_{i,p} < \infty$ . For  $d \geq 2$ , the reader should keep in mind the following two examples already given in [14] :

Linear random fields: Let  $(\varepsilon_i)_{i \in \mathbb{Z}^d}$  be i.i.d random variables with  $\varepsilon_i$  in  $\mathbb{L}^p$ ,  $p \geq 2$ . The linear random field  $X$  defined for any  $i$  in  $\mathbb{Z}^d$  by

$$X_i = \sum_{s \in \mathbb{Z}^d} a_s \varepsilon_{i-s}$$

with  $(a_s)_{s \in \mathbb{Z}^d}$  in  $\mathbb{R}^{\mathbb{Z}^d}$  such that  $\sum_{i \in \mathbb{Z}^d} a_i^2 < \infty$  is of the form (1) with a linear functional  $g$ . For any  $i$  in  $\mathbb{Z}^d$ ,  $\delta_{i,p} = |a_i| \|\varepsilon_0 - \varepsilon'_0\|_p$ . So,  $X$  is  $p$ -stable if  $\sum_{i \in \mathbb{Z}^d} |a_i| < \infty$ . Clearly, if  $H$  is a Lipschitz continuous function, under the above condition, the subordinated process  $Y_i = H(X_i)$  is also  $p$ -stable since  $\delta_{i,p} = O(|a_i|)$ .

Volterra field : Another class of nonlinear random field is the Volterra process which plays an important role in the nonlinear system theory (Casti [5], Rugh [21]): consider the second order Volterra process

$$X_i = \sum_{s_1, s_2 \in \mathbb{Z}^d} a_{s_1, s_2} \varepsilon_{i-s_1} \varepsilon_{i-s_2},$$

where  $a_{s_1, s_2}$  are real coefficients with  $a_{s_1, s_2} = 0$  if  $s_1 = s_2$  and  $(\varepsilon_i)_{i \in \mathbb{Z}^d}$  are i.i.d. random variables with  $\varepsilon_i$  in  $\mathbb{L}^p$ ,  $p \geq 2$ . Let

$$A_i = \sum_{s_1, s_2 \in \mathbb{Z}^d} (a_{s_1, i}^2 + a_{i, s_2}^2) \quad \text{and} \quad B_i = \sum_{s_1, s_2 \in \mathbb{Z}^d} (|a_{s_1, i}|^p + |a_{i, s_2}|^p).$$

By the Rosenthal inequality, there exists a constant  $C_p > 0$  such that

$$\delta_{i,p} = \|X_i - X_i^*\|_p \leq C_p A_i^{1/2} \|\varepsilon_0\|_2 \|\varepsilon_0\|_p + C_p B_i^{1/p} \|\varepsilon_0\|_p^2.$$

From now on, for any finite subset  $\Lambda$  of  $\mathbb{Z}^d$ , we denote  $|\Lambda|$  the number of elements in  $\Lambda$  and we observe  $(X_i)_{i \in \mathbb{Z}^d}$  on a sequence  $(\Lambda_n)_{n \geq 1}$  of finite subsets of  $\mathbb{Z}^d$  which only satisfies  $|\Lambda_n|$  goes to infinity as  $n$  goes to infinity. It is important to note that we do not impose any condition on the boundary of the regions  $\Lambda_n$ . The density estimator  $f_n$  of  $f$  is defined for any positive integer  $n$  and any  $x$  in  $\mathbb{R}$  by

$$f_n(x) = \frac{1}{|\Lambda_n| b_n} \sum_{i \in \Lambda_n} K\left(\frac{x - X_i}{b_n}\right)$$

where  $b_n$  is the bandwidth parameter and  $K$  is a probability kernel. Our aim is to provide sufficient conditions for the  $\mathbb{L}_1$ -distance between  $f_n$  and  $f$  to converge to zero in probability (Theorem 1) and for  $(|\Lambda_n|b_n)^{1/2}(f_n(x_i) - \mathbb{E}f_n(x_i))_{1 \leq i \leq k}$ ,  $(x_i)_{1 \leq i \leq k} \in \mathbb{R}^k$ ,  $k \in \mathbb{N}^*$ , to converge in law to a multivariate normal distribution (Theorem 2) under minimal conditions on the bandwidth parameter. We give also a Berry-Esseen's type central limit theorem for the considered estimator (Theorem 3). In the sequel, we denote  $|i| = \max_{1 \leq k \leq d} |i_k|$  for any  $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$  and we denote also  $\delta_i$  for  $\delta_{i,2}$ . The following assumptions are required.

- (A1) The kernel  $K$  is Lipschitzian and satisfies  $\int_{\mathbb{R}} K(u) du = 1$ ,  $\int_{\mathbb{R}} |K(u)| du < \infty$  and  $\int_{\mathbb{R}} K^2(u) du < \infty$ .
- (A2)  $b_n \rightarrow 0$  and  $|\Lambda_n| \rightarrow \infty$  such that  $|\Lambda_n|b_n \rightarrow \infty$ .
- (A3) The condition  $\sum_{i \in \mathbb{Z}^d} |i|^{\frac{5d}{2}} \delta_i < \infty$  is satisfied.

**Theorem 1** *If (A1), (A2) and (A3) hold then*

$$\lim_{n \rightarrow +\infty} \mathbb{E} \int_{\mathbb{R}} |f_n(x) - f(x)| dx = 0.$$

**Remark 1.** The above convergence result was obtained also by Hallin et al. ([17], Theorem 2.1) for rectangular region  $\Lambda_n$  and under a more restrictive condition on the bandwidth parameter related to the rate of convergence to zero of the so-called stability coefficients  $(v(m))_{m \geq 1}$  defined by  $v(m) = \|X_0 - \bar{X}_0\|_2^2$  where  $\bar{X}_0 = \mathbb{E}(X_0 | \mathcal{H}_m)$  and  $\mathcal{H}_m = \sigma(\varepsilon_s, |s| \leq m)$ . Arguing as in the proof of Lemma 9 below, one can notice that  $v(m) \leq \sum_{|i| > m} \delta_i^2$ .

In order to establish the asymptotic normality of  $f_n$ , we need additional assumptions:

- (A4) The marginal probability distribution of each  $X_k$  is absolutely continuous with continuous positive density function  $f$ .
- (A5) There exists  $\kappa > 0$  such that  $\sup_{(x,y) \in \mathbb{R}^2} f_{i|j}(y|x) \leq \kappa$  where  $f_{i|j}$  is the conditional density of  $X_i$  given  $X_j$  for any  $i$  and  $j$  in  $\mathbb{Z}^d$ .

**Remark 2.** In [2], the asymptotic normality of the frequency polygon estimator for strongly mixing random fields is established under Assumption (A5) among others. Moreover, in the case  $d = 1$ , it is shown that the stationary autoregressive process of order 1 satisfies Assumption (A5) (see example in [2], page 504).

**Theorem 2** Assume that **(A1)**, **(A2)**, **(A3)**, **(A4)** and **(A5)** hold. For any positive integer  $k$  and any distinct points  $x_1, \dots, x_k$  in  $\mathbb{R}$ ,

$$(|\Lambda_n|b_n)^{1/2} \begin{pmatrix} f_n(x_1) - \mathbb{E}f_n(x_1) \\ \vdots \\ f_n(x_k) - \mathbb{E}f_n(x_k) \end{pmatrix} \xrightarrow[n \rightarrow +\infty]{\text{Law}} \mathcal{N}(0, \Gamma) \quad (2)$$

where  $\Gamma$  is a diagonal matrix with diagonal elements  $\gamma_{ii} = f(x_i) \int_{\mathbb{R}} K^2(u) du$ .

**Remark 3.** A replacement of  $\mathbb{E}f_n(x_i)$  by  $f(x_i)$  for any  $1 \leq i \leq k$  in (2) is a classical problem in density estimation theory. For example, if  $f$  is assumed to be Lipschitz and if  $\int_{\mathbb{R}} |u| |K(u)| du < \infty$  then  $|\mathbb{E}f_n(x_i) - f(x_i)| = O(b_n)$  and thus the centering  $\mathbb{E}f_n(x_i)$  may be changed to  $f(x_i)$  without affecting the above result provided that  $|\Lambda_n|b_n^3$  converges to zero.

**Remark 4.** If  $(X_i)_{i \in \mathbb{Z}^d}$  is a linear random field of the form  $X_i = \sum_{j \in \mathbb{Z}^d} a_j \varepsilon_{i-j}$  where  $(a_j)_{j \in \mathbb{Z}^d}$  are real numbers such that  $\sum_{j \in \mathbb{Z}^d} a_j^2 < \infty$  and  $(\varepsilon_j)_{j \in \mathbb{Z}^d}$  are i.i.d. real random variables with zero mean and finite variance then  $\delta_i = |a_i| \|\varepsilon_0 - \varepsilon'_0\|_2$  and Theorem 2 holds provided that  $\sum_{i \in \mathbb{Z}^d} |i|^{\frac{5d}{2}} |a_i| < \infty$ . For  $\Lambda_n$  rectangular, Hallin et al. [16] obtained the same result when  $|a_j| = O(|j|^{-\gamma})$  with  $\gamma > \max\{d + 3, 2d + 0.5\}$  and  $|\Lambda_n|b_n^{(2\gamma-1+6d)/(2\gamma-1-4d)}$  goes to infinity. So, in the particular case of linear random fields, our assumption **(A3)** is more restrictive than the condition obtained by Hallin et al. [16] but our result is valid for a larger class of random fields (namely, the class of spatial processes of the form (1) observed on general regions) and under only minimal conditions on the bandwidth (see Assumption **(A2)**). Finally, for causal linear random fields, Wang and Woodroffe [25] obtained also a sufficient condition on the coefficients  $(a_j)_{j \in \mathbb{N}^d}$  for the kernel density estimator to be asymptotically normal. Their condition is less restrictive than the condition  $\sum_{i \in \mathbb{Z}^d} |i|^{\frac{5d}{2}} |a_i| < \infty$  but they assumed also  $\mathbb{E}(|\varepsilon_0|^p) < \infty$  for some  $p > 2$ .

Now, we are going to investigate the rate of convergence in (2). Recall that a Young function  $\psi$  is a real convex nondecreasing function defined on  $\mathbb{R}^+$  which satisfies  $\lim_{t \rightarrow \infty} \psi(t) = \infty$  and  $\psi(0) = 0$ . We define the Orlicz space  $\mathbb{L}_\psi$  as the space of real random variables  $Z$  defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  such that  $\mathbb{E}[\psi(|Z|/c)] < \infty$  for some  $c > 0$ . The Orlicz space  $\mathbb{L}_\psi$  equipped with the norm  $\|\cdot\|_\psi$  defined for any real random variable  $Z$  by  $\|Z\|_\psi = \inf\{c > 0; \mathbb{E}[\psi(|Z|/c)] \leq 1\}$  is a Banach space. For any positive integer  $n$  and any  $x$  in  $\mathbb{R}$  such that  $f(x) \neq 0$ , we denote  $D_n(x) = \sup_{t \in \mathbb{R}} |\mathbb{P}(U_n(x) \leq t) - \Phi(t)|$

where  $\Phi$  is the distribution function of the standard normal law and

$$U_n(x) = \frac{\sqrt{|\Lambda_n| b_n} (f_n(x) - \mathbb{E}f_n(x))}{\sqrt{f(x) \int_{\mathbb{R}} K^2(t) dt}}.$$

For any  $1 \leq \beta \leq 2$  we consider the Young function  $\psi_\beta$  defined for any  $x \in \mathbb{R}^+$  by  $\psi_\beta(x) = \exp(x^\beta) - 1$ . If  $X_i$  belongs to  $\mathbb{L}_{\psi_2}$ , we denote also  $\delta_{i, \psi_2} = \|X_i - X_i^*\|_{\psi_2}$  for any  $i$  in  $\mathbb{Z}^d$ .

**Theorem 3** *Let  $n$  be a positive integer and let  $x$  in  $\mathbb{R}$  be fixed such that  $f(x) \neq 0$ . Assume that  $\int_{\mathbb{R}} |K(t)|^\tau dt < \infty$  for some  $2 < \tau \leq 3$ .*

- (i) *Let  $\alpha > 0$  and  $p > 1$  such that  $\sum_{i \in \mathbb{Z}^d} |i|^{d(1+\alpha)} \delta_{i,p} < \infty$  then there exists  $\kappa > 0$  such that  $D_n(x) \leq \kappa |\Lambda_n|^{\theta_1(\alpha, \tau, p)} b_n^{\theta_2(\alpha, \tau, p)}$  where*

$$\theta_1(\alpha, \tau, p) = \frac{\alpha p(2 - \tau)}{2(\alpha p + (\tau - 1)(p + 1))} \quad \text{and} \quad \theta_2(\alpha, \tau, p) = \frac{p((\alpha + 3)(2 - \tau) - 3)}{2(\alpha p + (\tau - 1)(p + 1))}$$

- (ii) *Assume that  $X_0$  belongs to  $\mathbb{L}_{\psi_2}$  and let  $\alpha > 0$  such that  $\sum_{i \in \mathbb{Z}^d} |i|^{d(1+\alpha)} \delta_{i, \psi_2} < \infty$  then there exists  $\kappa > 0$  such that  $D_n(x) \leq \kappa |\Lambda_n|^{\theta_3(\alpha, \tau)} b_n^{\theta_4(\alpha, \tau)} \log \left( 1 + |\Lambda_n|^{-\theta_3(\alpha, \tau)} b_n^{-\theta_4(\alpha, \tau)} \right)$  where*

$$\theta_3(\alpha, \tau) = \frac{\alpha(2 - \tau)}{2(\alpha + \tau - 1)} \quad \text{and} \quad \theta_4(\alpha, \tau) = \frac{(\alpha + 3)(2 - \tau) - 3}{2(\alpha + \tau - 1)}.$$

## 2 Proofs

The proof of all lemmas of this section are postponed to the appendix. In the sequel, the letter  $\kappa$  denotes a positive constant which the value is not important and we consider the sequence  $(m_n)_{n \geq 1}$  defined by

$$m_n = \max \left\{ v_n, \left\lceil \left[ \left( \frac{1}{b_n^3} \sum_{|i| > v_n} |i|^{\frac{5d}{2}} \delta_i \right)^{\frac{1}{3d}} \right] + 1 \right\rceil \right\} \quad (3)$$

where  $v_n = \lceil b_n^{\frac{-1}{2d}} \rceil$  and  $\lceil \cdot \rceil$  denotes the integer part function. The following technical lemma is a spatial version of a result by Bosq, Merlevède and Peligrad ([1], pages 88-89).

**Lemma 1** *If (A3) holds then*

$$m_n \rightarrow \infty, \quad m_n^d b_n \rightarrow 0 \quad \text{and} \quad \frac{1}{(m_n^d b_n)^{3/2}} \sum_{|i| > m_n} |i|^{\frac{5d}{2}} \delta_i \rightarrow 0.$$

For any  $z$  in  $\mathbb{R}$  and any  $i$  in  $\mathbb{Z}^d$ , we denote

$$K_i(z) = K\left(\frac{z - X_i}{b_n}\right) \quad \text{and} \quad \bar{K}_i(z) = \mathbb{E}(K_i(z)|\mathcal{F}_{n,i}) \quad (4)$$

where  $\mathcal{F}_{n,i} = \sigma(\varepsilon_{i-s}; |s| \leq m_n)$ . So, denoting  $M_n = 2m_n + 1$ ,  $(\bar{K}_i(z))_{i \in \mathbb{Z}^d}$  is an  $M_n$ -dependent random field (i.e.  $\bar{K}_i(z)$  and  $\bar{K}_j(z)$  are independent as soon as  $|i - j| \geq M_n$ ).

**Lemma 2** For any  $p > 1$ , any  $x$  in  $\mathbb{R}$ , any positive integer  $n$  and any  $(a_i)_{i \in \mathbb{Z}^d}$  in  $\mathbb{R}^{\mathbb{Z}^d}$ ,

$$\left\| \sum_{i \in \Lambda_n} a_i (K_i(x) - \bar{K}_i(x)) \right\|_p \leq \frac{8m_n^d}{b_n} \left( p \sum_{i \in \Lambda_n} a_i^2 \right)^{1/2} \sum_{|i| > m_n} \delta_{i,p}.$$

## 2.1 Proof of Theorem 1

The proof follows the same lines of the proof of Theorem 2.1 in [17]. For any positive integer  $n$ , denote

$$J_n = \int_{\mathbb{R}} |f_n(x) - f(x)| dx.$$

For any positive real  $A$ , we have  $J_n = J_{n,1}(A) + J_{n,2}(A)$  where

$$J_{n,1}(A) = \int_{|x| > A} |f_n(x) - f(x)| dx \quad \text{and} \quad J_{n,2}(A) = \int_{|x| \leq A} |f_n(x) - f(x)| dx.$$

**Lemma 3** For any  $\varepsilon > 0$ , there exists  $A > 0$  such that  $\lim_{n \rightarrow +\infty} \mathbb{E}J_{n,1}(A) < \varepsilon$ .

Now,  $J_{n,2}(A) \leq J_{n,2}^{(1)}(A) + J_{n,2}^{(2)}$  where

$$J_{n,2}^{(1)}(A) = \int_{|x| \leq A} |f_n(x) - \mathbb{E}f_n(x)| dx \quad \text{and} \quad J_{n,2}^{(2)} = \int_{\mathbb{R}} |\mathbb{E}f_n(x) - f(x)| dx$$

**Lemma 4**  $J_{n,2}^{(2)}$  goes to zero as  $n$  goes to infinity.

So, it suffices to show that  $\mathbb{E}J_{n,2}^{(1)}(A)$  goes to zero as  $n$  goes to infinity. Keeping in mind the notation (4) and denoting

$$\bar{f}_n(x) = \frac{1}{|\Lambda_n|b_n} \sum_{i \in \Lambda_n} \bar{K}_i(x),$$

we have  $J_{n,2}^{(1)}(A) \leq I_{n,1}(A) + I_{n,2}(A)$  where

$$I_{n,1}(A) = \int_{|x| \leq A} |f_n(x) - \bar{f}_n(x)| dx \quad \text{and} \quad I_{n,2}(A) = \int_{|x| \leq A} |\bar{f}_n(x) - \mathbb{E}\bar{f}_n(x)| dx.$$

**Lemma 5**  $\mathbb{E}I_{n,i}(A) = O\left((|\Lambda_n|b_n)^{-1/2}\right)$  for any  $A > 0$  and any  $i \in \{1, 2\}$ .

Combining Lemma 3, Lemma 4 and Lemma 5, we obtain Theorem 1.



## 2.2 Proof of Theorem 2

Without loss of generality, we consider only the case  $k = 2$  and we refer to  $x_1$  and  $x_2$  as  $x$  and  $y$  ( $x \neq y$ ). Let  $\lambda_1$  and  $\lambda_2$  be two constants such that  $\lambda_1^2 + \lambda_2^2 = 1$  and note that

$$\begin{aligned} \lambda_1(|\Lambda_n|b_n)^{1/2}(f_n(x) - \mathbb{E}f_n(x)) + \lambda_2(|\Lambda_n|b_n)^{1/2}(f_n(y) - \mathbb{E}f_n(y)) &= \sum_{i \in \Lambda_n} \frac{\Delta_i}{|\Lambda_n|^{1/2}} \\ \lambda_1(|\Lambda_n|b_n)^{1/2}(\bar{f}_n(x) - \mathbb{E}\bar{f}_n(x)) + \lambda_2(|\Lambda_n|b_n)^{1/2}(\bar{f}_n(y) - \mathbb{E}\bar{f}_n(y)) &= \sum_{i \in \Lambda_n} \frac{\bar{\Delta}_i}{|\Lambda_n|^{1/2}} \end{aligned}$$

where  $\Delta_i = \lambda_1 Z_i(x) + \lambda_2 Z_i(y)$  and  $\bar{\Delta}_i = \lambda_1 \bar{Z}_i(x) + \lambda_2 \bar{Z}_i(y)$  and for any  $z$  in  $\mathbb{R}$ ,

$$Z_i(z) = \frac{1}{\sqrt{b_n}} (K_i(z) - \mathbb{E}K_i(z)) \quad \text{and} \quad \bar{Z}_i(z) = \frac{1}{\sqrt{b_n}} (\bar{K}_i(z) - \mathbb{E}\bar{K}_i(z))$$

where  $K_i(z)$  and  $\bar{K}_i(z)$  are defined by (4). Applying Lemma 1 and Lemma 2, we know that

$$\frac{1}{|\Lambda_n|^{1/2}} \left\| \sum_{i \in \Lambda_n} (\Delta_i - \bar{\Delta}_i) \right\|_2 \leq \frac{\kappa(|\lambda_1| + |\lambda_2|)}{(m_n^d b_n)^{3/2}} \sum_{|i| > m_n} |i|^{\frac{5d}{2}} \delta_i = o(1). \quad (5)$$

So, it suffices to prove the asymptotic normality of the sequence  $(|\Lambda_n|^{-1/2} \sum_{i \in \Lambda_n} \bar{\Delta}_i)_{n \geq 1}$ . We are going to follow the Lindeberg's type proof of Theorem 1 in [9]. We consider the notations

$$\eta = (\lambda_1^2 f(x) + \lambda_2^2 f(y)) \sigma^2 \quad \text{and} \quad \sigma^2 = \int_{\mathbb{R}} K^2(u) du. \quad (6)$$

**Lemma 6**  $\mathbb{E}(\bar{\Delta}_0^2)$  converges to  $\eta$  and  $\sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}|\bar{\Delta}_0 \bar{\Delta}_i| = o(M_n^{-d})$ .

Let  $N$  be fixed in  $\mathbb{N}^*$ . Let  $\varphi$  be a one to one map from  $[1, N] \cap \mathbb{N}^*$  to a finite subset of  $\mathbb{Z}^d$  and  $(\xi_i)_{i \in \mathbb{Z}^d}$  a real random field. For all integers  $k$  in  $[1, N]$ , we denote

$$S_{\varphi(k)}(\xi) = \sum_{i=1}^k \xi_{\varphi(i)} \quad \text{and} \quad S_{\varphi(k)}^c(\xi) = \sum_{i=k}^N \xi_{\varphi(i)}$$

with the convention  $S_{\varphi(0)}(\xi) = S_{\varphi(N+1)}(\xi) = 0$ . On the lattice  $\mathbb{Z}^d$  we define the lexicographic order as follows: if  $i = (i_1, \dots, i_d)$  and  $j = (j_1, \dots, j_d)$  are distinct elements of  $\mathbb{Z}^d$ , the notation  $i <_{\text{lex}} j$  means that either  $i_1 < j_1$  or for some  $k$  in  $\{2, 3, \dots, d\}$ ,  $i_k < j_k$  and  $i_l = j_l$  for  $1 \leq l < k$ . To describe the set  $\Lambda_n$ , we define the one to one map  $\varphi$  from  $[1, |\Lambda_n|] \cap \mathbb{N}^*$  to  $\Lambda_n$  by:  $\varphi$  is the unique function such that  $\varphi(k) <_{\text{lex}} \varphi(l)$  for  $1 \leq k < l \leq |\Lambda_n|$ . From now on, we consider a field  $(\xi_i)_{i \in \mathbb{Z}^d}$  of i.i.d. random

variables independent of  $(X_i)_{i \in \mathbb{Z}^d}$  such that  $\xi_0$  has the standard normal law  $\mathcal{N}(0, 1)$ . We introduce the fields  $Y$  and  $\gamma$  defined for any  $i$  in  $\mathbb{Z}^d$  by

$$Y_i = \frac{\bar{\Delta}_i}{|\Lambda_n|^{1/2}} \quad \text{and} \quad \gamma_i = \frac{\xi_i \sqrt{\eta}}{|\Lambda_n|^{1/2}}$$

where  $\eta$  is defined by (6). Note that  $Y$  is an  $M_n$ -dependent random field where  $M_n = 2m_n + 1$  and  $m_n$  is defined by (3). Let  $h$  be any function from  $\mathbb{R}$  to  $\mathbb{R}$ . For  $0 \leq k \leq l \leq |\Lambda_n| + 1$ , we introduce  $h_{k,l}(Y) = h(S_{\varphi(k)}(Y) + S_{\varphi(l)}^c(\gamma))$ . With the above convention we have that  $h_{k,|\Lambda_n|+1}(Y) = h(S_{\varphi(k)}(Y))$  and also  $h_{0,l}(Y) = h(S_{\varphi(l)}^c(\gamma))$ . In the sequel, we will often write  $h_{k,l}$  instead of  $h_{k,l}(Y)$ . We denote by  $B_1^4(\mathbb{R})$  the unit ball of  $C_b^4(\mathbb{R})$ :  $h$  belongs to  $B_1^4(\mathbb{R})$  if and only if it belongs to  $C^4(\mathbb{R})$  and satisfies  $\max_{0 \leq i \leq 4} \|h^{(i)}\|_\infty \leq 1$ . It suffices to prove that for all  $h$  in  $B_1^4(\mathbb{R})$ ,

$$\mathbb{E} \left( h \left( S_{\varphi(|\Lambda_n|)}(Y) \right) \right) \xrightarrow{n \rightarrow +\infty} \mathbb{E} \left( h \left( \xi_0 \sqrt{\eta} \right) \right).$$

We use Lindeberg's decomposition:

$$\mathbb{E} \left( h \left( S_{\varphi(|\Lambda_n|)}(Y) \right) - h \left( \xi_0 \sqrt{\eta} \right) \right) = \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left( h_{k,k+1} - h_{k-1,k} \right).$$

Now, we have  $h_{k,k+1} - h_{k-1,k} = h_{k,k+1} - h_{k-1,k+1} + h_{k-1,k+1} - h_{k-1,k}$  and by Taylor's formula we obtain

$$\begin{aligned} h_{k,k+1} - h_{k-1,k+1} &= Y_{\varphi(k)} h'_{k-1,k+1} + \frac{1}{2} Y_{\varphi(k)}^2 h''_{k-1,k+1} + R_k \\ h_{k-1,k+1} - h_{k-1,k} &= -\gamma_{\varphi(k)} h'_{k-1,k+1} - \frac{1}{2} \gamma_{\varphi(k)}^2 h''_{k-1,k+1} + r_k \end{aligned}$$

where  $|R_k| \leq Y_{\varphi(k)}^2 (1 \wedge |Y_{\varphi(k)}|)$  and  $|r_k| \leq \gamma_{\varphi(k)}^2 (1 \wedge |\gamma_{\varphi(k)}|)$ . Since  $(Y, \xi_i)_{i \neq \varphi(k)}$  is independent of  $\xi_{\varphi(k)}$ , it follows that

$$\mathbb{E} \left( \gamma_{\varphi(k)} h'_{k-1,k+1} \right) = 0 \quad \text{and} \quad \mathbb{E} \left( \gamma_{\varphi(k)}^2 h''_{k-1,k+1} \right) = \mathbb{E} \left( \frac{\eta}{|\Lambda_n|} h''_{k-1,k+1} \right)$$

Hence, we obtain

$$\begin{aligned} \mathbb{E} \left( h \left( S_{\varphi(|\Lambda_n|)}(Y) \right) - h \left( \xi_0 \sqrt{\eta} \right) \right) &= \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left( Y_{\varphi(k)} h'_{k-1,k+1} \right) \\ &\quad + \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left( \left( Y_{\varphi(k)}^2 - \frac{\eta}{|\Lambda_n|} \right) \frac{h''_{k-1,k+1}}{2} \right) \\ &\quad + \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left( R_k + r_k \right). \end{aligned}$$

Let  $1 \leq k \leq |\Lambda_n|$  be fixed. Since  $\mathbb{E}|\bar{\Delta}_0| = O(\sqrt{b_n})$  and  $\mathbb{E}(\bar{\Delta}_0^2)$  converges to  $\eta$  (by Lemma 6), we derive

$$\sum_{k=1}^{|\Lambda_n|} \mathbb{E}|R_k| \leq \mathbb{E} \left( \bar{\Delta}_0^2 \left( 1 \wedge \frac{|\bar{\Delta}_0|}{|\Lambda_n|^{1/2}} \right) \right) = o(1)$$

and

$$\sum_{k=1}^{|\Lambda_n|} \mathbb{E}|r_k| \leq \frac{\eta^{3/2} \mathbb{E}|\xi_0|^3}{|\Lambda_n|^{1/2}} = O(|\Lambda_n|^{-1/2}).$$

Consequently, we obtain

$$\sum_{k=1}^{|\Lambda_n|} \mathbb{E}(|R_k| + |r_k|) = o(1).$$

Now, it is sufficient to show

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^{|\Lambda_n|} \left( \mathbb{E}(Y_{\varphi(k)} h'_{k-1,k+1}) + \mathbb{E} \left( \left( Y_{\varphi(k)}^2 - \frac{\eta}{|\Lambda_n|} \right) \frac{h''_{k-1,k+1}}{2} \right) \right) = 0. \quad (7)$$

First, we focus on  $\sum_{k=1}^{|\Lambda_n|} \mathbb{E}(Y_{\varphi(k)} h'_{k-1,k+1})$ . Let the sets  $\{V_i^k; i \in \mathbb{Z}^d, k \in \mathbb{N}^*\}$  be defined as follows:  $V_i^1 = \{j \in \mathbb{Z}^d; j <_{\text{lex}} i\}$  and for  $k \geq 2$ ,  $V_i^k = V_i^1 \cap \{j \in \mathbb{Z}^d; |i - j| \geq k\}$ . For all  $n$  in  $\mathbb{N}^*$  and all integer  $k$  in  $[1, |\Lambda_n|]$ , we define

$$\mathbb{E}^{M_n} = \varphi([1, k] \cap \mathbb{N}^*) \cap V_{\varphi(k)}^{M_n} \quad \text{and} \quad S_{\varphi(k)}^{M_n}(Y) = \sum_{i \in \mathbb{E}_k^{M_n}} Y_i.$$

For any function  $\Psi$  from  $\mathbb{R}$  to  $\mathbb{R}$ , we define  $\Psi_{k-1,l}^{M_n} = \Psi \left( S_{\varphi(k)}^{M_n}(Y) + S_{\varphi(l)}^c(\gamma) \right)$ . We are going to apply this notation to the successive derivatives of the function  $h$ . Our aim is to show that

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left( Y_{\varphi(k)} h'_{k-1,k+1} - Y_{\varphi(k)} \left( S_{\varphi(k-1)}(Y) - S_{\varphi(k)}^{M_n}(Y) \right) h''_{k-1,k+1} \right) = 0. \quad (8)$$

First, we use the decomposition

$$Y_{\varphi(k)} h'_{k-1,k+1} = Y_{\varphi(k)} h'_{k-1,k+1}^{M_n} + Y_{\varphi(k)} \left( h'_{k-1,k+1} - h'_{k-1,k+1}^{M_n} \right).$$

Applying again Taylor's formula,

$$Y_{\varphi(k)} \left( h'_{k-1,k+1} - h'_{k-1,k+1}^{M_n} \right) = Y_{\varphi(k)} \left( S_{\varphi(k-1)}(Y) - S_{\varphi(k)}^{M_n}(Y) \right) h''_{k-1,k+1} + R'_k,$$

where

$$|R'_k| \leq 2 \left| Y_{\varphi(k)} \left( S_{\varphi(k-1)}(Y) - S_{\varphi(k)}^{M_n}(Y) \right) \left( 1 \wedge |S_{\varphi(k-1)}(Y) - S_{\varphi(k)}^{M_n}(Y)| \right) \right|.$$

Since  $(Y_i)_{i \in \mathbb{Z}^d}$  is  $M_n$ -dependent, we have  $\mathbb{E} \left( Y_{\varphi(k)} h'_{k-1, k+1} \right) = 0$  and consequently (8) holds if and only if  $\lim_{n \rightarrow +\infty} \sum_{k=1}^{|\Lambda_n|} \mathbb{E}|R'_k| = 0$ . In fact, considering the sets  $W_n = \{-M_n + 1, \dots, M_n - 1\}^d$  and  $W_n^* = W_n \setminus \{0\}$ , it follows that

$$\begin{aligned} \sum_{k=1}^{|\Lambda_n|} \mathbb{E}|R'_k| &\leq 2 \mathbb{E} \left( |\overline{\Delta}_0| \left( \sum_{i \in W_n} |\overline{\Delta}_i| \right) \left( 1 \wedge \frac{1}{|\Lambda_n|^{1/2}} \sum_{i \in W_n} |\overline{\Delta}_i| \right) \right) \\ &= 2 \mathbb{E} \left( \left( \overline{\Delta}_0^2 + \sum_{i \in W_n^*} |\overline{\Delta}_0 \overline{\Delta}_i| \right) \left( 1 \wedge \frac{1}{|\Lambda_n|^{1/2}} \sum_{i \in W_n} |\overline{\Delta}_i| \right) \right) \\ &\leq 2 \mathbb{E} \left( \overline{\Delta}_0^2 \left( 1 \wedge \frac{1}{|\Lambda_n|^{1/2}} \sum_{i \in W_n} |\overline{\Delta}_i| \right) \right) + 4M_n^d \sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}(|\overline{\Delta}_0 \overline{\Delta}_i|). \end{aligned}$$

Keeping in mind that  $\mathbb{E}|\overline{\Delta}_i| = O(\sqrt{b_n})$  and applying Lemma 1, we have

$$\mathbb{E} \left( \frac{1}{|\Lambda_n|^{1/2}} \sum_{i \in W_n} |\overline{\Delta}_i| \right) = \frac{|W_n| \mathbb{E}|\overline{\Delta}_0|}{|\Lambda_n|^{1/2}} \leq \frac{\kappa m_n^d b_n}{(|\Lambda_n| b_n)^{1/2}} = o(1).$$

Since  $\mathbb{E}(\overline{\Delta}_0^2)$  converges to  $\eta$ , we derive

$$\mathbb{E} \left( \overline{\Delta}_0^2 \left( 1 \wedge \frac{1}{|\Lambda_n|^{1/2}} \sum_{i \in W_n} |\overline{\Delta}_i| \right) \right) = o(1).$$

Applying Lemma 6, we obtain  $\sum_{k=1}^{|\Lambda_n|} \mathbb{E}|R'_k| = o(1)$ . In order to obtain (7) it remains to control

$$F_1 = \mathbb{E} \left( \sum_{k=1}^{|\Lambda_n|} h''_{k-1, k+1} \left( \frac{Y_{\varphi(k)}^2}{2} + Y_{\varphi(k)} \left( S_{\varphi(k-1)}(Y) - S_{\varphi(k)}^{M_n}(Y) \right) - \frac{\eta}{2|\Lambda_n|} \right) \right).$$

Keeping in mind that  $W_n = \{-M_n + 1, \dots, M_n - 1\}^d$  and applying again Lemma 6, we have

$$\begin{aligned} F_1 &\leq \left| \mathbb{E} \left( \frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h''_{k-1, k+1} \left( \overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2) \right) \right) \right| + \left| \eta - \mathbb{E}(\overline{\Delta}_0^2) \right| + 2 \sum_{j \in V_0^1 \cap W_n} \mathbb{E}|\overline{\Delta}_0 \overline{\Delta}_j| \\ &\leq \left| \mathbb{E} \left( \frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h''_{k-1, k+1} \left( \overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2) \right) \right) \right| + o(1), \end{aligned}$$

it suffices to prove that

$$F_2 = \left| \mathbb{E} \left( \frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h''_{k-1,k+1} \left( \overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2) \right) \right) \right|$$

goes to zero as  $n$  goes to infinity. In fact, we have  $F_2 \leq \frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} (J_k^1(n) + J_k^2(n))$  where  $J_k^1(n) = \left| \mathbb{E} \left( h''_{k-1,k+1} \left( \overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2) \right) \right) \right| = 0$  since  $h''_{k-1,k+1}$  is  $\sigma(\overline{\Delta}_i; i \in V_{\varphi(k)}^{M_n}) \vee \sigma(\xi_i; i \in \Lambda_n)$ -measurable and

$$\begin{aligned} J_k^2(n) &= \left| \mathbb{E} \left( \left( h''_{k-1,k+1} - h''_{k-1,k+1} \right) \left( \overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2) \right) \right) \right| \\ &\leq \mathbb{E} \left( \left( 2 \wedge \sum_{|i| < M_n} \frac{|\overline{\Delta}_i|}{|\Lambda_n|^{1/2}} \right) \overline{\Delta}_0^2 \right) \\ &= o(1). \end{aligned}$$

The proof of Theorem 2 is complete.

*Proof of Theorem 5.* Let  $n$  be a fixed positive integer and let  $x$  be fixed in  $\mathbb{R}$ . We have  $U_n(x) = \overline{U}_n(x) + R_n(x)$  where

$$\overline{U}_n(x) = \frac{\sqrt{|\Lambda_n|b_n} (\overline{f}_n(x) - \mathbb{E}\overline{f}_n(x))}{\sqrt{f(x) \int_{\mathbb{R}} K^2(t) dt}} \quad \text{and} \quad R_n(x) = \frac{\sqrt{|\Lambda_n|b_n} (f_n(x) - \overline{f}_n(x))}{\sqrt{f(x) \int_{\mathbb{R}} K^2(t) dt}}.$$

Denote  $\overline{D}_n(x) = \sup_{t \in \mathbb{R}} |\mathbb{P}(\overline{U}_n(x) \leq t) - \Phi(t)|$  and let  $\psi$  be a Young function. Arguing as in Theorem 2.2 in [12], we have

$$D_n(x) \leq \overline{D}_n(x) + \varphi(\|R_n\|_{\psi}) \quad (9)$$

where  $\varphi(x) = xh^{-1}(1/x)$  and  $h(x) = x\psi(x)$ . In the sequel we denote  $\sigma^2 = f(x) \int_{\mathbb{R}} K^2(t) dt$ . Since there exist  $\alpha > 0$  and  $p > 1$  such that  $\sum_{i \in \mathbb{Z}^d} |i|^{d(1+\alpha)} \delta_{i,p} < \infty$ , we derive from Lemma 2 that

$$\|R_n(x)\|_p \leq \frac{\kappa \sqrt{p}}{\sigma m_n^{d\alpha} b_n^{3/2}} \sum_{i \in \mathbb{Z}^d} |i|^{d(1+\alpha)} \delta_{i,p}. \quad (10)$$

Applying the Berry-Esseen's type theorem for  $m_n$ -dependent random fields established by Chen and Shao ([6], Theorem 2.6), we obtain

$$\overline{D}_n(x) \leq \frac{\kappa \int_{\mathbb{R}} |K(t)|^{\tau} f(x - tb_n) dt m_n^{(\tau-1)d}}{\sigma^{\tau} (|\Lambda_n| b_n)^{\frac{\tau}{2}-1}}. \quad (11)$$

Applying (9) with  $\psi(x) = x^p$ , we obtain  $\varphi(x) = x^{\frac{p}{p+1}}$  and

$$D_n(x) \leq \frac{\kappa \int_{\mathbb{R}} |K(t)|^\tau f(x - tb_n) dt m_n^{(\tau-1)d}}{\sigma^\tau (|\Lambda_n| b_n)^{\frac{\tau}{2}-1}} + \left( \frac{\kappa \sqrt{p}}{\sigma m_n^{d\alpha} b_n^{3/2}} \sum_{i \in \mathbb{Z}^d} |i|^{d(1+\alpha)} \delta_{i,p} \right)^{\frac{p}{p+1}}.$$

Optimizing in  $m_n$  the second part of the above inequality, we obtain the point (i) of Theorem 3. Now, we assume that  $X_0$  belongs to  $L_{\psi_2}$  and there exists  $\alpha > 0$  such that  $\sum_{i \in \mathbb{Z}^d} |i|^{d(1+\alpha)} \delta_{i,\psi_2} < \infty$ . We need the following lemma which can be obtained using the expansion of the exponential function (see [23]).

**Lemma 7** *Let  $\beta$  be a positive real number and  $Z$  be a real random variable. There exist positive universal constants  $A_\beta$  and  $B_\beta$  depending only on  $\beta$  such that*

$$A_\beta \sup_{p>2} \frac{\|Z\|_p}{p^{1/\beta}} \leq \|Z\|_{\psi_\beta} \leq B_\beta \sup_{p>2} \frac{\|Z\|_p}{p^{1/\beta}}.$$

Combining (10) and Lemma 7, we obtain

$$\|R_n(x)\|_{\psi_1} \leq \frac{\kappa}{\sigma m_n^{d\alpha} b_n^{3/2}} \sum_{i \in \mathbb{Z}^d} |i|^{d(1+\alpha)} \delta_{i,\psi_2}. \quad (12)$$

Applying (9) with  $\psi = \psi_1$ , we obtain

$$D_n(x) \leq \frac{\kappa \int_{\mathbb{R}} |K(t)|^\tau f(x - tb_n) dt m_n^{(\tau-1)d}}{\sigma^\tau (|\Lambda_n| b_n)^{\frac{\tau}{2}-1}} + \varphi \left( \frac{\kappa}{\sigma m_n^{d\alpha} b_n^{3/2}} \sum_{i \in \mathbb{Z}^d} |i|^{d(1+\alpha)} \delta_{i,\psi_2} \right).$$

Noting that  $\lim_{t \rightarrow 0} \frac{\varphi(t)}{t \log(1 + \frac{1}{t})} = 1$  and optimizing again in  $m_n$ , we obtain the point (ii) of Theorem 3. The Proof of Theorem 3 is complete.

### 3 Appendix

*Proof of Lemma 1.* We follow the proof by Bosq, Merlevède and Peligrad ([1], pages 88-89). First,  $m_n$  goes to infinity since  $v_n = \lceil b_n^{-\frac{1}{2d}} \rceil$  goes to infinity and  $m_n \geq v_n$ . For any positive integer  $m$ , we consider

$$r(m) = \sum_{|i|>m} |i|^{\frac{5d}{2}} \delta_i.$$

Since **(A3)** holds,  $r(m)$  converges to zero as  $m$  goes to infinity. Moreover,

$$m_n^d b_n \leq \max \left\{ \sqrt{b_n}, r(v_n)^{1/3} + b_n \right\} \xrightarrow{n \rightarrow +\infty} 0$$

and

$$m_n^d \geq \frac{1}{b_n} (r(v_n))^{1/3} \geq \frac{1}{b_n} (r(m_n))^{1/3} \quad \text{since } v_n \leq m_n.$$

Finally, we obtain

$$\frac{1}{(m_n^d b_n)^{3/2}} \sum_{|i| > m_n} |i|^{\frac{5d}{2}} \delta_i \leq \sqrt{r(m_n)} \xrightarrow{n \rightarrow +\infty} 0.$$

The proof of Lemma 1 is complete.

*Proof of Lemma 2.* Let  $p > 1$  be fixed. We follow the proof of Proposition 1 in [14]. For any  $i$  in  $\mathbb{Z}^d$  and any  $x$  in  $\mathbb{R}$ , we denote  $R_i = K_i(x) - \bar{K}_i(x)$ . Since there exists a measurable function  $H$  such that  $R_i = H(\varepsilon_{i-s}; s \in \mathbb{Z}^d)$ , we are able to define the physical dependence measure coefficients  $(\delta_{i,p}^{(n)})_{i \in \mathbb{Z}^d}$  associated to the random field  $(R_i)_{i \in \mathbb{Z}^d}$ . We recall that  $\delta_{i,p}^{(n)} = \|R_i - R_i^*\|_p$  where  $R_i^* = H(\varepsilon_{i-s}^*; s \in \mathbb{Z}^d)$  and  $\varepsilon_j^* = \varepsilon_j \mathbb{1}_{\{j \neq 0\}} + \varepsilon_0' \mathbb{1}_{\{j=0\}}$  for any  $j$  in  $\mathbb{Z}^d$ . In other words, we obtain  $R_i^*$  from  $R_i$  by just replacing  $\varepsilon_0$  by its copy  $\varepsilon_0'$  (see [26]). Let  $\tau : \mathbb{Z} \rightarrow \mathbb{Z}^d$  be a bijection. For any  $l \in \mathbb{Z}$ , for any  $i \in \mathbb{Z}^d$ ,

$$P_l R_i := \mathbb{E}(R_i | \mathcal{F}_l) - \mathbb{E}(R_i | \mathcal{F}_{l-1}) \quad (13)$$

where  $\mathcal{F}_l = \sigma(\varepsilon_{\tau(s)}; s \leq l)$ .

**Lemma 8** *For any  $l$  in  $\mathbb{Z}$  and any  $i$  in  $\mathbb{Z}^d$ , we have  $\|P_l R_i\|_p \leq \delta_{i-\tau(l),p}^{(n)}$ .*

*Proof of Lemma 8.* Let  $l$  in  $\mathbb{Z}$  and  $i$  in  $\mathbb{Z}^d$  be fixed.

$$\|P_l R_i\|_p = \|\mathbb{E}(R_i | \mathcal{F}_l) - \mathbb{E}(R_i | \mathcal{F}_{l-1})\|_p = \|\mathbb{E}(R_0 | T^i \mathcal{F}_l) - \mathbb{E}(R_0 | T^i \mathcal{F}_{l-1})\|_p$$

where  $T^i \mathcal{F}_l = \sigma(\varepsilon_{\tau(s)-i}; s \leq l)$ . Hence,

$$\begin{aligned} \|P_l R_i\|_p &= \left\| \mathbb{E} \left( H \left( (\varepsilon_{-s})_{s \in \mathbb{Z}^d} \right) | T^i \mathcal{F}_l \right) - \mathbb{E} \left( H \left( (\varepsilon_{-s})_{s \in \mathbb{Z}^d \setminus \{i-\tau(l)\}}; \varepsilon_{\tau(l)-i}' \right) | T^i \mathcal{F}_l \right) \right\|_p \\ &\leq \left\| H \left( (\varepsilon_{-s})_{s \in \mathbb{Z}^d} \right) - H \left( (\varepsilon_{-s})_{s \in \mathbb{Z}^d \setminus \{i-\tau(l)\}}; \varepsilon_{\tau(l)-i}' \right) \right\|_p \\ &= \left\| H \left( (\varepsilon_{i-\tau(l)-s})_{s \in \mathbb{Z}^d} \right) - H \left( (\varepsilon_{i-\tau(l)-s})_{s \in \mathbb{Z}^d \setminus \{i-\tau(l)\}}; \varepsilon_0' \right) \right\|_p \\ &= \|R_{i-\tau(l)} - R_{i-\tau(l)}^*\|_p \\ &= \delta_{i-\tau(l),p}^{(n)}. \end{aligned}$$

The proof of Lemma 8 is complete.

For all  $i$  in  $\mathbb{Z}^d$ ,  $R_i = \sum_{l \in \mathbb{Z}} P_l R_i$ . Consequently,  $\|\sum_{i \in \Lambda_n} a_i R_i\|_p = \|\sum_{l \in \mathbb{Z}} \sum_{i \in \Lambda_n} a_i P_l R_i\|_p$ . Applying the Burkholder inequality (cf. [15], page 23) for the martingale difference sequence  $(\sum_{i \in \Lambda_n} a_i P_l R_i)_{l \in \mathbb{Z}}$ , we obtain

$$\left\| \sum_{i \in \Lambda_n} a_i R_i \right\|_p \leq \left( 2p \sum_{l \in \mathbb{Z}} \left\| \sum_{i \in \Lambda_n} a_i P_l R_i \right\|_p^2 \right)^{\frac{1}{2}} \leq \left( 2p \sum_{l \in \mathbb{Z}} \left( \sum_{i \in \Lambda_n} |a_i| \|P_l R_i\|_p \right)^2 \right)^{\frac{1}{2}}$$

By the Cauchy-Schwarz inequality, we have

$$\left( \sum_{i \in \Lambda_n} |a_i| \|P_l R_i\|_p \right)^2 \leq \sum_{i \in \Lambda_n} a_i^2 \|P_l R_i\|_p \times \sum_{i \in \Lambda_n} \|P_l R_i\|_p$$

and by Lemma 8,  $\sum_{i \in \mathbb{Z}^d} \|P_l R_i\|_p \leq \sum_{j \in \mathbb{Z}^d} \delta_{j,p}^{(n)}$ . So, we obtain

$$\left\| \sum_{i \in \Lambda_n} a_i R_i \right\|_p \leq \left( 2p \sum_{j \in \mathbb{Z}^d} \delta_{j,p}^{(n)} \sum_{i \in \Lambda_n} a_i^2 \sum_{l \in \mathbb{Z}} \|P_l R_i\|_p \right)^{\frac{1}{2}}.$$

Applying again Lemma 8, for any  $i$  in  $\mathbb{Z}^d$ , we have  $\sum_{l \in \mathbb{Z}} \|P_l R_i\|_p \leq \sum_{j \in \mathbb{Z}^d} \delta_{j,p}^{(n)}$ . Finally, we derive

$$\left\| \sum_{i \in \Lambda_n} a_i R_i \right\|_p \leq \left( 2p \sum_{i \in \Lambda_n} a_i^2 \right)^{\frac{1}{2}} \sum_{i \in \mathbb{Z}^d} \delta_{i,p}^{(n)}.$$

Since  $\bar{K}_i^* = \mathbb{E}(K_i^*(x) | \mathcal{F}_{n,i}^*)$  where  $\mathcal{F}_{n,i}^* = \sigma(\varepsilon_{i-s}^*; |s| \leq m_n)$  and  $(K_i(x) - \bar{K}_i(x))^* = K_i^*(x) - \bar{K}_i^*(x)$ , we derive  $\delta_{i,p}^{(n)} \leq 2\|K_i(x) - K_i^*(x)\|_p$ . Since  $K$  is Lipschitzian, we obtain

$$\delta_{i,p}^{(n)} \leq \frac{2\delta_{i,p}}{b_n} \quad (14)$$

where  $\delta_{i,p} = \|X_i - X_i^*\|_p$ . Moreover, we have also  $\delta_{i,p}^{(n)} \leq 2\|K_0(x) - \bar{K}_0(x)\|_p$ .

**Lemma 9** For any  $p > 1$ , any positive integer  $n$  and any  $x$  in  $\mathbb{R}$ ,

$$\|K_0(x) - \bar{K}_0(x)\|_p \leq \frac{\sqrt{8p}}{b_n} \sum_{|j| > m_n} \delta_{j,p}.$$

*Proof of Lemma 9.* Let  $p > 1$  be fixed. We consider the sequence  $(\Gamma_n)_{n \geq 0}$  of finite subsets of  $\mathbb{Z}^d$  defined by  $\Gamma_0 = \{(0, \dots, 0)\}$  and for any  $n$  in  $\mathbb{N}^*$ ,  $\Gamma_n = \{i \in \mathbb{Z}^d; |i| = n\}$ . The cardinality of the set  $\Gamma_n$  is  $|\Gamma_n| = 2d(2n+1)^{d-1}$  for  $n \geq 1$ . Let  $\tau: \mathbb{N}^* \rightarrow \mathbb{Z}^d$  be the bijection defined by  $\tau(1) = (0, \dots, 0)$  and



- for any  $n$  in  $\mathbb{N}^*$ , if  $l \in ]a_{n-1}, a_n]$  then  $\tau(l) \in \Gamma_n$ ,
- for any  $n$  in  $\mathbb{N}^*$ , if  $(i, j) \in ]a_{n-1}, a_n]^2$  and  $i < j$  then  $\tau(i) <_{\text{lex}} \tau(j)$

where  $a_n = \sum_{j=0}^n |\Gamma_j|$  goes to infinity as  $n$  goes to infinity. Let  $(m_n)_{n \geq 1}$  be the sequence of positive integers defined by (3). For any  $n$  in  $\mathbb{N}^*$ , we recall that  $\mathcal{F}_{n,0} = \sigma(\varepsilon_{-s}; |s| \leq m_n)$  (see (4)) and we consider also the  $\sigma$ -algebra  $\mathcal{G}_n := \sigma(\varepsilon_{\tau(j)}; 1 \leq j \leq n)$ . By the definition of the bijection  $\tau$ , for any  $n$  in  $\mathbb{N}$ ,  $1 \leq j \leq a_n$  if and only if  $|\tau(j)| \leq n$ . So, we have  $\mathcal{G}_{a_{mn}} = \mathcal{F}_{n,0}$ . Consequently,  $K_0(x) - \bar{K}_0(x) = \sum_{l > a_{mn}} D_l$  with  $D_l = \mathbb{E}(K_0(x)|\mathcal{G}_l) - \mathbb{E}(K_0(x)|\mathcal{G}_{l-1})$  for any  $l$  in  $\mathbb{Z}$ . Let  $p \geq 2$  be fixed. Since  $(D_l)_{l \in \mathbb{Z}}$  is a martingale-difference sequence, applying Burkholder's inequality (cf. [15], page 23), we derive

$$\|K_0(x) - \bar{K}_0(x)\|_p \leq \left( 2p \sum_{l > a_{mn}} \|D_l\|_p^2 \right)^{1/2}.$$

Denoting  $K'_0(x) = K\left(b_n^{-1}\left(x - g\left((\varepsilon_{-s})_{s \in \mathbb{Z}^d \setminus \{-\tau(l)\}}; \varepsilon'_{\tau(l)}\right)\right)\right)$ , we obtain

$$\begin{aligned} \|D_l\|_p &= \|\mathbb{E}(K_0(x)|\mathcal{G}_l) - \mathbb{E}(K'_0(x)|\mathcal{G}_l)\|_p \leq \|K_0(x) - K'_0(x)\|_p \\ &\leq \frac{1}{b_n} \left\| g\left((\varepsilon_{-s})_{s \in \mathbb{Z}^d}\right) - g\left((\varepsilon_{-s})_{s \in \mathbb{Z}^d \setminus \{-\tau(l)\}}; \varepsilon'_{\tau(l)}\right) \right\|_p \\ &= \frac{1}{b_n} \left\| g\left((\varepsilon_{-\tau(l)-s})_{s \in \mathbb{Z}^d}\right) - g\left((\varepsilon_{-\tau(l)-s})_{s \in \mathbb{Z}^d \setminus \{-\tau(l)\}}; \varepsilon'_0\right) \right\|_p \\ &= \frac{1}{b_n} \|X_{-\tau(l)} - X_{-\tau(l)}^*\|_p = \frac{\delta_{-\tau(l),p}}{b_n} \end{aligned}$$

and finally

$$\|K_0(x) - \bar{K}_0(x)\|_p \leq \frac{1}{b_n} \left( 2p \sum_{l > a_{mn}} \delta_{-\tau(l),p}^2 \right)^{1/2} \leq \frac{\sqrt{2p}}{b_n} \sum_{|j| > m_n} \delta_{j,p}.$$

The proof of Lemma 9 is complete.

Applying Lemma 9, we derive

$$\delta_{i,p}^{(n)} \leq \frac{2\sqrt{8p}}{b_n} \sum_{|j| > m_n} \delta_{j,p}. \quad (15)$$

Combining (14) and (15), we obtain

$$\sum_{i \in \mathbb{Z}^d} \delta_{i,p}^{(n)} \leq 2 \left( \frac{m_n^d \sqrt{8p}}{b_n} \sum_{|j| > m_n} \delta_{j,p} + \frac{1}{b_n} \sum_{|j| > m_n} \delta_{j,p} \right).$$

The proof of Lemma 2 is complete.

*Proof of Lemma 3.* We follow the proof of Lemma 4.1 in [17]. Let  $\varepsilon > 0$  be fixed and let  $A > 0$  such that

$$\int_{|x|>A/2} f(x)dx \leq \frac{\varepsilon}{3}.$$

We have

$$\begin{aligned} \mathbb{E}J_{n,1}(A) &\leq \int_{|x|>A} \mathbb{E}|f_n(x)|dx + \int_{|x|>A} f(x)dx \leq \int_{|x|>A} \int_{\mathbb{R}} K(t)f(x - b_nt)dt dx + \frac{\varepsilon}{3} \\ &= \int_{|t|>\frac{A}{2}} K(t) \int_{|x|>A} f(x - b_nt)dx dt + \int_{|t|\leq\frac{A}{2}} K(t) \int_{|x|>A} f(x - b_nt)dx dt + \frac{\varepsilon}{3} \\ &\leq \int_{|t|>\frac{A}{2}} K(t) \int_{|y+b_nt|>A} f(y)dy dt + \int_{|t|\leq\frac{A}{2}} K(t) \int_{|y|>A(1-\frac{b_n}{2})} f(y)dy dt + \frac{\varepsilon}{3} \\ &\leq \frac{\varepsilon}{3} + \int_{|y|>A(1-\frac{b_n}{2})} f(y)dy + \frac{\varepsilon}{3}. \end{aligned}$$

Consequently, we obtain  $\lim_{n \rightarrow \infty} \mathbb{E}J_{n,1}(A) \leq \varepsilon$ . The proof of Lemma 3 is complete.

*Proof of Lemma 4.* For any integer  $n$  and any real  $x$ , we have

$$J_{n,2}^{(2)} = \int_{\mathbb{R}} \left| \left( f * \frac{1}{b_n} K \left( \frac{\cdot}{b_n} \right) \right) (x) - f(x) \right| dx$$

and the result follows from Theorem 1, page 6, in [11]. The proof of Lemma 4 is complete.

*Proof of Lemma 5.* Let  $n$  in  $\mathbb{N}^*$  and  $x$  in  $\mathbb{R}$  be fixed. Applying Lemmas 1 and 2, we have

$$\|f_n(x) - \bar{f}_n(x)\|_2 \leq \frac{\kappa \sum_{|i|>m_n} |i|^{\frac{5d}{2}} \delta_i}{\sqrt{|\Lambda_n| b_n (m_n^d b_n)^{3/2}}} = o\left((|\Lambda_n| b_n)^{-1/2}\right).$$

So, we obtain the result for  $\mathbb{E}I_{n,1}(A)$ . Now,  $\|\bar{f}_n(x) - \mathbb{E}\bar{f}_n(x)\|_2^2$  equals to

$$\frac{1}{|\Lambda_n|^2 b_n} \left( |\Lambda_n| \mathbb{E} \left( \bar{Z}_0^2(x) \right) + \sum_{\substack{j \in \mathbb{Z}^d \setminus \{0\} \\ |j| < M_n}} |\Lambda_n \cap (\Lambda_n - j)| \mathbb{E} \left( \bar{Z}_0(x) \bar{Z}_j(x) \right) \right) \quad (16)$$

where we recall that  $\bar{Z}_i(x) = \frac{1}{\sqrt{b_n}} (\bar{K}_i(x) - \mathbb{E}\bar{K}_i(x))$  and  $M_n = 2m_n + 1$ .

**Lemma 10** *Let  $x, s$  and  $t$  be fixed in  $\mathbb{R}$ . Then  $\mathbb{E}(\overline{Z}_0^2(x))$  converges to  $f(x) \int_{\mathbb{R}} K^2(u) du$  and  $\sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}|\overline{Z}_0(s)\overline{Z}_i(t)| = o(M_n^{-d})$ .*

*Proof of Lemma 10.* For any  $z$  in  $\mathbb{R}$ , we have

$$\mathbb{E}K_0^2(z) = b_n \int_{\mathbb{R}} K^2(v) f(z - vb_n) dv = O(b_n). \quad (17)$$

Let  $s$  and  $t$  be fixed in  $\mathbb{R}$ . Since  $\mathbb{E}(\overline{K}_0(s)\overline{K}_0(t)) = \mathbb{E}(K_0(s)\overline{K}_0(t))$ , we have

$$|\mathbb{E}(\overline{K}_0(s)\overline{K}_0(t)) - \mathbb{E}(K_0(s)K_0(t))| \leq \|K_0(s)\|_2 \|K_0(t) - \overline{K}_0(t)\|_2.$$

Using (17) and Lemma 9, we have

$$|\mathbb{E}(\overline{K}_0(s)\overline{K}_0(t)) - \mathbb{E}(K_0(s)K_0(t))| \leq \frac{\kappa}{\sqrt{b_n}} \sum_{|j| > m_n} \delta_j.$$

Since  $b_n |\mathbb{E}(Z_0(s)Z_0(t)) - \mathbb{E}(\overline{Z}_0(s)\overline{Z}_0(t))| = |\mathbb{E}(K_0(s)K_0(t)) - \mathbb{E}(\overline{K}_0(s)\overline{K}_0(t))|$ , we have

$$M_n^d |\mathbb{E}(Z_0(s)Z_0(t)) - \mathbb{E}(\overline{Z}_0(s)\overline{Z}_0(t))| \leq \frac{\kappa}{(m_n^d b_n)^{3/2}} \sum_{|j| > m_n} |j|^{\frac{5d}{2}} \delta_j. \quad (18)$$

Moreover, keeping in mind Assumptions **(A1)**, **(A2)** and **(A4)**, we have

$$\lim_n \frac{1}{b_n} \mathbb{E}(K_0(s)K_0(t)) = \lim_n \int_{\mathbb{R}} K(v) K\left(v + \frac{t-s}{b_n}\right) f(s - vb_n) dv = u(s, t) f(s) \int_{\mathbb{R}} K^2(u) du \quad (19)$$

where  $u(s, t) = 1$  if  $s = t$  and  $u(s, t) = 0$  if  $s \neq t$ . We have also

$$\lim_n \frac{1}{b_n} \mathbb{E}K_0(s)\mathbb{E}K_0(t) = \lim_n b_n \int_{\mathbb{R}} K(v) f(s - vb_n) dv \int_{\mathbb{R}} K(w) f(t - wb_n) dw = 0. \quad (20)$$

Let  $x$  be fixed in  $\mathbb{R}$ . Choosing  $s = t = x$  and combining (18), (19), (20) and Lemma 1, we obtain  $\mathbb{E}(\overline{Z}_0^2(x))$  goes to  $f(x) \int_{\mathbb{R}} K^2(u) du$  as  $n$  goes to infinity.

In the other part, let  $i \neq 0$  be fixed in  $\mathbb{Z}^d$  and let  $s$  and  $t$  be fixed in  $\mathbb{R}$ . We have

$$\mathbb{E}|\overline{Z}_0(s)\overline{Z}_i(t)| \leq \frac{1}{b_n} \mathbb{E}|\overline{K}_0(s)\overline{K}_i(t)| + \frac{3}{b_n} \mathbb{E}|K_0(s)| \mathbb{E}|K_0(t)|. \quad (21)$$

Keeping in mind that  $||\alpha| - |\beta|| \leq |\alpha - \beta|$  for any  $(\alpha, \beta)$  in  $\mathbb{R}^2$  and applying the Cauchy-Schwarz inequality, we obtain

$$|\mathbb{E}|\overline{K}_0(s)\overline{K}_i(t)| - \mathbb{E}|K_0(s)K_i(t)|| \leq \|\overline{K}_0(s)\|_2 \|\overline{K}_0(t) - K_0(t)\|_2 + \|K_0(t)\|_2 \|\overline{K}_0(s) - K_0(s)\|_2$$

Using (17) and Lemma 9, we obtain

$$\frac{M_n^d}{b_n} |\mathbb{E}|\bar{K}_0(s)\bar{K}_i(t)| - \mathbb{E}|K_0(s)K_i(t)|| \leq \frac{\kappa}{(m_n^d b_n)^{3/2}} \sum_{|j|>m_n} |j|^{\frac{5d}{2}} \delta_j. \quad (22)$$

Since Assumptions **(A1)** and **(A4)** hold and  $M_n^d b_n = o(1)$ , we have

$$\frac{M_n^d}{b_n} \mathbb{E}|K_0(s)| \mathbb{E}|K_0(t)| = M_n^d b_n \int_{\mathbb{R}} |K(u)| f(s - ub_n) du \int_{\mathbb{R}} |K(v)| f(t - vb_n) dv = o(1). \quad (23)$$

Moreover, using Assumption **(A5)**, we have

$$\begin{aligned} \mathbb{E}|K_0(s)K_i(t)| &= \iint_{\mathbb{R}^2} \left| K\left(\frac{s-u}{b_n}\right) K\left(\frac{t-v}{b_n}\right) \right| f(u) (f_{i|0}(v|u) - f(v)) dudv \\ &\quad + \iint_{\mathbb{R}^2} \left| K\left(\frac{s-u}{b_n}\right) K\left(\frac{t-v}{b_n}\right) \right| f(u) f(v) dudv \\ &\leq \kappa b_n^2 \int_{\mathbb{R}} |K(w)| f(s - wb_n) dw \times \int_{\mathbb{R}} |K(w)| dw \\ &\quad + b_n^2 \int_{\mathbb{R}} |K(w)| f(s - wb_n) dw \times \int_{\mathbb{R}} |K(w)| f(t - wb_n) dw. \end{aligned}$$

So, using again Assumptions **(A1)** and **(A4)** and  $M_n^d b_n = o(1)$ , we derive

$$\frac{M_n^d}{b_n} \mathbb{E}|K_0(s)K_i(t)| = o(1). \quad (24)$$

Combining (21), (22), (23), (24) and Lemma 1, we obtain

$$M_n^d \sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}|\bar{Z}_0(s)\bar{Z}_i(t)| = o(1). \quad (25)$$

The proof of Lemma 10 is complete.

Combining (16) and Lemma 10, we derive  $\|\bar{f}_n(x) - \mathbb{E}\bar{f}_n(x)\|_2^2 = O((|\Lambda_n| b_n)^{-1})$ . The proof of Lemma 5 is complete.

*Proof of Lemma 6.* Let  $x$  and  $y$  be two distinct real numbers. Noting that

$$\begin{aligned} \mathbb{E}(\Delta_0^2) &= \lambda_1^2 \mathbb{E}(Z_0^2(x)) + \lambda_2^2 \mathbb{E}(Z_0^2(y)) + 2\lambda_1 \lambda_2 \mathbb{E}(Z_0(x)Z_0(y)) \\ \mathbb{E}(\bar{\Delta}_0^2) &= \lambda_1^2 \mathbb{E}(\bar{Z}_0^2(x)) + \lambda_2^2 \mathbb{E}(\bar{Z}_0^2(y)) + 2\lambda_1 \lambda_2 \mathbb{E}(\bar{Z}_0(x)\bar{Z}_0(y)) \end{aligned}$$

and using (18) and Lemma 1, we obtain

$$\lim_{n \rightarrow +\infty} M_n^d |\mathbb{E}(\Delta_0^2) - \mathbb{E}(\bar{\Delta}_0^2)| = 0. \quad (26)$$

Combining (19) and (26), we derive that  $\mathbb{E}(\overline{\Delta}_0^2)$  converges to  $\eta = (\lambda_1^2 f(x) + \lambda_2^2 f(y)) \int_{\mathbb{R}} K^2(u) du$ . Let  $i \neq 0$  be fixed in  $\mathbb{Z}^d$ . Combining (25) and

$$\mathbb{E}|\overline{\Delta}_0 \overline{\Delta}_i| \leq \lambda_1^2 \mathbb{E}|\overline{Z}_0(x) \overline{Z}_i(x)| + \lambda_2^2 \mathbb{E}|\overline{Z}_0(y) \overline{Z}_i(y)| + \lambda_1 \lambda_2 \mathbb{E}|\overline{Z}_0(x) \overline{Z}_i(y)| + \lambda_1 \lambda_2 \mathbb{E}|\overline{Z}_0(y) \overline{Z}_i(x)|, \quad (27)$$

we obtain  $M_n^d \sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}|\overline{\Delta}_0 \overline{\Delta}_i| = o(1)$ . The proof of Lemma 6 is complete.

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