

# Minimal simple pairs in the cubic grid

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## Motivation

Topological properties are fundamental in many applications of image analysis. Topology-preserving operators, like homotopic skeletonisation, are used to transform an object while leaving unchanged its topological characteristics. In discrete grids ( $\mathbb{Z}^2$  or  $\mathbb{Z}^3$ ), such a transformation can be defined and efficiently implemented thanks to the notion of simple point.

A typical topology-preserving transformation based on simple points deletion, that we call *guided homotopic thinning*, may be described as follows. The input data consists of a set  $X$  of points in the grid (called object), and a subset  $K$  of  $X$  (called constraint set). Let  $X_0 = X$ . At each iteration  $i$ , choose a simple point  $x_i$  in  $X_i$  but not in  $K$  according to some criterion (e.g., a priority function) and set  $X_{i+1} = X_i \setminus \{x_i\}$ . Continue until reaching a step  $n$  such that no simple point for  $X_n$  remains in  $X_n \setminus K$ . We call the result of this process a *homotopic skeleton of  $X$  constrained by  $K$* .

In such a transformation, the result is expected to fulfil a property of minimality, as suggested by the term “skeleton”. This is indeed the case for the procedure described above, since the result  $X_n$  is minimal in the sense that it contains no simple point outside of  $K$ . However, we could formulate a stronger minimality requirement, which seems natural for this kind of transformation: informally, the result  $X_n$  should not strictly include any set  $Y$  which is “topologically equivalent” to  $X$ , and which contains  $K$ . We say that a homotopic skeleton of  $X$  constrained by  $K$  is *globally minimal* if it fulfills this condition.

Now, a fundamental question arises: is any homotopic skeleton globally minimal? Let us illustrate this problem in dimensions 2 and 3. In  $\mathbb{Z}^2$ , consider a full rectangle  $X$  of any size, and the constraint set  $K = \emptyset$ . Obviously, this object  $X$  is topologically equivalent to a single point, thus only homotopic skeletons which are singletons are globally minimal. A. Rosenfeld proved that any homotopic skeleton of  $X$  is indeed reduced to a single point.

But quite surprisingly, in dimension 3, this property does not hold: if  $X$  is e.g. a full  $10 \times 10 \times 10$  cube, we may find a homotopic skeleton of  $X$  (with empty constraint set) which is not reduced to a single point. A classical counter-example is the Bing’s house with two rooms (Fig. 1).

It could be argued that objects like Bing’s houses are unlikely to appear while processing real (noisy) images, because of their complex shape and their size. However, we found that there exists a large class of objects presenting similar properties, some of them being quite small (Fig. 2). Let us call a *lump relative to  $K$*  any object  $X$  which has no simple point outside of  $K$ , and which strictly includes a subset  $Y$  including  $K$  and topologically equivalent to  $X$  (i.e., a homotopic skeleton which is not globally minimal).

Now, two questions arise: is it possible to detect when a thinning procedure gets stuck on a lump, and then, is it possible to find a way towards a globally minimal homotopic skeleton? For performing the latter task, a solution consists of identifying a subset of  $X$  which can be removed without changing topology; we call such a subset a *simple set*. We are interested essentially by simple sets which are minimal, in the sense that they do not strictly include any other simple set, since it is sufficient to detect such sets in order to carry on thinning. Also, we hope that minimal simple sets have a specific structure which could make them easier to analyse.

This work (developed in the framework of abstract complexes) is dedicated to the study of the simplest ones among such minimal simple sets, called minimal simple pairs (MSPs), which are those composed of two non-simple points. These minimal simple sets are the ones which are most likely to appear in practical applications (Fig. 3), hence the interest in understanding their structure.

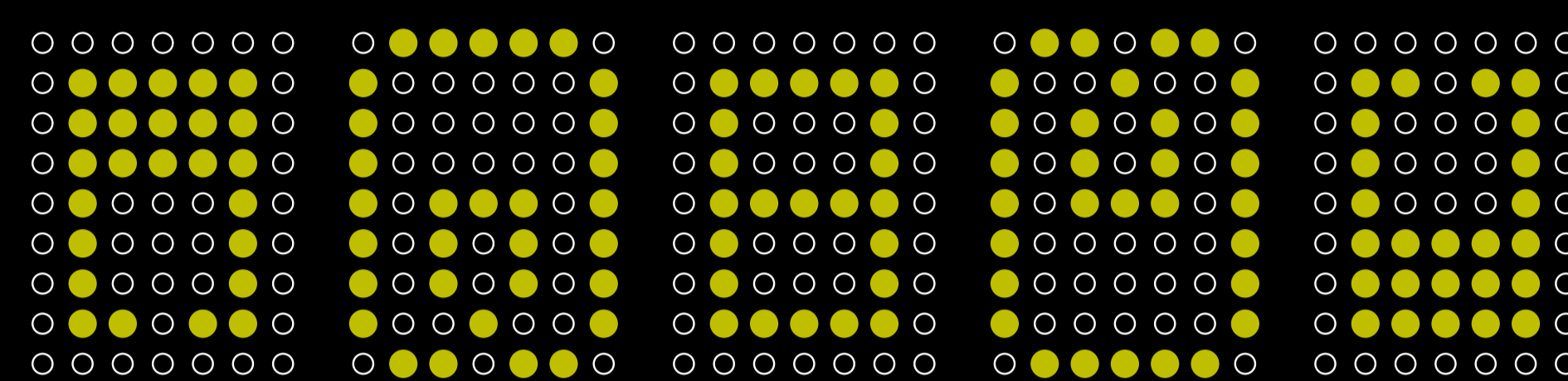
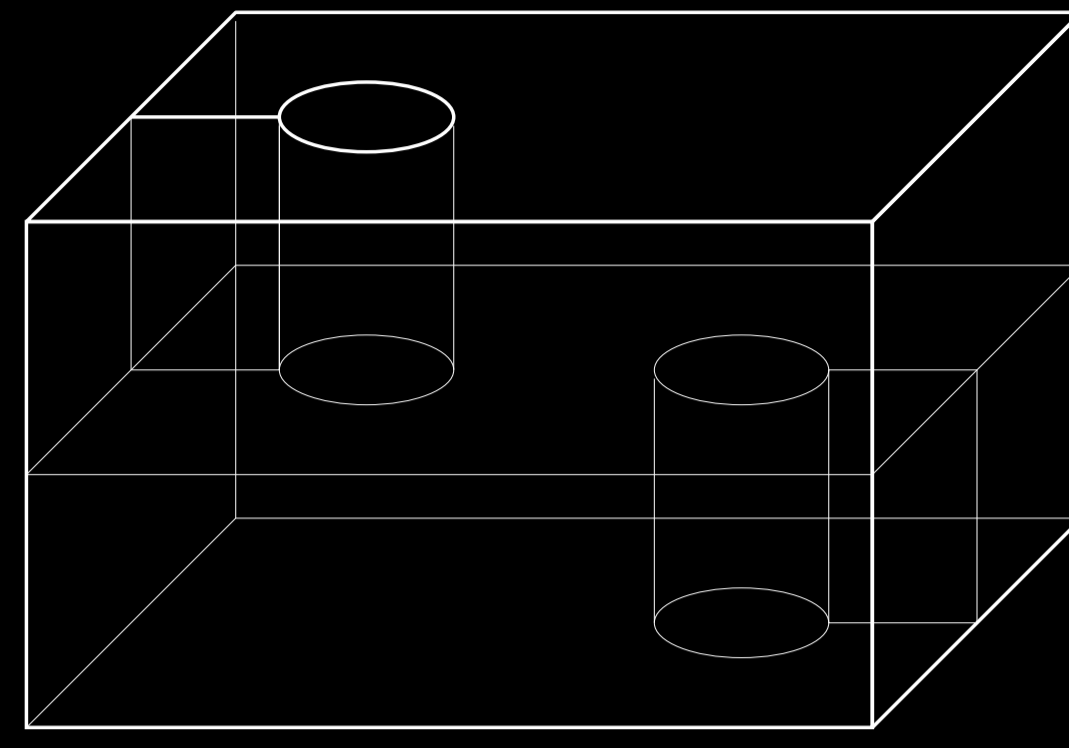


Fig. 1 - A Bing’s house in  $\mathbb{R}^3$  (top), and in  $\mathbb{Z}^3$  (2D “slices”, bottom).

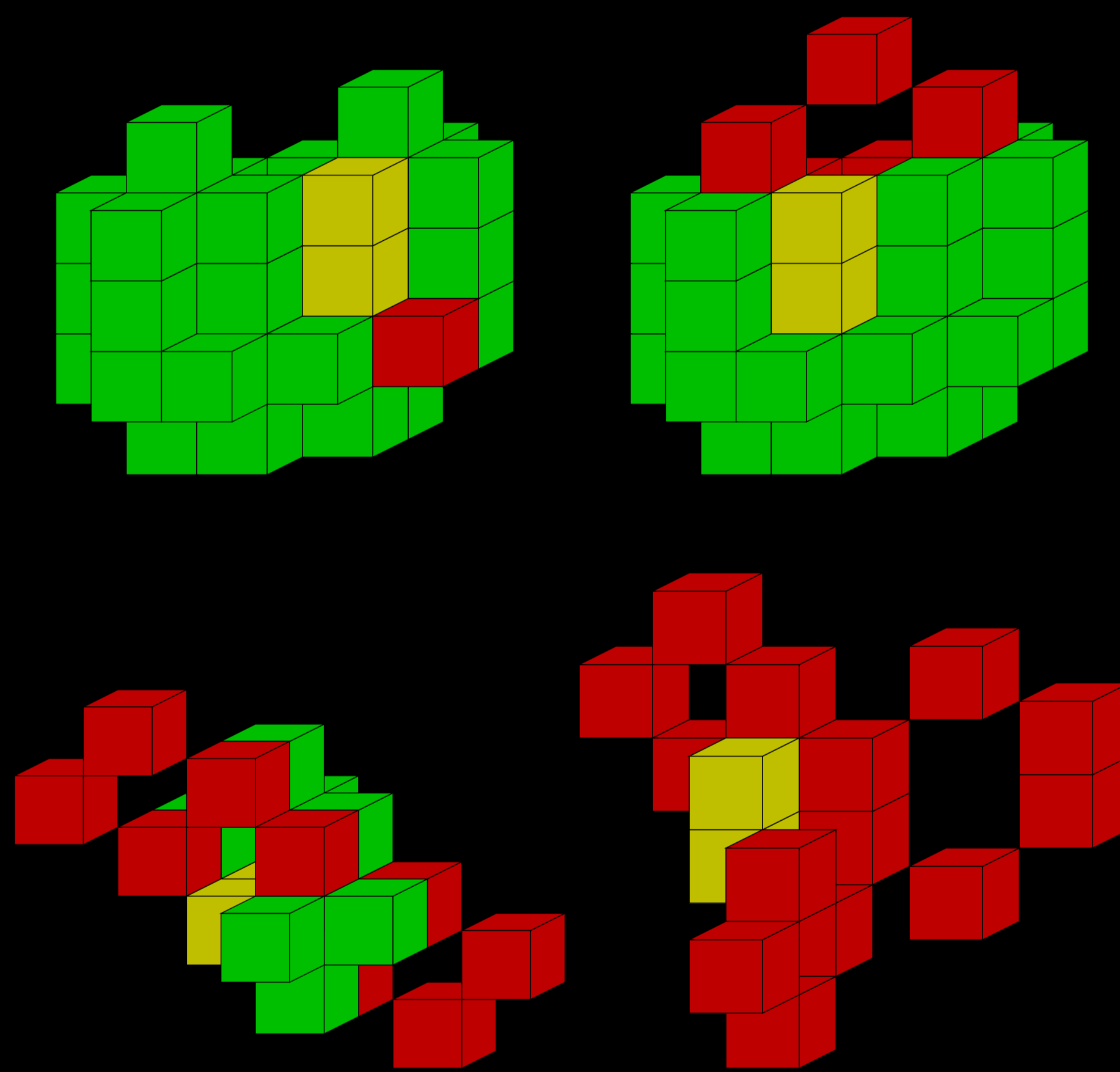


Fig. 2 - 3D view of sets  $X_i$  (from top-left to bottom-right:  $i = 1$  to 4) without simple points. In red: a smallest subset topologically equivalent to  $X_i$ . In yellow: an MSP for  $X_i$ .

Input :  $X \subset \mathbb{Z}^3$   
Output :  $Y \subseteq X$  ( $Y$  topologically equivalent to  $X$ )

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Algo. 1
Y = X
while  $\exists x \in Y$  such that  $x$  is a 26-simple point for  $Y$  do
  Y = Y \ {x}
end while
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Algo. 2
Y = X
repeat
  while  $\exists x \in Y$  such that  $x$  is a 26-simple point for  $Y$  do
    Y = Y \ {x}
  end while
  if  $\exists P = \{x, y\} \subset Y$  such that  $P$  is an MSP for  $Y$  then
    Y = Y \ P
  end if
until Y admits no simple point nor MSP
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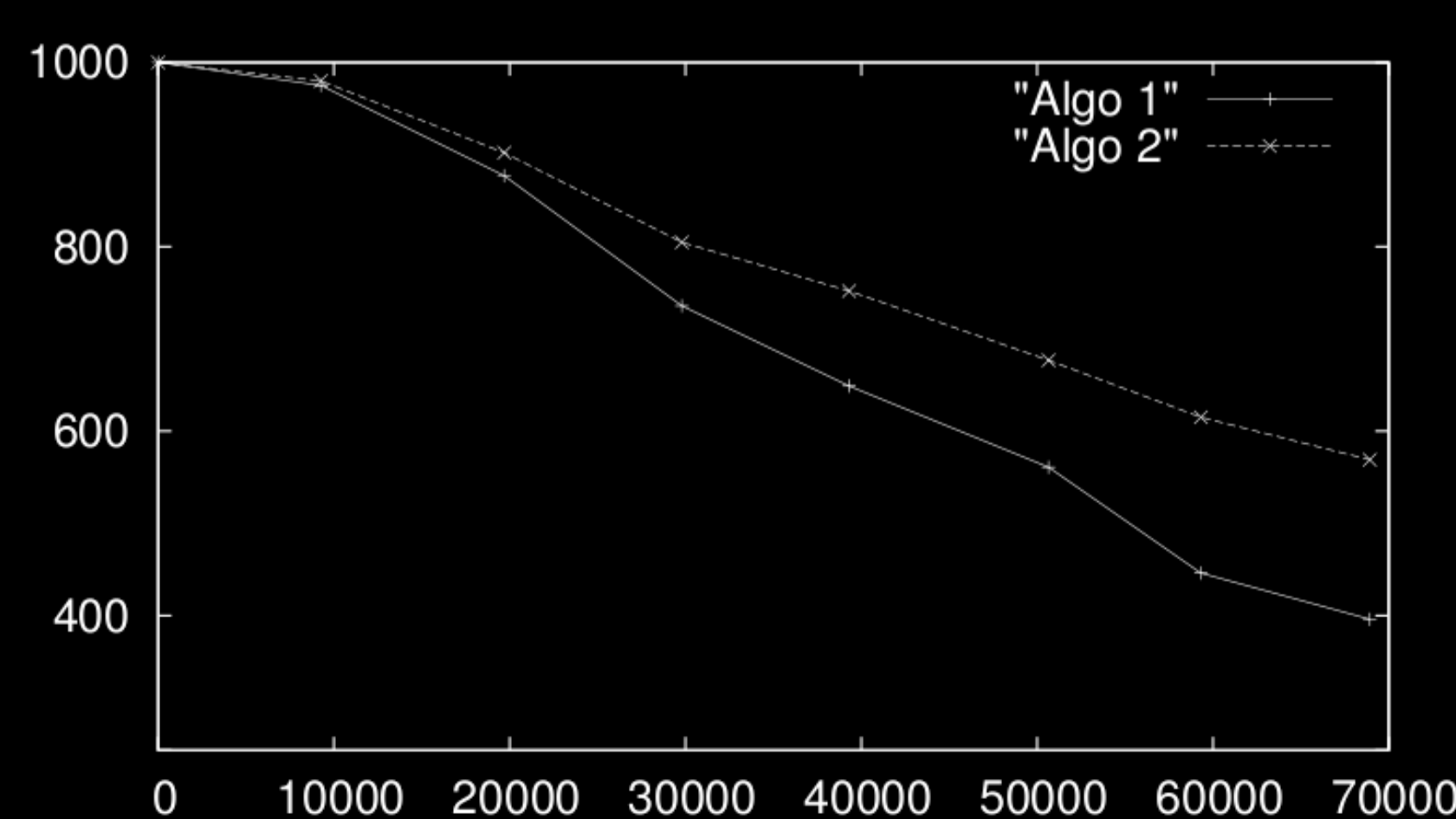


Fig. 3. Efficiency of reduction algorithms based on simple points (Algo 1) and both simple points and MSPs (Algo 2), respectively. Horizontal axis: size (number of voxels) of original objects (full cubes). Vertical axis: number of objects (over 1000 experiments) which were reduced to a single voxel.

## Properties of MSPs

**Proposition 1** Let  $P \subseteq F$  be an MSP for  $F$ . Then  $P$  is connected ( $|C[P]| = 1$ ).

The voxels constituting an MSP cannot be simple voxels. The attachment of a non-simple voxel  $\hat{f}$  can either: (i) be empty, (ii) be equal to the boundary of  $\hat{f}$ , (iii) be disconnected, (iv) have at least one tunnel. Some of these cases cannot appear in an MSP. First, (i) and (iii) cannot hold for such a voxel, i.e., the attachment of a voxel in an MSP is non-empty and connected.

**Proposition 2** Let  $P \subseteq F$  be an MSP for  $F$ . Then the attachment to  $F$  of any voxel of  $P$  is connected. ( $\forall g \in P^+, |C[Att(\hat{g}, F)]| = 1$ ).

Then, with the next proposition, we show that (ii) cannot hold, hence, the attachment to  $F$  of any voxel  $g$  in an MSP has no cavity.

**Proposition 3** Let  $P \subseteq F$  be an MSP for  $F$ . Then the attachment to  $F$  of any voxel of  $P$  is different from its boundary ( $\forall g \in P^+, Att(\hat{g}, F) \neq \hat{g}^*$ ).

The following proposition, (with Prop. 2 and Prop. 3) implies that the attachment to  $F$  of any voxel in an MSP has at least one tunnel.

**Proposition 4** Let  $P \subseteq F$  be an MSP for  $F$ . Then the Euler characteristic of the attachment to  $F$  of any voxel of  $P$  is negative ( $\forall g \in P^+, \chi(Att(\hat{g}, F)) \leq 0$ ).

From Prop. 1, we know that an MSP is necessarily connected. The following proposition tells us more about the intersection of the two voxels which compose any MSP.

**Proposition 5** Let  $P \subseteq F$  be an MSP for  $F$ , and let  $g_1, g_2$  be the two voxels of  $P$ . Then,  $g_1 \cap g_2$  is a 2-face.

The two following propositions are necessary conditions for an MSP, similar to the conditions which characterise simple voxels.

**Proposition 6** Let  $P \subseteq F$  be an MSP for  $F$ . Then the attachment to  $F$  of  $P$  is connected ( $|C[Att(P, F)]| = 1$ ).

**Proposition 7** Let  $P \subseteq F$  be an MSP for  $F$ . Then the attachment to  $F$  of  $P$  has a Euler characteristic equal to 1 ( $\chi(Att(P, F)) = 1$ ).

## Characterisation of MSPs

**Proposition 8** Let  $P \subseteq F$  be a pair. Then  $P$  is an MSP for  $F$  if and only if all the following conditions hold:

- the intersection of the two voxels of  $P$  is a 2-face, (1)
- $\forall g \in P^+, |C[Att(\hat{g}, F)]| = 1$ , (2)
- $\forall g \in P^+, \chi(Att(\hat{g}, F)) \leq 0$ , (3)
- $|C[Att(P, F)]| = 1$ , (4)
- $\chi(Att(P, F)) = 1$ . (5)

Actually, conditions (1), (3), (4) and (5) are sufficient to characterise an MSP, since condition (2) may be deduced from them.

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