A note on 3-D simple points and simple-equivalence

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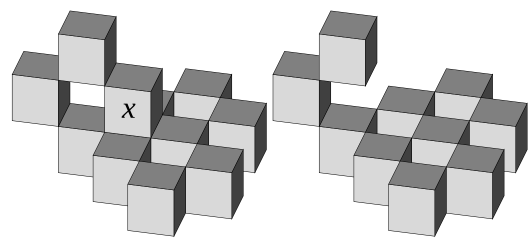
1. Introduction

Topology-preserving operators, like homotopic skeletonization, are used to transform the geometry of an object while leaving unchanged its topological characteristics. In discrete grids ($\mathbb{Z}^2$ or $\mathbb{Z}^3$), such a transformation can be defined thanks to the notion of simple point: intuitively, a point of an object is called simple if it can be deleted from this object without altering topology. This notion, pionneered by Duda, Hart, Munson, Golay and Rosenfeld [12], has since been the subject of an abundant literature. In particular, local characterizations of simple points have been proposed, which enable efficient implementations of thinning procedures.

In [11], the authors study some configurations where an object X strictly contains an object Y, topologically equivalent to X, and where X has no simple point. We call such a configuration a lump (an example of lump is given in Fig. 1a). Lumps cannot appear in $\mathbb{Z}^2$, but they are not uncommon in $\mathbb{Z}^3$, where they may "block" some homotopic thinning procedures and prevent to obtain globally minimal skeletons. The existence of certain such objects, like the one of Fig. 1a, illustrates a "counter-property" of simple points: deleting a simple point always preserves topology, but one can sometimes delete a non-simple point while preserving topology. In Fig. 1a, the point x is not simple (in Sec. 4 we give a definition and a characterization which enable the reader to verify this claim), but we can see that if we remove x, intuitively, one tunnel is destroyed and another one is created. As a consequence, objects of Fig. 1a and Fig. 1b have the same topology. Notice that this kind of configuration has been considered, in particular in [10].



(a) (b)

Figure 1. A lump, made of 11 voxels, is depicted in (a). It contains no simple voxel, and is simple-equivalent to the complex in (b), made of 10 voxels. Both objects have three tunnels.

In this paper, we prove that in a particular case, a 3-D point can be removed while preserving topology *if and only if* it is a simple point. This property holds in the case of simply connected objects, that is, connected objects which have no tunnels.

We develop this work in the framework of abstract complexes. In this framework, we retrieve the main notions and results of digital topology, such as the notion of simple point.

2. Cubical complexes

Intuitively, a cubical complex may be thought of as a set of elements having various dimensions (*e.g.* cubes, squares, edges, vertices) glued together according to certain rules. In this section, we recall briefly some basic definitions on complexes, see also [4,3] for more details. Let $\mathbb{Z}$ be the set of integers. We consider the families of sets $\mathbb{F}_0^1$, $\mathbb{F}_1^1$, such that $\mathbb{F}_0^1 = \{\{a\} \mid a \in \mathbb{Z}\}$, $\mathbb{F}_1^1 = \{\{a, a+1\} \mid a \in \mathbb{Z}\}$. A subset f of $\mathbb{Z}^n$ ($n \geq 1$) which is the Cartesian product of exactly m elements of $\mathbb{F}_1^1$ and $(n-m)$ elements of $\mathbb{F}_0^1$ is called a *face* or an *m-face* of $\mathbb{Z}^n$, m is the *dimension* of f, and we write $\dim(f) = m$.

We denote by $\mathbb{F}^n$ the set composed of all m-faces of $\mathbb{Z}^n$ ($m = 0$ to n). An m-face of $\mathbb{Z}^n$ is called a *point* if $m = 0$, a *(unit) interval* if $m = 1$, a *(unit) square* if $m = 2$, a *(unit) cube* if $m = 3$. In the sequel, we will focus on $\mathbb{F}^3$.

Let f be a m-face in $\mathbb{F}^3$, with $m \in \{0, \ldots, 3\}$. We set $\hat{f} = \{g \in \mathbb{F}^3 \mid g \subseteq f\}$, we say that $\hat{f}$ is a *cell* or an *m-cell*. Any $g \in \hat{f}$ is a *face* of f, and any $g \in \hat{f}$ such that $g \neq f$ is a *proper face* of f.

A finite set X of faces of $\mathbb{F}^3$ is a *complex* (in $\mathbb{F}^3$) if for any $f \in X$, we have $\hat{f} \subseteq X$. Any subset Y of a complex X which is also a complex is a *subcomplex* of X. If Y is a subcomplex of X, we write $Y \preceq X$. If X is a complex in $\mathbb{F}^3$, we also write $X \preceq \mathbb{F}^3$. In Fig. 2 and Fig. 3, some complexes are represented.

Let $X \preceq \mathbb{F}^3$, a face $f \in X$ is a *facet* of X if there is no $g \in X$ such that f is a proper face of g. We denote by X^+ the set composed of all facets of X. The *dimension* of a non-empty complex X in $\mathbb{F}^3$ is defined by $\dim(X) = \max\{\dim(f) \mid f \in X^+\}$. We say that X is an *m-complex* if $\dim(X) = m$.

Let $X \preceq \mathbb{F}^3$ be a complex. A sequence $\pi = (f_0, \ldots, f_k)$ of 0-faces of X is a *path in X (from f_0 to f_k)* if $f_{i-1} \cup f_i$ is a 1-face of X, for all $i \in \{1, \ldots, k\}$. The points f_0 and f_k are the *extremities* of the path; the path is said to be a *loop* if $f_0 = f_k$. The *inverse of* π is the path $\pi^{-1} = (g_0, \ldots, g_k)$ where $g_i = f_{k-i}$, for all $i \in \{0, \ldots, k\}$. If $\pi = (f_0, \ldots, f_k)$ and $\pi' = (h_0, \ldots, h_\ell)$ are two paths such that $h_0 = f_k$, the *concatenation of* π *and* π' is the path $\pi . \pi' = (f_0, \ldots, f_k, h_1, \ldots, h_\ell)$. If $k = 0$, *i.e.* $\pi = (f_0)$, the path π is called a *trivial loop*. We say that X *is connected* if, for any two points f, g in X, there is a path in X from f to g. We say that Y is a *connected component of* X if $Y \preceq X$, Y is connected and if Y is maximal for these two properties (*i.e.*, we have $Z = Y$ whenever $Y \preceq Z \preceq X$ and Z is connected).

3. Topological invariants

Euler characteristics. Let X be a complex in $\mathbb{F}^3$, and let us denote by n_i the number of i-faces of X, $i = 0, \ldots, 3$. The *Euler characteristic* of X, written $\chi(X)$, is defined by $\chi(X) = n_0 - n_1 + n_2 - n_3$. The Euler characteristic is a well-known topological invariant. If X and Y are two complexes, we have the following basic property: $\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y)$.

Fundamental group. The fundamental group, introduced by Poincaré, is another topological invariant which describes the structure of tunnels in an object. It is based on the notion of homotopy of loops. Briefly and informally, consider the relation between loops in a complex X, which links two loops π and π' whenever π can be "continuously deformed" (in X) into π' (we say that π and π' are homotopic in X). This relation is an equivalence relation, the equivalence classes of which form a group with the operation derived from the concatenation of loops: it is the fundamental group of X.

Let us now define precisely the fundamental group in the framework of cubical complexes (see [7] for a similar construction in the framework of digital topology). Let X be a complex in $\mathbb{F}^3$, and let p be any point in X (called base point). Let $\Lambda_p(X)$ be the set of all loops in X from p to p. Let $\pi, \pi' \in \Lambda_p(X)$, we say that π and π' are *directly homotopic (in X)* if they are of the form $\pi = \pi_1 . \gamma . \pi_2$ and $\pi' = \pi_1 . \gamma' . \pi_2$, with γ and γ' having the same extremities and being contained in a same face of X. We say that π and π' are *homotopic (in X)*, and we write $\pi \sim_X \pi'$, if there exists a sequence $\langle \pi_0, \ldots, \pi_\ell \rangle$ such that $\pi_0 = \pi$, $\pi_\ell = \pi'$, and π_i, π_{i-1} are directly homotopic in X, for all $i \in \{1, \ldots, \ell\}$. The relation $\sim_X$ is an equivalence relation over $\Lambda_p(X)$. Let us denote by $\Pi_p(X)$ the set of all equivalence classes for this relation. The concatenation of loops is compatible with the homotopy relation, *i.e.*, $\pi_1 . \pi_2 \sim_X \pi_3 . \pi_4$ whenever $\pi_1 \sim_X \pi_3$ and $\pi_2 \sim_X \pi_4$. Hence, it induces an operation on $\Pi_p(X)$ which, to the equivalence classes of $\pi_1, \pi_2 \in \Lambda_p(X)$, associates the equivalence class of $\pi_1 . \pi_2$. This operation (also denoted by $.$) provides $\Pi_p(X)$ with a group structure, that is, $(\Pi_p(X), .)$ satisfies the four following properties: closure (for all P, Q in $\Pi_p(X)$, $P . Q \in \Pi_p(X)$), associativity (for all P, Q, R in $\Pi_p(X)$, $P . (Q . R) = (P . Q) . R$), identity (there exists an *identity element* $I \in \Pi_p(X)$ such that for all P in $\Pi_p(X)$, $P . I = I . P = P$), and inverse (for all P in $\Pi_p(X)$, there exists an element $P^{-1} \in \Pi_p(X)$, called the *inverse of* P, such that $P . P^{-1} = P^{-1} . P = I$). The identity element is the equivalence class of the trivial loop (p), and the inverse of the equivalence class of a loop $\pi \in \Lambda_p(X)$ is the equivalence class of the inverse loop π^{-1}. The group $(\Pi_p(X), .)$ is called the *fundamental group of X with base point p*.

If X is connected, it can be shown that the fundamental groups of X with different base points are isomorphic, thus in the sequel we will not refer anymore to the base point unless necessary.

We say that a group is *trivial* if it is reduced to the identity element. It may be easily seen that the fundamental group of any single cell is trivial. A complex X

is said to be *simply connected* whenever it is connected and its fundamental group is trivial. Informally, simply connected objects are connected objects which do not have tunnels. Such objects may have cavities (like a hollow sphere).

Let X be an n-complex, with $n > 0$, and let m be an integer such that $0 \leq m \leq n$. We define the *m-skeleton of X*, denoted by $S_m(X)$, as the subcomplex of X composed of all the k-faces of X, for all $k \leq m$. The following property may be easily verified.

Proposition 1. *Let $X \preceq \mathbb{F}^n$, let π and π' be two loops in X having the same base point. The loops π and π' are homotopic in X if and only if they are homotopic in $S_2(X)$.*

Thus, the fundamental group of a complex $X \preceq \mathbb{F}^n$, for any $n \geq 2$, only depends on the 0-, 1- and 2-faces of X. The faces of higher dimension play no role in its construction.

4. Topology preserving operations

Collapse. The collapse, a well-known operation in algebraic topology [6], leads to a notion of homotopy equivalence in discrete spaces. To put it briefly, the collapse operation preserves topology.

Let X be a complex in $\mathbb{F}^3$ and let $f \in X^+$. If there exists one proper face g of f such that f is the only face of X which contains g, then we say that the pair (f,g) is a *free pair* for X. If (f,g) is a free pair for X, the complex $Y = X \setminus \{f,g\}$ is an *elementary collapse* of X. In this case, we write $X \searrow^e Y$.

Let X,Y be two complexes. We say that X *collapses onto* Y if $X = Y$ or if there exists a *collapse sequence from X to Y*, *i.e.*, a sequence of complexes $\langle X_0, \ldots, X_\ell \rangle$ such that $X_0 = X$, $X_\ell = Y$, and $X_{i-1} \searrow^e X_i$, for all $i \in \{1, \ldots, \ell\}$. Fig. 2 illustrates a collapse sequence. We say that X and Y are *collapse-equivalent* if $X = Y$ or if there exists a sequence of complexes $\langle X_0, \ldots, X_\ell \rangle$ such that $X_0 = X$, $X_\ell = Y$, and for any $i \in \{1, \ldots, \ell\}$, either $X_i \searrow^e X_{i-1}$ or $X_{i-1} \searrow^e X_i$ holds. Let X,Y such that $Y \preceq X$. Obviously, if X collapses onto Y then X and Y are collapse-equivalent, but the converse is not true in general (a classical counter-example is Bing's house, see [5,11]).

It is well known that, if two complexes X and Y are collapse-equivalent, then they have the same Euler characteristics and they have isomorphic fundamental groups.

Simplicity. Intuitively, a cell in a complex X is called simple if it can be "removed" from X while preserving topology. We recall here a definition of simplicity (see [2]) based on the collapse operation, which can be seen as a discrete counterpart of the one given by T.Y. Kong [9].

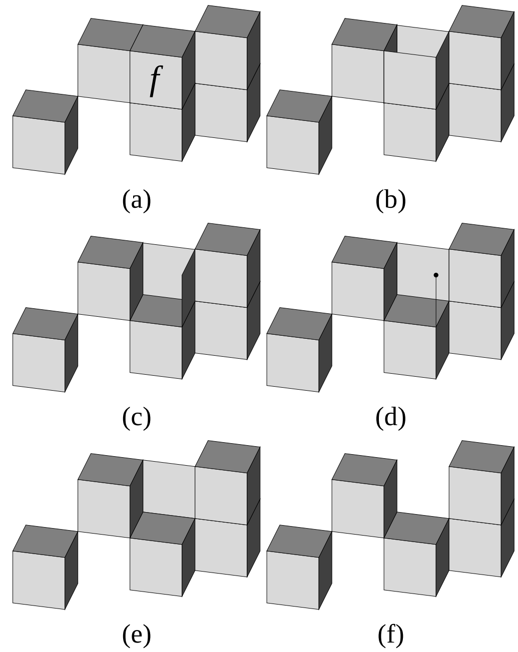


Figure 2. (a): a complex X, and a 3-face f. (f): a complex Y which is the detachment of $\hat{f}$ from X. (a-f): a collapse sequence from X to Y.

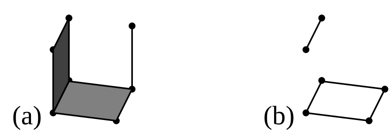


Figure 3. (a): the attachment of $\hat{f}$ for X (see Fig. 2a). (b): the attachment of $\hat{x}$ for the complex depicted in Fig. 1a.

The operation of detachment allows us to remove a subcomplex from a complex, while guaranteeing that the result is still a complex (see Fig. 2a,f). Let $Y \preceq X \preceq \mathbb{F}^3$. We set $X \oslash Y = \cup\{\hat{f} \mid f \in X \setminus Y\}$. The set $X \oslash Y$ is a complex which is the *detachment* of Y from X.

Definition 2. *Let $X \preceq \mathbb{F}^3$. Let $f \in X^+$, we say that f and $\hat{f}$ are* simple for X *if X collapses onto $X \oslash \hat{f}$.*

The notion of attachment leads to a local characterization [2] of simple facets, which follows easily from the definitions. Let $Y \preceq X \preceq \mathbb{F}^3$. The *attachment of Y for X* is the complex defined by $Att(Y,X) = Y \cap (X \oslash Y)$.

Proposition 3. *Let $X \preceq \mathbb{F}^3$, let $f \in X^+$. The facet f is simple for X if and only if $\hat{f}$ collapses onto $Att(\hat{f},X)$.*

In Fig. 2, we can check from the very definition of a simple face, that the 3-face f is indeed simple. As an illustration of Prop. 3, we can verify that the 3-face x of the complex depicted in Fig. 1a cannot collapse onto its attachment, shown in Fig. 3; thus x is not simple.

5. The new property

In the image processing literature, a digital image is often considered as a set of pixels in 2-D or voxels in 3-D. A voxel is an elementary cube, thus an easy correspondance can be made between this classical view and the framework of cubical complexes. In the sequel of the paper, we call *voxel* any 3-cell. If a complex $X \preceq \mathbb{F}^3$ is a union of voxels, we write $X \sqsubseteq \mathbb{F}^3$. If $X,Y \sqsubseteq \mathbb{F}^3$ and $Y \preceq X$, then we write $Y \sqsubseteq X$. From now on, we consider only complexes that are unions of voxels.

Notice that, if $X \sqsubseteq \mathbb{F}^3$ and if $\hat{f}$ is a voxel of X, then $X \oslash \hat{f} \sqsubseteq \mathbb{F}^3$. There is indeed an equivalence between the operation on complexes that consists of removing (by detachment) a simple voxel, and the removal of a 26-simple voxel in the framework of digital topology (see [8,3]).

Let us quote a characterization of 3-D simple voxels proposed by Kong in [9], which is equivalent to the following theorem for subcomplexes of $\mathbb{F}^3$; this characterization will be used in the proof of our main theorem.

Theorem 4 (Adapted from Kong [9])**.** *Let $X \sqsubseteq \mathbb{F}^3$. Let $f \in X^+$. Then $\hat{f}$ is a simple voxel for X if and only if $Att(\hat{f},X)$ is connected and $\chi(Att(\hat{f},X)) = 1$.*

Definition 5. *Let $X,Y \sqsubseteq \mathbb{F}^3$. We say that X and Y are* simple-equivalent *if $X = Y$ or if there exists a sequence of complexes $\langle X_0,\ldots,X_\ell\rangle$ such that $X_0 = X$, $X_\ell = Y$, and for any $i \in \{1,\ldots,\ell\}$, we have either*
$X_i = X_{i-1} \oslash x_i$, where x_i is a voxel that is simple for X_{i-1} ; or
$X_{i-1} = X_i \oslash x_i$, where x_i is a voxel that is simple for X_i.
We say that X is contractible *if X is simple-equivalent to a single voxel.*

Remark that, if X and Y are simple-equivalent then they are collapse-equivalent; hence they have the same Euler characteristics and their fundamental groups are isomorphic. We can now define the notion of lump evoked in the introduction.

Definition 6. *Let $Y \sqsubseteq X \sqsubseteq \mathbb{F}^3$, such that X and Y are simple-equivalent. If $X \neq Y$ and X does not contain any simple voxel, then we say that X is a* lump *relative to Y, or simply a* lump.

The following proposition will be used for the proof of Th. 8.

Proposition 7. *Let $X \sqsubseteq \mathbb{F}^3$ be a connected complex, let x be a voxel of X such that $X \oslash x$ is connected and $Att(x,X)$ is not connected. Then there exists a loop in X that is not homotopic in X to a trivial loop.*

In other words, under the conditions of Prop. 7 the complex X has a tunnel, more precisely it is not simply connected. A proof is given in the appendix, which follows the same main lines as the proof of Prop. 3 in [1]. Finally, let us state and prove our main result.

Theorem 8. *Let $X \sqsubseteq \mathbb{F}^3$, such that X is simply connected. Let x be a voxel of X. Then x is simple for X if and only if $X \oslash x$ is simple-equivalent to X.*

Proof. The forward implication is obvious, let us prove the converse.

Suppose that $X \oslash x$ is simple-equivalent to X and x is not simple for X. Remark that, since X is simply connected and $X \oslash x$ is simple-equivalent to X, $X \oslash x$ is also simply connected (for collapse preserves the fundamental group). From the very definition of the attachment, we have $\chi(X) = \chi([X \oslash x] \cup x) = \chi(X \oslash x) + \chi(x) - \chi(Att(x,X))$. Since X and $X \oslash x$ are simple-equivalent we have $\chi(X) = \chi(X \oslash x)$, furthermore $\chi(x) = 1$ since x is a cell. From this we deduce $\chi(Att(x,X)) = 1$, hence $Att(x,X)$ is non-empty, and from Th. 4, $Att(x,X)$ cannot be connected. From Prop. 7, there exists a loop in X that is not homotopic to a trivial loop, thus the fundamental group of X is not trivial, a contradiction with the fact that X is simply connected. □

Since any contractible set is obviously simply connected, we have the following corollary.

Corollary 9. *Let $X \sqsubseteq \mathbb{F}^3$, such that X is contractible. Let x be a voxel of X. Then x is simple for X if and only if $X \oslash x$ is simple-equivalent to X.*

6. Conclusion

We proved a new property about the notion of 3-D simple point, which has been extensively studied for fourty years and proved useful in many applications. The interest of this result is not only theoretical, since configurations of the same nature as the lump of Fig. 1a are likely to appear in practical image processing procedures (see [11]).

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Appendix: proof of Prop. 7

Proof. Let C_1 and C_2 be two distinct components of $Att(x,X)$. Remark that C_1 and C_2 are subcomplexes of $X \oslash x$. Since $X \oslash x$ is connected, there must exist a path γ_1 in $X \oslash x$ that links a point $p_1 \in C_1$ to a point $p_2 \in C_2$. Let γ_2 be a path from p_2 to p_1 in x; $\gamma = \gamma_1 . \gamma_2$ constitutes a loop in X. We have seen that, in order to define the fundamental group, the base point can be arbitrarily chosen; the choice of a loop having p_1 as its extremities may thus be made without loss of generality.

For any path π, let us define the number $\#(\pi, C_1)$ of pairs of consecutive points of π that are of the type (u,v) with u in C_1 and v not in x, or inversely. Obviously $\#(\gamma_2, C_1) = 0$, and since γ_1 lies in $X \oslash x$ and connects C_1 to C_2, it can be seen that $\#(\gamma_1, C_1)$ must be odd, hence $\#(\gamma, C_1)$ must also be odd.

Let us consider a loop γ' directly homotopic in X to γ. We will prove in the following that $\#(\gamma', C_1)$ is odd. By induction, this property will extend to any loop homotopic in X to γ. By definition, we have $\gamma = P_1 . Q . P_2$ and $\gamma' = P_1 . R . P_2$, with Q and R having the same extremities and being contained in a same face f of X. Observe that, by Prop. 1, we may suppose that f is a 1- or a 2-face. If $\hat{f} \preceq x$ or if $\hat{f} \cap C_1 = \emptyset$, then obviously $\#(\gamma', C_1) = \#(\gamma, C_1)$. Suppose now that $\hat{f} \not\preceq x$ and $\hat{f} \cap C_1 \neq \emptyset$.

Without loss of generality, we can write Q and R in the form $Q = Q_1 Q'_1 Q_2 Q'_2 \ldots Q_k Q'_k$, $k > 0$, and $R = R_1 R'_1 R_2 R'_2 \ldots R_\ell R'_\ell$, $\ell > 0$, with all subsequences Q_i and R_i being composed by points inside C_1, all subsequences Q'_i and R'_i being composed by points outside C_1, and all these subsequences being non-empty except possibly Q_1, R_1, Q'_k, and R'_ℓ.

Since $\hat{f} \cap C_1 \neq \emptyset$ we have $\hat{f} \cap x \neq \emptyset$, and since $\hat{f} \not\preceq x$ and $\hat{f} \preceq X$, we have $\hat{f} \cap x \preceq Att(x,X)$. Hence, since C_1 is a connected component of $Att(x,X)$, we must have $\hat{f} \cap x \preceq C_1$. From this, we deduce that in $\hat{f}$, all the points that are not in C_1 are outside x, thus all the points in the subsequences Q'_i and R'_i are outside x.

Thus, the pairs $Q'_1 Q_2$, $Q_2 Q'_2, \ldots, Q'_{k-1} Q_k$ each bring a contribution of one unit to $\#(\gamma, C_1)$. We have indeed: $\#(\gamma, C_1) = \#(P_1, C_1) + \delta(Q_1) + 2k - 3 + \delta(Q'_k) + \#(P_2, C_1)$, where $\delta(\pi) = 0$ whenever the path π is empty, and $\delta(\pi) = 1$ otherwise. Remark that $\delta(Q_1) = 1$ iff the first point of Q is in C_1, and $\delta(Q'_k) = 1$ iff the last point of Q is not in C_1. Remark also that if $k = 1$, then necessarily $\delta(Q_1) = \delta(Q'_k) = 1$. By the same reasonning, we have $\#(\gamma', C_1) = \#(P_1, C_1) + \delta(R_1) + 2\ell - 3 + \delta(R'_\ell) + \#(P_2, C_1)$, furthermore $\delta(Q_1) = \delta(R_1)$ and $\delta(Q'_k) = \delta(R'_\ell)$ because Q and R have the same extremities. Since $\#(\gamma, C_1)$ is odd, we see that $\#(\gamma', C_1)$ is also odd.

Hence the result, since for any trivial loop π we have $\#(\pi, C_1) = 0$. $\square$