Topological characterization of simple points by complex collapsibility

Yukiko Kenmochi

Topological characterization of simple points by complex collapsibility

Yukiko KENMOCHI†

† IGM, CNRS/Université de Marne-la-Vallée/ESIEE - Laboratoire A2SI, ESIEE, France

Abstract Thinning is an image operation whose goal is to reduce object points in a "topology-preserving" way. Such points whose removal does not change the topology are called simple points and they play an important role in any thinning process. For efficient computation, local characterizations have been already studied based on the concept of point connectivity for two- and three-dimensional digital images. In this paper, we introduce a new topological characterization of simple points based on collapsibility of polyhedral complexes. We also study their topological characteristics and propose a linear thinning algorithm.

Key words thinning, simple points, topological characterization, polyhedral complexes, collapsing

1. Introduction

Thinning is an image operation whose purpose is to reduce object points in a "topology-preserving" way. Such points whose removal does not change the topology are called simple points and they play an important role in any thinning process. Mathematically, the definition of simple points is given as follows. Let us consider the 3D lattice space $\mathbb{Z}^3$. A point $\boldsymbol{x}$ in a finite subset $\mathbf{V} \subset \mathbb{Z}^3$ is said to be simple if there is a one-to-one correspondence of each connected component of $\mathbf{V}$ and its complement $\overline{\mathbf{V}}$, and the holes of $\mathbf{V}$ and $\overline{\mathbf{V}}$, with each connected component of $\mathbf{V} \setminus \{\boldsymbol{x}\}$ and $\overline{\mathbf{V}} \cup \{\boldsymbol{x}\}$, and the holes of $\mathbf{V} \setminus \{\boldsymbol{x}\}$ and $\overline{\mathbf{V}} \cup \{\boldsymbol{x}\}$, respectively [2]. Because the above global definition is not appropriate for computation, many studies on their local characterization have been made: for example, in 3D, characterizations by using connected component numbers, genus, Euler numbers, and/or other numbers [2], [9], [14].

Here, we introduce one of the most simple characterizations of simple points in 3D by using topological numbers, proposed in [2]. We consider the following m-neighborhoods in $\mathbb{Z}^3$:

$$\mathbf{N}_6(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{Z}^3 : \|\boldsymbol{x} - \boldsymbol{y}\|_1 \leqq 1\},$$
$$\mathbf{N}_{26}(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{Z}^3 : \|\boldsymbol{x} - \boldsymbol{y}\|_\infty \leqq 1\}.$$

Let $\mathbf{V}$ be a subset in $\mathbb{Z}^3$, $C_m(\mathbf{V})$ be the set of all m-connected components of $\mathbf{V}$, and $C_m^a[\boldsymbol{x}, \mathbf{V}]$ be the set of all components in $C_m(\mathbf{V})$ which are m-adjacent to a point $\boldsymbol{x}$. Then, we define topological numbers

$$T_6(\boldsymbol{x}, \mathbf{V}) = \#C_6^a[\boldsymbol{x}, \mathbf{N}_{18}(\boldsymbol{x}) \setminus \{\boldsymbol{x}\} \cap \mathbf{V}],$$
$$T_{26}(\boldsymbol{x}, \mathbf{V}) = \#C_{26}^a[\boldsymbol{x}, \mathbf{N}_{26}(\boldsymbol{x}) \setminus \{\boldsymbol{x}\} \cap \mathbf{V}],$$

where $\#X$ is the cardinal of a set X and $\mathbf{N}_{18}(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{Z}^3 : \max(\|\boldsymbol{x} - \boldsymbol{y}\|_\infty, \lceil \frac{\|\boldsymbol{x} - \boldsymbol{y}\|_1}{2} \rceil) \leqq 1\}$. By using these topological numbers, we obtain the following proposition.

[**Proposition 1**] A point $\boldsymbol{x} \in \mathbf{V}$ is m-simple if and only if $T_m(\boldsymbol{x}, \mathbf{V}) = T_{\overline{m}}(\boldsymbol{x}, \overline{\mathbf{V}}) = 1$ for $(m, \overline{m}) = (6, 26), (26, 6)$.

In this paper, we present a new topological characterization of simple points based on collapsibility of polyhedral complexes (Section 2). We show that our characterisation is also local and only needs the connectivity m of $\mathbf{V}$ but not $\overline{m}$ of $\overline{\mathbf{V}}$ (Section 3). Therefore, we can avoid the well-known problem of how to choose a connectivity pair $(m, \overline{m})$ for $\mathbf{V}$ and $\overline{\mathbf{V}}$. Moreover, we show topological characteristics of simple points derived from collapsibility (Section 4) and correct the results in [7]. We also propose a linear thinning algorithm (Section 5) and discuss on the advantages of our method and on the problems which still exist (Section 6).

2. Complexes and collapsibility

2.1 Polyhedral complexes in $\mathbb{R}^3$

For the definitions of convex polyhedra and polyhedral complexes in $\mathbb{R}^3$, we follow the notions in [15]. Similar notations are also seen in [1], [13].

[**Definition 1**] A convex polyhedron σ is the convex hull of a finite set of points in some $\mathbb{R}^n$.

The dimension of a convex polyhedron σ is the dimension of its convex hull. An n-dimensional convex polyhedron σ is abbreviated to an n-polyhedron. For instance, a point is a 0-polyhedron, a line segment is a 1-polyhedron, a triangle or a square is a 2-polyhedron, and a tetrahedron or a hexahedron is a 3-polyhedron.

A linear inequality $\boldsymbol{a} \cdot \boldsymbol{x} \leqq z$ is said to be *valid* for σ if it is satisfied for all points $\boldsymbol{x} \in \sigma$. A *face* of σ is then defined by any set of the form

$$\delta = \sigma \cap \{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{a} \cdot \boldsymbol{x} = z\}$$

where $\boldsymbol{a} \cdot \boldsymbol{x} \leqq z$ is valid for σ. If a k-dimensional convex polyhedron τ is a face of σ, τ is called an k-face and such a binary relation is denoted by $\tau \prec \sigma$. Note that the binary relation is reflexive so that $\sigma \prec \sigma$ for any σ and also $\emptyset \prec \sigma$ for any σ.

[**Definition 2**] A polyhedral complex $\mathbf{K}$ is a finite collection of convex polyhedra such that

(1) the empty polyhedron is in $\mathbf{K}$,

(2) if $\sigma \in \mathbf{K}$ and $\tau \prec \sigma$, then $\tau \in \mathbf{K}$,

(3) if $\sigma, \tau \in \mathbf{K}$, then the intersection $\sigma \cap \tau$ is a common face of σ and τ.

The *dimension* of $\mathbf{K}$ is the largest dimension of a convex polyhedron in $\mathbf{K}$. It is known that $\mathbf{K}$ is a partially ordered set [1].

2.2 Collapsing

In this subsection, we introduce a deformation retraction of a polyhedral complex, called *collapsing* [12], [13].

Let $\mathbf{K}$ be an n-complex and σ be an r-polyhedron in $\mathbf{K}$ where $r < n$. If there is exactly one $(r+1)$-face $\tau \in \mathbf{K}$ such that $\sigma \prec \tau$, such a σ is called *free*. Then we say that there is an *elementary collapse* of $\mathbf{K}$ to a subcomplex $\mathbf{K}' = \mathbf{K} \backslash \{\sigma, \tau\}$, denoted by $\mathbf{K} \searrow^e \mathbf{K}'$.

We say that $\mathbf{K}$ *collapses* to a subcomplex $\mathbf{L}$ if there is a sequence of elementary collapses

$$\mathbf{K} = \mathbf{K}_0 \searrow^e \mathbf{K}_1 \searrow^e \ldots \searrow^e \mathbf{K}_k = \mathbf{L},$$

and write $\mathbf{K} \searrow \mathbf{L}$. It is well known that there is a homotopy equivalence between $\mathbf{K}$ and $\mathbf{L}$ if $\mathbf{K} \searrow \mathbf{L}$.

[**Definition 3**] An n-complex $\mathbf{K}$ is said to be collapsible if $\mathbf{K}$ collapses to a point, and we write $\mathbf{K} \searrow 0$ in this case.

3. Collapsibility and simple points

3.1 Complex construction from a point set

If we have a method to construct a polyhedral complex $\mathbf{K}$ from a finite point set $\mathbf{V}$ in $\mathbb{Z}^3$, satisfying the following two properties, we can derive a new local characterization of simple points based on such a polyhedral complex.

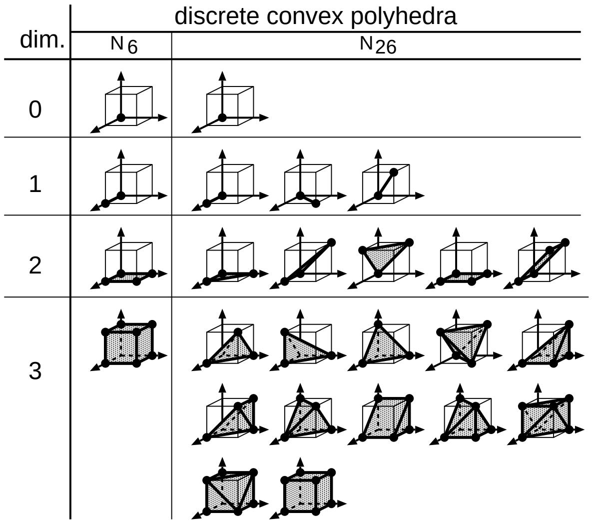
[**Property 1**] A polyhedral complex $\mathbf{K}$ is uniquely constructed from any finite subset $\mathbf{V} \subset \mathbb{Z}^3$, denoted by $\mathbf{K} = cmp(\mathbf{V})$.

Let $Sk_n(\mathbf{K})$ be the union of all n-polyhedra in $\mathbf{K}$, called an n-dimensional *skeleton* of $\mathbf{K}$. Therefore, $Sk_0(\mathbf{K})$ denotes the union of sets of vertices of all $\sigma \in \mathbf{K}$.

[**Property 2**] Let $\mathbf{K}$ be a polyhedral complex constructed from a finite subset $\mathbf{V} \subset \mathbb{Z}^3$. Then we have $\mathbf{V} = Sk_0(\mathbf{K})$.

Several methods can be found in the framework of Khalimsky Topology [8], partially ordered sets [3], [4], and discrete polyhedral complexes [5], [6]. In the following, we explain a

Table 1 All n-dimensional discrete convex polyhedra, $n = 0, 1, 2, 3$, for the m-neighborhood systems, $m = 6, 26$, up to rotations and symmetries.



method prosed in [5], [6] for construction of a discrete polyhedral complex $\mathbf{K}$ from a finite point set $\mathbf{V}$ in $\mathbb{Z}^3$.

A discrete polyhedral complex is constructed with respect to a chosen m-neighborhood where $m = 6, 26$. Let us first consider the case of $m = 26$. We consider a unit cube whose eight vertices are discrete points in $\mathbb{Z}^3$. Setting the value of each point at either 1 or 0, we make a convex hull of points whose value is 1. The dimension of such a convex hull can vary from 0 to 3 and we see that every pair of adjacent vertices of any discrete convex polyhedron are 26-neighboring, as illustrated in Table 1. After generating a discrete convex polyhedron in each unit cubic region, we compute the union of all discrete convex polyhedra and their faces, and obtain a discrete polyhedral complex $\mathbf{K}$.

If we consider discrete convex polyhedra such that every pair of adjacent vertices are 6-neighboring, we obtain only one type of discrete convex polyhedra for each dimension as shown in Table 1. Similarly to the case of $m = 26$, for the case of $m = 6$, considering the union of all discrete convex polyhedra and their faces, we obtain a polyhedral complex $\mathbf{K}$. The details and the precise algorithm can be found in [5].

3.2 Collapsibility and simple points

In order to present a new local characterization of simple points by using polyhedral complexes, we need to define the following sets.

[**Definition 4**] Let $\mathbf{K}$ be a polyhedral complex. The combinatorial closure of a subset $\mathbf{K}' \subseteqq \mathbf{K}$ is defined as $Cl(\mathbf{K}') = \{\tau \in \mathbf{K} : \tau \prec \sigma,\ \sigma \in \mathbf{K}'\}$.

[**Definition 5**] For a polyhedral complex $\mathbf{K}$, the star of $\sigma \in \mathbf{K}$ is defined so that $star(\sigma) = \{\tau \in \mathbf{K} : \sigma \prec \tau\}$.

[**Definition 6**] For a polyhedral complex $\mathbf{K}$, the link of a point $\sigma \in \mathbf{K}$ is defined so that $link(\sigma) = Cl(star(\sigma)) \setminus star(\sigma)$.

If we need to emphasize $\mathbf{K}$ where a star and a link are calculated, we denote them by $star(\sigma : \mathbf{K})$ and $link(\sigma : \mathbf{K})$ respectively. Figure 1 shows examples of star and link. Note that any link is a polyhedral complex while stars are not always polyhedral complexes.

[**Proposition 2**] Let $\mathbf{V}$ be a finite point set and $cmp(\mathbf{V})$ be a discrete polyhedral complex constructed from $\mathbf{V}$ for the m-neighborhood system where $m = 6, 26$ as described in Section 3.1. A point $\boldsymbol{x} \in \mathbf{V}$ is m-simple if and only if $link(\boldsymbol{x} : cmp(\mathbf{V}))$ is collapsible.

We have calculated all local point configurations such that $link(\boldsymbol{x} : cmp(\mathbf{V}))$ is collapsible, and have verified that they are the same as those of m-simple points [2]. More precisely, we obtain 550435 different local point configurations in a $3 \times 3 \times 3$ point region for either case $m = 6, 26$. Remark that this is not a coincidence; we can derive this result from Proposition 1 which is obtained by another local characterization. We easily see that any m-simple point for $\mathbf{V}$ is a $\overline{m}$-simple point for $\overline{\mathbf{V}}$ if we interchange $\mathbf{V}$ with $\overline{\mathbf{V}}$.

Similar characterizations of simple points can be found in [3], [10]. Note that the topological space in [10] is dual to a discrete polyhedral complex for 6-neighborhood [5], [6] so that we have an inclusion relation $\prec$ which is inverse.

4. Topological characteristics of simple points

4.1 Some notions on polyhedral complexes

We give some notions for polyhedral complexes [5], [6].

[**Definition 7**] An n-complex $\mathbf{K}$ is said to be pure if there is at least one n-polyhedron $\sigma \in \mathbf{K}$ for every s-polyhedron $\tau \in \mathbf{K}$ so that $\tau \prec \sigma$.

Figure 2 shows examples of pure and non-pure discrete complexes.

[**Definition 8**] Let $\mathbf{K}$ be a polyhedral complex, and σ, τ be arbitrary elements in $\mathbf{K}$. We say that $\mathbf{K}$ is connected, if we have a path $\sigma = a_1, a_2, \ldots, a_n = \tau$ in $\mathbf{K}$ that satisfies $Cl(\{a_i\}) \cap Cl(\{a_{i+1}\}) \neq \{\emptyset\}$ for every $i = 1, 2, \ldots, n-1$.

The dimension of $star(\sigma)$ is defined as the largest dimension of convex polyhedra belonging to $star(\sigma)$ and denoted by $dim(star(\sigma))$.

4.2 Topological characterisation by stars

For each 0-polyhedron, namely a point $\boldsymbol{x}$, in the 0-skeleton $Sk_0(\mathbf{K})$ of a polyhedral complex $\mathbf{K}$, we define topological

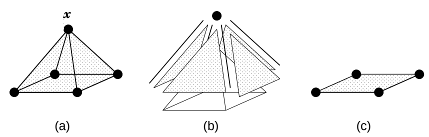


Fig. 1 (a) A 3-complex $\mathbf{K}$; (b) the star of $\boldsymbol{x} \in Sk_0(\mathbf{K})$; (c) the link of $\boldsymbol{x}$.

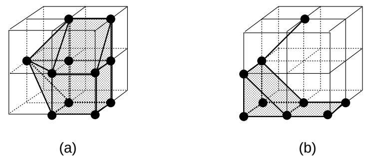


Fig. 2 Examples of (a) pure and (b) non-pure 3-complexes.

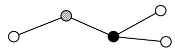


Fig. 3 One-dimensional topological characterisation of points whose stars are linear, semi-linear and neither of them, illustrated as grey, white and black points.

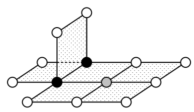


Fig. 4 Two-dimensional topological characterisation of points whose stars are cyclic, semi-cyclic and neither of them, illustrated as grey, white and black points.

characteristics of stars [1], [6].

[**Definition 9**] Let $\mathbf{K}$ be a polyhedral complex and $\boldsymbol{x} \in Sk_0(\mathbf{K})$. We say that $star(\boldsymbol{x})$ is linear if $link(\boldsymbol{x})$ consists of two 0-polyhedra.

[**Definition 10**] Let $\mathbf{K}$ be a polyhedral complex and $\boldsymbol{x} \in Sk_0(\mathbf{K})$. We say that $star(\boldsymbol{x})$ is semi-linear if $link(\boldsymbol{x})$ consists of one 0-polyhedron.

Figure 3 illustrates stars which are linear and semi-linear.

By using linear and semi-linear stars, we define combinatorial curves.

[**Definition 11**] Let $\mathbf{K}$ be a connected and pure 1-complex. We say that $\mathbf{K}$ is a combinatorial curve with endpoints if the star of every 0-polyhedron in $Sk_0(\mathbf{K})$ is either linear or semi-linear and there is at least one point whose star is semi-linear in $Sk_0(\mathbf{K})$.

[**Definition 12**] Let $\mathbf{K}$ be a connected and pure 1-complex. We say that $\mathbf{K}$ is a combinatorial closed curve if the star of every 0-polyhedron in $Sk_0(\mathbf{K})$ is linear.

By using the above definitions of combinatorial curves, we define topological characteristics of stars in two dimensions.

[**Definition 13**] Let $\mathbf{K}$ be a polyhedral complex and $\boldsymbol{x} \in Sk_0(\mathbf{K})$. We say that $star(\boldsymbol{x})$ is cyclic if $link(\boldsymbol{x})$ is a combinatorial closed curve.

[**Definition 14**] Let $\mathbf{K}$ be a polyhedral complex and $\boldsymbol{x} \in Sk_0(\mathbf{K})$. We say that $star(\boldsymbol{x})$ is semi-cyclic if $link(\boldsymbol{x})$ is a combinatorial curve with endpoints.

Figure 4 illustrates stars which are cyclic and semi-cyclic.

By using cyclic and semi-cyclic stars, we define combinatorial surfaces.

[**Definition 15**] Let $\mathbf{K}$ be a connected and pure 2-complex. We say that $\mathbf{K}$ is a combinatorial surface with

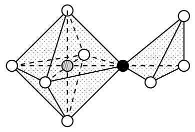


Fig. 5 Three-dimensional topological characterisation of points whose stars are spherical, semi-spherical and neither of them, illustrated as grey, white and black points.

edges if every 0-polyhedron in $Sk_0(\mathbf{K})$ has either a cyclic or semi-cyclic star, and there is at least one 0-polyhedron whose star is semi-cyclic in $Sk_0(\mathbf{K})$.

[**Definition 16**] Let $\mathbf{K}$ be a connected and pure 2-complex. We say that $\mathbf{K}$ is a combinatorial closed surface if every 0-polyhedron in $Sk_0(\mathbf{K})$ has a cyclic star.

By using combinatorial surfaces and the following combinatorial boundary, we define topological characteristics of stars in three dimensions.

[**Definition 17**] Let $\mathbf{K}$ be a pure n-complex where $n > 0$ and $\mathbf{H}$ be the set of all $(n-1)$-polyhedra in $\mathbf{K}$ each of which is a face of exactly one n-polyhedron in $\mathbf{K}$. The combinatorial boundary of $\mathbf{K}$ is then defined as the pure $(n-1)$-complex $\partial\mathbf{K} = Cl(\mathbf{H})$.

[**Definition 18**] Let $\mathbf{K}$ be a polyhedral complex and $\boldsymbol{x} \in Sk_0(\mathbf{K})$. We say that $star(\boldsymbol{x})$ is spherical if $link(\boldsymbol{x})$ is a combinatorial closed surface.

[**Definition 19**] Let $\mathbf{K}$ be a polyhedral complex and $\boldsymbol{x} \in Sk_0(\mathbf{K})$. We say that $star(\boldsymbol{x})$ is semi-spherical if $link(\boldsymbol{x})$ is a combinatorial surface with edges, and the edges, i.e., the combinatorial boundary $\partial(link(\boldsymbol{x}))$ is a combinatorial closed curve.

Figure 5 illustrates stars which are spherical and semi-spherical.

4.3 Point classification

Each 0-polyhedron, namely point $\boldsymbol{x}$, in the 0-skeleton $Sk_0(\mathbf{K})$ of an n-complex $\mathbf{K}$ where $n \leqq 3$ can be classified into one of the twelve types each of which satisfies one of the following conditions [6].

(1) $dim(star(\boldsymbol{x})) = 0$, that is $star(\boldsymbol{x}) = \{\boldsymbol{x}\}$;
(2) $star(\boldsymbol{x})$ is linear;
(3) $star(\boldsymbol{x})$ is semi-linear;
(4) $dim(star(\boldsymbol{x})) = 1$ and $star(\boldsymbol{x})$ is neither linear nor semi-linear;
(5) $star(\boldsymbol{x})$ is cyclic;
(6) $star(\boldsymbol{x})$ is semi-cyclic;
(7) $dim(star(\boldsymbol{x})) = 2$, $Cl(star(\boldsymbol{x}))$ is pure, and $star(\boldsymbol{x})$ is neither cyclic nor semi-cyclic;
(8) $dim(star(\boldsymbol{x})) = 2$ and $Cl(star(\boldsymbol{x}))$ is not pure;
(9) $star(\boldsymbol{x})$ is spherical;
(10) $star(\boldsymbol{x})$ is semi-spherical;
(11) $dim(star(\boldsymbol{x})) = 3$, $Cl(star(\boldsymbol{x}))$ is pure, and $star(\boldsymbol{x})$ is neither spherical nor semi-spherical;
(12) $dim(star(\boldsymbol{x})) = 3$ and $Cl(star(\boldsymbol{x}))$ is not pure.

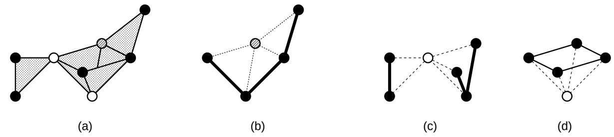


Fig. 6 (a) Three points of type 7 which are colored in grey and white, (b) the collapsible link of the grey point, and (c, d) the noncollapsible links of the white points.

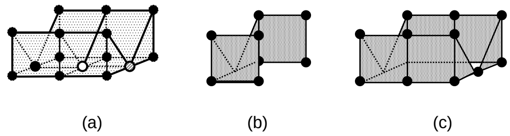


Fig. 7 (a) Two points of type 11 which are colored with grey and white, (b) the collapsible link of the grey point, and (c) the noncollapsible link of the white point.

4.4 Topological characteristics of simple points

In Section 4.3, we showed that each point in a point set $\mathbf{V}$ can be classified into one of the twelve types by using the complicial representation $cmp(\mathbf{V})$. In this subsection, we check which types of points are simple.

According to Proposition 2, we verify the collapsibility of $link(\boldsymbol{x})$ for every type of points $\boldsymbol{x}$ and then obtain the following theorem. In this paper, we omit the proof due to the page limitation.

[**Theorem 1**] Every point whose type is either 3, 6 or 10 is always a simple point. Contrarily, any point whose type is either 1, 2, 4, 5, 8 or 9 can never be a simple point.

From the above theorem, we see that points of types 7, 11 and 12, differing from the other types, have both cases which are simple and not simple. Figures 6, 7 and 8 show examples of simple and non-simple points for types 7, 11 and 12 respectively. The examples illustrate that the connectivity of $link(\boldsymbol{x})$ is a necessary condition but not a sufficient one for the collapsibility of $link(\boldsymbol{x})$.

From Theorem 1, we also see that simple points can be of the six different types 3, 6, 7, 10, 11, 12. Table 2 shows the

Table 2 The numbers of local point configurations of simple points for each point type with respect to 6- and 26-neighborhood systems.

	6-neighborhood	26-neighborhood
type 3	134 280	3
type 6	345 016	398
type 7	28 994	1 037
type 10	14 031	290 979
type 11	332	28 525
type 12	27 782	229 493
total	550 435	550 435

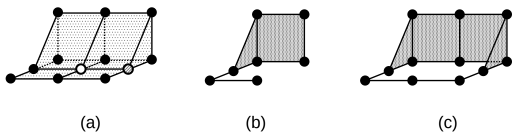


Fig. 8 (a) Two points of type 12 which are colored in grey and white, (b) the collapsible link of the grey point, and (c) the noncollapsible link of the white point.

numbers of all different local configurations of simple points for each point type.

5. Linear thinning algorithm

Given a finite subset $\mathbf{V}$ in $\mathbb{Z}^3$, we present a linear algorithm for thinning $\mathbf{V}$. In Algorithm 1, we require a list $\mathbf{P}$ for deletable point candidates and also a boolean function $A : \mathbf{V} \rightarrow \mathbb{B}$ for renewing the deletability of $\boldsymbol{x} \in \mathbf{V}$ after removing one of its neighboring points.

In Step 12 of Algorithm 1, we assume that points $\boldsymbol{y}$ whose types can be changed due to removing $\boldsymbol{x}$ from $\mathbf{V}$ are in a neighborhood of $\boldsymbol{x}$, i.e. $\mathbf{N}(\boldsymbol{x})$. If we use a method in the framework of either of a Khalimsky topology [8], a partially ordered set [3] or a discrete polyhedral complex [5], [6] for construction of a polyhedral complex $cmp(\mathbf{V})$, such a neighborhood $\mathbf{N}(\boldsymbol{x})$ can be considered to be the 26-neighborhood.

Obviously, the result of Algorithm 1 depends on a point

Algorithm 1: Thinning

input : a point set $\mathbf{V} \subset \mathbb{Z}^3$
output: a thinned set $\mathbf{V}$

1 **begin**
2 obtain a set of all simple points $\mathbf{P}$ which are not endpoints in $\mathbf{V}$;
3 **foreach** $\boldsymbol{x} \in \mathbf{V}$ **do**
4 **if** $\boldsymbol{x} \in \mathbf{P}$ **then**
5 $A(\boldsymbol{x}) \leftarrow True$;
6 **else**
7 $A(\boldsymbol{x}) \leftarrow False$;
8 **while** $\mathbf{P} \neq \emptyset$ **do**
9 select a point $\boldsymbol{x} \in \mathbf{P}$ and $\mathbf{P} \leftarrow \mathbf{P} \setminus \{\boldsymbol{x}\}$;
10 **if** $A(\boldsymbol{x}) = True$ **then**
11 $\mathbf{V} \leftarrow \mathbf{V} \setminus \{\boldsymbol{x}\}$ (change the value of $\boldsymbol{x}$ from 1 to 0);
12 **foreach** $\boldsymbol{y} \in \mathbf{N}(\boldsymbol{x}) \cap \mathbf{V}$ **do**
13 **if** $A(\boldsymbol{y}) = True$ *and* $\boldsymbol{y}$ *is not simple or is an endpoint* **then**
14 $A(\boldsymbol{y}) \leftarrow False$;
15 **else if** $A(\boldsymbol{y}) = False$ *and* $\boldsymbol{y}$ *is simple but not an endpoint* **then**
16 $\mathbf{P} \leftarrow \mathbf{P} \cup \{\boldsymbol{y}\}$ and $A(\boldsymbol{y}) \leftarrow True$;
17 **return** $\mathbf{V}$;
18 **end**

$\boldsymbol{x} \in \mathbf{P}$ selected in Step 9. If we set no endpoint, thinning results are topologically equivalent with respect to the initial set $\mathbf{V}$. Therefore, we can simply realize $\mathbf{P}$ as a queue in the case that we are interested in only topological results. However, if we set endpoints and our interests are not only topology but also geometry, we may need to realize $\mathbf{P}$ as a priority queue whose priorities depend on distances from the complement $\overline{\mathbf{V}}$, for example.

Applying Algorithm 1, we can obtain a curve or surface skeleton of an initial set $\mathbf{V}$, depending on the definition of endpoints. Thanks to the results of topological point classification in Subsection 4.3, we can set endpoints easily by dimensions and topological characteristics of stars. If we set endpoints to have type 3 (semi-linear), Algorithm 1 behaves as a curve thinning. Similarly, if we set endpoints to have type 3 and 6 (semi-linear and semi-cyclic), it behaves as a surface thinning.

6. Setting of endpoints

The above discussion on the endpoint setting is intuitively correct, but practically we cannot say that it always works. For example, it is very rare that we use the 6-neighborhood system for the curve/surface thinning because thinning results generally contain too many small parts because of too many configurations of endpoints; see in Table 2 that there are much more configurations of types 3 and 6 for the 6-neighborhood than those for the 26-neighborhood. For the 26-neighborhood system, the curve thinning works very well (see Fig. 10) while the surface thinning does not. This is because we do not have enough endpoints (type 6) for the 26-neighborhood system as shown in Table 2. To obtain those semi-cyclic points (type 6), we construct a polyhedral complex as a collection of convex polyhedra each of which is locally made from a set of points in $\mathbf{V}$ at a unit cubic region. Therefore, a constructed polyhedron tends to have three dimensions rather than less than three dimensions for any point configuration.

We therefore propose a simple method to obtain more configurations for semi-cyclic (type 6) points. In [6], we obtain all possible configurations of discrete surfaces which appear on the boundaries of 3D discrete objects and whose central points have cyclic stars on the surfaces: 6 and 6028 configu-

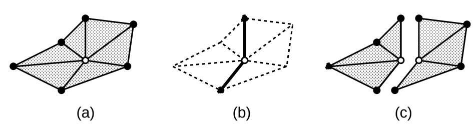


Fig. 9 An example of semi-cyclic star generation for the 26-neighborhood system; (a) a cyclic star, (b) a linear star having the common central point of (a), and (c) two new semi-cyclic stars made by cutting (a) by (b).

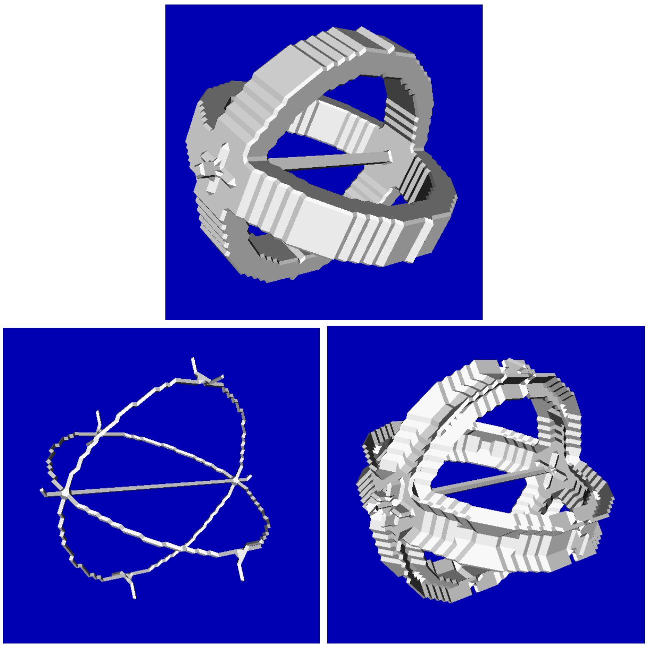


Fig. 10 The original 3D image (top), its curve thinning result (bottom left), and its surface thinning result (bottom right) for the 26-neighborhood system.

rations for the 6- and 26-neighborhood systems, respectively. We cut each cyclic star (a discrete surface) by a linear star (a discrete line) having the common central point and create additional semi-cyclic stars. Figure 9 illustrates an example for such a semi-cyclic star generation. Then we obtain 22399 configurations instead of 398 for type 6 in Table 2.

With these new semi-cyclic points, we obtain a surface thinning result in Figs. 10, 11. In these examples, priorities of **P** categorized by 26 directions are used. Figure 11 illustrates that we may dig a hole at the intersection of digitized planes depending on the rotation of digitized planes and the number of digitized planes. This is caused by the image discreteness: locally we cannot distinguish between a 3D part and an intersection of two 2D parts if they have the same local point configuration. In order to solve the problem, additional topological configurations for surface intersections [11] and/or supplementary geometrical concepts will be necessary.

7. Conclusions

In this paper, we presented a new topological characterization of simple points by using collapsibility of polyhedral complexes. By using the same framework, we showed that our topological characterization/classification of points by stars are useful for curve and surface thinning, but we also showed that even our approach faces the problem, caused by the discreteness of the space, such as thinning of surface intersections.

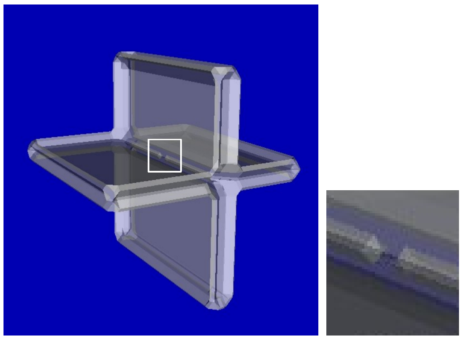


Fig. 11 A surface thinning result for two intersected digitized planes for the 26-neighborhood system (left) and the magnification of the white square (right). A transparency is given to make easy to see the interior such that there are two deep holes dug from the surface edges at the intersection and only one point is connected to surfaces.

References

[1] P. S. Alexandrov, *Combinatorial Topology*, Vol. 1, Graylock Press, Rochester, New York, 1956.

[2] G. Bertrand, "Simple points, topological numbers and geodesic neighborhoods in cubic grids", Pattern Recognition Letters, Vol. 15, pp. 1003–1011, October 1994.

[3] G. Bertrand, "New notions for discrete topology," in LNCS1568: Discrete Geometry for Computer Imagery, Proceedings of DGCI'99, pp. 218–228, 1999.

[4] G. Bertrand, M. Couprie, "A model for digital topoloty," in LNCS1568: Discrete Geometry for Computer Imagery, Proceedings of DGCI'99, pp. 229–241, 1999.

[5] Y. Kenmochi, A. Imiya, "Combinatorial boundary of a 3D lattice point set," Journal of Visual Communication and Image Representation, Vol. 17, No. 4, pp. 738–766, August 2006.

[6] Y. Kenmochi, Y. Nomura, "Local point configurations of discrete combinatorial surfaces", submited to Image and Vision Computing.

[7] Y. Kenmochi, K. Ushigusa, "Complicial thinning for three-dimensional digital images", in Proceedings of MIRU2005.

[8] E. Khalimsky, R. Kopperman, P. R. Meyer, "Computer graphics and connected topologies on finite ordered sets," Topology and its Applications, Vol. 36, pp. 1–17, July 1990.

[9] T. Y. Kong, A. Rosenfeld, "Digital topology: introduction and survey," Computer Vision, Graphics, and Image Processing, Vol. 48, pp. 357-393, 1989.

[10] T. Y. Kong, "Topology-Preserving Deletion of 1's from 2-, 3-, and 4-Dimensional Binary Images," in LNCS 1347: Discrete Geometry for Computer Imagery, Proceedings of DGCI'97, pp. 3-18, 1997.

[11] C. Lohou, G. Bertrand, "Poset approach to 3D parallel thinning," in SPIE Vision Geometry VIII, Vol. 3811, pp. 45–56, 1999.

[12] C. P. Rourke, B. J. Sanderson, *Introduction to Piecewise-Linear Topology*, Springer-Verlag, Berlin Heidelberg New York, 1972.

[13] J. Stillwell, *Classical topology and combinatorial group theory*, Springer-Verlag, New York, 1993.

[14] J. Toriwaki, *Three-dimensional digital image processing*, Shokodo, Tokyo, 2003.

[15] G. M. Ziegler, *Lectures on Polytopes*, Springer-Verlag, New York, 1998.