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# CHARACTERIZATIONS USING THE BIVARIATE FAILURE RATE FUNCTION

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### Abstract

A general procedure to characterize bivariate absolutely continuous distributions by using the bivariate failure (hazard) rate function is obtained. The theoretical results are illustrated by obtaining new characterizations of some probability models including the bivariate Gumbel exponential and the bivariate Pareto (Lomax) distributions.

*Key words:* characterization; bivariate failure rate function; bivariate probability models.

# 1 Introduction.

Several characterizations of probability models have been obtained in the last 30 years based on the univariate failure (hazard) rate or mean residual life functions. For example, it is well known that the exponential distribution is characterized in the class of absolutely continuous distributions by a constant failure rate (or mean residual life) function. It is well known that the univariate failure rate uniquely determines the distribution function, that is, it has all the information about the probability model. The same holds for the mean residual life. These two functions are used to describe the aging process in reliability and survival studies (see, e.g., Barlow and Proschan (1975)). Recent characterizations based on aging measures can be seen Belzunce *et al.* (2004), Navarro and Ruiz (2004a) and Nanda, Bhattacharjee and Alam (2006,2007).

If  $X$  is an absolutely continuous random variable (usually representing the lifetime of a ‘unit’ or a component in a system) with reliability function  $R(x) = \Pr(X \geq x)$  and density function  $f(x) = -R'(x)$ , the univariate failure rate is defined by

$$r(x) = \frac{f(x)}{R(x)} = -\frac{d}{dx} \ln R(x)$$

for all  $x$  such that  $R(x) > 0$ .

If  $(X_1, X_2)$  is a random vector (usually representing the lifetime of two ‘units’ or two components in a system) with absolutely continuous reliability function defined by  $R(x_1, x_2) = \Pr(X_1 \geq x_1, X_2 \geq x_2)$  and density function  $f(x_1, x_2)$ , the univariate failure rate was extended to the bivariate set up by using different ways.

The first one uses the bivariate failure rate function defined in Basu (1971) by

$$r(x_1, x_2) = \frac{f(x_1, x_2)}{R(x_1, x_2)}$$

for all  $(x_1, x_2)$  such that  $R(x_1, x_2) > 0$ . Puri and Rubin (1974) characterized a mixture of exponential distributions by  $r(x_1, x_2) = c$  for  $x_1 > 0$  and  $x_2 > 0$ . However, in general,  $r$  does not necessarily determine  $R$ . This fact was noted by Yang and Nachlas (2001), Finkelstein (2003) and Finkelstien and Esaulova (2005).

The second option is to use the hazard gradient defined in Johnson and Kotz (1975), Barlow and Proschan (1975) and Marshall (1975) by

$$h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2)),$$

where

$$h_i(x_1, x_2) = -\frac{\partial}{\partial x_i} \ln R(x_1, x_2)$$

for  $i = 1, 2$  and  $(x_1, x_2)$  such that  $R(x_1, x_2) > 0$ . It is well known that  $h$  uniquely determines  $R$  (see Marshall and Olkin (1979) and Shanbhag and Kotz (1987)). Some recent characterizations using the hazard gradient were given in Navarro and Ruiz (2004b), Kotz, Navarro and Ruiz (2007), and Navarro, Ruiz and Sandoval (2007). This function is also used in Navarro and Shaked (2006) and Navarro, Rychlik and Shaked (2007) to obtain ordering properties between order statistics and series systems with dependent components. Shaked and Shanthikumar (1987) gave a dynamic version of the multivariate hazard function.

The purpose of this paper is to study a general procedure to obtain characterizations of probability models by using the bivariate failure rate function of Basu (Section 2). The theoretical results are illustrated by some examples of characterizations of relevant probability models (Section 3). Some conclusions are given in Section 4.

## 2 The main result.

Let  $(X_1, X_2)$  be a random vector with absolutely continuous reliability function  $R(x_1, x_2) = \Pr(X_1 \geq x_1, X_2 \geq x_2)$ , density function  $f(x_1, x_2)$ , and bivariate failure rate

$$r(x_1, x_2) = \frac{f(x_1, x_2)}{R(x_1, x_2)}$$

for all  $(x_1, x_2)$  such that  $R(x_1, x_2) > 0$ . We shall denote the marginal reliability function of  $X_i$  by  $R_i(x_i) = \Pr(X_i \geq x_i)$ .

The main result of the paper is included in the next theorem which proves that the bivariate failure rate  $r$  of Basu (given the marginal distributions) uniquely determines  $R$  under some reasonable assumptions.

**Theorem 1.** If  $(X_1, X_2)$  has support  $S$ ,  $R(z_1, z_2) = 1$  for a fixed point  $(z_1, z_2) \in S$  such that  $[z_1, z_1 + \varepsilon) \times [z_2, z_2 + \varepsilon) \subset S$  for a positive real number  $\varepsilon$ , and  $\ln R(x_1, x_2)$  is analytical in  $S$ , that is, it can be written as

$$\ln R(x_1, x_2) = - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} \frac{(x_1 - z_1)^i}{i!} \frac{(x_2 - z_2)^j}{j!} \quad (1)$$

for all  $(x_1, x_2) \in S$ , where  $a_{i,j} \in \mathbb{R}$  for  $i = 0, 1, 2, \dots$  and  $j = 0, 1, 2, \dots$ , then the marginal distributions and  $r$  uniquely determine  $R$ .

*Proof.* From (1) and  $R(z_1, z_2) = 1$ , we have  $a_{0,0} = 0$ ,

$$R_1(x_1) = R(x_1, z_2) = \exp \left( - \sum_{i=0}^{\infty} a_{i,0} \frac{(x_1 - z_1)^i}{i!} \right)$$

and

$$R_2(x_2) = R(z_1, x_2) = \exp \left( - \sum_{j=0}^{\infty} a_{0,j} \frac{(x_2 - z_2)^j}{j!} \right)$$

for all  $(x_1, x_2) \in [z_1, z_1 + \varepsilon) \times [z_2, z_2 + \varepsilon)$ , where  $R_1$  and  $R_2$  are the marginal reliability functions. Hence,  $a_{i,0}$  and  $a_{0,j}$  are known for all  $i = 0, 1, 2, \dots$  and  $j = 0, 1, 2, \dots$ .

From (1), the bivariate failure rate function  $r$  of  $(X_1, X_2)$  defined by

$$r(x_1, x_2) = \frac{1}{R(x_1, x_2)} \frac{\partial^2}{\partial x_1 \partial x_2} R(x_1, x_2)$$

can be written as

$$\begin{aligned} r(x_1, x_2) &= \left( \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} a_{i,j} \frac{(x_1 - z_1)^{i-1}}{(i-1)!} \frac{(x_2 - z_2)^j}{j!} \right) \left( \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} a_{i,j} \frac{(x_1 - z_1)^i}{i!} \frac{(x_2 - z_2)^{j-1}}{(j-1)!} \right) \\ &\quad - \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} \frac{(x_1 - z_1)^{i-1}}{(i-1)!} \frac{(x_2 - z_2)^{j-1}}{(j-1)!} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(x_1 - z_1)^i}{i!} \frac{(x_2 - z_2)^j}{j!} \left( -a_{i+1,j+1} + \sum_{r=0}^i \sum_{s=0}^j \binom{i}{r} \binom{j}{s} a_{r,j+1-s} a_{i+1-r,s} \right) \end{aligned}$$

for all  $(x_1, x_2) \in S$ . Therefore,  $r$  is also analytical in  $S$  and it can be written as

$$r(x_1, x_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{i,j} \frac{(x_1 - z_1)^i}{i!} \frac{(x_2 - z_2)^j}{j!}$$

for all  $(x_1, x_2) \in S$ , where

$$b_{i,j} = -a_{i+1,j+1} + \sum_{r=0}^i \sum_{s=0}^j \binom{i}{r} \binom{j}{s} a_{r,j+1-s} a_{i+1-r,s} \quad (2)$$

for  $i = 0, 1, \dots$  and  $j = 0, 1, \dots$

Thus, if  $r$  is known, the coefficients  $b_{i,j}$  are known for  $i = 0, 1, \dots$  and  $j = 0, 1, \dots$ . Then, from (2),

$$a_{1,1} = -b_{0,0} + a_{0,1}a_{1,0}$$

is known.

By induction on  $k$ , let us suppose that the coefficients  $a_{i,j}$  are known for all  $i$  and  $j$  such that  $i + j < k$ . Then if  $i + j = k$ ,  $i \geq 1$  and  $j \geq 1$ , using (2),  $a_{i,j}$  can be obtained from

$$a_{i,j} = -b_{i-1,j-1} + \sum_{r=0}^{i-1} \sum_{s=0}^{j-1} \binom{i-1}{r} \binom{j-1}{s} a_{r,j-s} a_{i-r,s} \quad (3)$$

since  $r + j - s \leq i + j - 1$  and  $i - r + s \leq i + j - 1$ .

Therefore,  $R_1, R_2$  and  $r$  determine  $R$ .  $\square$

**Remark 2.** It is well known (see Theorem 1 in Puri and Rubin (1974)) that, if  $r$  is constant, then the distribution of  $(X_1, X_2)$  is the mixture of exponential distributions defined by the density

$$f(x_1, x_2) = \alpha \int_0^\infty \int_0^\infty \exp(-u_1 x_1 - u_2 x_2) G(du_1, du_2), \quad (4)$$

where  $\alpha = 1/E(X_1 X_2)$  and the probability measure  $G$  is concentrated on the set  $A = \{(u_1, u_2) : u_1 u_2 = \alpha, u_1 > 0, u_2 > 0\}$ . If we suppose that  $r(x_1, x_2) = \theta$  and that the marginal distributions are exponential with means  $\mu_1$  and  $\mu_2$ , then  $a_{1,0} = 1/\mu_1$ ,  $a_{0,1} = 1/\mu_2$ ,  $a_{1,1} = -\theta$ ,  $a_{i,0} = a_{0,i} = 0$  for  $i = 2, 3, \dots$  and  $a_{i,j}$  can be computed from (3). In the next section we give some new characterizations results.

### 3 Examples

In this section the theoretical results given in Section 2 are illustrated by some new characterization results. Firstly, we obtain a characterization of the bivariate Gumbel exponential distribution.



**Example 3.** If the marginal distributions are exponential with means  $\mu_1$  and  $\mu_2$  and  $r$  is given by

$$r(x_1, x_2) = \frac{1}{\mu_1 \mu_2} - \theta + \theta \frac{x_1}{\mu_1} + \theta \frac{x_2}{\mu_2} + \theta^2 x_1 x_2$$

for  $x_1 \geq 0$  and  $x_2 \geq 0$ , where  $0 \leq \theta \leq 1/(\mu_1 \mu_2)$ , then  $a_{1,0} = 1/\mu_1$ ,  $a_{0,1} = 1/\mu_2$ ,  $a_{i,0} = a_{0,i} = 0$  for  $i = 2, 3, \dots$ . Therefore,

$$a_{1,1} = -b_{0,0} + a_{0,1}a_{1,0} = \theta.$$

Moreover,

$$a_{2,1} = -b_{1,0} + a_{0,1}a_{2,0} + a_{1,1}a_{1,0} = -\frac{\theta}{\mu_1} + \frac{\theta}{\mu_1} = 0$$

Analogously,  $a_{1,2} = 0$ .

By induction on  $k$ , we suppose that  $a_{i,j} = 0$  for all  $i$  and  $j$  such that  $2 < i+j < k$ . Then, if  $i+j = k$ , from (3), we have

$$a_{2,2} = -b_{1,1} + a_{1,1}^2 = -\theta^2 + \theta^2 = 0$$

and, if  $(i, j) \neq (1, 1)$ , then

$$a_{i,j} = -b_{i-1,j-1} + \sum_{r=0}^{i-1} \sum_{s=0}^{j-1} \binom{i-1}{r} \binom{j-1}{s} a_{r,j-s} a_{i-r,s} = 0$$

since  $b_{i-1,j-1} = 0$  and either  $a_{r,j-s} = 0$  or  $a_{i-r,s} = 0$  (note that  $r+j-s \geq 1$ ,  $i-r+s \geq 1$  and  $r+j-s+i-r+s = i+j = k$ ). Hence,  $(X_1, X_2)$  has the following reliability function

$$R(x_1, x_2) = \exp \left( -\frac{x_1}{\mu_1} - \frac{x_2}{\mu_2} - \theta x_1 x_2 \right)$$

for  $x_1 \geq 0$  and  $x_2 \geq 0$ , that is,  $(X_1, X_2)$  has a bivariate Gumbel exponential distribution (see, e.g., Kotz, Balakrishnan and Johnson (2000, p. 351)). In particular, if  $\theta = 0$  then  $X_1$  and  $X_2$  are independent and

$$r(x_1, x_2) = \frac{1}{\mu_1 \mu_2} = r_1(x_1) r_2(x_2)$$

for  $x_1 \geq 0$  and  $x_2 \geq 0$ , where  $r_i(x_i) = 1/\mu_i$ ,  $i = 1, 2$ , are the marginal (univariate) failure rate functions. ◀

The following example shows that the main result can be applied without develop the series expression for  $\ln R$ .

**Example 4.** *If  $(X_1, X_2)$  is a random vector in the conditions of Theorem 1, then it has the bivariate Lomax (Pareto type II) distribution defined (see, e.g., Kotz, Balakrishnan and Johnson (2000, p. 382)) by the following reliability function*

$$R(x_1, x_2) = (1 + \alpha_1 x_1 + \alpha_2 x_2)^{-\theta}$$

for  $x_1 \geq 0$  and  $x_2 \geq 0$ , where  $\alpha_1 > 0$ ,  $\alpha_2 > 0$  and  $\theta > 0$  if, and only if, its bivariate failure rate function is given by

$$r(x_1, x_2) = \theta(\theta + 1)\alpha_1\alpha_2(1 + \alpha_1 x_1 + \alpha_2 x_2)^{-2}$$

for  $x_1 \geq 0$  and  $x_2 \geq 0$ , and the marginal distributions by

$$R_i(t) = (1 + \alpha_i t)^{-\theta}$$

for  $t \geq 0$  and  $i = 1, 2$ .

The ‘only if’ part can be proved by a straightforward computation of  $r$  and  $R_i$  from  $R$ .

The ‘if’ part is obtained from Theorem 1 taking into account that  $R(0, 0) = 1$  and

$$\ln R(x_1, x_2) = \theta \ln \left( 1 - \frac{\alpha_1 x_1 + \alpha_2 x_2}{1 + \alpha_1 x_1 + \alpha_2 x_2} \right)$$

and hence  $\ln R$  and  $r$  are analytical for  $x_1$  and  $x_2$  such that  $\alpha_1 x_1 + \alpha_2 x_2 \geq 0$ . ◀

The following example shows that the main result can also be applied to bivariate distributions without rectangular support.

**Example 5.** *If  $(X_1, X_2)$  is a random vector in the conditions of Theorem 1, then it has the following reliability function*

$$R(x_1, x_2) = (1 - \alpha_1 x_1 - \alpha_2 x_2)^\theta$$

for  $x_1 \geq 0$  and  $x_2 \geq 0$  such that  $\alpha_1 x_1 + \alpha_2 x_2 \leq 1$ , where  $\alpha_1 > 0$ ,  $\alpha_2 > 0$  and  $\theta > 0$  if, and only if, its bivariate failure rate function is given by

$$r(x_1, x_2) = \theta(\theta - 1)\alpha_1\alpha_2(1 - \alpha_1 x_1 - \alpha_2 x_2)^{-2}$$

for  $x_1 \geq 0$  and  $x_2 \geq 0$  such that  $\alpha_1 x_1 + \alpha_2 x_2 \leq 1$ , and the marginal distributions by

$$R_i(t) = (1 - \alpha_i t)^\theta$$

for  $0 \leq t \leq 1$  and  $i = 1, 2$ .

The ‘only if’ part can be proved by a straightforward computation of  $r, R_1$  and  $R_2$  from  $R$ .

The ‘if’ part is obtained from Theorem 1 taking into account that  $R(0, 0) = 1$  and

$$\ln R(x_1, x_2) = \theta \ln(1 - \alpha_1 x_1 - \alpha_2 x_2)$$

and hence  $\ln R$  and  $r$  are analytical for  $x_1$  and  $x_2$  such that  $\alpha_1 x_1 + \alpha_2 x_2 \leq 1$ . ◀

## 4 Conclusions

This paper shows that the bivariate failure rate of Basu (1971) and the marginal distributions uniquely determine the distribution function under some reasonable assumptions satisfy by the most common bivariate models. Hence, this function can be seen as a possible extension of the univariate failure rate function to the bivariate set-up and it can be used to characterize known bivariate models or to obtain new bivariate models. It is an open question if this property would hold by relaxing some of the hypothesis included in Theorem 1. It is also an open question to determine which functions can be the bivariate failure rate functions of bivariate probability models satisfying the assumptions of Theorem 1.

## 5 References.

- Barlow, R.E. and Proschan, F. (1975). *Statistical theory of reliability and life testing*. Holt, Rinehart and Winston, New York.
- Basu, A.P. (1971). Bivariate failure rate. *J. Amer. Statist. Assoc.* 66, 103–104.
- Belzunce, F., Navarro, J., Ruiz, J.M. and del Aguila, Y. (2004). Some results on residual entropy function. *Metrika* 59, 147–161.

- Finkelstein, M. (2003). On one class of bivariate distributions. *Statist. Probab. Lett.* 65 (1), 1–6.
- Finkelstein, M., Esaulova, V. (2005). On the weak IFR aging of bivariate lifetime distributions. *Appl. Stoch. Models Bus. Ind.* 21 (3), 265–272.
- Johnson, N.L. and Kotz, S. (1975). A vector of multivariate hazard rate. *J. Multivariate Anal.* 5, 53–66.
- Kotz, S., Balakrishnan, N. and Johnson, N. L. (2000). *Continuous Multivariate Distributions*. Wiley, New York.
- Kotz, S., Navarro, J. and Ruiz, J.M. (2007). Characterizations of Arnold and Strauss' and related bivariate models. *J. Multivariate Anal.* 98, 1494–1507.
- Marshall, A.W. (1975). Some comments on the hazard gradient. *Stochastic Processes and Their Applications* 3, 293–300.
- Marshall, A. W. and Olkin, I. (1979). *Inequalities: Theory of Majorization and its Applications*. Academic Press, San Diego, California.
- Nanda, A.K., Bhattacharjee, S. and Alam, S.S. (2006). Properties of proportional mean residual life model. *Statist. Probab. Lett.* 76 (9), 880–890.
- Nanda, A.K., Bhattacharjee, S. and Alam, S.S. (2007). Properties of aging intensity function. *Statist. Probab. Lett.* 77 (4), 365–373.
- Navarro, J. and Ruiz, J.M. (2004a). Characterizations from relationships between failure rate functions and conditional moments. *Comm. Statist. Theory Methods* 33 (12), 3159–3171.
- Navarro, J. and Ruiz, J.M. (2004b). A characterization of the multivariate normal distribution by using the hazard gradient. *Ann. Inst. Statist. Math.* 56, 361–367.
- Navarro, J., Ruiz, J.M. and Sandoval, C.J. (2007). Some characterizations of multivariate distributions using products of the hazard gradient and mean residual life components. *Statistics* 41 (1), 85–91.
- Navarro, J., Rychlik, T. and Shaked, M. (2007). Are the Order Statistics Ordered? A Survey of Recent Results. *Comm. Statist. Theory Methods* 36 (7), 1273–1290.
- Navarro, J. and Shaked, M. (2006). Hazard rate ordering of order statistics and systems. *J. Appl. Probab.* 43, 391–408.

Puri, P. S. and Rubin, H. (1974). On a characterization of the family of distributions with constant multivariate failure rates. *Ann. Probab.* 2, 738–740.

Shaked, M. and Shanthikumar, J.G. (1991). Dynamic multivariate mean residual life functions. *J. Appl. Probab.* 28, 613–629.

Shanbhag, D.N. and Kotz, S. (1987). Some new approaches to multivariate probability distributions. *J. Multivariate Anal.* 22, 189–211.

Yang, S.C. and Nachlas, J.A. (2001). Bivariate reliability and availability modeling. *IEEE Trans. Reliab.* 50 (1), 26–35.