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# Consistency of the regression estimator with functional data under long memory conditions

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## Abstract

We study the nonparametric regression estimation when the explanatory variable takes values in some abstract functional space. We establish some asymptotic results and we give the (pointwise and uniform) convergence of the kernel type estimator constructed from functional data under long memory conditions.

*Key words:* Nonparametric regression operator, Functional data, Long memory process

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## 1 Introduction

Since the beginning of the sixties, nonparametric statistical methods have been developed intensively where a lot of scientists today collect samples of curves and other functional observations. Functional data have become more and more popular in modern statistics. For an introduction and applications of this field, the books by Ramsay and Silverman (1997, 2002), Bosq (2000) and Ferraty and Vieu (2006) provide some basic methods of analysis along with diverse case studies in several areas including criminology, economics, archeology, and neurophysiology. It should be noticed that the extension of probability theory to random variables taking values in normed spaces (e.g. Banach and Hilbert spaces), including extensions of certain classical theorems, are very useful tools in the recent literature on functional data (Cf Chobanyan et al. (1987), among others).

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In this work, we are interested in the problem of estimating the regression operator when the explanatory data are of functional type, under long memory dependent structure. The phenomenon of long memory was known long before the development of the stochastic models. These processes are mainly of interest for modeling dependence structure of processes, for which the usual weak dependence assumptions (such as mixing or association conditions) are not satisfactory. In fact, in many statistical applications, the correlation between observations decreases to zero slower than in the classical models such as the ARMA models. Models of long memory were first introduced by Mandelbrot and Van Ness (1968) for fractional brownian motion. A good introduction of dependent processes can be found in Beran (1994), among others. For a more detailed discussion on long memory time series and their applications, one can refer for example to Hidalgo (1997) for econometric models and Turcotte (1994) for environmetrics or climatic studies.

Frequently, we say that a stationary process  $\{Z_t\}_{t \in \mathbb{Z}}$  is a long memory process if for some real  $\gamma \in ]0, 1]$ , and some positive constant  $\mathcal{C}$ , its autocorrelation function satisfies

$$|f_Z(j)| \sim \mathcal{C} |j|^{-\gamma} \text{ as } |j| \rightarrow +\infty \quad (1)$$

Other equivalent definitions of long memory process involving the spectral density can be found for instance in Cox (1984). In the finite dimensional framework, the problem of nonparametric regression estimation when the error is a long memory process, was extensively studied by many authors, such as Hall and Hart (1990), Csörgö and Mielniczuk (1995), Masry (2001), Estévez and Vieu (2003)...

In the infinite dimensional framework, the problem of functional estimation for nonparametric regression operator under some strong mixing conditions on the functional data was considered by Ferraty and Vieu (2004). They derived the rates of convergence for a functional version of the kernel type estimator. Rachdi and Vieu (2005, 2007) and Benhenni et al. (2007) constructed some curve data-driven criterion for choosing automatically the smoothing parameter for these estimators. More recently, Müller and Stadtmüller (2006) studied general functional linear models.

In this paper, we consider the similar problem of nonparametric estimation of the regression operator when the pairs  $(X_i, Y_i)$  for  $i \in \mathbb{Z}$  satisfy some long memory conditions. We give two different ways of convergence, pointwise and uniform convergence, for the estimator of the regression operator.

The organization of this paper is as follows. In section 2, we state the functional regression model and we define the nonparametric estimator. In section 3, we set up the different assumptions and we give the convergence results of the estimator through Theorem 3 and Theorem 4. The proofs of all the results are presented in section 4.

## 2 Problem of interest

Let  $(X, Y)$  be a random vector valued in  $H \times \mathbb{R}$  with  $\mathbb{E}|Y| < \infty$ , and where  $(H, d)$  is a semi-metric space equipped with the semi-metric  $d$ . The purpose of this paper is to study the estimation of the regression operator  $r$  when we observe identically distributed copies  $(X_1, Y_1), \dots, (X_n, Y_n)$  of  $(X, Y)$  in the following model

$$Y_i = r(X_i) + \varepsilon_i \text{ for } i \in \mathbb{Z}$$

The random errors  $\varepsilon_1, \dots, \varepsilon_n$  are assumed to be centered with finite second moments.

Most standard time series models assume that the non-summability covariance sequences  $\{f(i)\}_{i \in \mathbb{Z}}$  captures the intuition behind long range dependence (LRD) or *long memory* (see Estévez and Vieu, 2003):

$$\sum_{\tau=-\infty}^{+\infty} |f(\tau)| = +\infty$$

This assumption is not needed here.

The goal of this work is to estimate the unknown operator  $r(x) = \mathbb{E}(Y|X = x)$  for  $x \in H$ . For this aim, we use the functional version of Nadaraya-Watson kernel estimator (see Ferraty and Vieu, 2004)

$$\hat{r}_h(x) = \begin{cases} \frac{\sum_{i=1}^n Y_i K(d(x, X_i)/h_n)}{\sum_{i=1}^n K(d(x, X_i)/h_n)} & \text{if } K(d(x, X_i)/h_n) \neq 0 \\ 0 & \text{if } K(d(x, X_i)/h_n) = 0 \end{cases}$$

where  $K$  is a real valued function defined on  $\mathbb{R}_+$  and  $h = h_n$  is the bandwidth parameter, such that:  $h \in \mathbb{R}_+$  and  $\lim_{n \rightarrow +\infty} h = 0$ . In what follows, we will use the following notations

$$\hat{r}_h(x) = \frac{\hat{r}_{h,2}(x)}{\hat{r}_{h,1}(x)}$$

where

$$\hat{r}_{h,1}(x) = \frac{1}{n \mathbb{E}(\Delta_1(x))} \sum_{i=1}^n \Delta_i(x) \text{ and } \hat{r}_{h,2}(x) = \frac{1}{n \mathbb{E}(\Delta_1(x))} \sum_{i=1}^n Y_i \Delta_i(x)$$

with  $\Delta_i(x) = K(d(x, X_i)/h)$ .

The effect of the dependence of the pairs  $(X_i, Y_i)$  for  $i = 1, \dots, n$  is controlled by the following quantity:

$$Q(x) = \max(|F(x)|, |G(x)|), \text{ for } x \in H$$

where

$$F(x) = \sum_{i \neq j}^n f_{(i,j)}(x) \text{ and } G(x) = \sum_{i \neq j}^n g_{(i,j)}(x)$$

with

$$f_{(i,j)}(x) = \text{cov}(\Delta_i(x), \Delta_j(x)) \text{ and } g_{(i,j)}(x) = \text{cov}(\Gamma_i(x), \Gamma_j(x))$$

### 3 Main results

The following assumptions are needed for the statement of the results concerning the asymptotic performance of the regression operator estimate  $\hat{r}_h$ .

*About the kernel:* we assume that  $K$  is strictly decreasing on  $(0, 1)$  and there exist some positive constants  $c_1$  and  $c_2$  such that

$$c_1 \mathbb{1}_{(0,1)}(t) \leq K(t) \leq c_2 \mathbb{1}_{(0,1)}(t), \text{ for } t \in \mathbb{R} \quad (2)$$

The kernel  $K$  has a first order derivative  $K'$  on  $(0, 1)$  such that for some real constants  $c_3$  and  $c_4$

$$c_3 \leq K'(t) \leq c_4, \text{ for } t \in \mathbb{R} \quad (3)$$

*About the concentration of  $X$ :* we assume that the probability distribution of the functional variable  $X$  can be written as

$$c_5 \varphi(h) \leq \mathbb{P}(X \in \mathcal{B}(x, h)) \leq c_6 \varphi(h), \text{ for } x \in H \quad (4)$$

where  $c_5$  and  $c_6$  are real constants and  $\mathcal{B}(x, t)$  denotes the closed ball of center  $x$  and radius  $t$ , and  $\varphi(t)$  is a positive function such that

$$\lim_{t \rightarrow 0} \varphi(t) = 0 \text{ and } \lim_{n \rightarrow +\infty} n \varphi(h) = +\infty \quad (5)$$

*About the regression operator:* there exist some constants  $0 < c_7 < \infty$  and  $\beta > 0$ , such that

$$\forall x, y \in H, |r(x) - r(y)| \leq c_7 d(x, y)^\beta \quad (6)$$

*About the moments:* the response variable  $Y$  satisfies

$$\forall x \in H, \mathbb{E}(Y^2 | X = x) \leq \sigma(x) < \infty \quad (7)$$

where  $\sigma(\cdot)$  is a continuous function at  $x$ .

This assumption allows us to deal with unbounded variables (Cf page 63 of Ferraty and Vieu, 2006).

About the dependence: we assume that,

$$\lim_{n \rightarrow +\infty} n\varphi(h) = \lim_{n \rightarrow +\infty} \inf_{x \in S} \frac{n^2 \varphi^2(h)}{Q(x)} = +\infty \quad (8)$$

for some compact set  $S$  of  $H$

For instance, when  $f$  and  $g$  satisfy (1) with respectively some  $\gamma_1$  and  $\gamma_2$ , we have for all  $x \in H$ :

$$Q(x) \sim c_x \sum_{k=1}^n nk^{-\gamma} \sim c_x n^{2-\gamma} \text{ where } \gamma = \gamma_1 \text{ or } \gamma_2$$

for some positive constant  $c_x$  depending on the curve  $x$ . The condition (8) is true if  $\varphi(h) = O(n^{-\gamma/3})$ .

The following useful lemmas are needed in the proofs of Theorems 3 and 4.

**Lemma 1** Under condition (6), we have

$$\mathbb{E}(\hat{r}_{h,2}(x)) - r(x) = O(h^\beta), \text{ as } n \rightarrow +\infty$$

**Lemma 2** Assume that conditions (2), (4), (5), (6), (7) and (8) are satisfied, then we have as  $n \rightarrow +\infty$

- (i)  $\hat{r}_{h,2}(x) \rightarrow \mathbb{E}(\hat{r}_{h,2}(x))$ , in probability
- (ii)  $\hat{r}_{h,1}(x) \rightarrow 1$ , in probability

The main results are stated in the following theorems.

**Theorem 3** Under conditions of Lemma 2, we have for all  $x \in H$

$$\hat{r}_h(x) \rightarrow r(x), \text{ in probability, as } n \rightarrow +\infty$$

The uniform convergence in probability of the estimator  $\hat{r}_h(x)$  is obtained by the following result.

**Theorem 4** If condition (2) is replaced by condition (3) in Theorem 3 and if we assume that

$$\forall \epsilon_1 > 0, \exists c_8 > 0 \text{ and } \epsilon < \epsilon_1 \text{ such that } \int_0^\epsilon \varphi(u) du > c_8 \epsilon \varphi(\epsilon), \quad (9)$$

and for some compact set  $S$  of  $H$  there exist  $\alpha > 0$  such that :  $S \subset \cup_{l=1}^{l_n} \mathcal{B}_l$  where  $\mathcal{B}_l, l = 1, \dots, l_n$  are balls of the functional space  $H$ , all having the same radius  $t_n$  with  $l_n = t_n^{-\alpha}$ . (10)

Then we obtain uniformly on the compact set  $S$  that

$$\sup_{x \in S} |\hat{r}_h(x) - r(x)| \rightarrow 0, \text{ in probability, as } n \rightarrow +\infty \quad (11)$$

It should be noticed that these results showed the convergence of the estimators under long range. This result may be considered as an extension of the results given by Ferraty and Vieu (2004) for short range dependence (under strong mixing conditions) where rates of convergence are also obtained. However, the rates of convergence for long range dependence are not obtained by these results and are still an open problem in the infinite dimensional case. On the other hand, it may be possible to obtain rates of convergence by using adapted large deviation inequalities (see Estévez and Vieu, 2003), but nothing guarantees their optimality.

#### 4 Proofs

In what follows,  $\eta_i$ 's for  $i = 1, 2, 3$  denote some real positive constants.

*Proof of Lemma 1.* For a sample of identically distributed pairs  $(X_i, Y_i)$  for  $i = 1, \dots, n$ , we have

$$\begin{aligned} r(x) - \mathbb{E}(\hat{r}_{h,2}(x)) &= r(x) - \frac{1}{n \mathbb{E}(\Delta_1(x))} \sum_{i=1}^n \mathbb{E}(Y_i \Delta_i(x)) \\ &= \frac{\mathbb{E}((r(x) - r(X_1)) \Delta_1(x))}{\mathbb{E}(\Delta_1(x))} \end{aligned}$$

Because the support of the kernel function  $K$  is  $(0, 1)$ , and from the definition of  $\Delta_1(x)$ , we have

$$|r(X_1) - r(x)| \Delta_1(x) \leq \sup_{t \in \mathcal{B}(x,h)} |r(t) - r(x)| \Delta_1(x)$$

then it follows from condition (6) that

$$\begin{aligned} \mathbb{E}(|r(X_1) - r(x)| \Delta_1(x)) &\leq c_7 \sup_{t \in \mathcal{B}(x,h)} d(x,t)^\beta \mathbb{E}(\Delta_1(x)) \\ &\leq c_7 h^\beta \mathbb{E}(\Delta_1(x)) \end{aligned} \tag{12}$$

and as  $\lim_{n \rightarrow +\infty} h = 0$ , we obtain the result of Lemma 1.  $\square$

*Proof of Lemma 2.* The proof of the result (i) is based on the application of Chebychev's inequality. Indeed, because of:

$$\text{for } \epsilon > 0, \mathbb{P}(|\hat{r}_{h,2}(x) - \mathbb{E}(\hat{r}_{h,2}(x))| > \epsilon) \leq \frac{\text{var}(\hat{r}_{h,2}(x))}{\epsilon^2}$$



we need to show that

$$\text{var}(\widehat{r}_{h,2}(x)) \rightarrow 0 \text{ as } n \rightarrow +\infty$$

For that we have:

$$\begin{aligned} \text{var}(\widehat{r}_{h,2}(x)) &= \frac{1}{n \mathbb{E}^2(\Delta_1(x))} \text{var}(Y_1 \Delta_1(x)) \\ &\quad + \frac{1}{n^2 \mathbb{E}^2(\Delta_1(x))} \sum_{i \neq j} \text{cov}(Y_i \Delta_i(x), Y_j \Delta_j(x)) \\ &:= I_{1,n} + I_{2,n} \end{aligned}$$

For the first term, from (12) we have that

$$\begin{aligned} \mathbb{E}(\Delta_1(x) Y_1) &= \mathbb{E}(\mathbb{E}(Y_1 \Delta_1(x) | X_1)) \\ &= \mathbb{E}((r(X_1) - r(x)) \Delta_1(x)) + r(x) \mathbb{E}(\Delta_1(x)) \\ &= (r(x) + O(h^\beta)) \mathbb{E}(\Delta_1(x)) \end{aligned}$$

then

$$I_{1,n} = \frac{1}{n \mathbb{E}^2(\Delta_1(x))} \left( \mathbb{E}(\Delta_1^2(x) \mathbb{E}(Y_1^2 | X_1)) - (r(x) + O(h^\beta))^2 \mathbb{E}^2(\Delta_1(x)) \right)$$

From condition (7), we have

$$I_{1,n} \leq \frac{1}{n} \left( \frac{\mathbb{E}(\Delta_1^2(x))}{\mathbb{E}^2(\Delta_1(x))} \left( \sup_{t \in \mathcal{B}(x,h)} |\sigma(t) - \sigma(x)| + \sup_{t \in \mathcal{B}(x,h)} |\sigma(t)| \right) - (r(x) + O(h^\beta))^2 \right)$$

The lower and upper bounds of conditions (2) and (4) give

$$\mathbb{E}(\Delta_1(x)) \geq c_1 \mathbb{P}(X \in \mathcal{B}(x, h)) \geq c_1 c_5 \varphi(h)$$

and likewise  $\mathbb{E}(\Delta_1^2(x)) \leq c_2^2 c_6 \varphi(h)$ , and using condition (5) we deduce that

$$I_{1,n} \leq \frac{1}{n \varphi(h)} \left( \eta_1 - \varphi(h) (r(x) + O(h^\beta))^2 \right) \leq \frac{\eta_1}{n \varphi(h)} + o\left(\frac{1}{n \varphi(h)}\right)$$

so that

$$I_{1,n} = O\left(\frac{1}{n \varphi(h)}\right) \tag{13}$$

The second term  $I_{2,n}$ , can be directly written as

$$\begin{aligned}
I_{2,n} &= O\left(\frac{1}{n^2\varphi^2(h)} \sum_{i \neq j} \text{cov}(\Gamma_i(x), \Gamma_j(x))\right) \\
&= O\left(\frac{1}{n^2\varphi^2(h)} G(x)\right)
\end{aligned}$$

Finally, using the condition (8), we obtain

$$\text{var}(\hat{r}_{h,2}(x)) \rightarrow 0 \text{ as } n \rightarrow +\infty$$

The result (ii) can be derived similarly as (i) by taking  $Y_i = 1$  and using the fact that  $\mathbb{E}(\hat{r}_{h,1}(x)) = 1$ .  $\square$

*Proof of Theorem 3.* The result follows by a direct application of the previous Lemmas 1 and 2 together with the following decomposition

$$\begin{aligned}
\hat{r}_h(x) - r(x) &= \frac{1}{\hat{r}_{h,1}(x)} [(\hat{r}_{h,2}(x) - \mathbb{E}(\hat{r}_{h,2}(x))) - (r(x) - \mathbb{E}(\hat{r}_{h,2}(x)))] \\
&\quad - \frac{r(x)}{\hat{r}_{h,1}(x)} (\hat{r}_{h,1}(x) - 1)
\end{aligned} \tag{14}$$

and, it remains to show that, there exists  $\theta > 0$  such that

$$\mathbb{P}(|\hat{r}_{h,1}(x)| \leq \theta) \rightarrow 0. \tag{15}$$

For this aim, since  $\hat{r}_{h,1}(x)$  converges to 1, in probability as  $n \rightarrow +\infty$ , we have

$$\forall \epsilon > 0, \mathbb{P}(|\hat{r}_{h,1}(x) - 1| > \epsilon) \rightarrow 0 \tag{16}$$

and we can notice that

$$\mathbb{P}\left[|\hat{r}_{h,1}(x)| \leq \frac{1}{2}\right] \leq \mathbb{P}\left[|\hat{r}_{h,1}(x) - 1| > \frac{1}{2}\right]$$

By letting  $\theta = \epsilon = 1/2$ , the proof of (15) is achieved.  $\square$

*Proof of Theorem 4.* If assumptions (3) and (9) are satisfied then, for  $h$  small enough (see Lemma 4.4, page 44 in Ferraty and Vieu, 2006)

$$\text{for } x \in H, \eta_2\varphi(h) \leq \mathbb{E}\left(K\left(\frac{d(x, X)}{h}\right)\right) \leq \eta_3\varphi(h), \tag{17}$$

In order to prove (17), we can write

$$\mathbb{E} \left( K \left( \frac{d(x, X_i)}{h} \right) \right) = \int_0^1 K(t) d\mathbb{P}^{\frac{d(x, X_i)}{h}}(t)$$

where  $d\mathbb{P}^{\frac{d(x, X_i)}{h}}$  denotes the image measure of the real random variable  $d(x, X_i)/h$ .

Now, since the first derivative of  $K$  exists, we have  $K(t) = K(0) + \int_0^t K'(u)du$ ,

which implies, by applying the Fubini's Theorem, that

$$\begin{aligned} \mathbb{E} \left( K \left( \frac{d(x, X_i)}{h} \right) \right) &= \int_0^1 K(0) d\mathbb{P}^{\frac{d(x, X_i)}{h}}(t) + \int_0^1 \left( \int_0^t K'(u)du \right) d\mathbb{P}^{\frac{d(x, X_i)}{h}}(t) \\ &= K(0) \varphi(h) + \int_0^1 K'(u) \mathbb{P} \left( u \leq \frac{d(x, X_i)}{h} \leq 1 \right) du \end{aligned}$$

and the fact that  $K(1) = 0$  allows to write

$$\mathbb{E} \left( K \left( \frac{d(x, X_i)}{h} \right) \right) = - \int_0^1 K'(u) \varphi(hu) du$$

It suffices to use hypothesis (3) to show that, for  $h < \varepsilon_1$  and with  $\eta_2 = -c_1 c_{10}$

$$\mathbb{E} \left( K \left( \frac{d(x, X_i)}{h} \right) \right) \geq \eta_2 \varphi(h)$$

Concerning the upper bound, it suffices to remark that  $K$  is bounded with support  $(0, 1)$  and uses hypothesis (3) by putting  $\eta_3 = \sup_{t \in (0, 1)} K(t)$ .

In order to complete the proof of this Theorem, we use first the decomposition (14) and then (16). We notice that

$$\begin{aligned} &\sup_{x \in S} |\hat{r}_h(x) - r(x)| \\ &\leq \sup_{x \in S} |\hat{r}_h(x) - \mathbb{E}(\hat{r}_h(x))| + \sup_{x \in S} |\mathbb{E}(\hat{r}_h(x)) - r(x)| \\ &\leq \sup_{x \in S} |\hat{r}_{h,2}(x) - r(x)| + \sup_{x \in S} |\hat{r}_{h,2}(x) - \mathbb{E}(\hat{r}_{h,2}(x))| + \sup_{x \in S} |\hat{r}_{h,1}(x) - 1| \end{aligned}$$

According to Lemma 1, we claim immediately that

$$\sup_{x \in S} |\hat{r}_{h,2}(x) - r(x)| = O(h^\beta) \quad (18)$$

On the other hand, from condition (10), it suffices to look at the uniform convergence on the closest center to  $x$  among the balls centers  $\{c_1, \dots, c_{l_n}\}$ ,

$c_j(x) = \arg \min_{c_j \in \{c_1, \dots, c_{l_n}\}} d(c_j, x)$ , (Cf pages 75-76 in Ferraty and Vieu (2006)).  
We have

$$\begin{aligned} & \mathbb{P} \left( \sup_{x \in S} |\hat{r}_{h,2}(c_j(x)) - \mathbb{E}(\hat{r}_{h,2}(c_j(x)))| > \epsilon \right) \\ &= \mathbb{P} \left( \max_{j=1, \dots, l_n} |\hat{r}_{h,2}(c_j) - \mathbb{E}(\hat{r}_{h,2}(c_j))| > \epsilon \right) \\ &\leq l_n \max_{j=1, \dots, l_n} \mathbb{P} (|\hat{r}_{h,2}(c_j) - \mathbb{E}(\hat{r}_{h,2}(c_j))| > \epsilon) \end{aligned}$$

Choosing  $l_n = t_n^{-\alpha}$  with  $t_n \sim n^{-\zeta}$ ,  $\zeta > 0$ , and using the results of Lemma 2, we obtain in probability

$$\lim_{n \rightarrow \infty} \sup_{x \in S} |\hat{r}_{h,2}(x) - \mathbb{E}(\hat{r}_{h,2}(x))| = 0 \text{ and } \lim_{n \rightarrow \infty} \sup_{x \in S} |\hat{r}_{h,1}(x) - 1| = 0 \quad (19)$$

From (18) and (19), we claim the final result (11).  $\square$

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