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► **To cite this version:**

Laurent Najman. Euler Method for Mutational Equations. Journal of Mathematical Analysis and Applications, 1995, 196 (1), pp.814-822. 10.1006/jmaa.1995.1445 . hal-00622109

**HAL Id: hal-00622109**

**<https://hal.science/hal-00622109>**

Submitted on 11 Sep 2011

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## Euler Method for Mutational Equations

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*Submitted by Hélène Frankowska*

Received September 19, 1994

We adapt the Euler theorem for approximating solutions to differential equations to the new mathematical framework of mutational equations, which aims to compute derivatives of shape deformation in metric spaces. © 1995 Academic Press, Inc.

### 1. INTRODUCTION

Mutational equations [1, 2] are an extension of differential calculus for maps from one metric space to another. Basic theorems of differential equations (such as the Cauchy–Lipschitz and Peano equations) can be adapted to this mathematical framework.

Mutational equations seem to be a relevant tool for describing the dynamics of objects that are not naturally imbedded in a linear space, in particular for describing dynamics of sets. This is important in many areas of applied mathematics, such as *optimal shape* [12, 5], *visual servoing* [6, 4], and *mathematical morphology* [8–11]. We refer to [2] for numerous motivations. Accordingly, the issue of discretization and constructive approximation for mutational equations, which is the aim of this paper, is of particular interest. We adapt the Euler theorem, which allows us to approximate solutions by a sequence of points in metric space. We end this paper by giving an example of an application for computing the evolution of tubes.

### 2. TRANSITIONS ON METRIC SPACES

Definitions given in this section and those that follow are quoted from [2].

Transitions adapt to metric spaces the concept of half-line  $x + hv$  starting from  $x$  in the direction  $v$  by replacing it by “curved” half-lines  $\vartheta(h, x)$ .

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Indeed, the “linear” structure of half-lines in vector spaces is not really needed to build a differential calculus.

DEFINITION 1 [2]. Let  $X$  be a metric space for a distance  $d$ . A map  $\vartheta: [0, 1] \times X \rightarrow X$  satisfying

- (i)  $\vartheta(0, x) = x$
- (ii)  $\|\vartheta(x)\| := \sup_{h \neq k} \frac{d(\vartheta(h, x), \vartheta(k, x))}{|h - k|} < +\infty$
- (iii)  $\|\vartheta\|_\Delta := \sup_{h \in [0, 1], x \neq y} \frac{d(\vartheta(h, x), \vartheta(h, y))}{d(x, y)} < +\infty$
- (iv)  $\lim_{h \rightarrow 0^+} \frac{d(\vartheta(t + h, x), \vartheta(h, \vartheta(h, \vartheta(t, x))))}{h} = 0$

is called a *transition*.

We denote by  $\overline{\Theta}(X)$  the space of all transitions on  $X$ .

We define an equivalence relation  $\sim_x$  between transitions by

$$\vartheta_1 \sim_x \vartheta_2 \quad \text{if and only if} \quad \lim_{h \rightarrow 0^+} \frac{d(\vartheta_1(h, x), \vartheta_2(h, x))}{h} = 0.$$

We say that  $(X, \Theta(X))$  is a (complete) *mutational space* if  $X$  is a (complete) metric space and  $\Theta(X) \subset \overline{\Theta}(X)$  is a nontrivial subspace of transitions, closed in  $\mathcal{C}([0, 1] \times X, X)$  supplied with the pointwise convergence.

One observes that the transitions  $\vartheta(h, \cdot)$  are Lipschitz uniformly with respect to  $h \in [0, 1]$  and that for every  $x \in X$ , the maps  $\vartheta(\cdot, x)$  are Lipschitz.

We shall supply a space  $\Theta(X)$  of transitions with the distances  $d_\infty$  of uniform convergence and Lipschitz semidistance defined, respectively, by

$$d_\infty(\vartheta, \tau) := \sup_{h \in [0, 1], z \in X} d(\vartheta(h, z), \tau(h, z))$$

and

$$d_\Delta(\vartheta, \tau) := \sup_{h \in [0, 1], z \in X} \frac{d(\vartheta(h, z), \tau(h, z))}{h}.$$

### 3. MUTATIONAL EQUATIONS IN METRIC SPACES

Let  $x(\cdot)$  be an application from  $[0, 1]$  onto  $X$ , and  $\vartheta: [0, 1] \times X \rightarrow X$ . We say that  $\vartheta$  is a mutation of  $x(t)$  at time  $t$  if

$$\lim_{h \rightarrow 0^+} \frac{d(\vartheta(h, x(t)), x(t + h))}{h} = 0.$$



In this case, we note

$$\overset{\circ}{x}(t) \ni \vartheta.$$

Let us consider a mutational space  $(X, \Theta(X))$  and a single-valued map  $f: [0, \infty[ \times X \mapsto \Theta(X)$  from  $X$  to its space of transitions. We say that a function  $x(\cdot)$  from  $[0, T]$  to  $X$  is a *solution to the mutational equation*  $\overset{\circ}{x} \ni f(t, x)$  if for all  $t > 0$ ,  $f(t, x(t))$  is a mutation of  $x(t)$  at time  $t$ , i.e., if

$$\forall t \in [0, T], \quad \overset{\circ}{x}(t) \ni f(t, x(t)), \quad (1)$$

or, equivalently, if

$$\forall t \geq 0, \quad \lim_{h \rightarrow 0^+} \frac{d(f(t, x(t))(h, x(t)), x(t+h))}{h} = 0.$$

### 3.1. Examples of Mutational Equations

Let us point out, through two examples, that the uniqueness of the mutation is not ensured, which justifies the notation.

- First, consider the constant tube in  $X = \mathbb{R}^2$

$$K(t) = B,$$

where  $B$  is the Euclidean unit ball.

It is clear that

$$0 \in \overset{\circ}{K}(t), \quad \forall t.$$

But we can also check that the Lipschitz map defined by

$$\varphi(x, y) = (-y, x)$$

satisfies

$$\varphi \in \overset{\circ}{K}(t).$$

- Let  $F$  be a set-valued map from a vector space onto itself. The reachable map  $\vartheta$  associated to the differential inclusion  $x'(t) \in F(x(t))$ ,  $x(0) = x_0$ , defined by  $\vartheta(t, x_0) := \{x(s) \mid x'(s) \in F(x(s)), x(0) = x_0, s \in [0, t]\}$  is a good choice for being a mutation of  $x(\cdot)$  at time  $t = 0$ . Let us notice that in order for it to be mutation, it is not necessary that  $\vartheta$  belong to  $\Theta(X)$ . Let us apply this remark to mathematical morphology.

Let  $X$  be the set of compacts of  $\mathbb{R}^n$  (which is a metric space for the

Hausdorff distance),  $K$  be an element of  $X$ ,  $\vartheta_1(t, K) := \{x(s) \mid x'(s) \in B, x(0) \in K, s \in [0, t]\}$ , where  $B$  is the unit Euclidean ball, and  $\vartheta_2(t, K) := \{x(s) \mid x'(s) \in N_K(x(s)) \cap B, x(0) \in K, s \in [0, t]\}$ , where  $N_K(x)$  is the subnormal cone<sup>1</sup> of the set  $K$  at point  $x$  for the Euclidean norm.

In [7] it is shown that the morphological tube  $K(t) := K \oplus tB = \{k + tb \mid k \in K, b \in B\}$ , which corresponds to the dilation (the Minkowski sum) of a compact  $K$  with respect to the Euclidean unit ball  $B$ , satisfies the two mutational equations

$$\forall t \geq 0, \quad \overset{\circ}{K}(t) \ni \vartheta_1 \tag{2}$$

$$\forall t \geq 0, \quad \overset{\circ}{K}(t) \ni \vartheta_2, \tag{3}$$

with  $K(0) = K$ . Equation (2) is natural, and Eq. (3) clearly establishes, without any regularity assumptions on the compact set  $K$ , the intuitive idea that the dilation transforms the initial set  $K$  in the direction of the normal at any point of the set. Indeed, when the set  $K$  is a regular manifold, the subnormal cone  $N_K(x)$  is reduced to the half-line spanned by the outward normal  $n(x)$ . But if  $\vartheta_1$  is a transition, this is not the case for  $\vartheta_2$ , since the application  $\vartheta_2(t, \cdot)$  is not Lipschitz.

Peano's Theorem, which states the existence of a solution  $x(\cdot)$  to Eq. (1), can be adapted to the case of mutational equations.

### 3.2. Peano's Theorem for Mutational Equations

We give here Peano's theorem for mutational equations.

**THEOREM 2 [2].** *Let  $(X, \Theta(X))$  be a mutational space, and  $f: [0, \infty[ \times X \mapsto \Theta(X)$  be a uniformly continuous map bounded in the sense that*

$$\forall t \geq 0, \forall x \in X, \|f(t, x)\|_\Lambda := \sup_{h \in [0, 1], y \neq z} \frac{d(f(t, x)(h, y), f(t, x)(h, z))}{d(y, z)} \leq c_\Lambda$$

<sup>1</sup> The external circatangent cone of  $K$  at  $x$  is

$$C_K(x) := \{v \mid C_\uparrow d(x, K)(x)(v) \leq 0\},$$

and the external subnormal cone of  $K$  at  $x$  is the negative polar cone of  $C_K(x)$ , i.e.,

$$N_K(x) := C_K(x)^- = \{p \mid \forall v \in C_K(x), \langle p, v \rangle \leq 0\}.$$

See [3] for more details.

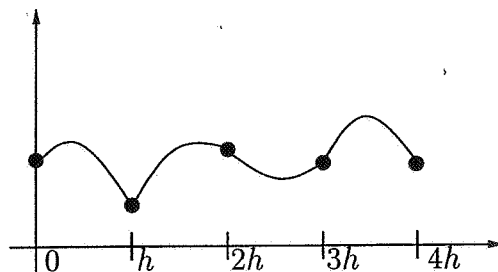


FIG. 1. Illustration of Euler's method for mutational equations.

and that

$$\forall t \geq 0, \forall x \in X, \forall y \in X, \|f(t, x)(y)\| := \sup_{k \neq h} \frac{d(f(t, x)(h, y), f(t, x)(k, y))}{|h - k|} \leq c.$$

Assume that the closed bounded balls of  $X$  are compact.

Then, from any initial state  $x_0 \in X$  starts one solution to the mutational equation  $\dot{x} \ni f(t, x)$ .

#### 4. EULER METHOD FOR MUTATIONAL EQUATIONS

For the sake of simplicity, we consider a mutational equation with non-explicit time dependence, but the following can be easily adapted to the explicit time-dependent case. We thus consider the mutational equation

$$\dot{x}(t) \ni f(x(t)). \quad (4)$$

A natural way for approximating solutions to differential mutations is the difference scheme

$$x_{j+1} := f(x_j)(h, x_j),$$

where  $h \in ]0, 1]$  is fixed.

We can interpolate the  $x_j$ 's on the nodes  $jh$  by setting (Fig. 1)

$$x_h(t) := f(x_j)(t - hj, x_j), \quad \forall t \in [jh, (j + 1)h].$$

We shall prove that the function  $x_h(\cdot)$  converges in some sense to a solution to the mutational equation (4).

THEOREM 3. We consider a mutational space  $(X, \Theta(X))$  and a single-valued map  $f: X \mapsto \Theta(X)$ . We posit the assumptions of Theorem 2.

For  $h \in ]0, 1]$ , we set

$$x_{j+1} := f(x_j)(h, x_j). \quad (5)$$

Let  $x_h(\cdot)$  be a piecewise defined function which interpolates the  $x_j$ 's on the nodes  $jh$  and on a finite time interval  $[0, T]$ :

$$x_h(t) := f(x_j)(t - hj, x_j), \quad \forall t \in [jh, (j+1)h[, t \leq T. \quad (6)$$

Then, starting from  $x_0 \in X$ , the solution to the explicit difference scheme (5) converges to a solution to the mutational equation

$$\dot{x}(t) \ni f(x(t)) \quad (7)$$

where  $h \rightarrow 0$ , in the sense that a subsequence of function  $x_h$  converges uniformly to a solution  $x(\cdot)$  to (7) with  $x(0) = x_0$ , on  $[0, T]$ .

We set

$$d_\infty(\tau, \hat{x}(t)) := \inf_{\sigma \in \hat{x}(t)} d_\infty(\tau, \sigma)$$

We need the following lemma:

LEMMA 4. We posit the assumptions of Theorem 3. Then, for all  $\varepsilon > 0$ , there exists  $H > 0$  such that for  $h < H$ , we have, for all  $t > 0$  and all  $s > 0$ ,

$$\begin{aligned} \text{(i)} \quad & d_\infty(f(x_h(t)), \hat{x}_h(t)) \leq \varepsilon \\ \text{(ii)} \quad & d(x_h(t), x_h(t+s)) \leq cs. \end{aligned} \quad (8)$$

*Proof.* We first note that

$$\hat{x}_h(t) \ni f(x_j), \quad \forall t \in [jh, (j+1)h[$$

because we have, for  $\varepsilon > 0$  and  $s$  small,

$$\begin{aligned} d(f(x_j)(s, x_h(t)), x_h(t+s)) &= d(f(x_j)(s, f(x_j)(t-hj, x_j)), f(x_j)(t-hj+s, x_j)) \\ &\leq \varepsilon \end{aligned}$$

by Definition 1(iv).

Then we have

$$\begin{aligned} d_\infty(f(x_h(t)), \hat{x}_h(t)) &\leq d_\infty(f(x_h(t)), f(x_j)) \\ &:= \sup_{s \in [0,1], z \in X} d(f(x_h(t))(s, z), f(x_j)(s, z)) \end{aligned}$$

and, by uniform continuity of  $f$ , for all  $\varepsilon > 0$ , there exists  $\eta > 0$  such that

$$\forall t \in [jh, (j+1)h[, d_\infty(f(x_h(t)), f(x_j)) \leq \varepsilon$$

as soon as  $d(x_h(t), x_j) \leq \eta$ . But for all  $t \in [jh, (j+1)h[$ ,

$$\begin{aligned} d(x_h(t), x_j) &= d(f(x_j)(t - hj, x_j), f(x_j)(0, x_j)) \\ &\leq c(t - hj) \\ &\leq ch, \end{aligned}$$

which proves inequality (8(i)).

Now, for  $t \in [jh, (j+1)h[$  and  $t + s \in [kh, (k+1)h[$ , we have

$$\begin{aligned} d(x_h(t+s), x_h(t)) &\leq d(f(x_k)(t+s-hk, x_k), f(x_k)(0, x_k)) \\ &\quad + \sum_{j < i < k} d(f(x_i)(h, x_i), f(x_i)(0, x_i)) \\ &\quad + d(f(x_j)(h, x_j), f(x_j)(t-hj, x_j)) \\ &\leq c(t+s-hk) + \sum_{j < i < k} ch + c(h-t+hj) \\ &\leq c(s+h(-k+(k-j-1)+j+1)) \\ &\leq cs, \end{aligned}$$

which proves inequality (8(ii)).

*Proof of Theorem 3.* We continue as in the proof of Theorem 2. For the sake of completeness, we give here the end of the proof.

Since the closed bounded balls of  $X$  are compact, and since the solutions remain in such closed bounded balls of  $X$ , we deduce that  $x_h(t)$  remains in a compact set of  $X$ .

Lemma 4, property (ii), implies that the sequence of continuous functions  $x_h(\cdot)$  is equicontinuous. Therefore Ascoli's theorem implies that a subsequence (again denoted by  $x_h(\cdot)$ ) converges uniformly to  $x(\cdot)$ .

This limit is a solution since for any  $t \in [jh, (j+1)h[$ ,



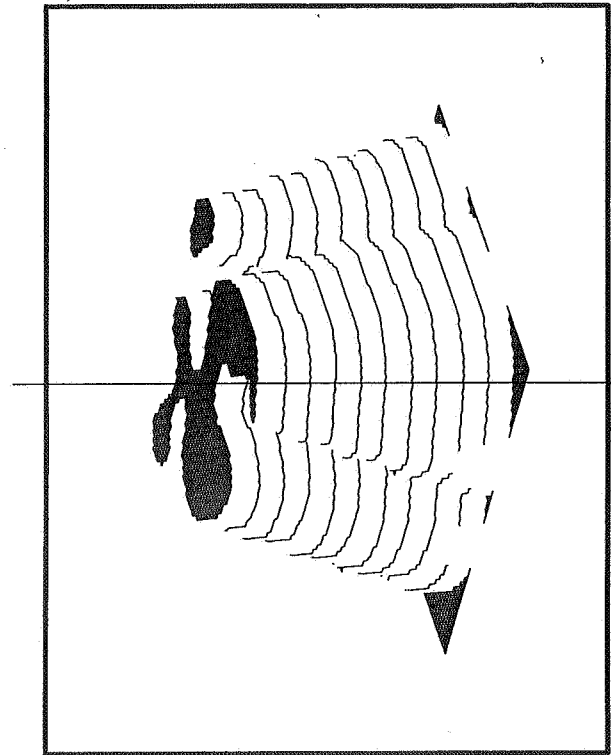


FIG. 2. Trajectory of the dilation by a segment whose orientation relies on time of a compact included in  $\mathbb{R}^2$ . The initial set  $K(0)$  is in grey.

$$d_\infty(f(x(t)), \hat{x}_h(t)) \leq d_\infty(f(x(t)), f(x_j)) + d_\infty(f(x_j), \hat{x}_h(t)).$$

But,  $f(x_j) \ni \hat{x}_h(t)$ , and  $f$  is uniformly continuous. Thus, for all  $\varepsilon > 0$ , there exists  $\eta > 0$  such that

$$d_\infty(f(x(t)), \hat{x}_h(t)) \leq \varepsilon$$

as soon as  $d(x_h(t), x_j) \leq ch \leq \eta$ .

Hence the theorem is proved.

### 5. AN EXAMPLE OF APPLICATION TO TUBES

The main example of application is the evolution of tubes, which are compact-valued maps  $K: \mathbb{R} \rightsquigarrow X = \mathbb{R}^n$ . Let us look at a morphological example, which is the dilation of a compact by a segment of variable angle.



We set  $f(t, K(t))(h, K(t)) = K(t) \oplus h\Theta(t) = \{k + h\theta \mid k \in K(t), \theta \in \Theta(t)\}$ , where  $\Theta(t)$  is a segment of angle  $t$ . Theorem 3 amounts to saying that we can approximate the solution to

$$\dot{K} \ni f(t, K(t)),$$

with  $K(0) = K_0$  by the sequence

$$K_{j+1} = K_j \oplus h\Theta(h_j).$$

Figure 2 shows an example of approximation of the solution to this mutational equation.

#### ACKNOWLEDGMENT

The author thanks Professor Jean-Pierre Aubin for helpful advices and suggestions, as well as the anonymous reviewer whose comments helped clarify the final version of this paper.

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