HAL
open science

Algebraic polynomials with random non-symmetric coefficients

K. Farahmand, C.T. Stretch

To cite this version:

K. Farahmand, C.T. Stretch. Algebraic polynomials with random non-symmetric coefficients. Statistics and Probability Letters, 2010, 78 (11), pp.1305. 10.1016/j.spl.2007.12.006 . hal-00622145

HAL Id: hal-00622145

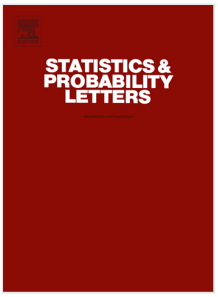
https://hal.science/hal-00622145v1

Submitted on 12 Sep 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Accepted Manuscript



Algebraic polynomials with random non-symmetric coefficients

K. Farahmand, C.T. Stretch

PII: S0167-7152(07)00410-5
DOI: 10.1016/j.spl.2007.12.006
Reference: STAPRO 4842

To appear in: *Statistics and Probability Letters*

Received date: 10 July 2006
Revised date: 18 September 2007
Accepted date: 11 December 2007

Please cite this article as: Farahmand, K., Stretch, C.T., Algebraic polynomials with random non-symmetric coefficients. *Statistics and Probability Letters* (2007), doi:10.1016/j.spl.2007.12.006

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.

ACCEPTED MANUSCRIPT

ALGEBRAIC POLYNOMIALS WITH RANDOM NON-SYMMETRIC COEFFICIENTS

K. FARAHMAND AND C. T. STRETCH

Abstract. This paper provides an asymptotic formula for the expected number of zeros of a polynomial of the form $a_0(\omega)+a_1(\omega)\binom{n}{1}^{1/2}x+a_2(\omega)\binom{n}{2}^{1/2}x^2+\cdots+a_n(\omega)\binom{n}{n}^{1/2}x^n$ for large n. The coefficients $\{a_j(\omega)\}_{j=0}^n$ are assumed to be a sequence of independent normally distributed random variables with fixed mean μ and variance one. It is shown that for μ non-zero this expected number is half of that for $\mu = 0$. This behavior is similar to that of classical random algebraic polynomials but differs from that of random trigonometric polynomials.

1. Introduction

There has been much interest in the study of the behavior of random algebraic polynomials. These are defined as

$$Q_n(x) = \sum_{j=0}^{n} a_j(\omega)x^j, \tag{1.1}$$

where $\{a_j(\omega)\}_{j=0}^n, \omega \in \Omega$, is a sequence of independent random variables defined on a fixed probability space $(\mathcal{A}, \Omega, \Pr)$. Suppose the $a_j(\omega)$ are identically distributed with $E(a_j(\omega)) = \mu$ and $\mathrm{var}(a_j(\omega)) = 1$. Let $N_n(a, b)$ be the number of real zeros of $Q_n(x)$ in (a, b). For many classes of distributions it has been shown that for $\mu = 0$ the expected number of real roots, $EN_n(-\infty, \infty)$, is asymptotic to $(2/\pi)\log n$. For μ non-zero Ibragimov and Maslova [13] show that this asymptotic value is reduced by half. Their results, based on [12], remain valid for a wide class of distributions of the coefficients. A recent work of Wilkins [18], shows that in fact the error term in the above asymptotic formula is small.

For the random trigonometric polynomial,

$$T_n(\theta) = \sum_{j=0}^{n} a_j(\omega)\cos j\theta,$$

the number of real roots behaves differently. From Dunnage [4] we know for $\mu = 0$ that $EN_n(0, 2\pi) \sim 2n/\sqrt{3}$. This shows they have significantly more zeros than algebraic polynomials. This number remains the same when we pass to the case of non-zero μ, see [7] or [17].

In this paper we consider random algebraic polynomials whose coefficients are independent but not identically distributed. Instead they can be written in the

2000 *Mathematics Subject Classification.* Primary 60G99, Secondary 60H99.

Key words and phrases. Number of real roots, random algebraic polynomials, Kac-Rice formula, random variables,

ACCEPTED MANUSCRIPT

form

$$P_n(x) = \sum_{j=0}^{n} \binom{n}{j}^{1/2} a_j(\omega) x^j, \tag{1.2}$$

where the $a_j(\omega)$ have identical distributions. Physical applications of these polynomials can be found in Ramponi [15]. The mathematical behaviour of $P_n(x)$ defined in (1.2) was presented for the first time by Edelman and Kostlan in their interesting work [5], which includes many of the original approaches to the study of random algebraic polynomials. In the latter it is shown that $P_n(x)$ has significantly fewer zeros than trigonometric polynomials but more than algebraic ones. In particular for normally distributed coefficients with $\mu = 0$ we have $EN_n(-\infty, \infty) \sim \sqrt{n}$. Our interest is in the case of normally distributed coefficients with non-zero μ. The mathematical significance of non-zero mean coefficients is explained in [2] or [8, page 52]. The latter includes a review of the recent developments of properties of $P_n(x)$, $Q_n(x)$, $T_n(x)$ and other related polynomials. In a similar direction the case of the polynomial $Q_n(x)$ with non-identically distributed coefficients is discussed in the recent work of [9] and [10].

In the following theorem we show that, like the case of classical algebraic random polynomials $Q_n(x)$ and unlike that of random trigonometric polynomials $T_n(x)$, the asymptotic value of $EN_n(-\infty, \infty)$ for $P_n(x)$ for non-zero μ is half of that for $\mu = 0$. We prove:

Theorem 1. *If the coefficients $a_j(\omega)$ of $P_n(x)$ in (1.2) have identical normal distributions with $\mu \neq 0$ and unit variance, then for large n*

$$EN_n(-\infty, \infty) \sim \frac{\sqrt{n}}{2}.$$

2. Primary Analysis

The Kac-Rice formula, Rice [16] or Kac [14], gives the expected number of real zeros of $P_n(x)$. We will use a generalization of this formula from [8, page 43]. Using the notation from this source we put

$$A^2 = \operatorname{var}(P_n(x)) = \sum_{j=0}^{n} \binom{n}{j} x^{2j} = (x^2+1)^n \tag{2.1}$$

$$B^2 = \operatorname{var}(P_n'(x)) = \sum_{j=1}^{n} j^2 \binom{n}{j} x^{2j-2} = n(x^2+1)^{n-2}(nx^2+1) \tag{2.2}$$

$$C = \operatorname{cov}(P_n(x), P_n'(x)) = \sum_{j=1}^{n} j \binom{n}{j} x^{2j-1} = nx(x^2+1)^{n-1} \tag{2.3}$$

$$\alpha = E(P_n(x)) = \mu \sum_{j=0}^{n} \binom{n}{j}^{1/2} x^j \tag{2.4}$$

$$\beta = E(P_n'(x)) = \mu \sum_{j=1}^{n} j \binom{n}{j}^{1/2} x^{j-1} \tag{2.5}$$

we also need

$$\Delta^2 = A^2B^2 - C^2 = n(x^2+1)^{2n-2} \tag{2.6}$$

and

ACCEPTED MANUSCRIPT

$$S^2 = \mathrm{var}(\beta P_n(x) - \alpha P_n'(x)) = \alpha^2 B^2 + \beta^2 A^2 - 2\alpha\beta C. \tag{2.7}$$

Note that all the terms defined as squares are positive. A and Δ are taken to be the positive square roots. The binomial theorem gives (2.1). (2.2) and (2.3) are obtained by differentiation of (2.1).

We can now give the extension of the Kac-Rice formula as in [8, page 43] by

$$\begin{aligned} EN_n(a,b) &= \int_a^b \frac{\Delta}{\pi A^2} \exp\left(-\frac{S^2}{2\Delta^2}\right)\, dx \\ &+ \int_a^b \frac{\sqrt{2}|\beta A^2 - C\alpha|}{\pi A^3} \exp\left(-\frac{\alpha^2}{2A^2}\right) \mathrm{erf}\left(\frac{|\beta A^2 - C\alpha|}{\sqrt{2}A\Delta}\right)\, dx \\ &= I_n(a,b) + J_n(a,b), \quad \text{say,} \end{aligned} \tag{2.8}$$

where $\mathrm{erf}(x) = \int_0^x \exp(-t^2)\, dt$.

In order to prove the theorem we need to estimate these two integrals, the first of which gives the required number of zeros while the second is small. To estimate I_n we need asymptotic expansions for the sums (2.4) and (2.5). These are quite delicate and the statement and proof is relegated to Appendix A.

To show J_n is small requires a change of variable to $u = \frac{\alpha}{A}$. To do this we need to identify the critical points of u, which is also delicate and provides the contents of Appendix A.

3. Proof of the theorem

We first consider $I_n(-\infty,\infty)$ defined in (2.8). From (2.1) and (2.6) we see

$$\frac{\Delta}{\pi A^2} = \frac{\sqrt{n}}{\pi(x^2+1)}.$$

Using (2.6), (2.7), (A.1) and (A.2) for $x > 0$ we obtain

$$\frac{S^2}{\Delta^2} = \mu^2 (8\pi n)^{1/2} \frac{x}{x^2+1} + O(n^{-1/2}).$$

In order to obtain this we need to use the first two terms of the expansions of α and β as the terms involving $n^{3/2}$ cancel.

For $x < 0$ using (2.6), (2.7), (A.3) and (A.4) we obtain for any ν

$$\frac{S^2}{\Delta^2} = o(n^{-\nu}).$$

Since the term in the exponential is always negative we see the integrand of $\frac{1}{\sqrt{n}} I_n(-\infty,\infty)$ is dominated by $\frac{1}{\pi(x^2+1)}$. Hence we can apply the dominated convergence theorem to obtain

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{\sqrt{n}} I_n(-\infty,\infty) &= \int_{-\infty}^{\infty} \lim_{n\to\infty} \frac{e^{-\frac{S^2}{2\Delta^2}}}{\pi(x^2+1)} dx \\ &= \int_{-\infty}^{0} \frac{1}{\pi(x^2+1)} dx = \frac{1}{2}. \end{aligned} \tag{3.1}$$

Now we wish to obtain an upper bound for J_n in (2.8). Since $\mathrm{erf}(x) < \frac{\sqrt{\pi}}{2}$ we see

$$J_n(-\infty,\infty) \le \int_{-\infty}^{\infty} \left(\frac{|\beta A^2 - C\alpha|}{\sqrt{2\pi} A^3}\right) \exp\left(-\frac{\alpha^2}{2A^2}\right)\, dx.$$

ACCEPTED MANUSCRIPT

This integral can be estimated by substituting $u = \dfrac{\alpha}{A}$. First note that

$$\frac{du}{dx} = \frac{\beta A^2 - C\alpha}{A^3} = \frac{\beta(x^2+1) - nx\alpha}{(x^2+1)^{n/2+1}}.$$

To complete the estimate we need to identify the critical points of u.

Lemma 1. *For $n > 0$, u can only have critical points at ± 1.*

Now substitution gives

$$\begin{aligned}&\int_{-\infty}^{\infty} \left(\frac{|\beta A^2 - C\alpha|}{\sqrt{2\pi}A^3}\right) \exp\left(-\frac{\alpha^2}{2A^2}\right) dx \\ &\leq \frac{3}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right) du = 3 \end{aligned} \tag{3.2}$$

which together with (3.1) completes the proof of the theorem.

As the location of the roots of u will take considerable effort we give an example to show that it is necessary. First note that the definitions in Section 2 can be carried out for any random polynomials $R(x)$ with independent normally distributed coefficients. In this generality u has a simple interpretation as

$$u(x) = \Phi^{-1}\mathrm{Prob}(R(x) > 0) \tag{3.3}$$

where Φ is the cdf of the standard normal distribution.

Suppose $f(x) = \sum_{j=0}^{n} \mu_j x^j$ is a polynomial of degree n with n real roots. Let $R(x)$ be a random polynomial where the coefficients $a_j(w)$ are independently normally distributed with mean μ_j and variance $\epsilon^2\binom{n}{j}$. If we take ϵ small enough the the expected number of roots will be close to n and (3.3) shows u has at least $n-1$ critical points.

We can now apply our previous analysis. We get an upper bound

$$I_n(-\infty, \infty) < \sqrt{n}$$

so most of the expected roots must come from the second integral.

The analysis of the second integral proceeds as before and shows

$$I_n(-\infty, \infty) < k + 1$$

where k is the number of turning points of u. Thus in this example the critical points of u and the integral $J_n(-\infty, \infty)$ give most of the expected real roots.

Appendix A. The Asymptotic Expansions

To complete the estimation of I_n we need the following lemma. This gives asymptotic estimates for large n to α and β for both $x > 0$ and $x < 0$.

Lemma 2. *For fixed $x > 0$ and any ν we have asymptotic expansions:*

$$\begin{aligned}&\sum_{j=0}^{n} \binom{n}{j}^{1/2} x^j \\ &\qquad = (8\pi n)^{1/4}(x^2+1)^{(n-1)/2} x^{1/2} \left(a_0 + a_1 n^{-1} + a_2 n^{-2} + \cdots\right) \end{aligned} \tag{A.1}$$

ACCEPTED MANUSCRIPT

where $a_0 = 1$, $a_1 = -\dfrac{x^4+1}{16x^2}$;

$$\sum_{j=0}^{n} j\binom{n}{j}^{1/2} x^j$$
$$= (8\pi)^{1/4} n^{5/4} (x^2+1)^{(n-3)/2} x^{5/2} \left(c_0 + c_1 n^{-1} + c_2 n^{-2} + \cdots\right) \quad \text{(A.2)}$$

where $c_0 = 1$, $c_1 = -\dfrac{x^4+8x^2-7}{16x^2}$;

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j}^{1/2} x^j = o\left((1+x^2)^{n/2} n^{-\nu}\right); \quad \text{(A.3)}$$

$$\sum_{j=0}^{n} (-1)^j j \binom{n}{j}^{1/2} x^j = o\left((1+x^2)^{n/2} n^{-\nu}\right). \quad \text{(A.4)}$$

Put

$$b_j = \binom{n}{j} p^j q^{n-j} \text{ with } q = 1-p.$$

If we put $p = \frac{x^2}{1+x^2}$ then $q = \frac{1}{1+x^2}$ and $x^2 = \frac{p}{q}$, thus

$$b_j = \binom{n}{j} |x|^{2j} (1+x^2)^{-n}.$$

We need to show

$$\sum_{j=0}^{n} b_j^{1/2} = (8\pi npq)^{1/4} \left(a_0 + a_1 n^{-1} + a_2 n^{-2} \cdots\right) \quad \text{(A.5)}$$

$$\sum_{j=0}^{n} j b_j^{1/2} = (8\pi npq)^{1/4} np \left(c_0 + c_1 n^{-1} + c_2 n^{-2} \cdots\right) \quad \text{(A.6)}$$

$$\sum_{j=0}^{n} (-1)^j b_j^{1/2} = o\left(n^{-\nu}\right) \quad \text{(A.7)}$$

$$\sum_{j=0}^{n} (-1)^j j b_j^{1/2} = o\left(n^{-\nu}\right). \quad \text{(A.8)}$$

We will apply the central limit theorem in the form of the normal approximation to the binomial. In order to get a sufficiently accurate estimate of the binomial coefficients we need to use a form of the central limit theorem from [3, page 231]. This result is for lattice valued distributions; we state the result for one dimensional lattices:

Lemma 3. *Suppose X_j are independent identically distributed random variables with unit variance and mean μ taking values in a discrete one dimensional lattice L. Suppose the expectation of $|X_1 - \mu|^s$ is finite for all s.*

ACCEPTED MANUSCRIPT

There are polynomials $P_r(y)$, with $P_0 = 1$, depending on the distribution so that if we put:

$$\begin{aligned} p_n &= \mathrm{P}(X_1 + \cdots + X_n = \alpha) \\ y &= n^{-1/2}(\alpha - n\mu) \\ \phi(y) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \\ q_n &= l n^{-1/2} \sum_{r=0}^{s-2} n^{-r/2} P_r(y)\phi(y) \qquad s \geq 2 \end{aligned}$$

where l is the determinant of the lattice, then:

$$\sup_{\alpha \in L} ((1 + |y(\alpha)|^s)|p_n - q_n(y)|) = o(n^{-\frac{s-1}{2}}).$$

We can drop the $|y(\alpha)|^s$. Take X_1 to be the distribution taking values 0 and $(pq)^{-1/2}$ with probabilities q and p. Then $L = \{j(pq)^{-1/2}\}$, $l = (pq)^{-1/2}$ and $\mu = \sqrt{\frac{p}{q}}$. Putting $\alpha = j(pq)^{-1/2}$ we get $p_n = b_j$. Also putting $L_n = \{j(pq)^{-1/2} | j = 0, \cdots, n\}$ and

$$\bar{q}_n = \begin{cases} q_n & q_n > 0 \\ 0 & q_n \leq 0 \end{cases}$$

gives

$$|p_n^{1/2} - \bar{q}_n^{1/2}| = \frac{|p_n - \bar{q}_n|}{|p_n^{1/2} + \bar{q}_n^{1/2}|} < \frac{K n^{-(s-1)/2}}{|p_n^{1/2} + \bar{q}_n^{1/2}|}$$

for some constant K. Let Y_ν be the subset of L_n where $|p_n^{1/2} + \bar{q}_n^{1/2}| < n^{-(\nu+2)}$, then

$$\begin{aligned} \sum_{\alpha \in L_n} |p_n^{1/2} - \bar{q}_n^{1/2}| &< \sum_{\alpha \in Y_\nu} |p_n^{1/2} - \bar{q}_n^{1/2}| + \sum_{\alpha \in L_n - Y_\nu} |p_n^{1/2} - \bar{q}_n^{1/2}| \\ &< n^{-(\nu+1)} + K n^{\nu+3-(s-1)/2}. \end{aligned}$$

Taking s large enough we get

$$\left| \sum_{j=0}^{n} b_j^{1/2} - \sum_{\alpha \in L_n} \bar{q}_n^{1/2} \right| < K_\nu n^{-(\nu+1)}.$$

Similar results hold for the other three sums. Now since the $P_r(y)$ are polynomials we can find positive C and w depending on s so that if $|y| < Cn^w$ we have $\left|\sum_{r=1}^{s-2} n^{-r/2} P_r(y)\right| < \frac{1}{2}$. The $\phi(y)$ term in the expression for q_n ensures $\sup_{|y|>Cn^w} |q_n| = o(n^{-k})$ for any k, and so $\sup_{|y|>Cn^w} |\bar{q}_n^{1/2}| = o(n^{-k})$ for any k. We now use an Euler-McLaurin summation formula [1]:

$$\begin{aligned} \sum_{j=A}^{B} f(j) &= \int_A^B f(t)dt + \sum_{k=1}^{m} (-1)^k \frac{B_k}{k!} (f^{(k-1)}(B) - f^{(k-1)}(A)) \\ &+ \frac{(-1)^{m-1}}{m!} \int_A^B B_m(t - [t]) f^{(m)}(t)dt \end{aligned} \tag{A.9}$$

ACCEPTED MANUSCRIPT

where

$$f(t) = (pqn)^{-1/4}\sqrt{\sum_{r=0}^{s-2} n^{-r/2}P_r(y)\phi(y)}$$

and $y = n^{-1/2}(\alpha - n\mu)$ and $\alpha = t(pq)^{-1/2}$. We take A and B to be the largest and smallest integers between 0 and n for which $|y| < Cn^w$. Changing variables in the first integral

$$\int_A^B f(t)dt = (npq)^{-1/4}\int_{y(A)}^{y(B)} \sqrt{\sum_{r=0}^{s-2} n^{-r/2}P_r(y)(2\pi)^{-1/4}e^{-\frac{1}{4}y^2}\sqrt{pqn}dy}$$

$$= (npq)^{1/4}(2\pi)^{-1/4}\int_{y(A)}^{y(B)} \sqrt{\sum_{r=0}^{s-2} n^{-r/2}P_r(y)e^{-\frac{1}{4}y^2}dy}.$$

$y(B)$ differs from Cn^w in that the corresponding j is an integer and not greater than n. The integrality changes the value by less than $(npq)^{-1/2}$ and the limit of n restricts it to $\sqrt{nq/p}$. So $y(B)$ tends to infinity with n. Similarly $y(A)$ tends to minus infinity.

By our choice of limits the term in the square root is bounded away from zero. Let $F_{s,t}(n,y)$ be the expansion of the square root in powers of n down to $n^{-t/2}$. Taking the limit of the integral as $n \to \infty$ using the dominated convergence theorem gives, using $P_0 = 1$,

$$\lim_{n\to\infty}\int_{y(A)}^{y(B)} \sqrt{\sum_{r=0}^{s-2} n^{-r/2}P_r(y)e^{-\frac{1}{4}y^2}dy}$$
$$= \int_{-\infty}^{\infty} F_{s,t}(n,y)e^{-\frac{1}{4}y^2}dy + O(n^{-(t+1)/2}).$$

The remaining terms of the summation formula tend to zero. This gives the asymptotic expansion in (A.5) except that the expansion is in powers of $n^{-1/2}$. We can get the other three expansions similarly. For (A.6) we put $j = \sqrt{pqn}y + np$ and

$$f(t) = (pqn)^{-1/4}\left(\sqrt{pqn}y + np\right)\sqrt{\sum_{r=0}^{s-2} n^{-r/2}P_r(y)\phi(y)}$$

in the summation formula. For the expansions of the alternating sums we split the sums into odd and even parts, apply the summation formula modified to go up in steps of two, and note that the expansions for the two parts are equal and opposite.

To proceed further we need to know the $P_r(y)$. The recipe for calculating them from the moments of the distribution is given in [3]. In our case they are:

$$\begin{aligned} P_0 &= 1 \\ P_1 &= \frac{q-p}{6\sqrt{pq}}\left(y^3 - 3y\right) \\ P_2 &= \frac{1-4pq}{72pq}\left(y^6 - 15y^4 + 45y^2 - 15\right) + \frac{1-6pq}{24pq}\left(y^4 - 6y^2 + 3\right). \end{aligned}$$

ACCEPTED MANUSCRIPT

The $P_r(y)$ are even functions of y for even r and odd for odd r. This ensures that the asymptotic expansions are in powers of n^{-1} rather than $n^{-1/2}$. We can now expand the square root

$$\begin{aligned}\sqrt{\sum_{r=0}^{s-2} n^{-r/2} P_r(y)} &= 1 + \frac{1}{2}P_1 n^{-\frac{1}{2}} + \left(\frac{1}{2}P_2 - \frac{1}{8}P_1^2\right) n^{-1} + O(n^{-\frac{3}{2}}) \\ &= F_{s,2}(n,y) + O(n^{-\frac{3}{2}}). \end{aligned} \tag{A.10}$$

Substituting and integrating gives

$$\int_{-\infty}^{\infty} F_{s,2}(n,y) e^{-\frac{1}{4}y^2} dy = 2\sqrt{\pi} - \frac{1-2pq}{8pq}\sqrt{\pi} n^{-1} + O(n^{-2})$$

and

$$\begin{aligned}&\int_{-\infty}^{\infty} \left(\sqrt{pqn}y + np\right) F_{s,2}(n,y) e^{-\frac{1}{4}y^2} dy \\ &\qquad = 2\sqrt{\pi}pn - \frac{p^2 + 8pq - 7q^2}{8q}\sqrt{\pi} + O(n^{-1}), \end{aligned} \tag{A.11}$$

which completes the proof of Lemma 2.

Appendix B. Proof of Lemma 1

Recall $\alpha_n(x) = \sum_{j=0}^{n} \binom{n}{j}^{1/2} x^j$ and $u_n(x) = \dfrac{\alpha_n(x)}{(x^2+1)^{n/2}}$. We first list some elementary properties of $u_n(x)$:

$$u_n(1/x) = \operatorname{sign}(x)^n u_n(x) \quad x \neq 0; \tag{B.1}$$

$$u_n'(x) = \frac{(x^2+1)\alpha_n'(x) - nx\alpha_n(x)}{(x^2+1)^{n/2+1}}; \tag{B.2}$$

$$= \frac{\sum_{j=0}^{n}(j+1)\binom{n}{j+1}^{1/2} - (n-j+1)\binom{n}{n-j+1}^{1/2}}{(x^2+1)^{n/2+1}}; \tag{B.3}$$

$$u_n'(0) = \sqrt{n}; \tag{B.4}$$

$$u_n'(1) = 0; \tag{B.5}$$

$$u_n'(-1) = 0 \quad n \text{ even}. \tag{B.6}$$

Using (B.1) it is sufficient to show that $u_n'(x) > 0$ for $x \in (-1,0)$ and $x \in (0,1)$. First consider $x \in (0,1)$. Put

$$c_j = \sqrt{(j+1)(n-j)} - \sqrt{j(n-j+1)}$$

in order to write

$$u_n'(x) = \sum_{j=0}^{n} \binom{n}{k}^{1/2} c_j x^j.$$

We can check $c_{n-j} = -c_j$ and $c_j > 0$ for $j < n/2$ which gives the required result.

For $x \in (-1,0)$ we consider the function

$$h(s) = \sum_{j=0}^{\infty} \frac{s^j}{\sqrt{j!}}.$$

ACCEPTED MANUSCRIPT

We will show that $h(-s)$ is completely monotonic, that is $h^{(k)}(s) > 0$ for all real x and $k \geq 0$. Equivalently $h(-s)$ is the Laplace transform of a probability distribution on $[0, \infty)$. We will construct this distribution starting from the standard Gumbel distribution with density function $\psi_1(y) = e^{-y-e^{-y}}$ and characteristic function $\Gamma(1 - iz)$.

First note from [6] that for $\Re(z) > 0$ we have

$$\log \Gamma(z) = \int_0^\infty \left[(z-1) - \frac{1 - e^{-(z-1)t}}{1 - e^{-t}} \right] \frac{e^{-t}}{t} \, dt.$$

Differentiating and putting $z = 1$ gives

$$-\gamma = \int_0^\infty \left[1 - \frac{t}{1 - e^{-t}} \right] \frac{e^{-t}}{t} \, dt$$

which by subtraction and substitution shows that for $\Im(z) > -1$ we have

$$\log \Gamma(1 - iz) = iz\gamma + \int_0^\infty \left(e^{izt} - 1 - izt \right) \frac{e^{-t}}{(1 - e^{-t})t} \, dt.$$

This is a Lévy-Kintchine equation and we conclude from [11] that the Gumbel distribution is infinitely divisible, that is for all $p > 0$ there are distributions Ψ_p with characteristic functions $\Gamma(1 - iz)^p$. As $|\Gamma(\frac{1}{2} + iy)|^2 \leq 2\pi e^{-|y|}$ we see Ψ_p has a density function $\psi_p(y)$. Put $\phi_p(x) = \psi_p(-\log x)/x$ and substitute $y = -\log x$ and $w = 1 - iz$ to get

$$\int_0^\infty x^{w-1} \phi_p(x) \, dx = \Gamma(w)^p,$$

that is $\phi_p(x)$ is a density function with Mellin transform $\Gamma(w)^p$. Taking $p = \frac{1}{2}$ we see that the moments of $\phi_{1/2}$ are $\mu'_j = \sqrt{j!}$ and the Laplace transform of $\phi_{1/2}$ is $h(-s)$. From the definition of $h(s)$ we can obtain the expansions

$$h(s)h(sx) = \sum_{n=0}^\infty \frac{1}{\sqrt{n!}} \alpha_n(x) s^n$$

and

$$h(\frac{s}{\sqrt{1+x^2}})h(\frac{sx}{\sqrt{1+x^2}}) = \sum_{n=0}^\infty \frac{1}{\sqrt{n!}} u_n(x) s^n.$$

Also from the Laplace transform description we obtain

$$h(\frac{s}{\sqrt{1+x^2}})h(\frac{sx}{\sqrt{1+x^2}}) = \int_0^\infty \int_0^\infty e^{s(v+xw)/\sqrt{1+x^2}} \phi_{1/2}(v)\phi_{1/2}(w) \, dvdw.$$

Differentiating with respect to x and taking the coefficient of s^n gives

$$\frac{u'_n(x)}{\sqrt{n!}} = \int_0^\infty \int_0^\infty \frac{(w - vx)(v + wx)^{n-1}}{(1 + x^2)^{n/2+1}(n-1)!} \phi_{1/2}(v)\phi_{1/2}(w) \, dvdw.$$

Symmetrizing it is sufficient to show that for $x \in (-1, 0)$

$$\int_0^\infty \int_0^\infty c_n \phi_{1/2}(v)\phi_{1/2}(w) \, dvdw > 0$$

where $c_n = (w - vx)(v + wx)^{n-1} + (v - wx)(w + vx)^{n-1}$. Note c_n satisfies the recurrence relation

$$c_{n+2} = (v + w)(1 + x)c_{n+1} - (v + xw)(w + xv)c_n.$$

ACCEPTED MANUSCRIPT

We have $c_1 = (v+w)(1-x)$ and $c_2 = 2vw(1-x^2)$. Suppose $v+w>0$ then $(v+xw)+(w+xv) = (v+w)(1-x) > 0$. Thus at least one of $v+xw$ and $w+xv$ is positive. If both are positive, the definition shows $c_n > 0$. If exactly one is positive the recurrence relation shows $c_n > 0$. Thus as $\phi_{1/2}$ is not supported at the origin the required integral is positive.

References

[1] G.E. Andrews, R. Askey, and R.R. Roy. *Special Functions*. Cambridge University Press, Cambridge, 2000.

[2] A.T. Bharucha-Reid and M. Sambandham. *Random Polynomials*. Academic Press, N.Y., 1986.

[3] Bhattacharya and Rao. *Normal Approximation and Asymptotic Expansions*. Krieger, Malabar, Florida, 1986.

[4] J.E.A. Dunnage. The number of real zeros of a random trigonometric polynomial. *Proc. London Math. Soc.*, 16:53–84, 1966.

[5] A. Edelman and E. Kostlan. How many zeros of a random polynomial are real? *Bull. Amer. Math. Soc.*, 32:1–37, 1995.

[6] A. Erdélyi. *Higher Transcendental Functions, Vol. 1*. McGraw-Hill Book Company, New York, Toronto, London, 1953.

[7] K. Farahmand. On the number of real zeros of a random trigonometric polynomial: coefficients with non-zero infinite mean. *Stoch. Anal. Appl.*, 5:379–386, 1987.

[8] K. Farahmand. *Topics in Random Polynomials*. Addison Wesley Longman, London, 1998.

[9] K. Farahmand, P. Flood, and P. Hannigan. Zeros of a random algebraic polynomial with coefficient means in geometric progression. *J. Math. Anal. Appl.*, 269:137–148, 2002.

[10] K. Farahmand and A. Nezakati. Algebraic polynomials with non-identical random coefficients. *Proc. Amer. Maths. Soc.*, 133:275–283, 2005.

[11] W. Feller. *An Introduction to Probability Theory and its Applications. Vol 2*. John Wiley and Sons, New York, London, Sydney, 1966.

[12] I.A. Ibragimov and N.B. Maslova. On the expected number of real zeros of random polynomials. I Coefficients with zero means. *Theory Probab. Appl.*, 16:228–248, 1971.

[13] I.A. Ibragimov and N.B. Maslova. On the expected number of real zeros of random polynomials.II.Coefficients with non-zero means. *Theory Probab. Appl.*, 16:485–493, 1971.

[14] M. Kac. On the average number of real roots of a random algebraic equation. *Bull.Amer.Math.Soc.*, 49:314–320, 1943.

[15] A. Ramponi. A note on the complex roots of complex random polynomials. *Statistics and Prob. Lett.*, 44:181–187, 1999.

[16] S.O. Rice. Mathematical theory of random noise. *Bell. System Tech. J.*, 25:46–156, 1945. Reprinted in: Selected Papers on Noise And Stochastic Processes (ed. N. Wax), Dover, New York, 1954, 133-294.

[17] M. Sambandham and N. Renganathan. On the number of real zeros of a random trigonometric polynomial: coefficients with non-zero mean. *J. Indian Math. Soc.*, 45:193–203, 1981.

[18] J.E. Wilkins. An asymptotic expansion for the expected number of real zeros of a random polynomial. *Proc. Amer. Math. Soc.*, 103:1249–1258, 1988.

Department of Mathematics, University of Ulster, Jordanstown, Co. Antrim BT37 0QB, U.K

E-mail address: K.Farahmand@ulster.ac.uk

E-mail address: ct.stretch@ulster.ac.uk

ACCEPTED MANUSCRIPT