



HAL
open science

On fake Brownian motions

Krzysztof Oleszkiewicz

► **To cite this version:**

| Krzysztof Oleszkiewicz. On fake Brownian motions. *Statistics and Probability Letters*, 2010, 78 (11), pp.1251. 10.1016/j.spl.2007.11.026 . hal-00622142

HAL Id: hal-00622142

<https://hal.science/hal-00622142>

Submitted on 12 Sep 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Accepted Manuscript

On fake Brownian motions

Krzysztof Oleszkiewicz

PII: S0167-7152(07)00401-4
DOI: 10.1016/j.spl.2007.11.026
Reference: STAPRO 4833

To appear in: *Statistics and Probability Letters*

Received date: 30 June 2007
Revised date: 12 September 2007
Accepted date: 6 November 2007

Please cite this article as: Oleszkiewicz, K., On fake Brownian motions. *Statistics and Probability Letters* (2007), doi:10.1016/j.spl.2007.11.026

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



On fake Brownian motions

Krzysztof Oleszkiewicz ^{*†}

Institute of Mathematics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland

Abstract

Recently Albin constructed an example of a continuous martingale different from the classical Brownian motion but with the same marginal distributions, thus improving on the result of Hamza and Klebaner. We present a simpler solution to this problem.

Keywords: continuous martingale, Gaussian marginals, Brownian motion

Let $(X_t)_{t \geq 0}$ be a real-valued random process and let $(\mathcal{M}_t)_{t \geq 0}$ be a natural filtration associated to the process, $\mathcal{M}_t = \sigma((X_s)_{0 \leq s \leq t})$. We will say that $(X_t)_{t \geq 0}$ is a *fake Brownian motion* if it satisfies the following four conditions: $(X_t, \mathcal{M}_t)_{t \geq 0}$ is a martingale, it is continuous, X_t is a Gaussian random variable with mean zero and variance t for every $t \geq 0$ and, finally, the process $(X_t)_{t \geq 0}$ does not have the same distribution as the classical Brownian motion $(B_t)_{t \geq 0}$. Clearly, instead of the last condition we can just require that $(X_t)_{t \geq 0}$ is not a Gaussian process, since the other conditions easily imply that it has the same covariance structure as $(B_t)_{t \geq 0}$. It is natural to ask whether such processes exist. Hamza and Klebaner proved recently that there exists a large family of non-Gaussian martingale processes having the same marginal distributions as $(B_t)_{t \geq 0}$. However, they were not able to ensure the continuity of these martingales, thus leaving the question about existence of fake Brownian motions open. The question was answered affirmatively by Albin soon, but his construction of a fake Brownian motion was rather involved. In this note we describe a new and quite simple construction.

Throughout the paper we assume that $G_1, G_2, (B_t)_{t \geq 0}$ and $(W_t)_{t \geq 0}$ are independent, and that G_1 and G_2 have distribution $\mathcal{N}(0, 1)$ whereas $(B_t)_{t \geq 0}$ and $(W_t)_{t \geq 0}$ are standard Brownian motions. By $\|\cdot\|$ we denote the standard Euclidean norm.

Given $a \geq 0$ let us define a filtration $\mathcal{F}_t^{(a)} = \sigma(G_1, G_2, (W_s)_{0 \leq s \leq a + \ln t})$ for $t \geq e^{-a}$. Let

$$X_t^{(a)} = \sqrt{t} \left(G_1 \cos W_{a + \ln t} + G_2 \sin W_{a + \ln t} \right)$$

for $t \geq e^{-a}$. Then $(X_t^{(a)}, \mathcal{F}_t^{(a)})_{t \geq e^{-a}}$ is a continuous martingale and $X_t^{(a)} \sim B_t$ for every $t \geq e^{-a}$. It is easy to check that $X_e^{(a)} - X_1^{(a)}$ is not Gaussian, so that $(X_t^{(a)})_{t \geq e^{-a}}$

^{*}Corresponding author at: Institute of Mathematics, Warsaw University, ul. Banacha 2, 02-097 Warsaw, Poland. *E-mail address:* koles@mimuw.edu.pl

[†]Research partially supported by the Polish MNiI Grant 1 PO3A 012 29

cannot be a Gaussian process. Another way to see that it cannot be extended to a Brownian motion is by noting that $\limsup_{t \rightarrow \infty} |X_t^{(a)}|/\sqrt{t} \leq |G_1| + |G_2| < \infty$ a.s., while $\limsup_{t \rightarrow \infty} |B_t|/\sqrt{t} = \infty$ a.s., by the law of iterated logarithm.

Let $a > b > 0$ and let $(V_s)_{s \geq 0}$ be given by $V_s = W_{a-b+s} - W_{a-b}$, so that $(V_s)_{s \geq 0}$ is a standard Brownian motion and $(V_s)_{s \geq 0}$, W_{a-b} , G_1 and G_2 are independent. Let $G'_1 = G_1 \cos W_{a-b} + G_2 \sin W_{a-b}$ and $G'_2 = G_2 \cos W_{a-b} - G_1 \sin W_{a-b}$. Then for $t \geq e^{-b} \geq e^{-a}$ we have

$$\begin{aligned} (X_t^{(a)})_{t \geq e^{-b}} &= \left(\sqrt{t}(G_1 \cos W_{a+\ln t} + G_2 \sin W_{a+\ln t}) \right)_{t \geq e^{-b}} = \\ &= \left(\sqrt{t}(G_1 \cos(W_{a-b} + V_{b+\ln t}) + G_2 \sin(W_{a-b} + V_{b+\ln t})) \right)_{t \geq e^{-b}} = \\ &= \left(\sqrt{t}((G_1 \cos W_{a-b} + G_2 \sin W_{a-b}) \cos V_{b+\ln t} + (G_2 \cos W_{a-b} - G_1 \sin W_{a-b}) \sin V_{b+\ln t}) \right)_{t \geq e^{-b}} \\ &= \left(\sqrt{t}(G'_1 \cos V_{b+\ln t} + G'_2 \sin V_{b+\ln t}) \right)_{t \geq e^{-b}} \end{aligned}$$

and the last process has the same distribution as $(X_t^{(b)})_{t \geq e^{-b}}$ since $(G'_1, G'_2, (V_s)_{s \geq 0})$ has the same distribution as $(G_1, G_2, (W_s)_{s \geq 0})$. Thus the above construction can be extended to all $t > 0$. The arguments are quite standard but we present them for the sake of completeness. For $t_1 < t_2 < \dots < t_n$ and $e^{-a} \leq t_1$ let

$$\mu_{t_1, t_2, \dots, t_n} = \mathcal{L}\left((X_{t_1}^{(a)}, X_{t_2}^{(a)}, \dots, X_{t_n}^{(a)})\right).$$

In view of the above $\mu_{t_1, t_2, \dots, t_n}$ does not depend on a . Also, it is clear that the measures $(\mu_{t_1, t_2, \dots, t_n})_{n \in \mathcal{N}, 0 < t_1 < t_2 < \dots < t_n}$ are consistent since for any two of them some $a > 0$ may be chosen such that both of them are finite dimensional distributions of the process $(X_t^{(a)})_{t \geq e^{-a}}$. By the Kolmogorov consistency (extension) theorem there exists a stochastic process $(\tilde{X}_t)_{t > 0}$ such that the measures introduced above are its finite dimensional distributions. Hence $(\tilde{X}_t)_{t > e^{-a}}$ has the same finite dimensional distributions as $(X_t^{(a)})_{t > e^{-a}}$ for every $a > 0$. Let $X_t = \tilde{X}_t$ for $t \in Q_+$ and $X_t = \lim_{s \rightarrow t, s \in Q_+} X_s$ for $t \in (0, \infty) \setminus Q_+$. Random events $\{\text{path of } (X_s^{(k)})_{s \in [e^{-k}, k] \cap Q_+} \text{ is uniformly continuous}\}$ and $\{\lim_{s \rightarrow t, s \in [e^{-k}, \infty) \cap Q_+} X_s^{(k)} = X_t^{(k)}\}$ for $k \in \mathcal{N}$ and $t \in (e^{-k}, \infty) \setminus Q_+$ depend only on countably many of the random variables and belong to the cylindrical σ -field of the process $(X_s^{(k)})_{s \geq e^{-k}}$. Thus

$$\begin{aligned} &\mathbb{P}(\text{path of } (\tilde{X}_s)_{s \in [e^{-k}, k] \cap Q_+} \text{ is uniformly continuous}) = \\ &= \mathbb{P}(\text{path of } (X_s^{(k)})_{s \in [e^{-k}, k] \cap Q_+} \text{ is uniformly continuous}) = 1 \end{aligned}$$

and

$$\mathbb{P}(X_t = \tilde{X}_t) = \mathbb{P}\left(\lim_{s \rightarrow t, s \in [e^{-k}, \infty) \cap Q_+} X_s^{(k)} = X_t^{(k)}\right) = 1.$$

There are only countably many $k \in \mathcal{N}$, so that $(X_t)_{t > 0}$ has paths continuous a.s. and it is a stochastic modification of (i.e. it has the same finite dimensional as) $(\tilde{X}_t)_{t > 0}$.

Now we set $X_0 = 0$. The continuity a.s. of paths at $t = 0$ (in fact, even some much stronger regularity condition) follows by a direct argument. Note that for $C > 0$ and $a > 0$ one has

$$\mathbb{P}\left(\sup_{t \geq e^{-a}} |X_t^{(a)}|/\sqrt{t} > C\right) \leq \mathbb{P}(|G_1| + |G_2| > C) \leq 2\mathbb{P}(|G_1| > C/2) \leq 4e^{-C^2/8}.$$

Hence, for $a > n + 1$,

$$\mathbb{P}(\exists t \in [e^{-n-1}, e^{-n}] |X_t| > C_n e^{-n/2}) \leq \mathbb{P}\left(\sup_{t \in [e^{-n-1}, e^{-n}]} |X_t^{(a)}|/\sqrt{t} > C_n\right) \leq 4e^{-C_n^2/8}.$$

Let us set $C_n = e^{\varepsilon n/2}$ for $\varepsilon \in (0, 1)$. The Borel-Cantelli lemma yields $|X_t|/t^{\frac{1}{2}-\varepsilon} \xrightarrow{t \rightarrow 0^+} 0$ a.s. for every $\varepsilon > 0$. Note that so far we have extended the process (X_t) only. Now we may also define a new filtration $(\mathcal{M}_t)_{t \geq 0}$ with $\mathcal{M}_t = \sigma((X_s)_{s \in [0, t]})$. To check that $\mathbb{E}(X_T | \mathcal{M}_t) = X_t$ for $0 < t < T$, let us consider a multiplicative system (π -system) \mathcal{A}_t consisting of random events of the form $\{X_{s_1} \in B_1, \dots, X_{s_k} \in B_k\}$ for $k = 1, 2, \dots$, $0 < s_1, \dots, s_k \leq t$ and $B_1, \dots, B_k \in \mathcal{B}(\mathbb{R})$. One can easily see that $\sigma(\mathcal{A}_t) = \mathcal{M}_t$, so by the lemma on π - and λ -systems it suffices to prove that $\mathbb{E}X_T 1_A = \mathbb{E}X_t 1_A$ for every $A \in \mathcal{A}_t$. For $a > 0$ such that $e^{-a} = \min_{i \leq k} s_i$ let $\tilde{A} = \{X_{s_1}^{(a)} \in B_1, \dots, X_{s_k}^{(a)} \in B_k\}$. Then we have

$$\mathbb{E}X_T 1_A = \mathbb{E}X_T^{(a)} 1_{\tilde{A}} = \mathbb{E}X_t^{(a)} 1_{\tilde{A}} = \mathbb{E}X_t 1_A.$$

We have proved that $(X_t)_{t \geq 0}$ is a fake Brownian motion.

To avoid technicalities of the preceding construction one may prefer to set $Y_t = B_t$ for $t \in [0, 1]$ and $Y_t = \sqrt{t} \left(B_1 \cos W_{\ln t} + G_2 \sin W_{\ln t} \right)$ for $t \geq 1$. Then $(Y_t)_{t \geq 0}$ is a fake Brownian motion as well - the only point which perhaps needs some extra explanation is the martingale structure: observe first that $(Y_t, \mathcal{F}_t)_{t \geq 0}$ is a martingale for a filtration given by $\mathcal{F}_t = \sigma((B_s)_{0 \leq s \leq t})$ for $t \in [0, 1]$ and $\mathcal{F}_t = \sigma(G_2, (B_s)_{0 \leq s \leq 1}, (W_s)_{s \in [0, \ln t]})$ for $t \geq 1$. Now just recall that if $(Y_t, \mathcal{G}_t)_{t \geq 0}$ is a martingale for some filtration $(\mathcal{G}_t)_{t \geq 0}$ then also $(Y_t, \mathcal{M}_t)_{t \geq 0}$ is a martingale, for $\mathcal{M}_t = \sigma((Y_s)_{0 \leq s \leq t})$.

There is also another simple way to construct fake Brownian motions. Let us consider a unit sphere $S^{n-1} \subseteq \mathbb{R}^n$, $n \geq 2$, and let $(U_t)_{t \in \mathbb{R}}$ be a stationary \mathbb{R}^n -valued Brownian motion on the sphere such that U_t is uniformly distributed on S^{n-1} for every $t \in \mathbb{R}$, see for example Skipper (2007), Ex. 3.3.2 of Hsu (2002) or Ito and McKean (1974), p.269. The time range as stated there is $t \geq 0$ but it may be easily extended to $t \in \mathbb{R}$ in a similar way as we did for the process X_t above. Clearly, $\mathbb{E}\left(U_T \mid \sigma((U_s)_{s \leq t})\right) = \psi(T-t)U_t$ for $T > t$, for some function $\psi : (0, \infty) \rightarrow (-1, 1)$ because of the strict convexity of the Euclidean ball (expectation is taken in \mathbb{R}^n), rotation invariance and time homogeneity. One easily observes that $\psi(x+y) = \psi(x)\psi(y)$ for all $x, y > 0$ and therefore $\psi(t) = e^{-\kappa t}$ for some $\kappa > 0$ - the noncontinuous solutions of the Cauchy functional equation are excluded here since ψ is bounded. Up to an appropriate linear change of time we may and will assume that $\kappa = 1$. Let R be a nonnegative random variable with χ_n distribution (standard chi distribution with n degrees of freedom, with a density $2^{1-\frac{n}{2}} \Gamma(n/2)^{-1} r^{n-1} e^{-r^2/2} 1_{(0, \infty)}(r)$), independent of the process $(U_t)_{t \in \mathbb{R}}$. A Gaussian $\mathcal{N}(0, Id_n)$ random vector $\mathbf{G} = \|\mathbf{G}\| \cdot (\mathbf{G}/\|\mathbf{G}\|)$ (note that the factors are independent) has the same distribution as $R \cdot U_t$ for every t , so that $R \cdot p(U_t) \sim \mathcal{N}(0, 1)$, where p denotes a projection to the first coordinate: $p(x_1, \dots, x_n) = x_1$. Hence $(Z_t)_{t \geq 0}$ defined by $Z_0 = 0$ and $Z_t = \sqrt{t} R \cdot p(U_{(\ln t)/2})$ for $t > 0$ is a fake Brownian motion (with obvious analogy to $(X_t)_{t \geq 0}$ if $n = 2$). Now it is also clear that $\limsup_{t \rightarrow 0^+} |Z_t|/\sqrt{t} < \infty$ a.s. Certainly, $(Z_{\lambda t})_{t \geq 0}$ has the same distribution as $(\sqrt{\lambda} Z_t)_{t \geq 0}$ for any fixed $\lambda > 0$. Since the process $(U_t)_{t \in \mathbb{R}}$ has the same distribution as $(U_{-t})_{t \in \mathbb{R}}$ we easily deduce that the process $(Z_t)_{t > 0}$ has the same distribution as $(tZ_{1/t})_{t > 0}$, which is another property it shares with the standard Brownian motion.

One can also consider a projection $q : \mathbb{R}^n \rightarrow \mathbb{R}^{n-2}$ to the first $n-2$ coordinates: $q(x_1, \dots, x_n) = (x_1, \dots, x_{n-2})$ (now $n \geq 3$). Then it is well known that $q(U_t)$ is

uniformly distributed on the unit ball of \mathbb{R}^{n-2} - it is obviously a simple exercise in analysis but it may also be proved in the following probabilistic way. Certainly, $q(U_t)$ is rotation invariant. Hence it suffices to prove that $\|q(U_t)\|$ has the same distribution as $\|J\|$, where J is uniformly distributed on the unit ball of \mathbb{R}^{n-2} . Since both $q(U_t)$ and J are bounded it is enough to prove that $\mathbb{E}\|q(U_t)\|^p = \mathbb{E}\|J\|^p$ for $p > 0$. Let \mathbf{G} be an \mathbb{R}^n -valued $\mathcal{N}(0, Id_n)$ Gaussian vector. Then $\mathbf{G} = \|\mathbf{G}\| \cdot (\mathbf{G}/\|\mathbf{G}\|)$ has the same distribution as $R \cdot U_t$, and consequently $q(\mathbf{G})$ has the same distribution as $R \cdot q(U_t)$, where R has distribution χ_n and is independent of $q(U_t)$. Note that $\|q(U_t)\|$ has distribution χ_{n-2} . Thus

$$\begin{aligned} \mathbb{E}\|q(U_t)\|^p &= \mathbb{E}\|q(\mathbf{G})\|^p / \mathbb{E}R^p = \frac{\Gamma((p+n-2)/2)}{\Gamma((n-2)/2)} \frac{\Gamma(n/2)}{\Gamma((p+n)/2)} = \\ &= \frac{n-2}{p+n-2} = \int_0^1 (n-2)r^{n-3}r^p dr = \mathbb{E}\|J\|^p \end{aligned}$$

which ends the proof.

Using the fact that every rotation invariant unimodal density may be expressed as a mixture of indicator functions of centrally symmetric Euclidean balls, i.e. as a density of $W \cdot q(U_0)$ for some nonnegative random variable W independent of the process $(U_t)_{t \geq 0}$, we can easily modify the reasoning we have used to construct a fake Brownian motion, to obtain other martingales with marginals being dilates of some rotation invariant unimodal distribution, and with paths continuous a.s. In this way we can construct a fake \mathbb{R}^{n-2} -valued rotation invariant α -stable process ($\alpha \in (0, 2]$). For $\alpha \in (0, 2)$ it also can be obtained in a simpler way, as $Q_t = \sqrt{S} \cdot \mathbf{B}_{t^{2/\alpha}}$, where S is a nonnegative $\alpha/2$ -stable random variable independent of the $(n-2)$ -dimensional rotation invariant Brownian motion $(\mathbf{B}_t)_{t \geq 0}$. The case $\alpha = 2$ easily follows from the first construction described in this note if we replace random variables G_1 and G_2 by $\mathcal{N}(0, Id_{n-2})$ Gaussian vectors.

The method seems to have potential for further extensions. For example, if $(X_t)_{t \geq 0}$ is a fake Brownian motion and $(X'_t)_{t \geq 0}$ is its independent copy then $(\hat{X}_t)_{t \geq 0}$ given by $\hat{X}_t = X_t \cos \beta + X'_t \sin \beta$ for some fixed $\beta \in [0, 2\pi)$ is a fake Brownian motion as well. Certainly, $\hat{X}_{t \geq 0}$ is not a Gaussian process, by Cramer's theorem.

Acknowledgements I would like to thank Professor Tomasz Bojdecki for introducing me with the problem. Professor Michel Ledoux pointed out the Hsu's book.

References

- [1] Albin, J.M.P., A continuous non-Brownian motion martingale with Brownian motion martingale distributions, to appear in *Statist. Probab. Lett.*
- [2] Hamza, K. and Klebaner, F.C., A family of non-Gaussian martingales with Gaussian marginals, to appear in *J. Appl. Math. Stoch. Anal.*
- [3] Hsu, E.P., *Stochastic Analysis on Manifolds*, Graduate Studies in Mathematics 38, AMS, Providence, 2002
- [4] Ito, K. and McKean, H.P., Jr., *Diffusion processes and their sample paths*, Springer-Verlag, Berlin, 1974
- [5] Skipper, M., *Limit Theorems for Projections of High Dimensional Spherical Brownian Motion and Random Walk on a Hypercube*, Transfer Thesis, available on-line at www.maths.ox.ac.uk/skipper/Research.htm, 2007