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Sobolev stability of plane wave solutions to the cubic nonlinear Schrödinger equation on a torus

Erwan Faou, Ludwig Gauckler and Christian Lubich

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Abstract

It is shown that plane wave solutions to the cubic nonlinear Schrödinger equation on a torus behave orbitally stable under generic perturbations of the initial data that are small in a high-order Sobolev norm, over long times that extend to arbitrary negative powers of the smallness parameter. The perturbation stays small in the same Sobolev norm over such long times. The proof uses a Hamiltonian reduction and transformation and, alternatively, Birkhoff normal forms or modulated Fourier expansions in time.

1 Introduction and statement of the result

Consider the cubic nonlinear Schrödinger equation (NLS) on the d -dimensional torus $\mathbb{T}^d = \mathbb{R}^d / (2\pi\mathbb{Z})^d$, for arbitrary dimension $d \geq 1$, in the defocusing ($\lambda = 1$) or focusing ($\lambda = -1$) case,

$$i\partial_t u = -\Delta u + \lambda|u|^2 u, \quad x \in \mathbb{T}^d, t \in \mathbb{R}. \quad (1.1)$$

For initial data made of a single Fourier mode, $u_*(x, 0) = \rho e^{im \cdot x}$, the equation has the plane-wave solution $u_*(x, t) = \rho e^{i(m \cdot x - \omega t)}$ with $\omega = |m|^2 + \lambda \rho^2$. We show that under *generic* perturbations of such initial data by functions with small H^s Sobolev norm, for sufficiently large Sobolev exponent s , the solution remains essentially localized in the m th Fourier mode over very long times, and the perturbation remains small in the same H^s norm. For the precise formulation of the result we decompose the solution in the Fourier basis, $u(x, t) = \sum_{j \in \mathbb{Z}^d} u_j(t) e^{ij \cdot x}$.

Theorem 1.1 *Let $\rho_0 > 0$ be such that $1 + 2\lambda\rho_0^2 > 0$, and let $N > 1$ be fixed arbitrarily. There exist $s_0 > 0$ and a set of full measure \mathcal{P} in the interval $(0, \rho_0]$ such that for every $s \geq s_0$ and every $\rho \in \mathcal{P}$, there exists $\varepsilon_0 > 0$ such that for every $m \in \mathbb{Z}^d$ the following holds: if the initial data $u(\bullet, 0)$ is such that*

$$\|u(\bullet, 0)\|_{L^2} = \rho \quad \text{and} \quad \|e^{-im\cdot\bullet}u(\bullet, 0) - u_m(0)\|_{H^s} = \varepsilon \leq \varepsilon_0,$$

then the solution of (1.1) with these initial data satisfies

$$\|e^{-im\cdot\bullet}u(\bullet, t) - u_m(t)\|_{H^s} \leq 2\varepsilon \quad \text{for} \quad t \leq \varepsilon^{-N}.$$

To our knowledge, this is the first long-time orbital stability result as well as long-time high-regularity result for the cubic NLS (1.1) in dimension $d > 1$. It is a contrasting counterpart to the instability and ill-posedness results for *rough* perturbations given by Christ, Colliander & Tao [7, 8].

In the 1D case, much is already known about orbital stability of periodic waves for the cubic NLS through the work by Cazenave & Lions [6] and Gallay & Haragus [13, 14]; see also further references therein.

Technically more closely related to the present work are long-time stability and high-regularity results by Bambusi & Grébert [4, 2, 18] and Gauckler & Lubich [15] for small solutions to modifications of the periodic cubic NLS (1.1) by the addition of a convolution term $V * u$, which eliminates the resonance of the frequencies of the linearization of (1.1) around 0. Such a resonance-removing modification by a convolution potential was previously studied also by Bourgain [5] and more recently by Eliasson & Kuksin [12], but we emphasize that such a modification of the equation is not required for the problem considered here, essentially because we do not consider small solutions of (1.1).

After the reductions and transformations of Section 2, the problem appears in a Hamiltonian form for which there are two known techniques to arrive at Theorem 1.1: *Birkhoff normal forms*, which use a sequence of nonlinear canonical coordinate transforms to transform the system to a form from which the dynamical properties can be read off, and *modulated Fourier expansions*, which embed the system into a larger modulation system having a Hamiltonian structure with a group invariance property that yields the existence of almost-invariants which allow us to infer the long-time properties. In Section 3 we reduce Theorem 1.1 to a known abstract result on the long-time near-conservation of super-actions, which was proved by Grébert [18] via Birkhoff normal forms and by Gauckler [16] via modulated Fourier expansions, under differing conditions which we verify for both approaches.

2 Reductions and Transformations

2.1 Reduction to the case $m = 0$

In terms of Fourier coefficients, (1.1) is given by

$$i\dot{u}_j = |j|^2 u_j + \lambda \sum_{j=j_1-j_2+j_3} u_{j_1} \bar{u}_{j_2} u_{j_3}, \quad (2.1)$$

where $|j|$ denotes the Euclidean norm of $j \in \mathbb{Z}^d$. With the given $m \in \mathbb{Z}^d$, we transform to

$$v_j = u_{j+m} e^{it(|m|^2 + 2j \cdot m)}. \quad (2.2)$$

Note that this transformation preserves the L^2 norm. The equation for v_j is

$$(|m|^2 + 2j \cdot m)v_j + i\dot{v}_j = |j+m|^2 v_j + \lambda \sum_{j=j_1-j_2+j_3} v_{j_1} \bar{v}_{j_2} v_{j_3}$$

or equivalently

$$i\dot{v}_j = |j|^2 v_j + \lambda \sum_{j=j_1-j_2+j_3} v_{j_1} \bar{v}_{j_2} v_{j_3},$$

so that $v(x, t) = \sum_{j \in \mathbb{Z}^d} v_j(t) e^{ij \cdot x}$ is a solution of (1.1) and is localized in the zero mode if u is localized in the m th mode. In other words, up to the transformation (2.2), we can restrict our attention to the case $m = 0$.

2.2 Elimination of the zero mode

We separate the 0-mode:

$$\begin{aligned} i\dot{v}_j &= (|j|^2 + 2\lambda|v_0|^2)v_j + \lambda v_0^2 \bar{v}_{-j} \\ &\quad + 2\lambda \sum_{\substack{j=-j_2+j_3 \\ j_2, j_3 \neq 0}} v_0 \bar{v}_{j_2} v_{j_3} + \lambda \sum_{\substack{j=j_1+j_3 \\ j_1, j_3 \neq 0}} v_{j_1} \bar{v}_0 v_{j_3} + \lambda \sum_{\substack{j=j_1-j_2+j_3 \\ j_1, j_2, j_3 \neq 0}} v_{j_1} \bar{v}_{j_2} v_{j_3} \\ i\dot{v}_0 &= \lambda|v_0|^2 v_0 + 2\lambda \sum_{j \neq 0} v_j \bar{v}_j v_0 + \lambda \sum_{j \neq 0} v_j \bar{v}_0 v_{-j} + \lambda \sum_{\substack{0=j_1-j_2+j_3 \\ j_1, j_2, j_3 \neq 0}} v_{j_1} \bar{v}_{j_2} v_{j_3} \end{aligned}$$

and make the change of variables $(v_0, v_j) \mapsto (a, \theta, w_j)$ defined by

$$v_0 = \sqrt{\rho^2 - a} e^{-i\theta}, \quad \text{and} \quad v_j = w_j e^{-i\theta}, \quad j \in \mathcal{Z} := \mathbb{Z}^d \setminus \{0\}.$$

By the conservation of the L^2 norm and Parseval's equality we have for all times t ,

$$\rho^2 = |v_0|^2 + \sum_{j \neq 0} |v_j|^2 = \rho^2 - a + \sum_{j \neq 0} |w_j|^2$$

which means

$$a = \sum_{j \neq 0} |w_j|^2. \quad (2.3)$$

Hence we can completely forget the dynamics of a : it will be controlled by the w_j using (2.3). The equation for w_j reads

$$\begin{aligned} i\dot{w}_j + \dot{\theta}w_j &= (|j|^2 + 2\lambda(\rho^2 - a))w_j + \lambda(\rho^2 - a)\bar{w}_{-j} \\ &+ 2\lambda \sum_{\substack{j=-j_2+j_3 \\ j_2, j_3 \neq 0}} \sqrt{\rho^2 - a} \bar{w}_{j_2} w_{j_3} + \lambda \sum_{\substack{j=j_1+j_3 \\ j_1, j_3 \neq 0}} w_{j_1} \sqrt{\rho^2 - a} w_{j_3} \\ &+ \lambda \sum_{\substack{j=j_1-j_2+j_3 \\ j_1, j_2, j_3 \neq 0}} w_{j_1} \bar{w}_{j_2} w_{j_3}. \end{aligned} \quad (2.4)$$

To obtain $\dot{\theta}$, let us write the equation for v_0 in terms of (a, θ) :

$$\begin{aligned} \dot{\theta} \sqrt{\rho^2 - a} - \frac{i\dot{a}}{2\sqrt{\rho^2 - a}} &= \lambda(\rho^2 - a)^{3/2} + 2\lambda \sqrt{\rho^2 - a} \sum_{j \neq 0} |w_j|^2 \\ &+ \lambda \sqrt{\rho^2 - a} \sum_{j \neq 0} w_j w_{-j} + \lambda \sum_{\substack{j_1-j_2+j_3=0 \\ j_1, j_2, j_3 \neq 0}} w_{j_1} \bar{w}_{j_2} w_{j_3}. \end{aligned}$$

The equation for $\dot{\theta}$ can therefore be written, using (2.3),

$$\dot{\theta} = \lambda(\rho^2 + a) + \lambda \operatorname{Re} \left(\sum_{j \neq 0} w_j w_{-j} + \frac{1}{\sqrt{\rho^2 - a}} \sum_{\substack{j_1-j_2+j_3=0 \\ j_1, j_2, j_3 \neq 0}} w_{j_1} \bar{w}_{j_2} w_{j_3} \right).$$

Inserting this equation and (2.3) into (2.4), we arrive at a system of differential equations for the w_j , $j \in \mathcal{Z} = \mathbb{Z}^d \setminus \{0\}$.

2.3 The reduced Hamiltonian system

This reduced system of differential equations turns out to be Hamiltonian (see Appendix A),

$$i\dot{w}_j = \frac{\partial \tilde{H}}{\partial \bar{w}_j}(w, \bar{w}), \quad j \in \mathcal{Z}, \quad (2.5)$$

with the real-valued Hamilton function

$$\begin{aligned}
\tilde{H}(w, \bar{w}) &= \sum_{j_1} (|j_1|^2 + \lambda \rho^2) w_{j_1} \bar{w}_{j_1} + \frac{\lambda}{2} \rho^2 \sum_{j_1} \bar{w}_{j_1} \bar{w}_{-j_1} + \frac{\lambda}{2} \rho^2 \sum_{j_1} w_{j_1} w_{-j_1} \\
&+ \frac{\lambda}{2} \sum_{j_1+j_2-j_3-j_4=0} w_{j_1} w_{j_2} \bar{w}_{j_3} \bar{w}_{j_4} - \frac{3\lambda}{2} \left(\sum_{j_1} w_{j_1} \bar{w}_{j_1} \right) \left(\sum_{j_2} w_{j_2} \bar{w}_{j_2} \right) \\
&- \frac{\lambda}{2} \left(\sum_{j_1} w_{j_1} w_{-j_1} \right) \left(\sum_{j_2} w_{j_2} \bar{w}_{j_2} \right) - \frac{\lambda}{2} \left(\sum_{j_1} \bar{w}_{j_1} \bar{w}_{-j_1} \right) \left(\sum_{j_2} w_{j_2} \bar{w}_{j_2} \right) \\
&+ \lambda \left(\sum_{j_1+j_2-j_3=0} w_{j_1} w_{j_2} \bar{w}_{j_3} + \sum_{j_1-j_2-j_3=0} w_{j_1} \bar{w}_{j_2} \bar{w}_{j_3} \right) \sqrt{\rho^2 - \sum_{j_4} w_{j_4} \bar{w}_{j_4}}.
\end{aligned} \tag{2.6}$$

Expanding $\sqrt{\rho^2 - x}$ into a convergent power series for $|x| < \rho^2$, we can write the Hamiltonian (2.6) as the infinite sum

$$\tilde{H}(w, \bar{w}) = \sum_{r \geq 2} \tilde{H}_r(w, \bar{w})$$

where $\tilde{H}_r(w, \bar{w})$ is a homogeneous polynomial of degree r in terms of (w_j, \bar{w}_j) , which is of the form

$$\tilde{H}_r(w, \bar{w}) = \sum_{p+q=r} \sum_{\substack{(k,l) \in \mathcal{Z}^p \times \mathcal{Z}^q \\ \mathcal{M}(k,l)=0}} \tilde{H}_{kl} w_{k_1} \cdots w_{k_p} \bar{w}_{l_1} \cdots \bar{w}_{l_q}$$

where

$$\mathcal{M}(k, l) = k_1 + \dots + k_p - l_1 - \dots - l_q \tag{2.7}$$

denotes the *momentum* of the multi-index (k, l) . We note that the Taylor expansion of \tilde{H} contains only terms with zero momentum, and its coefficients satisfy the bound

$$|\tilde{H}_{kl}| \leq \tilde{M} \tilde{L}^{p+q} \quad \text{for all } (k, l) \in \mathcal{Z}^p \times \mathcal{Z}^q, \tag{2.8}$$

where \tilde{M} and \tilde{L} depend on ρ .

2.4 Diagonalization and non-resonant frequencies

We study now the linear part of the system (2.5). As we will see, its eigenvalues are non-resonant for almost all parameters ρ . Moreover, we can control the diagonalization of this linear operator.

The linear part in the differential equation for w_j is $(|j|^2 + \lambda\rho^2)w_j + \lambda\rho^2\bar{w}_{-j}$. On taking the equation for w_j together with that for \bar{w}_{-j} , we are thus led to consider the matrix (for $n = |j|^2$)

$$A_n = \begin{pmatrix} n + \lambda\rho^2 & \lambda\rho^2 \\ -\lambda\rho^2 & -n - \lambda\rho^2 \end{pmatrix}.$$

Lemma 2.1 *For all $n \geq 1$, the matrix A_n is diagonalized by a 2×2 matrix S_n that is real symplectic and hermitian and has condition number smaller than 2:*

$$S_n^{-1} A_n S_n = \begin{pmatrix} \Omega_n & 0 \\ 0 & -\Omega_n \end{pmatrix} \quad \text{with} \quad \Omega_n = \sqrt{n^2 + 2n\lambda\rho^2}.$$

Proof. We obtain

$$S_n = \frac{1}{\sqrt{(\Omega_n + n)(\Omega_n + n + 2\lambda\rho^2)}} \begin{pmatrix} n + \lambda\rho^2 + \Omega_n & -\lambda\rho^2 \\ -\lambda\rho^2 & n + \lambda\rho^2 + \Omega_n \end{pmatrix}$$

and

$$S_n^{-1} = \frac{1}{\sqrt{(\Omega_n + n)(\Omega_n + n + 2\lambda\rho^2)}} \begin{pmatrix} n + \lambda\rho^2 + \Omega_n & \lambda\rho^2 \\ \lambda\rho^2 & n + \lambda\rho^2 + \Omega_n \end{pmatrix},$$

and the statements of the lemma then follow by direct verification. \blacksquare

Note that the condition $1 + 2\lambda\rho^2 > 0$ in Theorem 1.1 ensures that all the eigenvalues Ω_n are real, or equivalently, that the linearization of the system (2.5) at 0 is stable. The frequencies Ω_n turn out to satisfy Bambusi's non-resonance inequality [1, 4] for almost all norm parameters $\rho > 0$.

Lemma 2.2 *Let $r > 1$ and $\rho_0 > 0$ with $1 + 2\lambda\rho_0^2 > 0$. There exist $\alpha = \alpha(r) > 0$ and a set of full Lebesgue measure $\mathcal{P} \subset (0, \rho_0]$ such that for every $\rho \in \mathcal{P}$ there is a $\gamma > 0$ such that the following non-resonance condition is satisfied: for all positive integers p, q with $p + q \leq r$ and for all $m = (m_1, \dots, m_p) \in \mathbb{N}^p$ and $n = (n_1, \dots, n_q) \in \mathbb{N}^q$,*

$$|\Omega_{m_1} + \dots + \Omega_{m_p} - \Omega_{n_1} - \dots - \Omega_{n_q}| \geq \frac{\gamma}{\mu_3(m, n)^\alpha}, \quad (2.9)$$

except if the frequencies cancel pairwise. Here, $\mu_3(m, n)$ denotes the third-largest among the integers $m_1, \dots, m_p, n_1, \dots, n_q$.

Proof. Note that

$$\Omega_n = n + \lambda\rho^2 - \frac{\rho^4}{2(n + \lambda\rho^2)} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

We can then use the proof of [4, Section 5.1] (see in particular (5.13) in [4]). \blacksquare

2.5 The transformed Hamiltonian system

Applying the symplectic linear transformation

$$\begin{pmatrix} w_j \\ \bar{w}_{-j} \end{pmatrix} = S_n \begin{pmatrix} \xi_j \\ \bar{\xi}_{-j} \end{pmatrix} \quad \text{for } j \in \mathcal{Z} \text{ and } n = |j|^2,$$

with the matrices S_n of Lemma 2.1, to the Hamiltonian system (2.5) of equations for w_j , we end up with a Hamiltonian system

$$i \frac{d}{dt} \xi_j(t) = \frac{\partial H}{\partial \bar{\xi}_j}(\xi(t), \bar{\xi}(t)), \quad j \in \mathcal{Z} = \mathbb{Z}^d \setminus \{0\},$$

with the real-valued Hamilton function

$$H(\xi, \bar{\xi}) = \tilde{H}(w, \bar{w}),$$

with \tilde{H} of (2.6). This Hamiltonian is of the form

$$H(\xi, \bar{\xi}) = \sum_{j \in \mathcal{Z}} \omega_j |\xi_j|^2 + P(\xi, \bar{\xi}), \quad (2.10)$$

where the frequencies are $\omega_j = \Omega_n$ for $|j|^2 = n$ with $\Omega_n = \sqrt{n^2 + 2n\lambda\rho^2}$, and the non-quadratic term P is of the form

$$P(\xi, \bar{\xi}) = \sum_{p+q \geq 3} \sum_{\substack{k \in \mathcal{Z}^p, l \in \mathcal{Z}^q \\ \mathcal{M}(k,l)=0}} H_{kl} \xi_{k_1} \cdots \xi_{k_p} \bar{\xi}_{l_1} \cdots \bar{\xi}_{l_q}, \quad (2.11)$$

where the sum is still only over multi-indices with zero momentum (2.7), since the transformation mixes only terms that give the same contribution to the momentum. From (2.8) and Lemma 2.1 we obtain the following bound for the Taylor coefficients.

Lemma 2.3 *There exist $M > 0$ and $L > 0$ such that for all positive integers p, q with $p + q \geq 3$ the coefficients in (2.11) are bounded by*

$$|H_{kl}| \leq M L^{p+q} \quad \text{for all } k \in \mathcal{Z}^p, l \in \mathcal{Z}^q.$$

The Hamiltonian equations of motion are now

$$i \frac{d}{dt} \xi_j(t) = \omega_j \xi_j(t) + \frac{\partial P}{\partial \bar{\xi}_j}(\xi(t), \bar{\xi}(t)), \quad j \in \mathcal{Z}, \quad (2.12)$$

where the nonlinearity is of the form

$$\frac{\partial P}{\partial \bar{\xi}_j}(\xi, \bar{\xi}) = \sum_{p+q \geq 2} \sum_{\substack{k \in \mathcal{Z}^p, l \in \mathcal{Z}^q \\ \mathcal{M}(k,l)=j}} P_{j,k,l} \xi_{k_1} \cdots \xi_{k_p} \bar{\xi}_{l_1} \cdots \bar{\xi}_{l_q} \quad (2.13)$$

with $P_{j,k,l}$ an integral multiple (at most $(q+1)$ times) of $H_{k,(l,j)}$.

Note that after the change of variables, the weighted ℓ^2 -norm

$$\|\xi\|_s = \left(\sum_{j \in \mathcal{Z}} |j|^{2s} |\xi_j|^2 \right)^{\frac{1}{2}}$$

of the sequence ξ is equivalent to the Sobolev norm $\|e^{-im \cdot x} u - u_m\|_{H^s}$ of the corresponding function u ,

$$\hat{c} \|\xi\|_s \leq \|e^{-im \cdot x} u - u_m\|_{H^s} \leq \hat{C} \|\xi\|_s \quad (2.14)$$

with positive constants depending on λ and ρ . In particular, under the assumptions of Theorem 1.1, the system (2.12) has small initial values whose ℓ_s^2 norm is of order ε .

3 Long-time near-conservation of super-actions

In this section we give the proof of Theorem 1.1. After the transformations of the previous section, we can verify that the conditions required to apply existing results on the long-time near-conservation of so-called super-actions are fulfilled. A transformation back to the original variables then gives Theorem 1.1.

3.1 Super-actions

Without the nonlinearity $\frac{\partial P}{\partial \xi_j}$ in (2.12), the *actions*

$$I_j(\xi, \bar{\xi}) = |\xi_j|^2, \quad j \in \mathcal{Z},$$

would be exactly conserved along solutions of (2.12). In the presence of the nonlinearity and in view of the partial resonance $\omega_j = \Omega_n$ for all $j \in \mathcal{Z}$ with $|j|^2 = n$ and the non-resonance of the Ω_n as given by Lemma 2.2, there remains long-time near-conservation of *super-actions*

$$J_n(\xi, \bar{\xi}) = \sum_{|j|^2=n} I_j(\xi, \bar{\xi}), \quad n \in \mathbb{N}, \quad (3.1)$$

along solutions of (2.12) provided that the initial value is small. The precise result in our situation is the following.

Theorem 3.1 (Long-time near-conservation of super-actions) *Fix $N > 1$ arbitrarily. For every $\rho_0 > 0$ such that $1 + 2\lambda\rho_0^2 > 0$ there exist $s_0 > 0$ and a set of full measure \mathcal{P} in the interval $(0, \rho_0]$ such that for all $s \geq s_0$ and $\rho \in \mathcal{P}$*

the following holds: There exist $\varepsilon_0 > 0$ and C such that for small initial data satisfying

$$\|\xi(0)\|_s \leq \varepsilon \leq \varepsilon_0,$$

the super-actions of the solution of (2.12) starting with $\xi(0)$ at $t = 0$ are nearly conserved,

$$\sum_{n \geq 1} n^s \frac{|J_n(\xi(t), \overline{\xi(t)}) - J_n(\xi(0), \overline{\xi(0)})|}{\varepsilon^2} \leq C\varepsilon^{\frac{1}{2}},$$

over long times

$$0 \leq t \leq \varepsilon^{-N}.$$

Since $\|\xi\|_s^2 = \sum_{n \geq 1} n^s J_n(\xi, \overline{\xi})$, this theorem implies that $\|\xi(t)\|_s$ stays of order ε over long times $t \leq \varepsilon^{-N}$. When we transform this result back to the solution u of (1.1), we immediately get Theorem 1.1 on using (2.14).

There are two entirely different approaches to prove Theorem 3.1, *Birkhoff normal forms* and *modulated Fourier expansions*. Both approaches will be outlined in the following subsections. Each proof relies on a non-resonance condition on the frequencies ω_j describing the linear part in (2.12), a regularity condition on the nonlinearity in (2.12) and a condition on the interaction of modes (zero-momentum condition). Based on Lemmas 2.2 and 2.3, these assumptions will be verified in the following subsections, separately for each approach since they are not exactly the same for both proofs. Once the conditions are verified, we can directly apply results from Bambusi and Grébert [2, 18] (using Birkhoff normal forms) and Gauckler [16] (using modulated Fourier expansions) to obtain Theorem 3.1.

3.2 Proof of Theorem 3.1 via Birkhoff normal forms

We follow the Birkhoff normal form approach as developed in [1, 2, 4, 18]. We verify that the assumptions of [18, Theorem 7.2] are fulfilled by the system (2.12).

3.2.1 Regularity of the nonlinearity

For multi-indices $k = (k_1, \dots, k_p) \in \mathcal{Z}^p$ and $l = (l_1, \dots, l_p) \in \mathcal{Z}^p$ we denote by $\mu_i(k, l)$ the i -th largest integer among $|k_1|, \dots, |k_p|, |l_1|, \dots, |l_p|$, so that $\mu_1(j) \geq \mu_2(j) \geq \mu_3(j) \geq \dots$. Moreover, for a given positive radius r , we set

$$B_s(r) = \{(\xi, \overline{\xi}) \in \mathbb{C}^{\mathcal{Z}} \times \mathbb{C}^{\mathcal{Z}} : \|\xi\|_s \leq r\}.$$

To apply Theorem 7.2 in [18], the Hamilton function

$$H = H_0 + P$$

with

$$\begin{aligned}
H_0(\xi, \bar{\xi}) &= \sum_{j \in \mathcal{Z}} \omega_j |\xi_j|^2 \\
P(\xi, \bar{\xi}) &= \sum_{p+q \geq 3} \sum_{\substack{k \in \mathcal{Z}^p, l \in \mathcal{Z}^q \\ \mathcal{M}(k, l) = 0}} H_{kl} \xi_{k_1} \cdots \xi_{k_p} \bar{\xi}_{l_1} \cdots \bar{\xi}_{l_q},
\end{aligned}$$

needs to satisfy a non-resonance condition on the frequencies, as provided by Lemma 2.2, and the following conditions on the non-quadratic part (see [18, Definition 4.4]):

- (H1) There exists $s_0 \geq 0$ such that for all $s \geq s_0$, there exists $r > 0$ such that $P \in \mathcal{C}^\infty(B_s(r), \mathbb{C})$.
- (H2) The Taylor coefficients H_{kl} satisfy the following property: for all $(k, l) \in \mathcal{Z}^p \times \mathcal{Z}^q$ we have $\bar{H}_{kl} = H_{lk}$, and for all p, q with $p + q \geq 3$, there exists $\nu \geq 0$ such that for every $N \in \mathbb{N}$, there exists C depending on N, p and q such that for all $(k, l) \in \mathcal{Z}^p \times \mathcal{Z}^q$,

$$|H_{kl}| \leq C \mu_3(k, l)^\nu \left(\frac{\mu_3(k, l)}{\mu_3(k, l) + \mu_1(k, l) - \mu_2(k, l)} \right)^N. \quad (3.2)$$

The bound (3.2), used in many works on normal forms applied to nonlinear PDEs - see [10, 11, 1, 18, 3]- implies in particular that the nonlinearity acts on the ball $B_s(r)$. Moreover, it is preserved by the Poisson bracket of two functions and by the normal form construction under a non-resonance condition implying a control of the small denominator by the third largest integer.

We now show that (H1) and (H2) are implied by the coefficient estimates of Lemma 2.3 together with the fact that the Hamiltonian has only terms with zero momentum. Let us consider a fixed multi-index (k, l) satisfying $\mathcal{M}(k, l) = 0$. Following the proof of [18, Lemma 5.2], we see that we always have

$$|\mu_1(k, l) - \mu_2(k, l)| \leq |\mathcal{M}(k, l)| + \sum_{n=3}^{p+q} \mu_n(k, l) \leq (p+q-2)\mu_3(k, l).$$

From this relation, we infer

$$\left(\frac{\mu_3(k, l)}{\mu_3(k, l) + \mu_1(k, l) - \mu_2(k, l)} \right)^N \geq (p+q-1)^{-N}$$

Using Lemma 2.3, we thus see that the coefficients H_{kl} satisfy the bound (3.2) with the constant $C = ML^{p+q}(p+q-1)^N$ and $\nu = 0$. This yields (H2). The assertion (H1) results from the fact that $\sqrt{\rho^2 - \sum_{j \in \mathcal{Z}} w_j \bar{w}_j}$ is analytic on $B_s(r)$ for $r < \rho$, and that the monomials with zero momentum terms define a smooth Hamiltonian as soon as $s_0 > d/2$ (see for instance [4]). This ensures that $P \in \mathcal{C}^\infty(B_s(r), \mathbb{C})$.

3.2.2 A normal form result

For a given Hamiltonian $K \in C^\infty(B_s(r), \mathbb{C})$ satisfying $K(\xi, \bar{\xi}) \in \mathbb{R}$ we denote by $X_K(\xi, \bar{\xi})$ the Hamiltonian vector field

$$X_K(\xi, \bar{\xi})_j = \left(i \frac{\partial K}{\partial \xi_j}, -i \frac{\partial K}{\partial \bar{\xi}_j} \right), \quad j \in \mathcal{Z},$$

associated with the Poisson bracket

$$\{K, G\} = i \sum_{j \in \mathcal{Z}} \frac{\partial K}{\partial \xi_j} \frac{\partial G}{\partial \bar{\xi}_j} - \frac{\partial K}{\partial \bar{\xi}_j} \frac{\partial G}{\partial \xi_j},$$

which is well defined for Hamiltonian functions K and G in the class of Hamiltonians defined above.

We are now ready to apply Theorem 7.2 of [18] to the Hamiltonian (2.10). We obtain the following result:

Theorem 3.2 *Let ρ be in the set \mathcal{P} of full measure as given by Lemma 2.2 for some $N \geq 3$. There exists s_0 and for any $s \geq s_0$ there exist two neighborhoods \mathcal{U} and \mathcal{V} of the origin in $B_s(\rho)$ and an analytic canonical transformation $\tau : \mathcal{V} \rightarrow \mathcal{U}$ which puts $H = H_0 + P$ in normal form up to order N , i.e.,*

$$H \circ \tau = H_0 + Z + R$$

where

- (i) Z is a polynomial of degree N which commutes with all the J_n , $n \geq 1$, i.e., $\{Z, J_n\} = 0$ for all $n \geq 1$,
- (ii) $R \in C^\infty(\mathcal{V}, \mathbb{R})$ and $\|X_R(\xi, \bar{\xi})\|_s \leq C_s \|\xi\|_s^N$ for $\xi \in \mathcal{V}$,
- (iii) τ is close to the identity: $\|\tau(\xi, \bar{\xi}) - (\xi, \bar{\xi})\|_s \leq C_s \|\xi\|_s^2$ for all $\xi \in \mathcal{V}$.

Let us recall the principle underlying the proof of this result: the construction of the transformation τ is made by induction by cancelling iteratively the polynomials of growing degree in the Hamiltonian $H_0 + P$. As τ is determined as the flow at time one of a polynomial Hamiltonian $\chi = \sum_{n \geq 3} \chi_n$, where χ_n are homogeneous polynomials of degree n , we are led to solve by induction homological equations of the form

$$\{H_0, \chi_n\} = Z_n + Q_n$$

where Z_n is the n -th component of the normal form term, and Q_n a homogeneous polynomial of degree n depending on P and the terms constructed at the previous iterations. Writing the equation in terms of coefficients, this equation can be written in the form

$$(\omega_{k_1} + \dots + \omega_{k_p} - \omega_{l_1} - \dots - \omega_{l_q}) \chi_{kl} = Z_{kl} + Q_{kl}$$

where $(k, l) \in \mathcal{Z}^p \times \mathcal{Z}^q$ with $p + q = n$. Using the non-resonance condition (2.9), we see that we can solve this equation for χ_{kl} and set $Z_{kl} = 0$ without losing too much regularity (i.e., χ will satisfy (H2) for some ν), except for the multi-indices (k, l) having equal length $p = q$ and after permutation, $|k_1|^2 = |l_1|^2, \dots, |k_p|^2 = |l_p|^2$. This yields that the normal form term Z contains only terms of the form $\xi_{k_1} \cdots \xi_{k_p} \bar{\xi}_{l_1} \cdots \bar{\xi}_{l_q}$ with $p = q$ and $|k_1|^2 = |l_1|^2, \dots, |k_p|^2 = |l_p|^2$. We then check that these terms Poisson-commute with J_n for all n , and hence $\{J_n, Z\} = 0$ for all n .

The proof of Theorem 3.1 can then be done using $\|\xi\|_s^2 = \sum_{n \geq 1} n^s J_n(\xi, \bar{\xi})$ and following the proof of Corollary 4.9 in [18].

3.3 Proof of Theorem 3.1 via modulated Fourier expansions

Modulated Fourier expansions are an analytical technique for studying long-time properties of nonlinear perturbations to oscillatory linear systems. They were originally introduced in [19] to explain the long-time behaviour of numerical methods for oscillatory ordinary differential equations; see also [20, Chap. XIII]. In [9, 17] and [15] they were used to study long-time properties of small solutions of nonlinear wave equations and nonlinear Schrödinger equations with an external potential, respectively, and in [16] for general classes of Hamiltonian partial differential equations.

Let us assume for the moment that there are no resonances among the frequencies, in particular $\omega_j \neq \omega_{j'}$ also for $|j| = |j'|$. A modulated Fourier expansion of the solution ξ of (2.12) is an approximation of ξ in terms of products of propagators $e^{-i\omega_j t}$ for the linear equation with slowly varying coefficient functions,

$$\xi_j(t) \approx \tilde{\xi}_j(t) = \sum_{\mathbf{k}} z_j^{\mathbf{k}}(\varepsilon t) e^{-i(\mathbf{k} \cdot \boldsymbol{\omega})t}, \quad j \in \mathcal{Z}, \quad (3.3)$$

where the sum runs over a finite set of sequences of integers $\mathbf{k} = (\mathbf{k}(\ell))_{\ell \in \mathcal{Z}} \in \mathbb{Z}^{\mathcal{Z}}$ with finitely many nonzero entries, and where $\mathbf{k} \cdot \boldsymbol{\omega} = \sum_{\ell \in \mathcal{Z}} \mathbf{k}(\ell) \omega_\ell$.

Inserting the ansatz (3.3) into the equations of motion (2.12) and equating terms with the same exponential $e^{-i(\mathbf{k} \cdot \boldsymbol{\omega})t}$ leads to the *modulation system*

$$\begin{aligned} i\varepsilon \dot{z}_j^{\mathbf{k}} + (\mathbf{k} \cdot \boldsymbol{\omega}) z_j^{\mathbf{k}} \\ = \omega_j z_j^{\mathbf{k}} + \sum_{p+q \geq 2} \sum_{\substack{\mathbf{k}^1 + \dots + \mathbf{k}^p \\ -\mathbf{l}^1 - \dots - \mathbf{l}^q = \mathbf{k}}} \sum_{\substack{k \in \mathcal{Z}^p, l \in \mathcal{Z}^q \\ \mathcal{M}(k, l) = j}} P_{j, k, l} z_{k_1}^{\mathbf{k}^1} \cdots z_{k_p}^{\mathbf{k}^p} \bar{z}_{l_1}^{\mathbf{l}^1} \cdots \bar{z}_{l_q}^{\mathbf{l}^q} \end{aligned}$$

with the coefficients $P_{j, k, l}$ of (2.13). Modulated Fourier expansions can hence be seen as embedding the original system of equations in a larger system. The non-

linearity is the partial derivative with respect to $\bar{z}_j^{\mathbf{k}}$ of the modulation potential

$$\mathcal{P}(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{p+q \geq 3} \sum_{\substack{\mathbf{k}^1 + \dots + \mathbf{k}^p \\ -\mathbf{1}^1 - \dots - \mathbf{1}^{q+1} = \mathbf{0}}} \sum_{\substack{k \in \mathcal{Z}^p, l \in \mathcal{Z}^{q+1} \\ \mathcal{M}(k, l) = 0}} H_{kl} z_{k_1}^{\mathbf{k}^1} \dots z_{k_p}^{\mathbf{k}^p} \bar{z}_{l_1}^{\mathbf{1}^1} \dots \bar{z}_{l_{q+1}}^{\mathbf{1}^{q+1}}$$

with the coefficients H_{kl} of the non-quadratic part of the Hamiltonian $H(\xi, \bar{\xi})$. The modulation potential is invariant under transformations $z_j^{\mathbf{k}} \mapsto e^{i\mathbf{k}(\ell)\theta} z_j^{\mathbf{k}}$ for $\theta \in \mathbb{R}$ and fixed $\ell \in \mathcal{Z}$. The modulation system thus inherits the Hamiltonian structure from the original equations of motions for ξ , and its transformation invariance leads to formal invariants

$$\mathcal{I}_\ell(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{j, \mathbf{k}} \mathbf{k}(\ell) |z_j^{\mathbf{k}}|^2, \quad \ell \in \mathcal{Z},$$

with $\mathbf{z} = (z_j^{\mathbf{k}})_{j, \mathbf{k}}$ of the modulation system, see [16, Sect. 3.1]. These formal invariants form the cornerstone for the study of long time intervals.

On a short time interval of length ε^{-1} it is possible to construct an approximate solution of the modulation system in an iterative way, such that—under certain assumptions to be verified below—the ansatz (3.3) describes the solution ξ up to a small error,

$$\|\xi(t) - \tilde{\xi}(t)\|_s \leq C\varepsilon^{N+3} \quad \text{for } 0 \leq t \leq c\varepsilon^{-1}$$

with $\tilde{\xi}$ given by (3.3) with the approximate solution of the modulation system, see [16, Sects. 3.2–3.4]. The constants depend on N and s but not on ε . Along this approximate solution \mathbf{z} of the modulation system, the formal invariants \mathcal{I}_ℓ then become almost-invariants,

$$\sum_{\ell \in \mathcal{Z}} |\ell|^{2s} \left| \frac{d}{dt} \mathcal{I}_\ell(\mathbf{z}(\varepsilon t), \bar{\mathbf{z}}(\varepsilon t)) \right| \leq C\varepsilon^{N+2},$$

which are close to the actions $I_\ell(\xi, \bar{\xi}) = |\xi_\ell|^2$,

$$\sum_{\ell \in \mathcal{Z}} |\ell|^{2s} |\mathcal{I}_\ell(\mathbf{z}(\varepsilon t), \bar{\mathbf{z}}(\varepsilon t)) - I_\ell(\xi(t), \bar{\xi}(t))| \leq C\varepsilon^{\frac{5}{2}}.$$

These almost-invariants allow us to repeat the construction of modulated Fourier expansions on short time intervals of length ε^{-1} and patch ε^{-N+1} of those short intervals together, see [16, Sect. 3.5]. On a long time interval of length ε^{-N} we then get near-conservation of actions as stated in Theorem 3.1 (with actions instead of super-actions).

Compared to the above description, the modulated Fourier expansion for our problem (2.12) has some subtleties that are caused by the partial resonances

$\omega_j = \omega_{j'}$ for $|j| = |j'|$. Since all sums in the nonlinearity of (2.12) involve only products of the form $\xi_{k_1} \cdots \xi_{k_p} \bar{\xi}_{l_1} \cdots \bar{\xi}_{l_q}$ with $k_1 + \cdots + k_p - l_1 - \cdots - l_q = j$ (by the zero momentum condition in the Hamiltonian), only modulation functions $z_j^{\mathbf{k}}$ with

$$j = j(\mathbf{k}) = \sum_{\ell \in \mathcal{Z}} \mathbf{k}(\ell) \ell$$

can be different from zero. Moreover, since the frequencies ω_j in (2.12) are partially resonant, $\omega_j = \omega_{j'}$ for $|j| = |j'|$, we can distinguish exponentials $e^{-i(\mathbf{k}^1 \cdot \boldsymbol{\omega})t}$ and $e^{-i(\mathbf{k}^2 \cdot \boldsymbol{\omega})t}$ only if

$$\mathbf{k}^1 - \mathbf{k}^2 \notin \left\{ \mathbf{k} : \sum_{|\ell|^2=n} \mathbf{k}(\ell) = 0 \text{ for all } n \in \mathbb{N} \right\}.$$

For this reason, the sum in (2.12) is in our situation only over a set of representatives of sequences \mathbf{k} where $j(\mathbf{k})$ or $\mathbf{k} \cdot \boldsymbol{\omega}$ are distinguishable (in the above sense). The main consequence is that the quantities \mathcal{I}_ℓ from above are no longer invariants of the modulation system, but only certain sums of them:

$$\mathcal{J}_n(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\ell \in \mathcal{Z}: |\ell|^2=n} \mathcal{I}_\ell(\mathbf{z}, \bar{\mathbf{z}}), \quad n \in \mathbb{N}.$$

Along the approximate solution of the modulation system, they are close to the corresponding sums of the actions I_ℓ , the super-actions J_n . In this way we get long-time near-conservation of super-actions as in Theorem 3.1.

We finally state and verify the assumptions needed for the iterative construction of modulation functions. The first lemma below summarises the assumptions on the nonlinearity in (2.12), whereas the second lemma below deals with the non-resonance condition on the frequencies describing the linear part of (2.12). The properties stated in these lemmas are precisely the assumptions under which Theorem 3.1 has been shown in [16, Theorem 2.7].

Lemma 3.3 *The expansion (2.13) of the nonlinearity in (2.12) has the following properties.*

(i) *It fulfills the zero momentum condition*

$$P_{j,k,l} = 0 \quad \text{if } j \neq \mathcal{M}(k,l)$$

for $j \in \mathcal{Z}$, $k \in \mathcal{Z}^p$ and $l \in \mathcal{Z}^q$.

(ii) *There exist constants $C_{p,q,s}$ depending only on p , q , s and ρ such that for*

$$|P_j^{p,q}(\xi^1, \dots, \xi^p, \bar{\xi}^1, \dots, \bar{\xi}^q)| = \sum_{k \in \mathcal{Z}^p, l \in \mathcal{Z}^q} |P_{j,k,l}| \xi_{k_1}^1 \cdots \xi_{k_p}^p \bar{\xi}_{l_1}^1 \cdots \bar{\xi}_{l_q}^q$$

the estimate

$$\begin{aligned} & \| |P|^{p,q}(\xi^1, \dots, \xi^p, \bar{\xi}^1, \dots, \bar{\xi}^q) \|_s \\ & \leq C_{p,q,s} \| \xi^1 \|_s \cdots \| \xi^p \|_s \| \bar{\xi}^1 \|_s \cdots \| \bar{\xi}^q \|_s \end{aligned} \quad (3.4)$$

holds for $\xi^1, \dots, \xi^p, \bar{\xi}^1, \dots, \bar{\xi}^q \in \ell_s^2$ if $s > \frac{d}{2}$.

(iii) There exist $r_0 > 0$ depending only on ρ , and C_s depending in addition on $s > \frac{d}{2}$ such that

$$\sum_{p+q \geq 2} C_{p,q,s} |z|^{p+q-2} \leq C_s \quad \text{for all } z \in \mathbb{C} \text{ with } |z| \leq r_0.$$

Proof. Property (i) is obvious, see (2.13).

For property (ii) we recall that (3.4) was verified in [16, Subsect. 2.6] in the situation $P_{j,k,l} = 0$ for $j \neq \mathcal{M}(k,l)$ and $P_{j,k,l} = 1$ else. The proof is just a repeated application of the Cauchy-Schwarz inequality, and the corresponding constants $C_{p,q,s}$ are given by C^{p+q} with C depending only on s . In our situation here, the coefficients $P_{j,k,l}$ vanish for $j \neq \mathcal{M}(k,l)$ and can be bounded with Lemma 2.3. This implies that the second property (ii) is satisfied with constants $C_{p,q,s} = (q+1)ML^{p+q+1}C^{p+q}$.

These constants satisfy (iii) with r_0 and C_s depending only on ρ and s . \blacksquare

Lemma 3.4 *The frequencies Ω_n grow like n , $c_1 n \leq \Omega_n \leq C_1 n$ with positive constants c_1 and C_1 depending only on ρ .*

Moreover, for all $\rho_0 > 0$ such that $1 + 2\lambda\rho_0^2 > 0$ and for all positive integers N , there exist s_0 and a set of full measure \mathcal{P} in the interval $(0, \rho_0]$ such that for all $s \geq s_0$ and all $\rho \in \mathcal{P}$ the following non-resonance condition holds: There exist $\varepsilon_0 > 0$ and C_0 such that for all $r \leq 2N + 2d + 6$ and all $0 < \varepsilon \leq \varepsilon_0$,

$$\left(\frac{n}{n_1 \cdots n_r} \right)^{s - \frac{d+1}{2}} \varepsilon^r \leq C_0 \varepsilon^{2N+2d+8}$$

whenever a near-resonance

$$|\Omega_n \pm \Omega_{n_1} \pm \cdots \pm \Omega_{n_r}| < \varepsilon^{\frac{1}{2}}$$

occurs with frequencies that do not cancel pairwise.

Proof. The asymptotic growth behaviour of the frequencies is obvious, and the non-resonance condition is implied by the non-resonance condition of Lemma 2.2 as shown in [9, Lemma 1]. \blacksquare

A Hamiltonian of the reduced system

We verify that the function \tilde{H} given in Subsect. 2.2 is indeed a Hamiltonian function for the system for w_j . We have

$$\begin{aligned}
\frac{\partial \tilde{H}}{\partial \bar{w}_j}(w, \bar{w}) &= (|j|^2 + \lambda \rho^2)w_j + \lambda \rho^2 \bar{w}_{-j} \\
&+ \lambda \sum_{j_1+j_2-j_3=j} w_{j_1} w_{j_2} \bar{w}_{j_3} - 3\lambda \left(\sum_{j_1} w_{j_1} \bar{w}_{j_1} \right) w_j \\
&- \frac{\lambda}{2} \left(\sum_{j_1} w_{j_1} w_{-j_1} \right) w_j - \frac{\lambda}{2} \left(\sum_{j_1} \bar{w}_{j_1} \bar{w}_{-j_1} \right) w_j - \lambda \left(\sum_{j_1} w_{j_1} \bar{w}_{j_1} \right) \bar{w}_{-j} \\
&+ \lambda \left(\sum_{j_1+j_2=j} w_{j_1} w_{j_2} \right) \sqrt{\rho^2 - \sum_{j_4} w_{j_4} \bar{w}_{j_4}} \\
&+ \lambda \left(\sum_{j_1+j_2-j_3=0} w_{j_1} w_{j_2} \bar{w}_{j_3} \right) \frac{-w_j}{2\sqrt{\rho^2 - \sum_{j_4} w_{j_4} \bar{w}_{j_4}}} \\
&+ 2\lambda \left(\sum_{j_1-j_2=j} w_{j_1} \bar{w}_{j_2} \right) \sqrt{\rho^2 - \sum_{j_4} w_{j_4} \bar{w}_{j_4}} \\
&+ \lambda \left(\sum_{j_1-j_2-j_3=0} w_{j_1} \bar{w}_{j_2} \bar{w}_{j_3} \right) \frac{-w_j}{2\sqrt{\rho^2 - \sum_{j_4} w_{j_4} \bar{w}_{j_4}}}.
\end{aligned}$$

Using $a = \sum_{j_1} w_{j_1} \bar{w}_{j_1}$ we get

$$\begin{aligned}
\frac{\partial \tilde{H}}{\partial \bar{w}_j}(w, \bar{w}) &= (|j|^2 + \lambda \rho^2)w_j + \lambda \rho^2 \bar{w}_{-j} + \lambda \sum_{j_1+j_2-j_3=j} w_{j_1} w_{j_2} \bar{w}_{j_3} \\
&- 3\lambda a w_j - \lambda \operatorname{Re} \left(\sum_{j_1} w_{j_1} w_{-j_1} \right) w_j - \lambda a \bar{w}_{-j} \\
&+ \lambda \left(\sum_{j_1+j_2=j} w_{j_1} w_{j_2} \right) \sqrt{\rho^2 - a} + 2\lambda \left(\sum_{j_1-j_2=j} w_{j_1} \bar{w}_{j_2} \right) \sqrt{\rho^2 - a} \\
&- \lambda \operatorname{Re} \left(\sum_{j_1+j_2-j_3=0} w_{j_1} w_{j_2} \bar{w}_{j_3} \right) \frac{w_j}{\sqrt{\rho^2 - a}},
\end{aligned}$$

and finally with $\dot{\theta}$ from Subsect. 2.2,

$$\begin{aligned}
\frac{\partial \tilde{H}}{\partial \bar{w}_j}(w, \bar{w}) &= (|j|^2 + 2\lambda(\rho^2 - a))w_j + \lambda(\rho^2 - a)\bar{w}_{-j} + \lambda \sum_{j_1+j_2-j_3=j} w_{j_1} w_{j_2} \bar{w}_{j_3} \\
&- \dot{\theta} w_j + \lambda \left(\sum_{j_1+j_2=j} w_{j_1} w_{j_2} \right) \sqrt{\rho^2 - a} + 2\lambda \left(\sum_{j_1-j_2=j} w_{j_1} \bar{w}_{j_2} \right) \sqrt{\rho^2 - a} \\
&= i\dot{w}_j.
\end{aligned}$$

References

- [1] D. Bambusi, *Birkhoff normal form for some nonlinear PDEs*. Comm. Math. Phys. **234** (2003), 253–283.
- [2] D. Bambusi, *A Birkhoff normal form theorem for some semilinear PDEs*. Hamiltonian Dynamical Systems and Applications, Springer, 2007, pp. 213–247.
- [3] D. Bambusi, J.-M. Delort, B. Grébert and J. Szeftel, *Almost global existence for Hamiltonian semilinear Klein-Gordon equations with small Cauchy data on Zoll manifolds*. Comm. Pure Appl. Math. **60** (2007), 1665–1690.
- [4] D. Bambusi and B. Grébert, *Birkhoff normal form for partial differential equations with tame modulus*. Duke Math. J. **135** (2006), 507–567.
- [5] J. Bourgain, *Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations*. Ann. of Math. (2), **148** (1998), 363–439.
- [6] T. Cazenave and P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*. Comm. Math. Phys. **85** (1982), 549–561.
- [7] M. Christ, J. Colliander and T. Tao, *Ill-posedness for nonlinear Schrödinger and wave equations*. Annales de l’IHP, to appear. <http://arxiv.org/abs/math/0311048>
- [8] M. Christ, J. Colliander and T. Tao, *Instability of the periodic nonlinear Schrödinger equation*. Preprint, 2003. <http://arxiv.org/abs/math/0311227>
- [9] D. Cohen, E. Hairer and Ch. Lubich, *Long-time analysis of nonlinearly perturbed wave equations via modulated Fourier expansions*. Arch. Ration. Mech. Anal. **187** (2008), 341–368.
- [10] J. M. Delort and J. Szeftel, *Long-time existence for small data nonlinear Klein-Gordon equations on tori and spheres*. Internat. Math. Res. Notices **37** (2004) 1897–1966
- [11] J. M. Delort and J. Szeftel, *Long-time existence for semi-linear Klein-Gordon equations with small Cauchy data on Zoll manifolds*. Amer. J. Math. **128** (2006) 1187–1218.
- [12] L. H. Eliasson and S. B. Kuksin, *KAM for the nonlinear Schrödinger equation*. Ann. of Math. (2) **172** (2010), 371–435.
- [13] T. Gallay and M. Haragus, *Stability of small periodic waves for the nonlinear Schrödinger equation* J. Diff. Equations **234** (2007), 544–581.
- [14] T. Gallay and M. Haragus, *Orbital stability of periodic waves for the nonlinear Schrödinger equation* J. Dyn. Diff. Eqns. **19** (2007), 825–865.

- [15] L. Gauckler and Ch. Lubich, *Nonlinear Schrödinger equations and their spectral semi-discretizations over long times*. Found. Comput. Math. **10** (2010), 141–169.
- [16] L. Gauckler, *Long-time analysis of Hamiltonian partial differential equations and their discretizations*. Dissertation (doctoral thesis), Univ. Tübingen, 2010. <http://nbn-resolving.de/urn:nbn:de:bsz:21-opus-47540>.
- [17] L. Gauckler, E. Hairer, Ch. Lubich and D. Weiss, *Metastable energy cascades in weakly nonlinear wave equations*, Preprint, 2011. <http://na.uni-tuebingen.de/preprints.shtml>
- [18] B. Grébert, *Birkhoff normal form and Hamiltonian PDEs*. Sémin. Cong. **15** (2007), 1–46.
- [19] E. Hairer and Ch. Lubich, *Long-time energy conservation of numerical methods for oscillatory differential equations*. SIAM J. Numer. Anal. **38** (2000), 414–441.
- [20] E. Hairer, Ch. Lubich and G. Wanner, *Geometric numerical integration. Structure-preserving algorithms for ordinary differential equations*. 2nd ed., Springer Series in Computational Mathematics 31, Springer-Verlag, Berlin, 2006.

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