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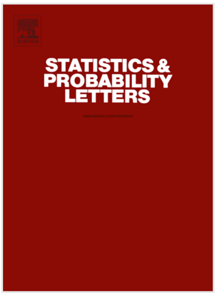

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Precise large deviation results for the total claim amount under subexponential claim sizes

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Abstract

The paper deals with the renewal risk model. A precise large deviation result in the case of subexponential claim sizes is proved. As a special case, the example of Pareto distributed claim sizes and inter-occurence times is investigated.

Keywords: renewal risk model; subexponential distribution; large deviations

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1 Introduction

In this paper we investigate the precise large deviations for the renewal risk model (total claim amount process), having the following structure:

Assumption H_1 The claim sizes $Z_1, Z_2, \ldots$ form a sequence of independent identically distributed (i.i.d.) nonnegative random variables with a common distribution function (d.f.) $B(u) = \mathrm{P}(Z_1 \le u)$, which has a finite mean $a = \mathrm{E}Z_1$ and a finite second moment $\mathrm{E}Z_1^2 < \infty$.

Assumption H_2 The inter-occurrence times $\theta_1 = T_1, \theta_2 = T_2 - T_1, \theta_3 = T_3 - T_2, \ldots$ are i.i.d. non-negative random variables with mean $1/\lambda$ and finite second moment $\mathrm{E}\theta_1^2 < \infty$. In addition, $\theta_1, \theta_2, \ldots$ are mutually independent of $Z_1, Z_2, \ldots$.

In the special case, where $\theta_1, \theta_2, \ldots$ have exponential distribution, this model is called a Poisson model.

The random variables $T_k = \sum_{i=1}^{k} \theta_i,\ k = 1, 2, \ldots$ constitute a renewal counting process $N(t) = \#\{k = 1, 2, \ldots : T_k \in (0, t]\},\ t \ge 0$ with a mean function $\lambda(t) = \mathrm{E}N(t)$, for which $\lambda(t) \sim \lambda t$ as $t \to \infty$. Define a random walk process $S_n = \sum_{k=1}^{n} Z_k,\ n \ge 1,\ S_0 = 0$.

We are interested in a precise large deviation result for random sums (total claim amount process) $S_{N(t)}$ under the assumption that claim sizes' distribution B is *heavy-tailed*. A natural class of heavy-tailed distributions is the class of subexponential distributions. Recall that the d.f. $B(x)$ on $[0, \infty)$ is called subexponential and is denoted $B \in \mathscr{S}$) if the tail $\overline{B} = 1 - B$ satisfies equality

$$\lim_{u \to \infty} \overline{B * B}(u)/\overline{B}(u) = 2,$$

where $B * B$ denotes the Stieltjes convolution of B with itself. The precise large deviations for random sums $S_{N(t)}$ in special cases of $\mathscr{S}$ were studied in Klüppelberg and Mikosch (1997), Tang et al. (2001) (the tails $\overline{B}$ are of extended regular variation), Ng et al. (2004) (B has a consistent variation). They prove that, under corresponding regularity conditions, it holds

$$\mathrm{P}(S_{N(t)} > x + a\lambda(t)) \sim \lambda \overline{B}(x), \quad t \to \infty$$

uniformly for $x \ge \gamma\lambda(t)$ for every $\gamma > 0$, i.e.

$$\lim_{t \to \infty} \sup_{x \ge \gamma\lambda(t)} \Big| \frac{\mathrm{P}(S_{N(t)} > x + \mu\lambda(t))}{\lambda(t)\overline{B}(x)} - 1 \Big| = 0. \tag{1.1}$$

For applications of the precise deviation results in insurance and finance see, e.g., Klüppelberg and Mikosch (1997), Mikosch and Nagaev (1998), among others.

In our paper we prove that, under mild additional assumptions on the subexponential d.f. of the claim size Z_1 and on the d.f. of the inter-occurrence time θ_1, for every nonnegative $\mu \ge 0$, the relation

$$\mathrm{P}(S_{N(t)} > x + (a + \mu)\lambda t) \sim \lambda t \overline{B}(x + \mu\lambda t), \quad x \to \infty \tag{1.2}$$

holds uniformly for all $t \in [f(x), \gamma x/Q(x)]$ in case $\mu > 0$, and for all $t \in [f_1(x), o(x/Q(x))]$ in case $\mu = 0$, where $f(x)$, $f_1(x)$ are arbitrary infinitely increasing functions and $\gamma > 0$ is an arbitrary positive constant. Note that, in general, the zones of uniform convergence in (1.1) and

(1.2) are different. The relation in (1.2) with respect to $x \to \infty$ is more natural for studying the asymptotics of the finite time ruin probabilities, where the initial capital of an insurance company, x, tends to infinity.

The rest of the paper is organized as follows. Additional assumptions on the distribution B and on the renewal process $N(t)$, together with the main theorem are formulated in Section 2. The proof of the theorem is given in Section 3. An example of the Pareto distributed claim sizes and inter-occurence times is presented in Section 4.

2 Additional assumptions and main result

To formulate our main result we need to introduce some additional notations and assumptions.

Let $Q(u) = -\log \overline{B}(u)$, $u \in \mathbb{R}_+$ be the hazard function of distribution B. We assume also that there exists a non-negative function $q : \mathbb{R}_+ \to \mathbb{R}_+$ such that $Q(u) = \int_0^u q(v)dv$, $u \in \mathbb{R}_+$. The function q is called the hazard rate of d.f. B. Denote by

$$r := \limsup_{u\to\infty} uq(u)/Q(u) \tag{2.1}$$

a hazard ratio index.

The next two assumptions A and B are essential for our purposes.

Assumption A The distribution B is subexponential and satisfies the following conditions:

$$r < 1/2; \quad \liminf_{u\to\infty} uq(u) \geq \begin{cases} 2 & \text{if } r = 0, \\ 4/(1-r) & \text{if } r \neq 0. \end{cases}$$

Assumption B For any positive $\delta > 0$ there exists a positive $\epsilon > 0$ such that

$$\sum_{k>(1+\delta)\lambda t} \mathrm{P}(N(t) \geq k)(1+\epsilon)^k \to 0 \quad \text{as} \quad t \to \infty. \tag{2.2}$$

Remark 2.1 Condition (2.2) is satisfied for the Poisson model. Indeed, if $N(t)$ is the homogeneous Poisson process with intensity λ, then

$$\begin{aligned}
\sum_{k>(1+\delta)\lambda t} \mathrm{P}(N(t) \geq k)(1+\epsilon)^k &\leq \sum_{k>(1+\delta)\lambda t} \frac{(\lambda(1+\epsilon)t)^k}{k!}\Big(1 + \frac{1}{1+\delta} + \frac{1}{(1+\delta)^2} + \dots\Big) \\
&\leq \frac{1+\delta}{\delta}\frac{(\lambda(1+\epsilon)t)^{[(1+\delta)\lambda t]+1}}{([(1+\delta)\lambda t]+1)!}\Big(1 + \frac{1+\epsilon}{1+\delta} + \Big(\frac{1+\epsilon}{1+\delta}\Big)^2 + \dots\Big) \\
&\ll \frac{(1+\delta)^2}{\delta(\delta-\epsilon)} \exp\Big\{\Big(\delta + (1+\delta)\ln\frac{1+\epsilon}{1+\delta}\Big)\lambda t\Big\} \frac{1}{\sqrt{\lambda t(1+\delta)}},
\end{aligned}$$

if $\epsilon < \delta$. In general case, the verification of Assumption B is more complicated. As conjectured by Klüppelberg and Mikosch (1997) (see Lemma 2.3 therein) Assumption B can be satisfied if the following stochastic ordering relation holds:

$$\mathrm{P}(\theta_1 \leq u) \leq \mathrm{P}(E_1 \leq u), \quad u \in \mathbb{R}_+,$$

for some exponential random variable E_1, implying

$$\mathrm{P}(N(t) \geq k) = \mathrm{P}(\theta_1 + \ldots + \theta_k \leq t) \leq \mathrm{P}(E_1 + \ldots + E_k \leq t) = \mathrm{P}(\tilde{N}(t) \geq k), \qquad (2.3)$$

where $\tilde{N}(t)$, $t \geq 0$ is a homogenous Poisson process. However, the last inequality is not sufficient for Assumption B to hold. For example, take i.i.d. Pareto(2) random variables $\theta_1, \theta_2, \ldots$, i.e.

$$P(\theta_1 > u) = \Big(\frac{1}{1+u}\Big)^2, \quad u \geq 0.$$

Obviously,

$$P(\theta_1 \leq u) = 1 - \mathrm{e}^{-2\log(1+u)} \leq 1 - \mathrm{e}^{-2u} = P(E_1 \leq u), \quad u \geq 0,$$

where $E_1 \sim \mathrm{Exp}(2)$. Then inequality (2.3) reads as

$$\mathrm{P}(N(t) \geq k) \leq \mathrm{P}(\tilde{N}(t) \geq k) = \mathrm{e}^{-2t}\Big(\frac{(2t)^k}{k!} + \frac{(2t)^{k+1}}{(k+1)!} + \ldots\Big). \qquad (2.4)$$

Since $\lambda = (E\theta_1)^{-1} = 1$, for fixed $0 < \delta \leq 1/2$ and every $\epsilon > 0$ we have

$$\begin{aligned}\sum_{k>(1+\delta)t} \mathrm{P}(\tilde{N}(t) \geq k)(1+\epsilon)^k &= e^{-2t} \sum_{k>(1+\delta)t} \Big(\frac{(2t)^k}{k!} + \frac{(2t)^{k+1}}{(k+1)!} + \ldots\Big)(1+\epsilon)^k \\ &> \mathrm{e}^{-2t}\frac{(2t)^{[2t]}}{[2t]!}(1+\epsilon)^{[2t]} \sim \mathrm{e}^{-2t}\frac{(2t)^{[2t]}}{[2t]^{[2t]}\mathrm{e}^{-[2t]}\sqrt{[2t]}}\,(1+\epsilon)^{[2t]} \to \infty\end{aligned}$$

by Stirling formula, so that (2.2) does not hold when $N(t)$ is replaced by $\tilde{N}(t)$. This difficulty was also noted in Tang et al. (2001, p. 92). The reason for this discrepancy lies in the "shifted" set of summation indices. In this example, instead of summation with respect to $k > (1+\delta)2t$ (because $\mathrm{E}\tilde{N}(t) = 2t$) as in (2.2), the set of summation indices is $k > (1+\delta)t$. More precise verification of (2.2) in the Pareto(β) ($\beta > 2$) case is given in Section 4.

The main result of our paper is the following theorem.

Theorem 2.1 *Let assumptions* H_1, H_2, A *and* B *be satisfied. Then, for every positive* $\mu > 0$, *it holds*

$$\mathrm{P}(S_{N(t)} > x + (a+\mu)\lambda t) \sim \lambda t\overline{B}(x + \mu\lambda t)$$

as $x \to \infty$ *uniformly for all* $t \in [f(x), \gamma x/Q(x)]$, *where* $f(x)$ *is an arbitrary infinitely increasing function and* $\gamma > 0$ *is arbitrary positive constant. (Note that* $x/Q(x)$ *is nondecreasing function for sufficiently large* x, *see Lemma 3.3* (i).*)*

The proof of the theorem is given in Section 3. In case $\mu = 0$ we have:

Corollary 2.1 *Under assumptions* H_1, H_2, A *and* B *it holds*

$$\mathrm{P}(S_{N(t)} > x + a\lambda t) \sim \lambda t\overline{B}(x),$$

as $x \to \infty$ *uniformly for all* $t \in [f_1(x), f_2(x)]$, *where* $f_1(x)$ *is an arbitrary infinitely increasing function and* $f_2(x) = o(x/Q(x))$.

PROOF. It follows from Theorem 2.1 and Lemma 3.4 that with some fixed positive μ

$$\begin{aligned}
P(S_{N(t)} > x + a\lambda t) &\geq P(S_{N(t)} > x + (a+\mu)\lambda t) \\
&\geq (1 - o(1))\lambda t \overline{B}(x) \; \frac{\overline{B}(x + \mu\lambda t)}{\overline{B}(x)} \\
&= (1 - o(1))\lambda t \overline{B}(x) \; \exp\left\{ - \int_x^{x+\mu\lambda t} q(u)\mathrm{d}u \right\} \\
&\geq (1 - o(1))\lambda t \overline{B}(x) \left(1 - \int\limits_x^{x+\mu\lambda t} \frac{uq(u)}{Q(u)} \frac{Q(u)}{u} \; \mathrm{d}u \right) \\
&\geq (1 - o(1))\lambda t \overline{B}(x) \left(1 - O\left(\frac{tQ(x)}{x} \right) \right) \\
&\geq (1 - o(1))\lambda t \overline{B}(x)
\end{aligned}$$

as $x \to \infty$ and $t \in [f_1(x), f_2(x)]$.

On the other hand, for such fixed positive μ, $P\left(S_{N(t)} > x + a\lambda t\right) \leq P\left(S_{N(t)} > x^* + (a+\mu)\lambda t\right)$, where $x^* = x - \mu\lambda f_2(x) = x - o(1)x/Q(x) = x(1 - o(1))$. Hence, similarly we can obtain that for sufficiently large x

$$\begin{aligned}
P(S_{N(t)} > x + a\lambda t) &\leq (1 + o(1))\lambda t \overline{B}(x^* + \mu\lambda t) \\
&\leq (1 + o(1))\lambda t \overline{B}(x) \frac{\overline{B}(x^*)}{\overline{B}(x)} \frac{\overline{B}(x^* + \mu\lambda t)}{\overline{B}(x^*)} \\
&\leq (1 + o(1))\lambda t \overline{B}(x) \exp\left\{ \frac{Q(x^*)}{x^*}(x - x^* + \mu\lambda t) \right\} \\
&\leq (1 + o(1))\lambda t \overline{B}(x)
\end{aligned}$$

uniformly for t from interval $[f_1(x), f_2(x)]$. □

3 Proof of main theorem

The statement of Theorem 2.1 follows from Theorem 3.1, Lemma 3.3 below and the estimate

$$\frac{\overline{B}(x + \mu\lambda t(1-\Delta))}{\overline{B}(x + \mu\lambda t)} = \exp\left\{ \int_{x+\mu\lambda t(1-\Delta)}^{x+\mu\lambda t} \frac{q(u)u}{Q(u)} \frac{Q(u)}{u} \; \mathrm{d}u \right\} \leq \exp\left\{ \frac{\Delta\mu\lambda t Q(x)}{x} \right\}$$

provided $0 < \Delta < 1$ and x is sufficiently large.

Theorem 3.1 *Under assumptions* H_1, H_2, A *and* B, *for every positive* $\mu > 0$, Δ_1, Δ_2 ($\Delta_2 < 1$) *and any domain* $D \subset [f_1(x), \infty)$, *where* $f_1(x)$ *is an infinitely increasing function, it holds*

$$\limsup_{x\to\infty} \sup_{t\in D} \frac{P(S_{N(t)} > x + (a+\mu)\lambda t)}{\lambda t \overline{B}(x + \mu\lambda t)} \leq 1 + \Delta_1 \limsup_{x\to\infty} \sup_{t\in D} \frac{\overline{B}(x + \mu\lambda t(1-\Delta_2))}{\overline{B}(x + \mu\lambda t)}$$

and

$$\liminf_{x\to\infty} \inf_{t\in D} \frac{P(S_{N(t)} > x + (a+\mu)\lambda t)}{\lambda t \overline{B}(x + \mu\lambda t)} \geq 1 - \Delta_1 \liminf_{x\to\infty} \inf_{t\in D} \frac{\overline{B}(x + \mu\lambda t(1-\Delta_2))}{\overline{B}(x + \mu\lambda t)}.$$

The statement of this theorem follows from lemmas 3.1 and 3.4 below.

Lemma 3.1 *Let assumptions* H_1, H_2, A *and* B *be satisfied and let* $\epsilon(t)$ *be a monotonically vanishing function satisfying* $\lim_{t\to\infty} t\epsilon^2(t) = \infty$. *Then for every* $\mu > 0$ *and* $0 < \Delta < 1$

$$\sum_{|k-\lambda t|>\epsilon(t)\lambda t} \mathrm{P}(N(t) = k)\mathrm{P}(S_k > x + \lambda(\mu + a)t) = o(1)t\overline{B}(x + \mu\lambda t(1 - \Delta))$$

as $x \to \infty$ *uniformly for* $t \in [f_3(x), \infty)$, *where* $f_3(x)$ *is an infinitely increasing function.*

Proof. Write

$$\begin{aligned}
&\sum_{|k-\lambda t|>\varepsilon(t)\lambda t} \mathrm{P}(N(t) = k)\mathrm{P}(S_k > x + \lambda(\mu + a)t) \\
&= \Big(\sum_{k\in\mathcal{D}_1} + \sum_{k\in\mathcal{D}_2} + \sum_{k\in\mathcal{D}_3}\Big)\mathrm{P}(N(t) = k)\mathrm{P}(S_k > x + \lambda(\mu + a)t) =: I_1 + I_2 + I_3, \qquad (3.1)
\end{aligned}$$

where $\mathcal{D}_1 = [0, (1 - \epsilon(t))\lambda t)$, $\mathcal{D}_2 = ((1 + \epsilon(t))\lambda t, (1 + \delta)\lambda t]$, $\mathcal{D}_3 = ((1 + \delta)\lambda t, \infty)$.

First we estimate the therm I_1. For this we use the next large deviation result (see Theorem 4.1 of Baltrūnas et al. (2004)) and auxiliary Lemma 3.3.

Lemma 3.2 *Assume that random variables* $Z_1, Z_2, \ldots$ *satisfy assumptions* H_1 *and* A. *Then*

$$\mathrm{P}(S_n - \mathrm{E}S_n > \tau) \sim n\overline{B}(\tau)$$

as $n \to \infty$ *uniformly for* $\tau \geq \tau_n$, *where* τ_n *is any infinitely increasing sequence satisfying*

$$\lim_{n\to\infty} \sqrt{n} \sup_{\tau\geq\tau_n} \frac{Q(\tau)}{\tau} = 0.$$

Lemma 3.3 *Suppose that the hazard function* $Q(u)$ *satisfies condition* $r < 1$. *Then:*
(i) $Q(u)/u$ *does not increase for sufficiently large* u;
(ii) *for every* $\epsilon > 0$ *there exists positive* u_ϵ *and* c_ϵ, *such that*

$$Q(u) \leq c_\epsilon u^{r+\epsilon} \quad \textit{for } u > u_\epsilon. \qquad (3.2)$$

Proof of Lemma 3.3. (i) The proof easily follows from the proof of Lemma 2.1 in Baltrūnas (2005).

(ii) We have

$$(\log Q(u))' = \frac{q(u)}{Q(u)} = \frac{uq(u)}{Q(u)}\frac{1}{u} \leq (r + \epsilon)\frac{1}{u}$$

for $u > u_\epsilon$. Hence,

$$\begin{aligned}
\log Q(u) - \log Q(u_\epsilon) &= \int_{u_\epsilon}^{u} (\log Q(v))' \mathrm{d}v \\
&\leq (r + \epsilon)\int_{u_\epsilon}^{u} \frac{1}{v}\, \mathrm{d}v = (r + \epsilon)\log\frac{u}{u_\epsilon},
\end{aligned}$$

implying $Q(u) \leq \frac{Q(u_\epsilon)}{u_\epsilon^{r+\epsilon}}\ u^{r+\epsilon} = c_\epsilon u^{r+\epsilon}$ for large u. $\square$

Since $Q(u)/u$ does not increase for sufficiently large u, it follows from lemmas 3.2–3.3 and Assumption A that for $t = t(x) \in [f_2(x), \infty)$,

$$\begin{aligned} I_1 &\leq \mathrm{P}(S_{[(1-\varepsilon(t))\lambda t]} > x + (\mu + a)\lambda t) \sum_{k \in \mathcal{D}_1} \mathrm{P}(N(t) = k) \\ &\sim (1 - \epsilon(t))\lambda t \overline{B}(x + \mu\lambda t + \lambda t a \epsilon(t)) \sum_{k \in \mathcal{D}_1} \mathrm{P}(N(t) = k) \\ &\leq \lambda t \overline{B}(x + \mu\lambda t)\ \mathrm{P}(N(t) < (1 - \epsilon(t))\lambda t) \\ &= \lambda t \overline{B}(x + \mu\lambda t)\ \mathrm{P}\left\{ \frac{N(t) - \lambda t}{\sqrt{\lambda^3 t \mathrm{Var}\ \theta_1}} < -\sqrt{\frac{t\epsilon^2(t)}{\lambda \mathrm{Var}\ \theta_1}} \right\} \\ &= o(1) t \overline{B}(x + \mu\lambda t) \end{aligned} \tag{3.3}$$

by assumption $t\epsilon^2(t) \to \infty$ and the central limit theorem for the renewal process (see, e.g., Theorem 2.5.13 in Embrechts et al. (1997)):

$$\frac{N(t) - \lambda t}{\sqrt{\lambda^3 t \mathrm{Var}\ \theta_1}} \xrightarrow{d} \mathrm{N}(0, 1). \tag{3.4}$$

Consider the term I_2. We can apply the large deviation Lemma 3.2 once again for term I_2 to obtain, by the dominated convergence theorem, that for $t \in [f_3(x), \infty)$

$$\begin{aligned} I_2 &= \sum_{k \in \mathcal{D}_2} P(N(t) = k) \mathrm{P}(S_k - ES_k > x - ka + (\mu + a)\lambda t) \\ &\sim \sum_{k \in \mathcal{D}_2} \mathrm{P}(N(t) = k) k \overline{B}(x - (k - \lambda t)a + \mu\lambda t) \\ &\leq \overline{B}(x - \delta a \lambda t + \mu\lambda t) \sum_{k \in \mathcal{D}_2} \mathrm{P}(N(t) = k) k. \end{aligned}$$

Let the positive $\delta > 0$ be such that $\delta a < \Delta\mu$. Similarly as in the estimate of term I_1 we have

$$\begin{aligned} I_2 &\leq (1 + o(1))(1 + \delta)\lambda t \overline{B}(x + \mu\lambda t(1 - \Delta)) \mathrm{P}(N(t) > (1 + \epsilon(t))\lambda t) \\ &\leq (1 + o(1))(1 + \delta) \mathrm{P}\left\{ \frac{N(t) - \lambda t}{\sqrt{\lambda^3 t \mathrm{Var}\ \theta}} > \sqrt{\frac{t\epsilon^2(t)}{\lambda\ \mathrm{Var}\ \theta}} \right\} \lambda t \overline{B}(x + \mu\lambda t(1 - \Delta)) \\ &= o(1) t \overline{B}(x + \mu\lambda t(1 - \Delta)). \end{aligned} \tag{3.5}$$

Using the property of subexponential distribution B (see, e.g., Lemma 1.3.5 in Embrechts et al. (1997)) we have that for each $\epsilon > 0$ there exists a constant $K(\epsilon)$ such that

$$\mathrm{P}(S_n > x) \leq K(\epsilon)(1 + \epsilon)^n \overline{B}(x), \quad x \geq 0.$$

Therefore, applying Assumption B and taking into account the equivalence of (2.2) to (see Remark (ii) on p. 296 in Klüppelberg and Mikosch (1997))

$$\sum_{k > (1+\delta)\lambda t} \mathrm{P}(N(t) = k)(1 + \epsilon)^k \to 0,$$

we obtain

$$\begin{aligned} I_3 &\leq K(\epsilon) \sum_{k \in \mathcal{D}_3} \mathrm{P}(N(t)=k)(1+\epsilon)^k \overline{B}(x+\lambda(\mu+a)t) \\ &\leq K(\epsilon)\overline{B}(x+\mu\lambda t) \sum_{k \in \mathcal{D}_3} \mathrm{P}(N(t)=k)(1+\epsilon)^k \\ &= o(1)\overline{B}(x+\mu\lambda t). \end{aligned} \tag{3.6}$$

The proof of the lemma follows from (3.1) and the estimates (3.3), (3.5), (3.6). □

Lemma 3.4 *Assume that assumptions* H_1, H_2 *and* A *are satisfied. Then for every positive* $\mu > 0$

$$\sum_{|k-\lambda t| \leq \epsilon(t)\lambda t} \mathrm{P}(N(t)=k)\mathrm{P}(S_k > x+\lambda(\mu+a)t) \sim \lambda t \overline{B}(x+\mu\lambda t)$$

as $x \to \infty$ *uniformly for* $t > f_4(x)$, *where* $f_4(x)$ *is arbitrary infinitely increasing function,* $\epsilon(t) = c_1 \log t/\sqrt{t}$ *and* $c_1 > 0$ *is any positive constant.*

Proof. Rewrite

$$\begin{aligned} &\sum_{|k-\lambda t| \leq \epsilon(t)\lambda t} \mathrm{P}(N(t)=k)\mathrm{P}(S_k > x+\lambda(\mu+a)t) \\ &= \overline{B}(x+\mu\lambda t) \sum_{|k-\lambda t| \leq \epsilon(t)\lambda t} \mathrm{P}(N(t)=k)\varphi(x,t) \, \frac{\mathrm{P}(S_k > x+\lambda(\mu+a)t)}{\overline{B}(x-(k-\lambda t)a+\mu\lambda t)}, \end{aligned}$$

where $\varphi(x,t) = \frac{\overline{B}(x-(k-\lambda t)a+\mu\lambda t)}{\overline{B}(x+\mu\lambda t)}$.

Using Lemma 3.2 we obtain

$$\begin{aligned} \mathrm{P}(S_k > x+\lambda(\mu+a)t) &= \mathrm{P}(S_k - \mathrm{E}S_k > x-(k-\lambda t)a+\mu\lambda t) \\ &\sim k\overline{B}(x-(k-\lambda t)a+\mu\lambda t) \end{aligned} \tag{3.7}$$

uniformly for $|k-\lambda t| \leq \epsilon(t)\lambda t$.

We will show that

$$\lim_{x\to\infty} \varphi(x,t) = 1 \tag{3.8}$$

uniformly for $t \in [f_4(x), \infty)$.

Let $(k-\lambda t)a \geq 0$. Then, applying the mean value theorem, rewrite

$$\varphi(x,t) = \exp\{Q(x+\mu\lambda t) - Q(x-(k-\lambda t)a+\mu\lambda t)\} = \exp\{(k-\lambda t)aq(\xi)\} \geq 1,$$

where $x+\lambda t(\mu-\varepsilon(t)) \leq x+\mu\lambda t-(k-\lambda t)a \leq \xi \leq x+\mu\lambda t$. By (2.1), we have

$$q(\xi) = \frac{\xi q(\xi)}{Q(\xi)} \frac{Q(\xi)}{\xi} \leq (r+\epsilon)\frac{Q(\xi)}{\xi}$$

for every $\epsilon > 0$ and sufficiently large x. Therefore

$$\varphi(x,t) \leq \exp\left\{(r+\epsilon)(k-\lambda t)a\frac{Q(\xi)}{\xi}\right\} \tag{3.9}$$

for sufficiently large x. The function $Q(u)/u$ does not increase for sufficiently large u. Hence, by inequality $(k-\lambda t)a \le \lambda t\epsilon(t) = \lambda c_1\sqrt{t}\log t$, we obtain from (3.9) that

$$\varphi(x,t) \quad \le \quad \exp\left\{c_1(r+\epsilon)\sqrt{t}\log t\,\frac{Q(x+\lambda t(\mu-\epsilon(t)))}{x+\lambda t(\mu-\epsilon(t))}\right\} \le \exp\left\{c_2\log t\,\frac{Q(\lambda\mu t/2)}{\sqrt{t}}\right\}$$

for sufficiently large x. Since $r < 1/2$, we obtain for large t

$$\frac{Q(\lambda\mu t/2)\log t}{\sqrt{t}} \le c_\epsilon(\lambda\mu t/2)^{r+\epsilon}\frac{\log t}{\sqrt{t}} = c_\epsilon^*\,\frac{\log t}{t^{1/2-r-\epsilon}} \to 0$$

by (3.2), choosing $\epsilon > 0$ such that $r+\epsilon < 1/2$. If $(k-\lambda t)a < 0$, the proof of (3.8) is similar.

Hence, by the dominated convergence theorem, we have from (3.7)–(3.8) that

$$\begin{aligned}\sum_{|k-\lambda t|\le\epsilon(t)\lambda t} & \mathrm{P}(N(t)=k)\mathrm{P}(S_k > x+\lambda(\mu+a)t) \\ & \sim \quad \lambda t\overline{B}(x+\mu\lambda t)\mathrm{P}(|N(t)-\lambda t| \le \epsilon(t)\lambda t) \sim \lambda t\overline{B}(x+\mu\lambda t)\end{aligned}$$

according to relation (3.4). The proof of the lemma is complete. □

4 Example of Pareto distribution

In this section we will show that assumptions H_1, H_2, A and B are satisfied in case of claim sizes and inter-occurence times having heavy-tailed Pareto distribution. Assume that i.i.d. claim sizes $Z_1, Z_2, \ldots$ and inter-occurence times $\theta_1, \theta_2, \ldots$ have the Pareto(α) and Pareto(β) distributions, respectively:

$$\overline{B}(u) = \mathrm{P}(Z_1 > u) = \Big(\frac{1}{1+u}\Big)^{\alpha}, \quad \mathrm{P}(\theta_1 > u) = \Big(\frac{1}{1+u}\Big)^{\beta}, \quad u \ge 0$$

with $\alpha > 2$ and $\beta > 2$. Obviously, assumptions H_1 and H_2 are satisfied and $a = \mathrm{E}Z_1 = (\alpha-1)^{-1}$, $\mathrm{E}Z_1^2 = 2(\alpha-1)^{-1}(\alpha-2)^{-1}$, $\lambda = (\mathrm{E}\theta_1)^{-1} = \beta-1$, $E\theta_1^2 = 2(\beta-1)^{-1}(\beta-2)^{-1}$.

It is well-known that Pareto distribution is subexponential. Moreover, in our case $Q(u) = \alpha\log(1+u)$, $q(u) = \alpha(1+u)^{-1}$, implying

$$r = \limsup_{u\to\infty}\frac{uq(u)}{Q(u)} = 0 < \frac{1}{2} \ \text{ and } \liminf_{u\to\infty} uq(u) = \alpha > 2.$$

Hence, Assumption A is satisfied.

The most complicated is the verification of Assumption B. First we will estimate $\mathrm{P}(N(t) \ge k) = \mathrm{P}(\theta_1 + \ldots + \theta_k \le t)$ showing that for $k > (1+\delta)(\beta-1)t$ and every $\delta > 0$ the following inequality holds:

$$\mathrm{P}(\theta_1 + \ldots + \theta_k \le t) \le C_{\sigma_*,\delta,\beta}\exp\Big\{-\frac{\sigma_*\delta k}{(1+\delta)(2+\delta)(\beta-1)}\Big\}, \tag{4.1}$$

where σ_* is specified in (4.3) below.

The Laplace–Stieltjes transform of d.f. $P(\theta_1 \leq u)$ is

$$\hat{\theta}(s) = \int_0^\infty \mathrm{e}^{-us} \mathrm{dP}(\theta_1 \leq u) = \beta \int_0^\infty \frac{\mathrm{e}^{-us}}{(1+u)^{\beta+1}}\, \mathrm{d}u.$$

Obviously, $\hat{\theta}(s)$ is analytic function for $\mathrm{Re}\, s > 0$. The properties of the Laplace-Stieltjes transform imply

$$\hat{\theta}^k(s) = \int_0^\infty \mathrm{e}^{-us} \mathrm{dP}(\theta_1 + \ldots + \theta_k \leq u) \quad \forall s : \mathrm{Re}\, s > 0,$$

so that the inverse formula gives

$$\mathrm{P}(\theta_1 + \ldots + \theta_k \leq t) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} \frac{\mathrm{e}^{ts} - 1}{s}\, \hat{\theta}^k(s) \mathrm{d}s \quad \text{for any} \quad \sigma > 0. \tag{4.2}$$

For any real $s > 0$ we have

$$\begin{aligned} \hat{\theta}(s) &= \int_0^\infty \mathrm{e}^{-us} \mathrm{dP}(\theta_1 \leq u) = 1 - s\mathrm{E}\theta_1 + O(s^2 \mathrm{E}\theta_1^2) \\ &= 1 - \frac{s}{\beta - 1} + O(s^2) = \exp\Big\{ -\frac{s}{\beta-1} + O(s^2) \Big\}. \end{aligned}$$

Therefore, for any real positive s

$$\hat{\theta}(s) \leq \exp\Big\{ -\frac{s}{\beta-1} + \frac{c_\beta s^2}{\beta - 1} \Big\} = \exp\Big\{ -\frac{2s}{(2+\delta)(\beta-1)} \Big\} \exp\Big\{ \frac{s}{\beta-1} \Big(\frac{2}{2+\delta} - 1 + c_\beta s \Big) \Big\}.$$

Since $\frac{2}{2+\delta} - 1 + c_\beta s \to -\frac{\delta}{2+\delta} < 0$, $s \to 0$, there exists a small number $\sigma_* > 0$, such that

$$\hat{\theta}(\sigma_*) \leq \exp\Big\{ -\frac{2\sigma_*}{(2+\delta)(\beta-1)} \Big\}. \tag{4.3}$$

From (4.2), for such σ_* we obtain

$$\begin{aligned} \mathrm{P}(\theta_1 + \ldots + \theta_k \leq t) &\leq \frac{1}{2\pi} \int_{-\infty}^\infty \frac{|\mathrm{e}^{t(\sigma_* + \mathrm{i}y)} - 1|}{\sqrt{\sigma_*^2 + y^2}}\, |\hat{\theta}(\sigma_* + \mathrm{i}y)|^k \mathrm{d}y \\ &\leq \frac{1}{2\pi} \int_{-\infty}^\infty \frac{|\mathrm{e}^{t(\sigma_* + \mathrm{i}y)} - 1|}{\sqrt{\sigma_*^2 + y^2}}\, \hat{\theta}^{k-1}(\sigma_*) |\hat{\theta}(\sigma_* + \mathrm{i}y)| \mathrm{d}y \\ &\leq \frac{1}{2\pi}\, (\mathrm{e}^{t\sigma_*} + 1) \hat{\theta}^{k-1}(\sigma_*) \int_{-\infty}^\infty \frac{|\hat{\theta}(\sigma_* + \mathrm{i}y)|}{\sqrt{\sigma_*^2 + y^2}}\, \mathrm{d}y \\ &\leq \frac{1}{\pi}\, \mathrm{e}^{t\sigma_* - \frac{2\sigma_*(k-1)}{(\delta+2)(\beta-1)}} \int_{-\infty}^\infty \frac{|\hat{\theta}(\sigma_* + \mathrm{i}y)|}{\sqrt{\sigma_*^2 + y^2}}\, \mathrm{d}y. \end{aligned} \tag{4.4}$$

To estimate the integral in (4.4), note that equality (here Re $s > 0$)

$$\hat{\theta}(s) = \frac{\beta}{s} - \frac{\beta(\beta+1)}{s} \int_0^\infty \frac{\mathrm{e}^{-us}}{(1+u)^{\beta+2}}\, \mathrm{d}u$$

implies $|\hat{\theta}(\sigma_* + \mathrm{i}y)| \leq \frac{C_\beta}{|\sigma_* + \mathrm{i}y|} = \frac{C_\beta}{\sqrt{\sigma_*^2 + y^2}}$ and therefore

$$\int_{-\infty}^\infty \frac{|\hat{\theta}(\sigma_* + \mathrm{i}y)|}{\sqrt{\sigma_*^2 + y^2}}\, \mathrm{d}y \leq C_\beta \int_{-\infty}^\infty \frac{\mathrm{d}y}{\sigma_*^2 + y^2} = \frac{C_\beta \pi}{\sigma_*}. \tag{4.5}$$

For $k > (1+\delta)(\beta-1)t$, from (4.4) and (4.5) we obtain

$$\begin{aligned} \mathrm{P}(\theta_1 + \ldots + \theta_k \le t) &\le \frac{C_\beta}{\pi} \exp\Big\{ \frac{\sigma_* k}{(1+\delta)(\beta-1)} - \frac{2\sigma_*(k-1)}{(2+\delta)(\beta-1)} \Big\} \\ &= \frac{C_\beta}{\pi} \exp\Big\{ \frac{\sigma_* k}{\beta-1} \Big(\frac{1}{1+\delta} - \frac{2}{2+\delta} \Big) \Big\} \exp\Big\{ \frac{2\sigma_*}{(2+\delta)(\beta-1)} \Big\} \\ &= C_{\sigma_*,\delta,\beta} \exp\Big\{ -\frac{\sigma_* \delta k}{(1+\delta)(2+\delta)(\beta-1)} \Big\}, \end{aligned}$$

which is needed inequality (4.1).

Now we have

$$\sum_{k>(1+\delta)(\beta-1)t} \mathrm{P}(N(t) \ge k)(1+\epsilon)^k \le C_{\sigma_*,\delta,\beta} \sum_{k>(1+\delta)(\beta-1)t} (1+\epsilon)^k \exp\Big\{ -k \frac{\sigma_* \delta}{(1+\delta)(2+\delta)(\beta-1)} \Big\} \to 0$$

as $t \to \infty$, since

$$\frac{\sigma_* \delta}{(1+\delta)(2+\delta)(\beta-1)} - \log(1+\epsilon) \to \frac{\sigma_* \delta}{(1+\delta)(2+\delta)(\beta-1)} > 0, \quad \text{when } \epsilon \downarrow 0.$$

Therefore, Assumption B is satisfied.

To conclude the section, we recall that the main theorem of the paper says that, for every positive μ,

$$\mathrm{P}\left(S_{N(t)} > x + \left(\frac{\beta-1}{\alpha-1} + \mu \right) t \right) \sim \frac{(\beta-1)t}{\left(1 + x + \frac{\mu}{\alpha-1} t\right)^\alpha}, \quad x \to \infty,$$

uniformly for $t \in [f(x), \gamma x / \ln x]$ with arbitrary infinitely increasing function $f(x)$ and arbitrary positive constant γ.

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