HAL
open science

How to compute the extremal index of stationary random fields

H. Ferreira, L. Pereira

To cite this version:

H. Ferreira, L. Pereira. How to compute the extremal index of stationary random fields. Statistics and Probability Letters, 2010, 78 (11), pp.1301. 10.1016/j.spl.2007.11.025 . hal-00622144

HAL Id: hal-00622144

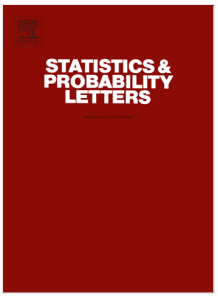
https://hal.science/hal-00622144v1

Submitted on 12 Sep 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Accepted Manuscript



How to compute the extremal index of stationary random fields

H. Ferreira, L. Pereira

PII: S0167-7152(07)00409-9
DOI: 10.1016/j.spl.2007.11.025
Reference: STAPRO 4841

To appear in: *Statistics and Probability Letters*

Received date: 16 November 2005
Revised date: 10 July 2007
Accepted date: 6 November 2007

Please cite this article as: Ferreira, H., Pereira, L., How to compute the extremal index of stationary random fields. *Statistics and Probability Letters* (2007), doi:10.1016/j.spl.2007.11.025

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.

ACCEPTED MANUSCRIPT

How to compute the extremal index of stationary random fields

H. Ferreira* and L. Pereira
Department of Mathematics
University of Beira Interior
Portugal

Abstract: We present local dependence conditions for stationary random fields under which the extremal index and the asymptotic distribution of the maximum $M_{(n_1,\ldots,n_d)}$ can be calculated from the joint distribution of a finite number $s_1 s_2$ of variables.
keywords: Extremal index, local and long range dependence, random field.

1 Introduction

Let $\mathbf{X} = \{X_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ be a stationary random field on $\mathbb{Z}_+^d$, where $\mathbb{Z}_+$ is the set of all positive integers and $d \geq 2$. We shall consider the conditions and results for $d = 2$ since it is notationally simplest and the proofs for higher dimensions follow analogous arguments. For $\mathbf{i} = (i_1, i_2)$ and $\mathbf{j} = (j_1, j_2)$, $\mathbf{i} \leq \mathbf{j}$ means $i_k \leq j_k$, $k = 1, 2$.

For a family of real levels $\{u_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ and a subset $\mathbf{I}$ of the rectangle of points $\mathbf{R_n} = \{1, \ldots, n_1\} \times \{1, \ldots, n_2\}$, we will denote the event $\{X_{\mathbf{i}} \leq u_{\mathbf{n}} : \mathbf{i} \in \mathbf{I}\}$ by $M_{\mathbf{n}}(\mathbf{I})$ or simply by $M_{\mathbf{n}}$ when $\mathbf{I} = \mathbf{R_n}$. If $\mathbf{I} = \emptyset$ then $M_{\mathbf{n}}(\mathbf{I}) = -\infty$.

For each $i = 1, 2$, we say the pair $\mathbf{I}$ and $\mathbf{J}$ is in $\mathcal{S}_i(l)$ if the distance between $\Pi_i(\mathbf{I})$ and $\Pi_i(\mathbf{J})$ is great or equal to l, where Π_i, $i = 1, 2$ denote the cartesian projections.

*Helena Ferreira, Department of Mathematics, University of Beira Interior, 6200 Covilhã, Portugal. E-mail:helena@mat.ubi.pt

ACCEPTED MANUSCRIPT

The distance $d(\mathbf{I},\mathbf{J})$ between sets $\mathbf{I}$ and $\mathbf{J}$ of $\mathbb{Z}_+^d$, $d \geq 1$, is the minimum of distances $d(\mathbf{i},\mathbf{j}) = max\{|i_s - j_s|, s = 1,\ldots,d\}$, $\mathbf{i} \in \mathbf{I}$ and $\mathbf{j} \in \mathbf{J}$.

Suppose that $\mathbf{X}$ satisfies the coordinatewise-mixing $\Delta(u_{\mathbf{n}})$-condition introduced in Leadbetter and Rootzén (1998) which exploit the past and future separation one coordinate at a time. Therefore, we suppose that for $\mathbf{X}$ there exist sequences of integer valued constants $\{k_{n_i}\}$, $\{l_{n_i}\}$, $i = 1,2$, such that, as $\mathbf{n} = (n_1,n_2) \to \boldsymbol{\infty}$, we have $(k_{n_1},k_{n_2}) \to \boldsymbol{\infty}$, $(\frac{k_{n_1}l_{n_1}}{n_1}, \frac{k_{n_2}l_{n_2}}{n_2}) \to \mathbf{0}$ and $(k_{n_1}\Delta_1, k_{n_1}k_{n_2}\Delta_2) \to \mathbf{0}$, where Δ_i are the components of the mixing coefficient defined as follows:

$$\Delta_1 = sup|P(M_{\mathbf{n}}(\mathbf{I}_1) \leq u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{I}_2) \leq u_{\mathbf{n}}) - P(M_{\mathbf{n}}(\mathbf{I}_1) \leq u_{\mathbf{n}})P(M_{\mathbf{n}}(\mathbf{I}_2) \leq u_{\mathbf{n}})|,$$

where the supremum is taken over pairs of $\mathbf{I}_1$ and $\mathbf{I}_2$ in $\mathcal{S}_1(l_{n_1})$ such that $|\Pi_1(\mathbf{I}_2)| \leq \frac{n_1}{k_{n_1}}$,

$$\Delta_2 = sup|P(M_{\mathbf{n}}(\mathbf{I}_1) \leq u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{I}_2) \leq u_{\mathbf{n}}) - P(M_{\mathbf{n}}(\mathbf{I}_1) \leq u_{\mathbf{n}})P(M_{\mathbf{n}}(\mathbf{I}_2) \leq u_{\mathbf{n}})|,$$

where the supremum is taken over pairs of $\mathbf{I}_1$ and $\mathbf{I}_2$ in $\mathcal{S}_2(l_{n_2})$ such that $\Pi_1(\mathbf{I}_1) = \Pi_1(\mathbf{I}_2)$ and $|\Pi_2(\mathbf{I}_2)| \leq \frac{n_2}{k_{n_2}}$.

Under the coordinatewise-mixing $\Delta(u_{\mathbf{n}})$-condition we have the following asymptotic independence for maxima over $\{(i-1)[\frac{n_1}{k_{n_1}}]+1,\ldots,i[\frac{n_1}{k_{n_1}}]\} \times \{(j-1)[\frac{n_2}{k_{n_2}}]+1,\ldots,j[\frac{n_2}{k_{n_2}}]\}$, $i = 1,\ldots,k_{n_1}$, $j = 1,\ldots,k_{n_2}$:

$$P(M_{\mathbf{n}} \leq u_{\mathbf{n}}) - P^{k_{n_1}k_{n_2}}(M_{([\frac{n_1}{k_{n_1}}],[\frac{n_2}{k_{n_2}}])} \leq u_{\mathbf{n}}) \longrightarrow_{\mathbf{n}\to\boldsymbol{\infty}} 0. \tag{1.1}$$

Accordingly Choi (2002), we shall say that $\mathbf{X}$ has extremal index θ if for each $\tau > 0$ there exists $\{u_{\mathbf{n}}^{(\tau)}\}_{\mathbf{n}\geq\mathbf{1}}$ such that, as $\mathbf{n} \to \boldsymbol{\infty}$, $n_1n_2P(X_{\mathbf{1}} > u_{\mathbf{n}}^{(\tau)}) \to \tau$ and $P(M_{\mathbf{n}} \leq u_{\mathbf{n}}^{(\tau)}) \to e^{-\theta\tau}$.

This paper is concerned with describing how to compute the extremal index of the stationary random field under local mixing conditions analogous to those considered in Chernick et al. (1991).

Those local restrictions on the oscillations of the values of the random field enable to compute the extremal index from the joint distribution of a finite number s_1s_2 of variables.

ACCEPTED MANUSCRIPT

2 Condition $D^{(\mathbf{s})}(u_{\mathbf{n}})$

In the following consider $\mathbf{R}^*_{\mathbf{i},\mathbf{j}} = \{i_1, i_1+1, \ldots, j_1\} \times \{i_2, i_2+1, \ldots, j_2\} - \{\mathbf{i}\}$. In particular, for $\mathbf{i} = \mathbf{1}$ we write simply $\mathbf{R}^*_{\mathbf{j}}$. For sake of simplicity we write $[\mathbf{n}/\mathbf{k}]$ for $([\frac{n_1}{k_{n_1}}], [\frac{n_2}{k_{n_2}}])$.

Definition The random field $\mathbf{X}$ satisfies the condition $D^{(\mathbf{s})}(u_{\mathbf{n}})$, for some $\mathbf{s} \in \mathbb{Z}^2_+$, if there exist sequences of integer valued constants $\{k_{n_i}\}$, $\{l_{n_i}\}$, $i = 1, 2$, such that, as $\mathbf{n} \to \boldsymbol{\infty}$, we have $(k_{n_1}, k_{n_2}) \to \boldsymbol{\infty}$, $(\frac{k_{n_1} l_{n_1}}{n_1}, \frac{k_{n_2} l_{n_2}}{n_2}) \to \mathbf{0}$ and

$$n_1 n_2 \max_{\mathbf{i} \le [\mathbf{n}/\mathbf{k}]} \sum_{\mathbf{j} \in \mathbf{R}_{[\mathbf{n}/\mathbf{k}]} \setminus \mathbf{R}_{\mathbf{i}}} P(X_{\mathbf{i}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{\mathbf{i},\mathbf{i}+\mathbf{s}-\mathbf{1}}) \le u_{\mathbf{n}}, X_{\mathbf{j}} > u_{\mathbf{n}}) \to 0.$$

For the cases $\mathbf{s} = \mathbf{1} = (1, 1)$ and $\mathbf{s} = \mathbf{2} = (2, 2)$ we find in the above definition the local conditions considered in Pereira and Ferreira (2005) and Pereira and Ferreira (2006). We shall pursue the direction of this dependence conditions and extend to spatial processes some known formulas to obtain the extremal index of time series.

By applying stationarity and the $D^{(\mathbf{s})}(u_{\mathbf{n}})$-condition, the asymptotic distribution of the maximum $M_{\mathbf{n}}$ can be calculated from the joint distribution of the $s_1 s_2$ variables $X_{\mathbf{1}}$ and $X_{\mathbf{i}}$, $i \in \mathbf{R}^*_{\mathbf{s}}$. Different local dependence conditions can be considered by changing the definition of the region $\mathbf{R}^*_{\mathbf{s}}$. Therefore the parameter ν in the following proposition could be computed from a different set of variables.

Proposition 2.1 *Let $\mathbf{X}$ be a stationary random field satisfying the $\Delta(u_{\mathbf{n}})$-condition and $D^{(\mathbf{s})}(u_{\mathbf{n}})$-condition for some $\mathbf{s}$. As $\mathbf{n} \to \boldsymbol{\infty}$, we have*

$$n_1 n_2 P(X_{\mathbf{1}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{\mathbf{s}}) \le u_{\mathbf{n}}) \to \nu > 0$$

if only if

$$P(M_{\mathbf{n}} \le u_{\mathbf{n}}) \to e^{-\nu}, \ \nu > 0.$$

Proof: From (1.1) and the stationarity we get

$$P(M_{\mathbf{n}} \le u_{\mathbf{n}}) - \Big(P(M_{[\mathbf{n}/\mathbf{k}]} \le u_{\mathbf{n}})\Big)^{k_{n_1} k_{n_2}} =$$

ACCEPTED MANUSCRIPT

$$P(M_{\mathbf{n}} \leq u_{\mathbf{n}}) - exp\Big(-(1+o(1))k_{n_1}k_{n_2}P(M_{[\mathbf{n}/\mathbf{k}]} > u_{\mathbf{n}})\Big) =$$

$$P(M_{\mathbf{n}} \leq u_{\mathbf{n}}) - exp\Big(-(1+o(1))k_{n_1}k_{n_2}\sum_{\mathbf{i}\leq[\mathbf{n}/\mathbf{k}]} P(X_{\mathbf{i}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{\mathbf{i},[\mathbf{n}/\mathbf{k}]}) \leq u_{\mathbf{n}}) + o(1)\Big) =$$

$$P(M_{\mathbf{n}} \leq u_{\mathbf{n}}) - exp\Big(-(1+o(1))k_{n_1}k_{n_2}\sum_{\mathbf{i}\leq[\mathbf{n}/\mathbf{k}]} P(X_{\mathbf{1}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{[\mathbf{n}/\mathbf{k}]-\mathbf{i}+\mathbf{1}}) \leq u_{\mathbf{n}}) + o(1)\Big) = o(1).$$

The result follows now by applying the $D^{(\mathbf{s})}(u_{\mathbf{n}})$-condition, since for $[\frac{n_1}{k_{n_1}}] - i_1 + 1 \geq s_1$ or $[\frac{n_2}{k_{n_2}}] - i_2 + 1 \geq s_2$ it holds

$$P(X_{\mathbf{1}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{\mathbf{s}}) \leq u_{\mathbf{n}}) - P(X_{\mathbf{1}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{[\mathbf{n}/\mathbf{k}]-\mathbf{i}+\mathbf{1}}) \leq u_{\mathbf{n}}) \leq$$

$$\sum_{\substack{\mathbf{j}\leq[\mathbf{n}/\mathbf{k}] \\ d(\mathbf{1},\mathbf{j})\geq\min\{s_1,s_2\}}} P(X_{\mathbf{1}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{\mathbf{s}}) \leq u_{\mathbf{n}}, X_{\mathbf{j}} > u_{\mathbf{n}}) = o(n_1 n_2).$$

Therefore

$$P(M_{\mathbf{n}} \leq u_{\mathbf{n}}) - exp\Big(-(1+o(1))n_1 n_2 P(X_{\mathbf{1}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_s) \leq u_{\mathbf{n}}) + o(1)\Big) = o(1).$$

□

As a corollary we provide a relation between the limit of $n_1 n_2 P(X_{\mathbf{1}} > u^{(\tau)}_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{\mathbf{s}}) \leq u^{(\tau)}_{\mathbf{n}})$ and the extremal index.

Proposition 2.2 *Let* $\mathbf{X}$ *be a stationary random field satisfying the* $\Delta(u^{(\tau)}_{\mathbf{n}})$*-condition and* $D^{(\mathbf{s})}(u^{(\tau)}_{\mathbf{n}})$*-condition for each* $\tau > 0$*. The extremal index of* $\mathbf{X}$ *exists and is* θ *if and only if, as* $\mathbf{n} \to \boldsymbol{\infty}$*, it holds*

$$n_1 n_2 P(M_{\mathbf{n}}(\mathbf{R}^*_{\mathbf{s}}) \leq u_{\mathbf{n}} | X_{\mathbf{1}} > u_{\mathbf{n}}) \to \theta.$$

ACCEPTED MANUSCRIPT

From the Normal comparison lemma we can prove that the stationary normal random fields such that $\rho_{\mathbf{n}} = E(X_{\mathbf{j}} X_{\mathbf{j+n}})$ satisfies $\rho_{\mathbf{n}} log(n_1 n_2) \to 0$, as $\mathbf{n} \to \boldsymbol{\infty}$, satisfy $\Delta(u_{\mathbf{n}}^{(\tau)})$-condition and $D^{(\mathbf{1})}(u_{\mathbf{n}}^{(\tau)})$-condition (Pereira e Ferreira (2005)).

The m-dependent random fields satisfy $\Delta(u_{\mathbf{n}}^{(\tau)})$-condition and $D^{(\mathbf{s})}(u_{\mathbf{n}}^{(\tau)})$-condition for some $\mathbf{s}$.

It is an open and interesting question to know if some autoregressive random fields can satisfy $D^{(\mathbf{s})}(u_{\mathbf{n}}^{(\tau)})$-condition for some $\mathbf{s}$ with different coordinates.

Here we present a simple example to apply the results.

Example: For each $\mathbf{i} = (i_1, i_2) \in Z_+^2$, let $b_s(\mathbf{i}), s = 1, 2, \ldots, 8$, be the neighbors of $\mathbf{i}$ defined as $b_1(\mathbf{i}) = (i_1 + 1, i_2)$, $b_2(\mathbf{i}) = \mathbf{i} + \mathbf{1}$, $b_3(\mathbf{i}) = (i_1, i_2 + 1)$, $b_4(\mathbf{i}) = (i_1 - 1, i_2 + 1)$, $b_5(\mathbf{i}) = (i_1 - 1, i_2)$, $b_6(\mathbf{i}) = \mathbf{i} - \mathbf{1}$, $b_7(\mathbf{i}) = (i_1, i_2 - 1)$, $b_8(\mathbf{i}) = (i_1 + 1, i_2 - 1)$ and $X_{b_s(\mathbf{i})} = -\infty$ if $b_s(\mathbf{i}) \notin Z_+^2$.
Let $\mathbf{Y} = \{Y_{\mathbf{n}}\}_{\mathbf{n} \geq \mathbf{0}}$ be an i.i.d. random field with common distribution function $F_{\mathbf{Y}}$ and define

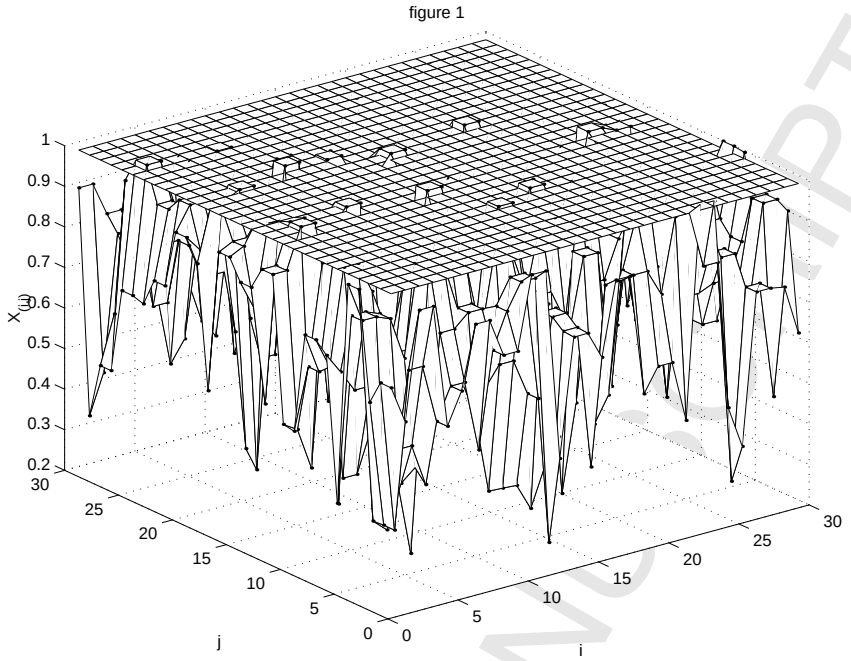
$$X_{\mathbf{n}} = max\left\{Y_{\mathbf{n}}, Y_{b_5(\mathbf{n})}, Y_{b_6(\mathbf{n})}, Y_{b_7(\mathbf{n})}\right\}, \ \mathbf{n} \geq \mathbf{1}.$$

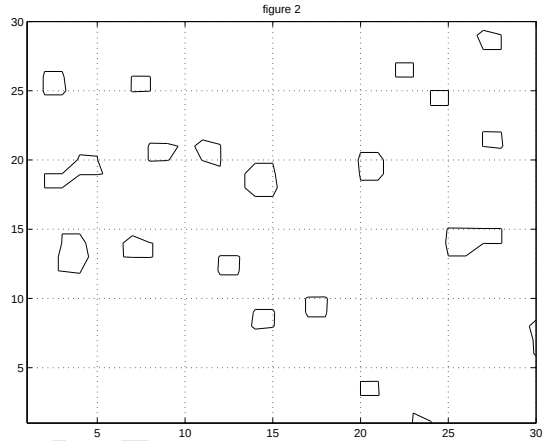
Let $\{u_{\mathbf{n}}\}_{\mathbf{n} \geq \mathbf{1}}$ be such that $n_1 n_2 \left(1 - F_{\mathbf{Y}}\left(u_{\mathbf{n}}\right)\right) \xrightarrow[\mathbf{n} \to \infty]{} \tau > 0$. Then, we have $u_{\mathbf{n}} \equiv u_{\mathbf{n}}^{(\tau_1)}$ for

$\mathbf{X}$ with $\tau_1 = 4\tau$.
$\mathbf{X}$ satisfies the $\Delta(u_{\mathbf{n}}^{(\tau)})$-condition and $D^{(\mathbf{2})}(u_{\mathbf{n}}^{(\tau)})$-condition for each $\tau > 0$ and $n_1 n_2 P(M_{\mathbf{n}}(\mathbf{R}_{\mathbf{2}}^*) \leq u_{\mathbf{n}} | X_{\mathbf{1}} > u_{\mathbf{n}}) \to \frac{1}{4}$. Therefore $\mathbf{X}$ has extremal index $\theta = \frac{1}{4}$.

We present a simulation of this random field to illustrate its predisposition to form clusters of high values. In the figure 1 we present the values of $X_{\mathbf{i}}$ and in figure 2 the contours of values higher than the 97-percentile.

ACCEPTED MANUSCRIPT





Acknowledgements: We are grateful to the referee for his corrections. Work partially supported by FCT/POCI2010/FEDER (EXMIXMOD project).

ACCEPTED MANUSCRIPT

References

Chernick, M.R., Hsing, T. and McCormick, W.P. (1991). Calculating the extremal index for a class of stationary sequences. *Adv. Appl. Prob.* **23**, 835-850.

Choi, H. (2002) *Central limit theory and extremes of random fields.* PhD Dissertation, Univ. of North Carolina at Chapel Hill.

Leadbetter and Rootzén, H. (1998) On extreme values in stationary random fields *Stochastic processes and related topics*, 275-285, *Trends Math.* Birkhauser Boston, Boston.

Pereira, L. and Ferreira, H. (2005) On extreme values in non stationary random fields. *Preprint.* Univ. of Lisbon.

Pereira, L. and Ferreira, H. (2006) Limiting crossing probabilities of random fields. *Journal of Applied Probability* **43**, 3.

ACCEPTED MANUSCRIPT