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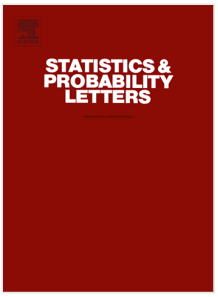


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PII: S0167-7152(07)00392-6
DOI: 10.1016/j.spl.2007.11.013
Reference: STAPRO 4824

To appear in: *Statistics and Probability Letters*

Received date: 8 February 2007
Revised date: 4 September 2007
Accepted date: 6 November 2007

Please cite this article as: Ho, L.P., Stoyan, D., Modelling marked point patterns by intensity-marked cox processes. *Statistics and Probability Letters* (2007), doi:10.1016/j.spl.2007.11.013

# Modelling Marked Point Patterns by Intensity-marked Cox Processes

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**Abstract**

This paper introduces two models of marked Cox point processes where the marks are constructed by means of the intensity function in order to obtain correlations between local point density and marks. Explicit expressions for various functional second-order characteristics are derived.

*Key words:* Marked point processes; Cox process.

*PACS:*

[1] This research was supported by grants from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project Nos. HKBU2048/02P and HKBU200503) and an FRG grant of the Hong Kong Baptist University.

## 1 Introduction

Marked point processes provide a very useful tool in spatial statistics. They represent a natural approach to the analysis of data where at random positions random variables are observed. A sucessful model for marked point processes is the random field model introduced by Mase (1996). It has two completely independent components, a point process and a random field $\{Z(\mathbf{x})\}$, and the mark of the point at location $\mathbf{x}$ is simply $Z(\mathbf{x})$. But this model does not assume any correlation between point density and marks which would e.g. mean that in regions of high point density the marks are systematically large. This may be a result of interaction between the points, in the biological context of competition.

The present paper introduces two simple models with a close relationship between point density and marks. They are density-dependent marked Cox processes. For both models explicit expressions for the second-order characteristics are given.

## 2 Summary characteristics for marked point processes

The fundamentals of the theory of marked point processes can be found in Daley & Vere-Jones (2004) and Stoyan et al. (1995). Here only the facts needed in the present paper are given. A marked point process is a random sequence

$\Psi = \{[\mathbf{x}_n; m_n]\}$ with points $\mathbf{x}_n \in \mathbb{R}^d$ and marks $m_n$ in some mark space; here only real-valued marks are considered. Throughout the paper it is assumed that $\Psi$ is stationary and isotropic.

The character $\Psi$ also denotes the random measure, i.e. for any Borel set $A$ in $\mathbb{R}^d$ and any Borel set $L$ in $\mathbb{R}$, $\Psi(A \times L)$ denotes the number of points in $A$ with mark in $L$. The corresponding mean satisfies

$$E\{\Psi(A \times L)\} = \lambda \nu(A) \mathcal{M}(L)\,, \tag{1}$$

where $\lambda$ is the intensity of the process, $\nu$ denotes the volume or Lebesgue measure and $\mathcal{M}$ is the mark distribution. In the given case of real-valued marks, $\mathcal{M}$ is described by the mark distribution function $F_{\mathcal{M}}(m)$, $F_{\mathcal{M}}(m) = \mathcal{M}((-\infty, m])$. The mean corresponding to $\mathcal{M}$ or $F_{\mathcal{M}}(m)$ is denoted by $\mu$ and called the mean mark.

Various second-order summary characteristics describe the variability and correlations of $\Psi$, see Schlather (2001), Schlather et al. (2004), Stoyan (1984) and Stoyan & Stoyan (1994). The first is the pair correlation function $g(r)$, which describes the variability of point distribution, ignoring the marks. The further characteristics $k_{mm}(r), k_{m\cdot}(r)$ and here $\gamma(r)$ describe correlations between the marks. The function $k_{mm}(r)$ is called the mark correlation function and can be heuristically explained as

$$k_{mm}(r) = \frac{E_{or}\{m(o)m(\mathbf{r})\}}{\mu^2} \quad \text{for } r > 0\,, \tag{2}$$

where $E_{or}$ denotes the conditional mean under the condition that $\Psi$ has points in two deterministic points of distance $r$, which can be assumed to be the origin $o$ and any point $\mathbf{r}$ of distance $r$ from $o$, and $m(o)$ and $m(\mathbf{r})$ are the corresponding marks. Thus $k_{mm}(r)$ is the normalized mean of the product of the marks of two points of distance $r$. A more rigorous definition can be found in Stoyan & Stoyan (1994, pp. 262-263). As discussed in Schlather (2001), the name "mark correlation function" is a bit misleading as there are other second-order characteristics which are closer to the idea of a correlation function. Furthermore, the function $k_{m\cdot}(r)$ is defined as

$$k_{m\cdot}(r) = \frac{E_{or}\{m(o)\}}{\mu} \quad \text{for } r > 0\,. \tag{3}$$

It gives the mean of the mark of a point which has another point in distance $r$. If there are correlations between points and marks in $\Psi$, the existence of another point in distance $r$ may have influence on the mean mark of a given point. Schlather et al. (2004) showed that for Mase's random field model $k_{m\cdot}(r) \equiv 1$ and thus this function offers an excellent tool for testing the goodness-of-fit of that model.

Finally, the mark variogram $\gamma(r)$ is given by

$$\gamma(r) = \frac{1}{2} E_{or}\{m(o) - m(\mathbf{r})\}^2 \quad \text{for } r > 0\,. \tag{4}$$

In the case of the random field model, $\gamma(r)$ equals the variogram of the random

field $\{Z(x)\}$.

The above definitions are valid for positive $r$ only. Schlather (2001) extended the definition to include the case $r = 0$ such that

$$k_{mm}(0) = \int_{-\infty}^{\infty} m^2 \mathrm{d}F_{\mathcal{M}}(m)/\mu^2\,,\ k_{m\cdot}(0) = 1 \quad \text{and} \quad \gamma(0) = 0\,.$$

## 3 The models

*Log Gaussian Cox process.* The starting point of model building is the log Gaussian Cox process (LGCP), see Møller et al. (1998) and Møller & Waagepetersen (2004). This is a special Cox process where the leading measure has as density function a positive random field $\{\Lambda(\mathbf{x})\}$. Each point process realization is a realization of an inhomogeneous Poisson process with intensity function $\{\lambda(\mathbf{x})\}$, where $\{\lambda(\mathbf{x})\}$ is a realization of $\{\Lambda(\mathbf{x})\}$. The name 'log Gaussian' results from the assumption that $\Lambda(\mathbf{x}) = \exp\{S(\mathbf{x})\}$, where the 'background field' $\{S(\mathbf{x})\}$ is a Gaussian field with mean $\mu_S$, variance $\sigma_S^2$ and correlation function $\rho_S(r)$.

It is known that the intensity and the pair correlation function of the LGCP are

$$\lambda = \exp\left(\mu_S + \frac{\sigma_S^2}{2}\right) \quad \text{and} \quad g(r) = \exp\{\sigma_S^2 \rho_S(r)\} \quad \text{for } r \geq 0\,.$$

The first formula results from $\lambda = E\{\Lambda(o)\}$ and the second from $\lambda^2 g(r) =$

$E\{\Lambda(o)\Lambda(\mathbf{r})\}$, where $\mathbf{r}$ is as in (2).

*Intensity-marked Cox process.* The log Gaussian assumption makes the correlation functions of the following marked point process mathematically tractable. Consider the intensity-marked Cox process (IMCP) with the points $\mathbf{x}_n$ of the LGCP above and marks $m(\mathbf{x}_n)$ given as

$$m(\mathbf{x}_n) = a + b\Lambda(\mathbf{x}_n) + \varepsilon(\mathbf{x}_n), \tag{5}$$

where $\Lambda(\mathbf{x}_n)$ is the value of the intensity measure $\{\Lambda(\mathbf{x})\}$ at $\mathbf{x}_n$ and $\varepsilon(\mathbf{x}_n)$ is a random error with Gaussian distribution of mean 0 and variance $\tau^2$. The sequence $\{\varepsilon(\mathbf{x}_n)\}$ consists of i.i.d. variables and is also independent of $\{\Lambda(\mathbf{x})\}$, while $a$ and $b$ are model constants. The case $b = 0$ means independent Gaussian marks, and $b > 0$ models the case that the marks are large in regions of high point density; conversely, $b < 0$ yields small marks.

*Geostatistical model for preferential sampling.* This model (GMPF), closely related to IMCP, was developed independently by Menezes (2005) and is of particular value for geostatistics, in cases where the sampling points depend on the observed random field. Again the points of this model form an LGCP, but $\Lambda(\mathbf{x})$ has the form

$$\Lambda(\mathbf{x}) = \exp\{\alpha + \beta S(\mathbf{x})\}, \tag{6}$$

with real parameters $\alpha$ and $\beta$. The case $\beta = 0$ corresponds to a Poisson process of intensity $e^{\alpha}$. This LGCP has

$$\lambda = \exp\left(\alpha + \beta\mu_S + \frac{\beta^2\sigma_S^2}{2}\right) \quad \text{and} \quad g(r) = \exp\{\beta^2\sigma_S^2\rho_S(r)\} \quad \text{for} \quad r \geq 0\,.$$

For GMPF, the marks $m(\mathbf{x}_n)$ are simply the values of the background field perturbed by i.i.d. Gaussian errors:

$$m(\mathbf{x}_n) = S(\mathbf{x}_n) + \varepsilon(\mathbf{x}_n)\,, \tag{7}$$

which corresponds to the case of geostatistical analysis of the background random field $\{S(\mathbf{x})\}$. The definition of the marks is the main difference between both models, since for Gaussian $S(\mathbf{x})$ also $\alpha + \beta S(\mathbf{x}_n)$ is Gaussian.

## 4 Formulas for the two models

For both models, IMCP as well as GMPF, formulas can be given for all second-order summary characteristics above. They are presented here together with short proofs for some of them. Details can be found in Ho (2006).

• *IMCP* First and second moments of marks:

$$\mu = a + b\lambda\exp(\sigma_S^2) \tag{8}$$

and

$$\mu_2 = a^2 + b^2\lambda^2\exp(3\sigma_S^2) + \tau^2 + 2ab\lambda\exp(\sigma_S^2). \tag{9}$$

Formula (8) shows clearly the effect of intensity marking. While the mean of the intensity field $\{\Lambda(\mathbf{x})\}$ is $\lambda$ and a formal calculation of mean mark by (5)

would yield $a + b\lambda$, the true mean mark is $\mu = a + b\lambda \exp(\sigma_S^2)$, which is for $b > 0$ larger and for $b < 0$ smaller than $a + b\lambda$. This bias is quite natural since in regions of high point density the marks are large for $b > 0$ and small for $b < 0$.

*Proof of (8) and (9):* Consider first the particular case of $a = 0$, $b = 1$ and $\tau = 0$, with $m(\mathbf{x}_n) = \Lambda(\mathbf{x}_n)$. Denote the lognormal distribution function of $\Lambda(o)$ by $F(m)$, whose parameters are $\mu_S$ and $\sigma_S^2$. Then the corresponding mark distribution function $F_{\mathcal{M}}(m)$ is obtained by the following calculation:

$$\begin{aligned} E\{\Psi(A \times (-\infty, m])\} &= E\left\{E \sum_{\mathbf{x}\in\Phi} 1_A(\mathbf{x}) 1_{(-\infty,m]}(\Lambda(\mathbf{x}))|\Lambda\right\} \\ &= E\left\{\int 1_A(\mathbf{x}) 1_{(-\infty,m]}(\Lambda(\mathbf{x}))\Lambda(\mathbf{x})\mathrm{d}\mathbf{x}\right\} \\ &= \int 1_A(\mathbf{x}) E\{1_{(-\infty,m]}(\Lambda(\mathbf{x}))\Lambda(\mathbf{x})\}\mathrm{d}\mathbf{x}\,, \end{aligned}$$

where $\Phi$ is the non-marked point process obtained by stripping off the marks of $\Psi$, $A$ is a Borel set with $\nu(A) > 0$. Because of stationarity the mean in the last integral is independent of $x$, which leads to

$$\begin{aligned} \int 1_A(\mathbf{x}) E\{1_{(-\infty,m]}(\Lambda(o))\Lambda(o)\}\mathrm{d}\mathbf{x} &= \nu(A) E\{1_{(-\infty,m]}(\Lambda(o))\Lambda(o)\} \\ &= \nu(A) \int_{-\infty}^{m} x\mathrm{d}F(x)\,. \end{aligned}$$

Formula (1) yields $F_{\mathcal{M}}(m) = \frac{1}{\lambda}\int_{-\infty}^{m} x\mathrm{d}F(x)$, and the corresponding mean is

$$\mu = \frac{E\{\Lambda(o)\}^2}{E\Lambda(o)} \;=\; \lambda \exp\left(\sigma_S^2\right)\,.$$

More generally, if $m(\mathbf{x}_n) = a + b\Lambda(\mathbf{x}_n) + \varepsilon(\mathbf{x}_n)$, the corresponding mean is $\mu = E(a + bX + \varepsilon)$, where $X$ has the distribution function $F_{\mathcal{M}}(m)$ above, and $\varepsilon$ is an independent zero-mean error. Hence, (8) follows.

For calculating $\mu_2$, we use (5) to obtain $\mu_2 = E(a + bX + \varepsilon)^2$ with $\varepsilon \sim N(0, \tau^2)$ and $X$ with distribution function $F_{\mathcal{M}}(m)$, both random variables being independent.

• Mark correlation functions:

$$k_{mm}(r) = \begin{cases} \dfrac{a^2 + 2ab\lambda \exp\{\sigma_S^2 + \sigma_S^2\rho_S(r)\} + b^2\lambda^2 \exp\{2\sigma_S^2 + 3\sigma_S^2\rho_S(r)\}}{\{a + b\lambda \exp(\sigma_S^2)\}^2}, & r > 0, \\ \dfrac{a^2 + 2ab\lambda \exp(\sigma_S^2) + b^2\lambda^2 \exp(3\sigma_S^2) + \tau^2}{\{a + b\lambda \exp(\sigma_S^2)\}^2}, & r = 0. \end{cases} \tag{10}$$

*Proof:* The numerator of $k_{mm}(r)$ as given by (2) satisfies

$$E_{or}\{m(o)m(\mathbf{r})\} = \frac{E\{m(o)m(\mathbf{r})\Lambda(o)\Lambda(\mathbf{r})\}}{E(\Lambda(o)\Lambda(\mathbf{r}))}, \tag{11}$$

where

$$\begin{aligned} E\{m(o)m(\mathbf{r})\Lambda(o)\Lambda(\mathbf{r})\} &= E\Big([a + b\exp\{S(o)\} + \varepsilon(o)][a + b\exp\{S(\mathbf{r})\} \\ &\quad + \varepsilon(\mathbf{r})]\exp\{S(o) + S(\mathbf{r})\}\Big) \\ &= a^2 E\Big[\exp\{S(o) + S(\mathbf{r})\}\Big] \\ &\quad + b^2 E\Big[\exp\{2S(o) + 2S(\mathbf{r})\}\Big] \\ &\quad + abE\Big[\exp\{2S(o) + S(\mathbf{r})\}\Big] \\ &\quad + abE\Big[\exp\{S(o) + 2S(\mathbf{r})\}\Big] \\ &= a^2\lambda^2 \exp\{\sigma_S^2\rho_S(r)\} \\ &\quad + b^2\lambda^4 \exp\{2\sigma_S^2 + 4\sigma_S^2\rho_S(r)\} \end{aligned}$$

$$+ 2ab\lambda^3 \exp\{\sigma_S^2 + 2\sigma_S^2\rho_S(r)\}.$$

Hence, (11) becomes

$$\begin{aligned} E_{or}\{m(o)m(\mathbf{r})\} = a^2 &+ 2ab\lambda \exp\{\sigma_S^2 + \sigma_S^2\rho_S(r)\} \\ &+ b^2\lambda^2 \exp\{2\sigma_S^2 + 3\sigma_S^2\rho_S(r)\}. \end{aligned}$$

On the other hand, it is known that $k_{mm}(0) = \frac{\mu_2}{\mu^2}$. Now, (10) can be obtained immediately by formulas (8) and (9).

• The function $k_{m.}(r)$:

$$k_{m.}(r) = \begin{cases} \dfrac{a + b\lambda \exp\{\sigma_S^2 + \sigma_S^2\rho_S(r)\}}{a + b\lambda \exp(\sigma_S^2)}, & r > 0, \\ 1, & r = 0. \end{cases} \tag{12}$$

*Proof:* The numerator of $k_{m.}$ as given in (3) satisfies

$$E_{or}\{m(o)\} = \frac{E\{m(o)\Lambda(o)\Lambda(\mathbf{r})\}}{E(\Lambda(o)\Lambda(\mathbf{r}))},$$

where

$$\begin{aligned} E\{m(o)\Lambda(o)\Lambda(\mathbf{r})\} &= E\Big([a + b\exp\{S(o)\} + \varepsilon(o)]\exp\{S(o) + S(\mathbf{r})\}\Big) \\ &= aE\Big[\exp\{S(o) + S(\mathbf{r})\}\Big] + abE\Big[\exp\{2S(o)\}\Big] \\ &= a\lambda^2 \exp\{\sigma_S^2\rho_S(r)\} + b\lambda^3 \exp\{\sigma_S^2 + 2\sigma_S^2\rho_S(r)\}. \end{aligned}$$

Hence, (12) follows.

• Mark variogram function:

$$\gamma(r) = \begin{cases} b^2\lambda^2 \exp\{2\sigma_S^2 + 2\sigma_S^2\rho_S(r)\}[\exp(\sigma_S^2) \\ \qquad - \exp\{\sigma_S^2\rho_S(r)\}] + \tau^2, \;\; r > 0 \\ 0, \qquad\qquad\qquad\qquad\qquad\qquad r = 0. \end{cases} \tag{13}$$

*Proof:* By definition of $\gamma(r)$ in (4), $E_{or}\{m(o) - m(\mathbf{r})\}^2$ is needed, which can be obtained via

$$E_{or}\{m(o) - m(\mathbf{r})\}^2 = \frac{E[\{m(o)^2 + m(\mathbf{r})^2 - 2m(o)m(\mathbf{r})\}\Lambda(o)\Lambda(\mathbf{r})]}{E\{\Lambda(o)\Lambda(\mathbf{r})\}}, \tag{14}$$

where

$$\begin{aligned} E\{m(\mathbf{r})^2\Lambda(o)\Lambda(\mathbf{r})\} &= E\{m(o)^2\Lambda(o)\Lambda(\mathbf{r})\} \\ &= E\Big([a + b\exp\{S(o)\} + \varepsilon(o)]^2 \exp\{S(o) + S(\mathbf{r})\}\Big) \\ &= a^2\lambda^2 \exp\{\sigma_S^2\rho_S(r)\} + b^2\lambda^4 \exp\{3\sigma_S^2 + 3\sigma_S^2\rho_S(r)\} \\ &\quad + 2ab\lambda^3 \exp\{\sigma_S^2 + 2\sigma_S^2\rho_S(r)\} + \tau^2\lambda^2 \exp\{\sigma_S^2\rho_S(r)\} \end{aligned}$$

This together with the formula above for $E_{or}\{m(o)m(\mathbf{r})\}$ yields (13).

*GMPF* • First and second moments of marks:

$$\mu = \mu_S + \beta\sigma_S^2 \tag{15}$$

and

$$\mu_2 = \sigma_S^2 + (\mu_S + \beta\sigma_S^2)^2 + \tau^2. \tag{16}$$

• Mark correlation functions:

$$k_{mm}(r) = \begin{cases} \dfrac{\sigma_S^2\rho_S(r) + \{\mu_S + \beta\sigma_S^2 + \beta\sigma_S^2\rho_S(r)\}^2}{(\mu_S + \beta\sigma_S^2)^2}, \; r > 0, \\ \\ \dfrac{\sigma_S^2 + (\mu_S + \beta\sigma_S^2)^2 + \tau^2}{(\mu_S + \beta\sigma_S^2)^2}, \qquad r = 0. \end{cases} \tag{17}$$

$$k_{m.}(r) = \begin{cases} \dfrac{\mu_S + \beta\sigma_S^2 + \beta\sigma_S^2\rho_S(r)}{\mu_S + \beta\sigma_S^2)}, \; r > 0, \\ \\ 1 \qquad\qquad\qquad r = 0. \end{cases} \tag{18}$$

• Mark variogram function:

$$\gamma(r) = \begin{cases} \sigma_S^2 + \tau^2 - \sigma_S^2\rho_S(r), \; r > 0, \\ \\ 0, \qquad\qquad r = 0. \end{cases} \tag{19}$$

All proofs of formulas (15) − (18) (which are not given in Menezes, 2005) follow the pattern of the proofs for the IMCP model and use the formula of the moment generating function of a bivariate normal distribution as

$$E\{\exp(t_1X + t_2Y)\} = \exp\{(t_1 + t_2)\mu + \frac{(t_1^2 + t_2^2)\sigma^2}{2} + t_1t_2\sigma^2\rho(r)\},$$

for all real values of $t_1$ and $t_2$, where $(X, Y)$ is a bivariate normal distribution with mean $\mu$, variance $\sigma^2$ and correlation function $\rho(\cdot)$. In order to help to unterstand the idea, here (15) is proved. It is

$$\lambda\mu = E\{S(o)\Lambda(o)\} = E\Big\{S(o)\exp\{\alpha + \beta S(o)\}\Big\}.$$

The right-hand mean is obtained as the derivative with respect to $\beta$ of $E\Big\{\exp\{\alpha + \beta S(o)\}\Big\}$, which is by means of the moment generation function $S(o)$ equal to

$$\Big(\mu_S + \beta\sigma_S^2\Big)\exp\Big\{\alpha + \beta\mu_S + \frac{\beta}{2}\sigma_S^2\Big\}$$

Since the exponential term is the intensity $\lambda$, (15) is obtained.

As formula (19) shows, the mark variogram coincides (up to $\tau^2$) with the variogram of the background field $\{S(x)\}$. In contrast, for IMCP the mark variogram differs from the field variogram. The formulas show that for both models all correlation functions are controlled by the correlation function $\rho_S(r)$ of the background $\{S(\mathbf{x})\}$. This makes that all correlation functions have the same range of correlation.

The paper Ho (2006) shows that the IMCP model can be fitted to a sample of forestry data with clusters of trees with small stem disameter marks.

**Acknowledgements**

The authors thank Prof. P. Diggle and Dr. R. Menezes for discussion and for sending us her PhD thesis and Dr. S.N. Chiu for helpful discussions. This research was partially supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. HKBU200605) and a Faculty Research Grant of the Hong Kong Baptist University.

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