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A uniqueness property of perfect fusion grids on $\mathbb{Z}^d$

Jean Cousty[1,2] and Gilles Bertrand[2]

[1] Asclepios Research Project, INRIA Sophia Antipolis, France
[2] Université Paris-Est, LABINFO-IGM, UMR CNRS 8049, A2SI-ESIEE, France
E-mail addresses: Jean.Cousty@sophia.inria.fr, g.bertrand@esiee.fr

Abstract Region merging methods consist of improving an initial segmentation by merging some pairs of neighboring regions. In a graph, merging two regions is not straightforward. The perfect fusion graphs defined in a previous paper verify all the properties requested by region merging algorithms. In this paper, we present a theorem which states that, in any dimension, the perfect fusion grids introduced previously are the only perfect fusion graphs between the adjacency relations which are the most frequently used in image analysis.

Keywords region merging, perfect fusion graph, perfect fusion grid, adjacency relation

Introduction

Image segmentation is the task of delineating objects of interest that appear in an image. In many cases, the result of such a process, also called a segmentation, is a set of connected regions lying in a background which constitutes the separation between regions. To define regions, an image is often considered as a graph whose vertex set is made of the pixels and whose edge set is given by an adjacency relation between them. Then, the regions are simply the connected components of foreground pixels (see for instance Fig. 1).

A popular approach to image segmentation, called region merging [6], consists of progressively merging pairs of regions, starting from an initial segmentation that contains too many regions (see, for instance, Figs. 1a and b). Given a subset S of an image equipped with an adjacency relation, merging two neighboring regions (connected components) of S is not straightforward. A problem occurs when we want to merge a pair of neighboring regions A and B of S and when each point adjacent to these two regions is also adjacent to a third one, which is not wanted in the merging. Fig. 1c illustrates such a situation, where x is adjacent to regions A, B, C and y to A, B, D. This problem has been identified in particular by T. Pavlidis (see [6], section 5.6: "When three regions meet"), and, as far as we know, has not been solved in general. A major contribution of [2] is the definition of a merging operation and the study of a class of graphs, called *perfect fusion graphs*, which verify all the properties required by region merging algorithms.

Unfortunately, the graphs which are the most frequently used in image analysis (namely, those induced by direct and indirect adjacency relations on $\mathbb{Z}^d$) are not perfect fusion graphs (see Section 6 in [2]). In [2], we introduced a graph on $\mathbb{Z}^d$ that we call the *perfect fusion grid*, which is indeed a perfect fusion graph, and which is "between" the graphs induced by the direct and indirect adjacencies. The main result (whose proof will be given in a forthcoming extended version [1]) in this paper, is that the perfect fusion grid is the only such graph, in any dimension $d \in \mathbb{N}$.

1 Perfect fusion graphs

1.1 Basic notions on graphs

Let E be a set, we denote by 2^E the set composed of all subsets of E. Let $X \subseteq E$, we write $\overline{X}$ the *complementary set* of X in E, *i.e.*, , $\overline{X} = E \setminus X$. Let E' be a set. The *Cartesian product* of E by E', denoted by $E \times E'$, is the set made of all pairs (x, y) such that $x \in E$ and $y \in E'$.

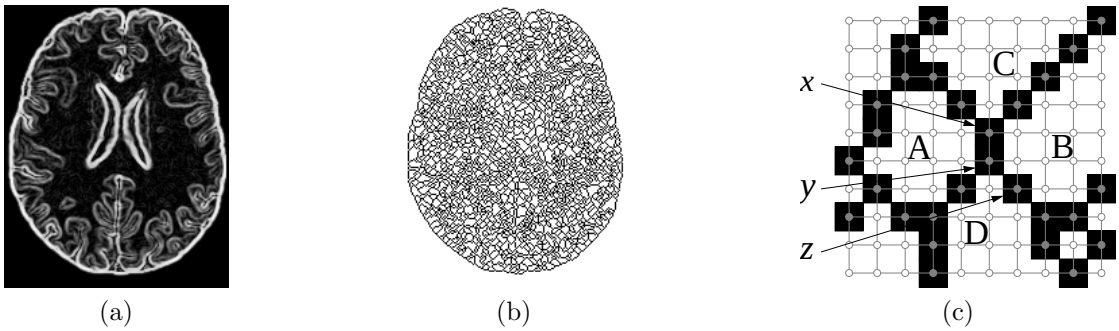

Figure 1: (a): Cross-section of a brain, after applying a gradient operator. (b): A segmentation of (a) (obtained by a watershed algorithm [5] using the adjacency Γ_1^2). (c): A zoom on a part of (b); the graph induced by Γ_1^2 is superimposed in gray.

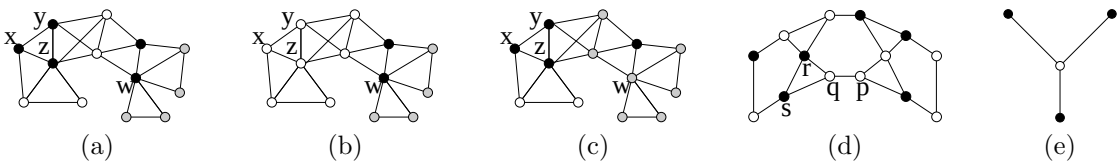

Figure 2: (a) A graph (E, Γ) (which is not a PFG) and a set $S \subset E$ (gray and white vertices). (b,c) Illustrations of region merging operations and the problem encountered by this operation [see text]. (d) Example of a PFG. (c) The graph $G^{\blacktriangle}$ used in Th. 2 to characterize the PFGs.

A *graph* is a pair (E, Γ) where E is a set and Γ is a binary relation on E (*i.e.*, $\Gamma \subseteq E \times E$) which is anti-reflexive (for any $x \in E$, $(x, x) \notin \Gamma$) and symmetric (for any x and y in E, $(y, x) \in \Gamma$ whenever $(x, y) \in \Gamma$). Each element of E (resp. Γ) is called a *vertex or a point* (resp. an *edge*). We will also denote by Γ the map from E to 2^E such that, for any $x \in E, \Gamma(x) = \{y \in E \mid (x, y) \in \Gamma\}$. Let $x \in E$, the set $\Gamma(x)$ is called the *neighborhood* of x and if $y \in \Gamma(x)$, we say that y is *adjacent to* x. If $X \subseteq E$, the *neighborhood* of X, denoted by $\Gamma(X)$, is the set $[\cup_{x \in X} \Gamma(x)] \setminus X$.

Let (E, Γ) be a graph and $X \subseteq E$. A *path* in X is a sequence $\langle x_0, \ldots, x_\ell \rangle$ such that $x_i \in X$, $i \in [0, \ell]$, and $(x_{i-1}, x_i) \in \Gamma$, $i \in [1, \ell]$. The set X is *connected* if, for any $x, y \in X$, there exists a path in X from x to y. Let $Y \subseteq X$, we say that Y is a *(connected) component* of X if Y is connected and maximal for this property, *i.e.*, if $Z = Y$ whenever $Y \subseteq Z \subseteq X$ and Z connected.

Important remark. From now on, when speaking about a graph $G = (E, \Gamma)$, we assume that E is connected and that G is *locally finite*, *i.e.*, the set $\Gamma(x)$ is finite for any $x \in E$.

1.2 Region merging and perfect fusion graphs

Consider the graph (E, Γ) depicted in Fig. 2a, where a subset S of E (white and gray vertices) is composed of four components. If we replace the set S by, for instance, the set $S \cup T$ where $T = \{x, y, z\}$, we obtain a set composed of three regions (see Fig. 2b). We can also say that we "merged two components of S through T". This operation may be seen as an "elementary merging" in the sense that only two regions of S were merged while all other regions of S were preserved. On the opposite, replacing the set S by the set $S \cup T$ where $T = \{w\}$ (see Fig. 2c), would merge three components of S. This section recalls the definitions introduced in [2] related to such merging operations in graphs. Then, we remind the definition of the perfect fusion graphs, which are the graphs in which any two neighboring regions can be merged through their common neighborhood while preserving all other regions.

Let (E, Γ) be a graph and let $S \subseteq E$. Let A and B be two distinct components of S and $T \subseteq \overline{S}$. We say that *A and B can be merged (for S) through T* if A and B are the only connected components of S adjacent to T and if $A \cup B \cup T$ is connected.

In other words (see Property 21 in [2] for a formal proof), the two regions A and B can be merged through T if and only if $A \cup B \cup T$ is a component of $S \cup T$. More precisely, they can be merged if and only if the components of $S \cup T$ are the same as the components of S excepting that A and B are replaced by $A \cup B \cup T$. For instance, in Fig. 2a the two white components can be merged through $\{x, y, z\}$ but the two gray components cannot be merged through $\{z\}$.

Let (E, Γ) be a graph, $S \subseteq E$, and let A and B be two distinct connected components of S. We set $\Gamma(A, B) = \Gamma(A) \cap \Gamma(B)$ and we say that $\Gamma(A, B)$ is the *common neighborhood of A and B*. If the common neighborhood of A and B is nonempty, A and B are said to be *neighbors*.

Definition 1 (perfect fusion graph). *Let (E, Γ) be a graph. We say that* (E, Γ) is a perfect fusion graph (PFG) *if, for any $S \subseteq E$, any two connected components A and B of S which are neighbors can be merged through* $\Gamma(A, B)$.

The graph of Fig. 2a is not a PFG since the two gray components cannot be merged through their common neighborhood $\{w\}$. On the other hand, the graph of Fig. 2d is a PFG.

The definition of the PFGs is based on a condition which must be verified for all subsets of the vertex sets. This means, if we want to check whether a graph is a PFG, then, using the straightforward method based on the definition, this will cost an exponential time with respect to the number of vertices. In fact, the PFGs can be recognized in a simpler way thanks to the following conditions which can be checked independently in the neighborhood of each vertex.

Let $G = (E, \Gamma)$ be a graph and let $X \subseteq E$. The *subgraph of G induced by X* is the graph $G_X = (X, \Gamma \cap [X \times X])$. In this case, we also say that G_X is a *subgraph of G*.

We denote by $G^{\blacktriangle}$ the graph of Fig. 2e.

Theorem 2 (from Th. 41 in [2]). *The three following statements are equivalent:*
i) (E, Γ) is a PFG;
ii) the graph $G^{\blacktriangle}$ is not a subgraph of (E, Γ);
iii) for any $x \in E$, any $X \subseteq \Gamma(x)$ contains at most two connected components.

Thanks to Th. 2, it can be verified that the graph (E, Γ) depicted in Fig. 2 is a PFG. Indeed, $G^{\blacktriangle}$ is not a subgraph of (E, Γ). Remark in particular that the subgraph induced by $\{p, q, r, s\}$ is not $G^{\blacktriangle}$ since it contains the edge (r, s).

We finish this section by reminding that, through the notion of a *line graph*, the region merging approaches based on "separations" composed of edges (*e.g.*, inter-pixel elements) also fall in the scope of perfect fusion graphs (see Prop. 29 in [2]).

2 Cubical grids in arbitrary dimensions

Digital images are defined on (hyper-) rectangular subsets of $\mathbb{Z}^d$ (with $d \in \mathbb{N}^\star$). Therefore, for region merging applications, $\mathbb{Z}^d$ must be equipped with an adjacency relation reflecting the geometrical relationship between its elements. We provide, in this section, a set of definitions that allows for recovering the adjacency relations which are the most frequently used in 2- and 3-dimensional image analysis and permit to extend them to images of arbitrary dimension.

Let $\mathbb{Z}$ be the set of integers. We define the families of sets H_0^1 and H_1^1 such that $H_0^1 = \{\{a\} \mid a \in \mathbb{Z}\}$ and $H_1^1 = \{\{a, a+1\} \mid a \in \mathbb{Z}\}$. Let $m \in [0, d]$. A subset C of $\mathbb{Z}^d$ which is the Cartesian product of exactly m elements of H_1^1 and $(d - m)$ elements of H_0^1 is called a *(m-)cube of $\mathbb{Z}^d$*.

Observe that an m-cube of $\mathbb{Z}^d$ is a point if $m = 0$, a (unit) interval if $m = 1$, a (unit) square if $m = 2$ and a (unit) cube if $m = 3$.

Let $\mathcal{C}$ be a set of cubes of $\mathbb{Z}^d$. The *binary relation induced by $\mathcal{C}$* is the set of all pairs (x, y) of $\mathbb{Z}^d$ such that there exists a cube in $\mathcal{C}$ which contains both x and y.

Definition 3 (m-adjacency). *Let $m \in [1, d]$. The m-*adjacency on $\mathbb{Z}^d$ *is the binary relation Γ_m^d induced by the set of all m-cubes of $\mathbb{Z}^d$. If $(x, y) \in \Gamma_m^d$, we say that x and y are m-*adjacent.

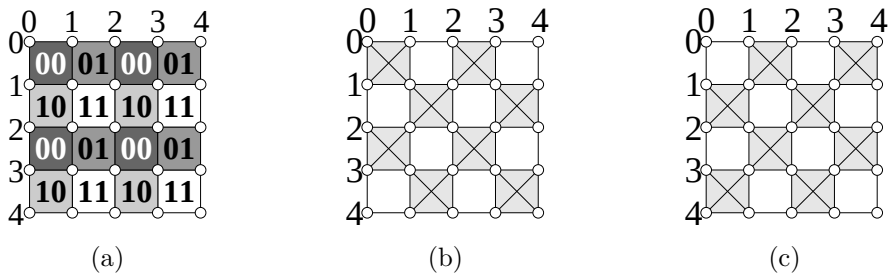

Figure 3: Chessboards and perfect fusion grids on $\mathbb{Z}^2$. (a): The map ψ; (b,c): two subgraphs of $(\mathbb{Z}^2, \Lambda^2_{11})$ and $(\mathbb{Z}^2, \Lambda^2_{10})$ induced by $\{0, \ldots, 4\} \times \{0, \ldots, 4\}$ with in gray the associated chessboards.

Observe that two points of $\mathbb{Z}^d$ are m-adjacent if and only if $|x_i - y_i| \leq 1$ for any $i \in [1, d]$ and $(\sum_{i=1}^d |x_i - y_i|) \leq m$. In the literature [4], Γ^2_1, Γ^2_2, Γ^3_1, Γ^3_2 and Γ^3_3 are often referred to as the 4-, 8-, 6- 18- and 26-adjacencies. On $\mathbb{Z}^d$, Γ^d_1 and Γ^d_d are sometimes called direct and indirect adjacencies.

The graphs induced by Γ^2_1 are not in general PFGs. For instance, in Fig. 1c the two neighboring regions A and B cannot be merged through $\Gamma^2_1(A, B) = \{x, y\}$ and thus the graph is not a PFG. More generally, the graphs which are the most frequently used in image analysis (namely, those induced by Γ^d_1 and Γ^d_d, with $d = 2, 3$) are not PFGs (see Sec. 6 in [2]).

3 Perfect fusion grids

In [2], we introduced a family of graphs on $\mathbb{Z}^d$, called the *perfect fusion grids*, which are indeed PFGs and which are "between" Γ^d_1 and Γ^d_d. In this section, we recall the definition of the perfect fusion grids, give a local characterization and then show that they are the only PFGs "between" the direct and indirect adjacency relations.

The perfect fusion grids can be defined by the mean of *chessboards*. Intuitively, a chessboard $\mathcal{C}$ on $\mathbb{Z}^d$ is a spanning set of d-cubes (*i.e.*, $\cup\{C \in \mathcal{C}\} = \mathbb{Z}^d$) such that the intersection of any two cubes in $\mathcal{C}$ is either empty or reduced to a point. For instance, the gray cubes shown in Figs. 3b and c constitute subsets of the two chessboards on $\mathbb{Z}^2$. The perfect fusion grids are the graphs induced by the chessboards on $\mathbb{Z}^d$ (see, *e.g.*, the graphs of Figs. 3b and c). Fig. 4a shows a set of regions obtained in this grid by a watershed algorithm [3]. Note that the problems pointed out in the introduction do not exist in this case: any pair of neighboring regions can be merged by simply removing from the black vertices the points which are adjacent to both regions (see Fig. 4b,c).

Let $\mathcal{C}^d$ be the set of all d-cubes of $\mathbb{Z}^d$, we define the map φ from $\mathcal{C}^d$ to $\mathbb{Z}^d$, such that for any $C \in \mathcal{C}^d$, $\varphi(C)_i = \min\{x_i \mid x \in C\}$, where $\varphi(C)_i$ is the i-th coordinate of $\varphi(C)$. It may be seen that C is equal to the Cartesian product: $\{\varphi(C)_1, \varphi(C)_1 + 1\} \times \cdots \times \{\varphi(C)_d, \varphi(C)_d + 1\}$. Thus, clearly φ is a bijection and allows for indexing the cubes of $\mathbb{Z}^d$.

Let $\mathbb{B} = \{0, 1\}$. We set $\overline{0} = 1$ and $\overline{1} = 0$. A *binary word of length d* is an element in $\mathbb{B}^d$. If $b = (b_1, \ldots, b_d)$ is in $\mathbb{B}^d$, we define $\overline{b}$ as the binary word of $\mathbb{B}^d$ such that for any $i \in [1, d]$, $(\overline{b})_i = (\overline{b_i})$.

We define the map ψ from $\mathcal{C}^d$ to $\mathbb{B}^d$ such that for any $C \in \mathcal{C}^d$ and any $i \in [1, d]$, $\psi(C)_i$ is equal to $[\varphi(C)_i \mod 2]$, that is the remainder in the integer division of $\varphi(C)_i$ by 2.

Definition 4 (chessboard & perfect fusion grid). *Let $b \in \mathbb{B}^d$.*
We set $\mathcal{C}^d_b = \{C \in \mathcal{C}^d \mid \psi(C) = b\}$ and we say that the set $\mathcal{C}^d_b \cup \mathcal{C}^d_{\overline{b}}$ is a (global) chessboard on $\mathbb{Z}^d$. *Let $\mathcal{C}$ be the chessboard on $\mathbb{Z}^d$ defined by $\mathcal{C}^d_b \cup \mathcal{C}^d_{\overline{b}}$. We denote by Λ^d_b the binary relation induced by $\mathcal{C}$ and we say that the pair $(\mathbb{Z}^d, \Lambda^d_b)$ is a* perfect fusion grid on $\mathbb{Z}^d$.

Fig. 3 illustrates these definitions on $\mathbb{Z}^2$.

Since the cardinality of $\mathbb{B}^d$ is 2^d, from the previous definition, it can be deduced that there are exactly 2^{d-1} distinct perfect fusion grids on $\mathbb{Z}^d$. However, any two perfect fusion grids are

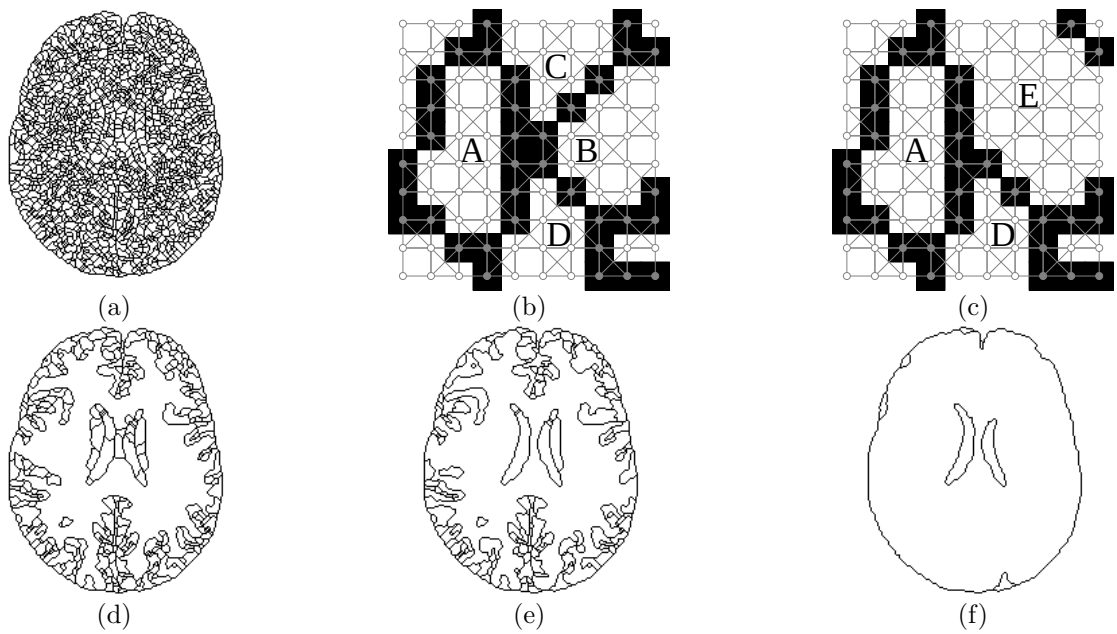

Figure 4: (a) A segmentation of Fig. 1a obtained on a perfect fusion grid. (b) A zoom on a part of (a); the regions A, B, C and D correspond to the ones of Fig. 1c; the corresponding perfect fusion grid is shown in gray. (c) Same as (b) after having merged B and C to form a new region E. (b,c,d) Several steps, starting from (a), of a region merging process on a perfect fusion grid.

equivalent up to a "binary translation". More precisely, it is proved in [2] that for any b and b' in $\mathbb{B}^d$, there exists $t \in \mathbb{B}^d$ such that, for any x and y in $\mathbb{Z}^d$, $y \in \Lambda_b^d(x)$ if and only if $y+t \in \Lambda_{b'}^d(x+t)$.

3.1 Local characterization

Certain classes of graphs can be locally characterized, *i.e.*, we can test if an arbitrary graph belongs to such a class by independently checking a condition in the neighborhood of each point. The next theorem states that the chessboards (hence the perfect fusion grids) can be locally characterized.

We set $\mathbb{U}_\star = \{1, -1\}$. Let $x \in \mathbb{Z}^d$ and $u \in \mathbb{U}^d$, we denote by $C(u, x)$ the d-cube of $\mathbb{Z}^d$ defined by $\{x_1, x_1 + u_1\} \times \cdots \times \{x_d, x_d + u_d\}$ and we set $\hat{C}(u, x) = C(-u, x)$. In other words, $C(u, x)$ is the set of all points y such that, for any $i \in [1, d]$, $y_i = x_i$ or $y_i = x_i + u_i$.

Definition 5 (local chessboard). *Let $\mathcal{C}$ be a set of d-cubes of $\mathbb{Z}^d$. We say that $\mathcal{C}$ is a* local chessboard *on $\mathbb{Z}^d$ if, for any $x \in \mathbb{Z}^d$, there exists $u \in \mathbb{U}_\star^d$ such that $C(u, x)$ and $\hat{C}(u, x)$ belong to $\mathcal{C}$ and such that they are the only two elements in $\mathcal{C}$ which contain x.*

On $\mathbb{Z}^2$ (resp. $\mathbb{Z}^3$), a local chessboard $\mathcal{C}$ is a set of 2-cubes (resp. 3-cubes) such that, for any point x, the cubes of $\mathcal{C}$ which contain x match one of two (resp. four) configurations depicted in the first (resp. second) column of Fig. 5. Observe that this notion of a local chessboard corresponds exactly to the intuitive idea given in the introduction of the section. As assessed by the following theorem, we can indeed prove the equivalence between global and local chessboards.

Theorem 6. *Let $\mathcal{C}$ be a set of d-cubes of $\mathbb{Z}^d$. The set $\mathcal{C}$ is a chessboard on $\mathbb{Z}^d$ if and only if $\mathcal{C}$ is a local chessboard on $\mathbb{Z}^d$.*

Furthermore, if $\mathcal{C} = \mathcal{C}_b^d \cup \mathcal{C}_{\bar{b}}^d$ (with $b \in \mathbb{B}^d$), then for any $x \in \mathbb{Z}^d$, the only two d-cubes of $\mathcal{C}$ which contain x are defined by $C(u, x)$ and $\hat{C}(u, x)$ with $u \in \mathbb{U}_\star^d$, and $u_i = (-1)^{(x_i - b_i)}$ for any $i \in [1, d]$.

From this local characterization of chessboards and by Th. 2.iii, it can be deduced that any perfect fusion grid $(\mathbb{Z}^d, \Lambda_b^d)$ is indeed a PFG between Γ_1^d and Γ_d^d (in the sense $\Gamma_1^d \subseteq \Lambda_b^d \subseteq \Gamma_d^d$). Fur-

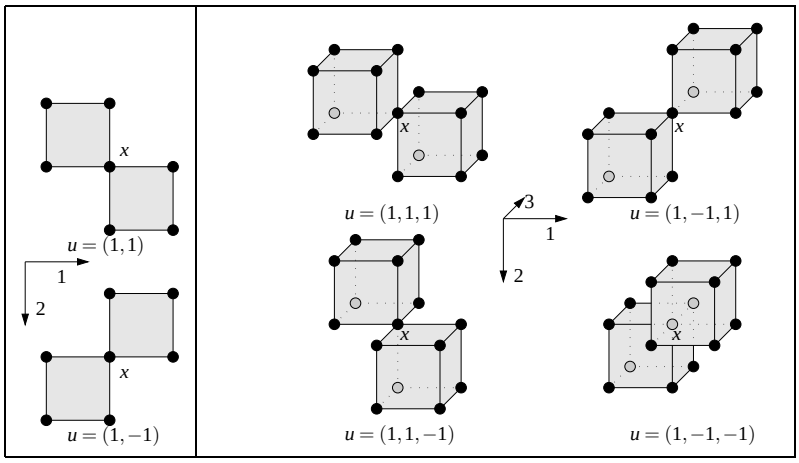

Figure 5: First (resp. second) column: local configurations in a local chessboard on $\mathbb{Z}^2$ (resp. $\mathbb{Z}^3$). In each column, we assume that $\mathbb{Z}^d$ is oriented as shown by the arrows. Then, the element of $u \in \mathbb{U}^d_\star$ which "generates" each configuration as $C(u, x \cup \hat{C}(u, x))$ is written under the configuration.

thermore, we point out that the second assertion of Th. 6 constitutes a practical way for computing the neighborhood of a point in a perfect fusion grid, as it is often required in applications.

3.2 Unicity theorem

The following theorem asserts that the only PFGs "between" Γ^d_1 and Γ^d_d are the perfect fusion grids. Since any two perfect fusion grids are equivalent up to a binary translation, this result establishes the uniqueness of the perfect fusion grid in any dimension $d \in \mathbb{N}$.

Theorem 7. *The graph $(\mathbb{Z}^d, \Gamma^d)$ is a PFG such that $\Gamma^d_1 \subseteq \Gamma^d \subseteq \Gamma^d_d$ if and only if it is a perfect fusion grid on $\mathbb{Z}^d$.*

In other words, the perfect fusion grid is, in any dimension, the only graph, "between" the direct and indirect adjacencies, which guarantees that any two neighboring regions can be merged through their common neighborhood while preserving all other regions. Fig. 4a,d,e,f show an example in image segmentation of such a region merging procedure in a perfect fusion grid.

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