Characterizing and Detecting Loops in n-Dimensional Discrete Toric Spaces

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Abstract. Toric spaces being non-simply connected, it is possible to find in such spaces some loops which are not homotopic to a point: we call them *toric loops*. Some applications, such as the study of the relationship between the geometrical characteristics of a material and its physical properties, rely on three-dimensional discrete toric spaces and require detecting objects having a toric loop.

In this work, we study objects embedded in discrete toric spaces, and propose a new definition of loops and equivalence of loops. Moreover, we introduce a characteristic of loops that we call *wrapping vector*: relying on this notion, we propose a linear time algorithm which detects whether an object has a toric loop or not.

1 Introduction

Topology is used in various domains of image processing in order to perform geometric analysis of objects. In porous material analysis, different topological transformations, such as skeletonisation, are used to study the relationships between the geometrical characteristics of a material and its physical properties.

When simulating a fluid flow through a porous material, the whole material can be approximated by the tessellation of the space made up by copies of one of its samples, under the condition that the volume of the sample exceeds the so-called Representative Elementary Volume (REV) of the material [1]. When the whole Euclidean space is tiled this way, one can remark that the result of the fluid flow simulation is itself the tessellation of the local flow obtained inside any copy of the sample (see Fig. 1-**a**). When considering the flow obtained inside the sample, it appears that the flow leaving the sample by one side comes back by the opposite side (see Fig. 1-**b**). Thus, it is possible to perform the fluid flow simulation only on the sample, under the condition that its opposite sides are joined: with this construction, the sample is embedded inside a toric space [2] [3]. In order to perform geometric analysis of fluid flow through porous materials, we therefore need topological tools adapted to toric spaces.

Considering the sample inside a toric space leads to new difficulties. In a real fluid flow, grains of a material (pieces of the material which are not connected with the borders of the sample) do not have any effect on the final results, as

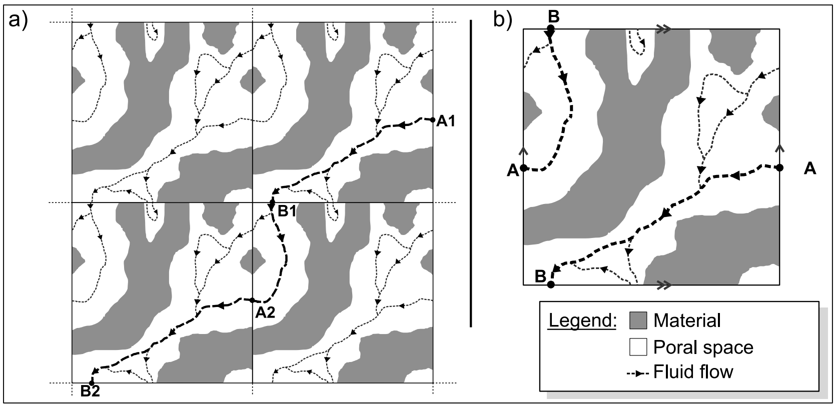

Fig. 1. Simulating a fluid flow - When simulating a fluid flow, a porous material (in gray) can be approximated by the tessellation of one of its samples (see **a**). When the results of the simulation are obtained (the dotted lines), one can see that the fluid flow through the mosaic is the tessellation of the fluid flow simulation results obtained in one sample. For example, one can look at the bold dotted line in **a)**: the flow going from $A1$ to $B1$ is the same than the flow going from $A2$ to $B2$. It is therefore possible to perform the fluid flow simulation through only one sample and, in order to obtain the same results than in **a)**, connect the opposite sides of the sample (see **b**): the sample is embedded inside a toric space.

these grains eventually either evacuate the object with the flow or get blocked and connect with the rest of the material. Thus, before performing a fluid flow simulation on a sample, it is necessary to remove its grains (typically, in a finite subset S of $\mathbb{Z}^n$, a grain is a connected component which does not 'touch' the borders of S). However, characterizing a grain inside a toric space, which does not have any border, is more difficult than in $\mathbb{Z}^n$. On the contrary of the discrete space $\mathbb{Z}^n$, n-dimensional discrete toric spaces are not simply connected spaces [3]: some loops, called *toric loops*, are not homotopic to a point (this can be easily seen when considering a 2D torus). In a toric space, a connected component may be considered as a grain if it contains no toric loop. Indeed, when considering a sample embedded inside a toric space, and a tessellation of the Euclidean space made up by copies of this sample, one can remark that the connected components of the sample which do not contain toric loops produce grains in the tessellation, while the connected components containing toric loops cannot be considered as grains in the tiling (see Fig. 2).

In this work, we give a new definition of loops and homotopy class, adapted to n-dimensional discrete toric spaces. Relying on these notions, we also introduce *wrapping vectors*, a new characteristic of loops in toric spaces which is the same for all homotopic loops. Thanks to wrapping vectors, we give a linear time algorithm which allows us to decide whether an n-dimensional object contains a toric loop or not.

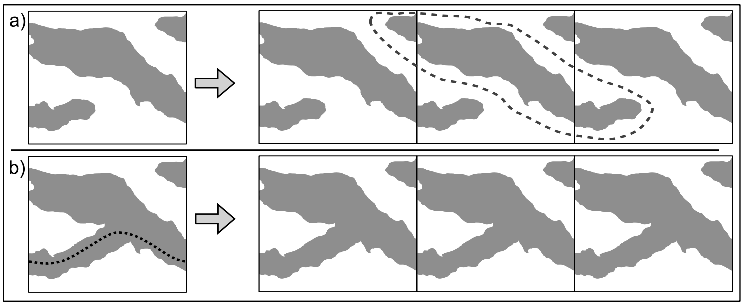

Fig. 2. Grains in toric spaces - The image in **a)** contains no grain based on the 'border criterion'; when the Euclidean space is tessellated with copies of the image, grains appear (the circled connected component is an example of grain). In **b)**, the connected component has toric loops (e.g. the dotted line) and when the Euclidean space is tessellated with copies of the image, no grain appear.

2 Basic Notions

2.1 Discrete Toric Spaces

A n-dimensional torus is classically defined as the direct product of n circles (see [2]). In the following, we give a discrete definition of toric space, based on modular arithmetic (see [4]).

Given d a positive integer. We set $\mathbb{Z}_d = \{0, ..., d-1\}$. We denote by $\oplus_d$ the operation such that for all $a, b \in \mathbb{Z}$, $(a \oplus_d b)$ is the element of $\mathbb{Z}_d$ congruent to $(a+b)$ modulo d. We point out that $(\mathbb{Z}_d, \oplus_d)$ forms a cyclic group of order d.

Let n be a positive integer, $\boldsymbol{d} = (d_1, ..., d_n) \in \mathbb{N}^n$, and $\mathbb{T}^n = \mathbb{Z}_{d_1} \times ... \times \mathbb{Z}_{d_n}$, we denote by $\oplus_{\boldsymbol{d}}$ the operation such that for all $\boldsymbol{a} = (a_1, ..., a_n) \in \mathbb{Z}^n$ and $\boldsymbol{b} = (b_1, ..., b_n) \in \mathbb{Z}^n$, $\boldsymbol{a} \oplus_{\boldsymbol{d}} \boldsymbol{b} = (a_1 \oplus_{d_1} b_1, ..., a_n \oplus_{d_n} b_n)$. The group $(\mathbb{T}^n, \oplus_d)$ is the direct product of the n groups $(\mathbb{Z}_{d_i}, \oplus_{d_i})_{(1 \leq i \leq n)}$, and is an *n-dimensional discrete toric space* [2].

The scalar d_i is *the size of the i-th dimension* of $\mathbb{T}^n$, and $\boldsymbol{d}$ is the *size (vector)* of $\mathbb{T}^n$. For simplicity, the operation $\oplus_{\boldsymbol{d}}$ will be also denoted by $\oplus$.

2.2 Neighbourhoods in Toric Spaces

As in $\mathbb{Z}^n$, various adjacency relations may be defined in a toric space.

Definition 1. *A vector* $\boldsymbol{s} = (s_1, ..., s_n)$ *of* $\mathbb{Z}^n$ *is an* m-step $(0 < m \leq n)$ *if, for all* $i \in [1; n], s_i \in \{-1, 0, 1\}$ *and* $\sum_{i=1}^{n} |s_i| \leq m$.

Two points $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{T}^n$ *are* m-adjacent *if there exists an m-step* $\boldsymbol{s}$ *such that* $\boldsymbol{a} \oplus \boldsymbol{s} = \boldsymbol{b}$.

In 2D, the 1- and 2-adjacency relations respectively correspond to the 4- and 8-neighbourhood adapted to bidimensional toric spaces. In 3D, the 1-, 2- and 3-adjacency relations can be respectively seen as the 6-, 18- and 26-neighbourhood adapted to three-dimensional toric spaces.

Based on the *m-adjacency relation* previously defined, we can introduce the notion of *m-connectedness.*

Definition 2. *A set of points X of $\mathbb{T}^n$ is* m-connected *if, for all $\boldsymbol{a}, \boldsymbol{b} \in X$, there exists a sequence $(\boldsymbol{x_1}, ..., \boldsymbol{x_k})$ of elements of X such that $\boldsymbol{x_1} = \boldsymbol{a}$, $\boldsymbol{x_k} = \boldsymbol{b}$ and for all $i \in [1; k-1]$, $\boldsymbol{x_i}$ and $\boldsymbol{x_{i+1}}$ are m-adjacent.*

2.3 Loops in Toric Spaces

Classically, in $\mathbb{Z}^n$, an m-loop is defined as a sequence of m-adjacent points such that the first point and the last point of the sequence are equal [5]. However, this definition does not suit discrete toric spaces: in small discrete toric spaces, two different loops can be written as the same sequence of points, as shown in the following example.

Example 3. Let us consider the bidimensional toric space $\mathbb{T}^2 = \mathbb{Z}_3 \times \mathbb{Z}_2$, and the 2-adjacency relation on $\mathbb{T}^2$. Let us also consider $\boldsymbol{x_1} = (1, 0)$ and $\boldsymbol{x_2} = (1, 1)$ in $\mathbb{T}^2$.

There are two ways of interpreting the sequence of points $\mathcal{L} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_1})$ as a loop of $\mathbb{T}^2$: either $\mathcal{L}$ is the loop passing by $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ and doing a 'u-turn' to come back to $\boldsymbol{x_1}$, or $\mathcal{L}$ is the loop passing by $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$, and 'wrapping around' the toric space in order to reach $\boldsymbol{x_1}$ without any 'u-turn', as shown on Fig. 3.

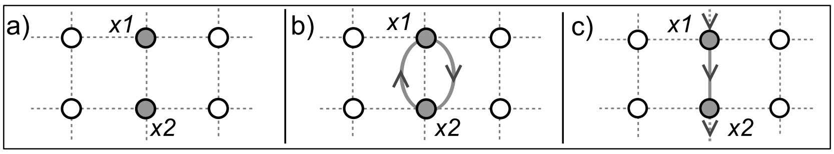

Fig. 3. Loops in toric spaces - In the toric space $\mathbb{Z}_3 \times \mathbb{Z}_2$ (see **a**), the sequence of points $(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_1})$ can be interpreted in two different ways: **b)** and **c)**.

Thus, when considering discrete toric spaces, loops cannot be considered as sequences of points since it can lead to such ambiguities. This is why we propose the following definition.

Definition 4. *Given $\boldsymbol{p} \in \mathbb{T}^n$, an* m-loop (of base point $\boldsymbol{p}$) *is a pair $\mathcal{L} = (\boldsymbol{p}, V)$, where $V = (\boldsymbol{v_1}, ..., \boldsymbol{v_k})$ is a sequence of m-steps such that $(\boldsymbol{p} \oplus \boldsymbol{v_1} \oplus ... \oplus \boldsymbol{v_k}) = \boldsymbol{p}$.*

We call i-th point of $\mathcal{L}$, *with $1 \leq i \leq k$, the point $(\boldsymbol{p} \oplus \boldsymbol{v_1} \oplus ... \oplus \boldsymbol{v_{i-1}})$.*

The loop $(\boldsymbol{p}, ())$ is called the trivial loop of base point $\boldsymbol{p}$.

The ambiguity pinpointed in Ex. 3 is removed with this definition of loops: let $\boldsymbol{v}$ be the vector $(0, 1)$, the loop passing by $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ and making a u-turn is $(\boldsymbol{x_1}, (\boldsymbol{v}, -\boldsymbol{v}))$ (see Fig. 3-**b**), while the loop wrapping around the toric space is $(\boldsymbol{x_1}, (\boldsymbol{v}, \boldsymbol{v}))$ (see Fig. 3-**c**).

3 Loop Homotopy in Toric Spaces

3.1 Homotopic Loops

In this section, we define an equivalence relation between loops, corresponding to an homotopy, inside a discrete toric space. An equivalence relation between loops inside $\mathbb{Z}^2$ and $\mathbb{Z}^3$ has been defined in [5], however, it cannot be adapted to discrete toric spaces (see [6]). Observe that the following definition does not constrain the loops to lie in a subset of the space, on the contrary of the definition given in [5].

Definition 5. *Let $\mathcal{K} = (\boldsymbol{p}, U)$ and $\mathcal{L} = (\boldsymbol{p}, V)$ be two m-loops of base point $\boldsymbol{p} \in \mathbb{T}^n$, with $U = (\boldsymbol{u_1}, ..., \boldsymbol{u_k})$ and $V = (\boldsymbol{v_1}, ..., \boldsymbol{v_l})$. The two m-loops $\mathcal{K}$* and *$\mathcal{L}$* are directly homotopic *if one of the three following conditions is satisfied:*

1. *There exists $j \in [1; l]$ such that $\boldsymbol{v_j} = 0$ and $U = (\boldsymbol{v_1}, ..., \boldsymbol{v_{j-1}}, \boldsymbol{v_{j+1}}, ..., \boldsymbol{v_l})$.*
2. *There exists $j \in [1; k]$ such that $\boldsymbol{u_j} = 0$ and $V = (\boldsymbol{u_1}, ..., \boldsymbol{u_{j-1}}, \boldsymbol{u_{j+1}}, ..., \boldsymbol{u_k})$.*
3. *There exists $j \in [1; k-1]$ such that*
   - *$V = (\boldsymbol{u_1}, ..., \boldsymbol{u_{j-1}}, \boldsymbol{v_j}, \boldsymbol{v_{j+1}}, \boldsymbol{u_{j+2}}, ..., \boldsymbol{u_k})$, and*
   - *$\boldsymbol{u_j} + \boldsymbol{u_{j+1}} = \boldsymbol{v_j} + \boldsymbol{v_{j+1}}$, and*
   - *$(\boldsymbol{u_j} - \boldsymbol{v_j})$ is an n-step.*

Remark 6. The last condition ($(\boldsymbol{u_j} - \boldsymbol{v_j})$ is an n-step) is not necessary for proving the results presented in this paper. However, it is needed when comparing the loop homotopy and the loop equivalence (see [5]), as done in [6].

Definition 7. *Two m-loops $\mathcal{K}$ and $\mathcal{L}$ of base point $\boldsymbol{p} \in \mathbb{T}^n$* are homotopic *if there exists a sequence of m-loops $(\mathcal{C}_1, ..., \mathcal{C}_k)$ such that $\mathcal{C}_1 = \mathcal{K}$, $\mathcal{C}_k = \mathcal{L}$ and for all $j \in [1; k-1], \mathcal{C}_j$ and $\mathcal{C}_{j+1}$ are directly homotopic.*

Example 8. In the toric space $\mathbb{Z}_4 \times \mathbb{Z}_2$, let us consider the point $\boldsymbol{p} = (0, 0)$, the 1-steps $\boldsymbol{v_1} = (1, 0)$ and $\boldsymbol{v_2} = (0, 1)$, and the 1-loops $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ and $\mathcal{L}_d$ (see Fig. 4). The loops $\mathcal{L}_a$ and $\mathcal{L}_b$ are homotopic, the loops $\mathcal{L}_b$ and $\mathcal{L}_c$ are directly homotopic, and the loops $\mathcal{L}_c$ and $\mathcal{L}_d$ are also directly homotopic.

On the other hand, it may be seen that the 1-loops depicted on Fig. 3-**b** and on Fig. 3-**c** are not homotopic.

3.2 Fundamental Group

Initially defined in the continuous space by Henri Poincaré in 1895 [7], the fundamental group is an essential concept of topology, based on the homotopy relation, which has been transposed in different discrete frameworks (see e.g. [5], [8], [9]).

Given two m-loops $\mathcal{K} = (\boldsymbol{p}, (\boldsymbol{u_1}, ..., \boldsymbol{u_k}))$ and $\mathcal{L} = (\boldsymbol{p}, (\boldsymbol{v_1}, ..., \boldsymbol{v_l}))$ of same base point $\boldsymbol{p} \in \mathbb{T}^n$, the *product of $\mathcal{K}$ and $\mathcal{L}$* is the m-loop $\mathcal{K}.\mathcal{L} = (\boldsymbol{p}, (\boldsymbol{u_1}, ..., \boldsymbol{u_k}, \boldsymbol{v_1}, ..., \boldsymbol{v_l}))$. We set $\mathcal{K}^{-1} = (\boldsymbol{p}, (-\boldsymbol{u_k}, ..., -\boldsymbol{u_1}))$.

The symbol $\prod$ will be used for the iteration of the product operation on loops. Given a positive integer w, we set $\mathcal{K}^w = \prod_{i=1}^{w} \mathcal{K}$ and $\mathcal{K}^{-w} = \prod_{i=1}^{w} \mathcal{K}^{-1}$. We also define $\mathcal{K}^0 = (\boldsymbol{p}, ())$.

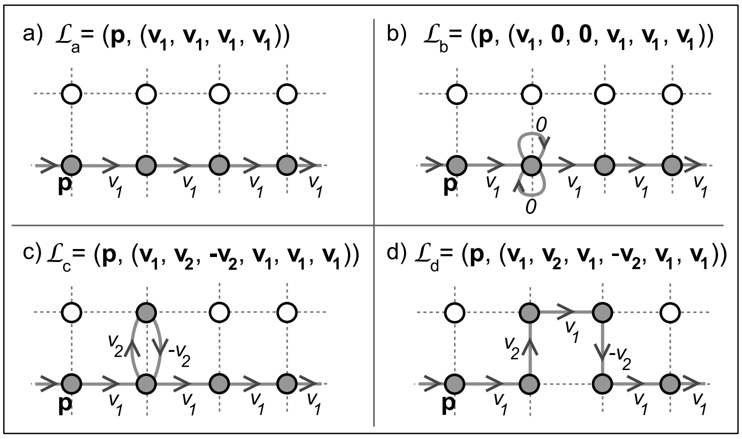

Fig. 4. Homotopic Loops - The 1-loops $\mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_c$ and $\mathcal{L}_d$ are homotopic.

The homotopy of m-loops is an equivalence relation and the equivalence class, called *homotopy class*, of an m-loop $\mathcal{L}$ is denoted by $[\mathcal{L}]$. The product operation can be extended to the homotopy classes of m-loops of same base point: the product of $[\mathcal{K}]$ and $[\mathcal{L}]$ is $[\mathcal{K}].[\mathcal{L}] = [\mathcal{K}.\mathcal{L}]$. We now define the fundamental group of $\mathbb{T}^n$.

Definition 9. *Given an m-adjacency relation on* $\mathbb{T}^n$ *and a point* $\boldsymbol{p} \in \mathbb{T}^n$*, the* m-fundamental group of $\mathbb{T}^n$ with base point $\boldsymbol{p}$ *is the group formed by the homotopy classes of all m-loops of base point* $\boldsymbol{p} \in \mathbb{T}^n$ *under the product operation.*

The identity element of this group is the homotopy class of the trivial loop.

4 Toric Loops in Subsets of $\mathbb{T}^n$

The toric loops, informally evoked in the introduction, can now be formalised using the definitions given in the previous sections.

Definition 10. *In* $\mathbb{T}^n$*, we say that an m-loop is a* toric m-loop *if it does not belong to the homotopy class of a trivial loop. A connected subset of* $\mathbb{T}^n$ *is* wrapped in $\mathbb{T}^n$ *if it contains a toric m-loop.*

Remark 11. The notion of grain introduced informally in Sec. 1 may now be defined: a connected component of $\mathbb{T}^n$ is as *a grain* if it is not wrapped in $\mathbb{T}^n$.

4.1 Algorithm for Detecting Wrapped Subsets of $\mathbb{T}^n$

In order to know whether a connected subset of $\mathbb{T}^n$ is wrapped or not, it is not necessary to build all the m-loops which can be found in the subset: the Wrapped Subset Descriptor (WSD) algorithm (see Alg. 1) answers this question in linear time (more precisely, in $O(N.M)$, where N is the number of points of $\mathbb{T}^n$, and M is the number of distinct m-steps), as stated by the following proposition.

Algorithm 1: WSD(n,m,$\mathbb{T}^n$,d,X)

```
   Data: An n-dimensional toric space T^n of dimension vector d and a non-empty
         m-connected subset X of T^n.
   Result: A boolean telling whether X has a toric m-loop or not.
1  Let p ∈ X; Coord(p) = 0^n; S = {p };
2  foreach x ∈ X do HasCoord(x) = false; HasCoord(p) = true;
3  while there exists x ∈ S do
4      S = S \ {x};
5      foreach non-null n-dimensional m-step v do
6          y = x ⊕_d v;
7          if y ∈ X and HasCoord(y) = true then
8              if Coord(y) ≠ Coord(x) + v then
9                  return true;
10         else if y ∈ X and HasCoord(y) = false then
11             Coord(y) = Coord(x) + v; S = S ∪ {y}; HasCoord(y) = true;
12 return false;
```

Proposition 12. *Let $\mathbb{T}^n$ be an n-dimensional toric space of size vector $\boldsymbol{d}$. A non-empty m-connected subset X of $\mathbb{T}^n$ is wrapped in $\mathbb{T}^n$ if and only if WSD(n,m, $\mathbb{T}^n$,$\boldsymbol{d}$,X) is true.*

Before proving Prop.12 (see Sec. 5.4), new definitions and theorems must be given: in particular, Th. 25 establishes a very important result on homotopic loops in toric spaces. Let us study an example of execution of Alg. 1 on an object.

Example 13. Let us consider a subset X of points of $\mathbb{Z}_4 \times \mathbb{Z}_4$ (see Fig. 5-**a**) and the 2-adjacency relation. In Fig. 5-**a**, one element $\boldsymbol{p}$ of X is given the coordinates of the origin (see l. 1 of Alg. 1); then we set $\boldsymbol{x} = \boldsymbol{p}$. In Fig. 5-**b**, every neighbour $\boldsymbol{y}$ of $\boldsymbol{x}$ which is in X is given coordinates depending on its position relative to $\boldsymbol{x}$ (l. 11) and is added to the set S (l. 11).

Then, in Fig. 5-**c**, one element of S is chosen as $\boldsymbol{x}$ (l. 3). Every neighbour $\boldsymbol{y}$ of $\boldsymbol{x}$ is scanned: if $\boldsymbol{y}$ is in X and has already coordinates (l. 7), it is compared with $\boldsymbol{x}$: as the coordinates of $\boldsymbol{x}$ and $\boldsymbol{y}$ are compatible in $\mathbb{Z}^2$ (the test achieved l. 8 returns false), the algorithm continues. Else (l. 10), $\boldsymbol{y}$ is given coordinates depending on its position relative to $\boldsymbol{x}$ (l. 11) and added to S (see Fig. 5-**d**).

Finally, in Fig. 5-**e**, an element of S is chosen as $\boldsymbol{x}$. The algorithm tests one of the neighbours $\boldsymbol{y}$ of $\boldsymbol{x}$ (the left neighbour) which has already coordinates (l. 7). The coordinates of $\boldsymbol{y}$ and $\boldsymbol{x}$ are incompatible in $\mathbb{Z}^2$ (the points $(-1, 1)$ and $(2, 1)$ are not neighbours in $\mathbb{Z}^2$), the algorithm returns true (l. 9): according to Prop. 12, the subset X is wrapped in $\mathbb{T}^n$.

To summarize, Alg. 1 'tries to embed' the subset X of $\mathbb{T}^n$ in $\mathbb{Z}^n$: if some incompatible coordinates are detected by the test achieved on l. 8 of Alg. 1, then the object has a feature (a toric loop) which is incompatible with $\mathbb{Z}^n$. A toric 2-loop lying in X is depicted in Fig. 5-**f**.

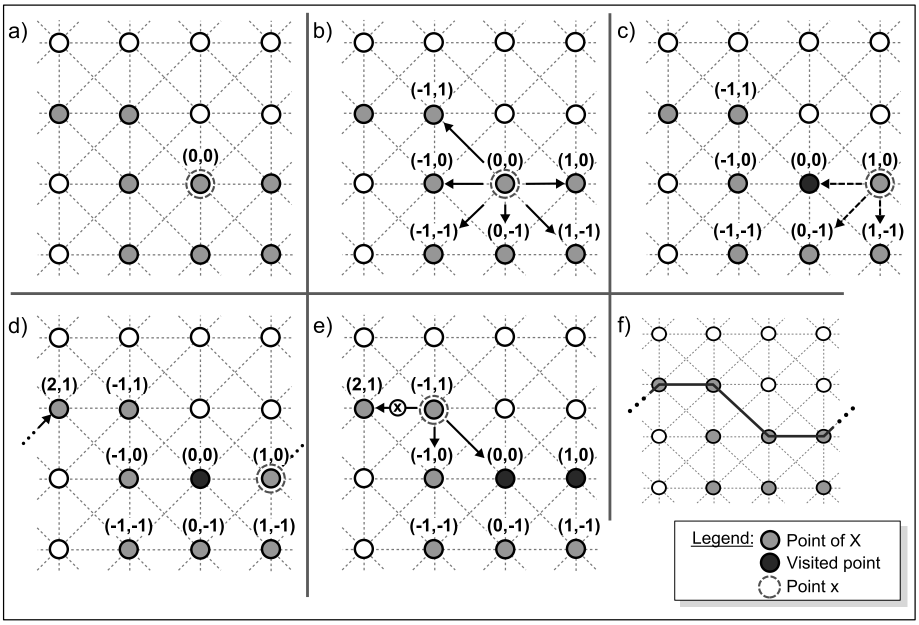

Fig. 5. Example of execution of WSD - see Ex. 13 for a detailed description.

5 Wrapping Vector and Homotopy Classes in $\mathbb{T}^n$

Deciding if two loops $\mathcal{L}_1$ and $\mathcal{L}_2$ belong to the same homotopy class can be difficult if one attempts to do this by building a sequence of directly homotopic loops which 'link' $\mathcal{L}_1$ and $\mathcal{L}_2$. However, this problem may be solved using the *wrapping vector*, a characteristic which can be easily computed on each loop.

5.1 Wrapping Vector of a Loop

The *wrapping vector* of a loop is the sum of all the elements of the m-step sequence associated to the loop.

Definition 14. *Let $\mathcal{L} = (\boldsymbol{p}, V)$ be an m-loop, with $V = (\boldsymbol{v_1}, ..., \boldsymbol{v_k})$. Then the* wrapping vector of $\mathcal{L}$ *is $\boldsymbol{v}_{\mathcal{L}} = \sum_{i=1}^{k} \boldsymbol{v_i}$.*

Remark 15. In Def. 14, the symbol $\sum$ stands for the iteration of the classical addition operation on $\mathbb{Z}^n$, not of the operation $\oplus$ defined in Sec. 2.1.

The notion of 'basic loops' will be used in the proof of Prop. 18 and in Def. 22.

Definition 16. *Let $\mathbb{T}^n$ be an n-dimensional toric space of size vector $\boldsymbol{d} = (d_1, ..., d_n)$. We denote, for each $i \in [1; n]$, by $\boldsymbol{b_i}$ the 1-step whose i-th coordinate is equal to 1, and by B_i the 1-step sequence composed of exactly d_i 1-steps $\boldsymbol{b_i}$.*

Given $\boldsymbol{p} \in \mathbb{T}^n$, for all $i \in [1; n]$, we define the i-th basic loop of base point $\boldsymbol{p}$ *as the 1-loop $(\boldsymbol{p}, B_i)$.*

Remark 17. For all $i \in [1; n]$, the wrapping vector of the i-th basic loop of base point $\boldsymbol{p}$ is equal to $(d_i.\boldsymbol{b_i})$.

The next property establishes that the wrapping vector of any m-loop can only take specific values in $\mathbb{Z}^n$. The proof can be found in Sec. 7.

Proposition 18. *Let $\mathbb{T}^n$ be an n-dimensional toric space of size vector $\boldsymbol{d} = (d_1, ..., d_n)$. A vector $\boldsymbol{w} = (w_1, ..., w_n)$ of $\mathbb{Z}^n$ is the wrapping vector of an m-loop of $\mathbb{T}^n$ if and only if, for all $i \in [1; n]$, w_i is a multiple of d_i.*

Definition 19. *Given $\mathbb{T}^n$ of size vector $\boldsymbol{d} = (d_1, ..., d_n)$, let $\mathcal{L}$ be an m-loop of wrapping vector $\boldsymbol{w} = (w_1, ..., w_n)$. The* normalized wrapping vector *of $\mathcal{L}$ is $\boldsymbol{w^*} = (w_1/d_1, ..., w_n/d_n)$.*

Example 20. The normalized wrapping vector gives information on how a loop 'wraps around' each dimension of a toric space. For example, let $\mathbb{T}^3 = \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_7$: a loop with normalized wrapping vector (2,1,0) wraps two times in the first dimension, one time in the second, and does not wrap in the third dimension.

5.2 Equivalence Between Homotopy Classes and Wrapping Vector

It can be seen that two directly homotopic m-loops have the same wrapping vector, as their associated m-step sequences have the same sum. Therefore, we have the following property.

Proposition 21. *Two homotopic m-loops of $\mathbb{T}^n$ have the same wrapping vector.*

The following definition is necessary in order to understand Prop. 24 and its demonstration, leading to the main theorem of this article.

Definition 22. *Let $\boldsymbol{p}$ be an element of $\mathbb{T}^n$, and $\boldsymbol{w^*} = (w_1^*, ..., w_n^*) \in \mathbb{Z}^n$.*

The canonical loop of base point $\boldsymbol{p}$ and normalized wrapping vector $\boldsymbol{w^*}$ *is the 1-loop $\prod_{i=1}^{n} (\boldsymbol{p}, B_i)^{w_i^*}$, where $(\boldsymbol{p}, B_i)$ is the i-th basic loop of base point $\boldsymbol{p}$.*

Example 23. Consider $(\mathbb{T}^3, \oplus)$, with $\mathbb{T}^3 = \mathbb{Z}_3 \times \mathbb{Z}_1 \times \mathbb{Z}_2$, $\boldsymbol{w} = (3, 1, -4)$ and $\boldsymbol{p} = (0, 0, 0)$. The canonical loop of base point $\boldsymbol{p}$ and wrapping vector $\boldsymbol{w}$ is the 1-loop $(\boldsymbol{p}, V)$ with:

$$\mathrm{V}=\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right)$$

Proposition 24. *Any m-loop of base point $\boldsymbol{p} \in \mathbb{T}^n$ and wrapping vector $\boldsymbol{w} \in \mathbb{Z}^n$ is homotopic to the canonical loop of base point $\boldsymbol{p}$ and wrapping vector $\boldsymbol{w}$.*

The proof of the previous proposition can be found in Sec. 7. We can now state the main theorem of this article, which is a direct consequence of Prop. 21 and Prop. 24.

Theorem 25. *Two m-loops of $\mathbb{T}^n$ of same base point are homotopic if and only if their wrapping vectors are equal.*

Remark 26. According to Th. 25, the homotopy class of the trivial loop $(\boldsymbol{p}, ())$ is the set of all m-loops of base point $\boldsymbol{p}$ that have a null wrapping vector.

5.3 Wrapping Vector and Fundamental Group

Given a point $\boldsymbol{p} \in \mathbb{T}^n$, we set $\Omega = \{\boldsymbol{w^*} \in \mathbb{Z}^n /$ *there exists an m-loop in* $\mathbb{T}^n$ *of base point* $\boldsymbol{p}$ *and of normalized wrapping vector* $\boldsymbol{w^*}\}$. From Prop. 18, it is plain that $\Omega = \mathbb{Z}^n$. Therefore, $(\Omega, +)$ is precisely $(\mathbb{Z}^n, +)$

Theorem 25 states that there exists a bijection between the set of the homotopy classes of all m-loops of base point $\boldsymbol{p}$ and Ω. The product (see Sec. 3.2) of two m-loops $\mathcal{K}$ and $\mathcal{L}$ of same base point $\boldsymbol{p}$ and of respective wrapping vectors $\boldsymbol{w_k}$ and $\boldsymbol{w_l}$ is the loop $(\mathcal{K}.\mathcal{L})$ of base point $\boldsymbol{p}$. The wrapping vector of $(\mathcal{K}.\mathcal{L})$ is $(\boldsymbol{w_k} + \boldsymbol{w_l})$, therefore we can state that there exists an isomorphism between the fundamental group of $\mathbb{T}^n$ and $(\Omega, +)$. Consequently, we retrieve in our discrete framework a well-known property of the fundamental group of toric spaces [2].

Proposition 27. *The fundamental group of* $\mathbb{T}^n$ *is isomorphic to* $(\mathbb{Z}^n, +)$.

5.4 Proof of Alg. 1

Proof. (of Prop. 12) For all $\boldsymbol{y} \in X$ such that $\boldsymbol{y} \neq \boldsymbol{p}$, there exists a point $\boldsymbol{x}$ such that the test performed on l. 10 of Alg. 1 is true: we call $\boldsymbol{x}$ *the label predecessor of* $\boldsymbol{y}$.

• If the algorithm returns false, then the test performed l. 8 of Alg. 1 was never true. Let $\mathcal{L} = (\boldsymbol{p}, V)$ be an m-loop contained in X, with $V = (\boldsymbol{v_1}, ..., \boldsymbol{v_k})$, and let us denote by $\boldsymbol{x_i}$ the i-th point of $\mathcal{L}$. As the test performed l. 8 was always false, we have the following:

$$\begin{cases} \text{for all } i \in [1; k-1], \boldsymbol{v_i} = Coord(\boldsymbol{x_{i+1}}) - Coord(\boldsymbol{x_i}) \\ \boldsymbol{v_k} = Coord(\boldsymbol{x_1}) - Coord(\boldsymbol{x_k}) \end{cases}$$

The wrapping vector of $\mathcal{L}$ is

$$\boldsymbol{w} = \textstyle\sum_{i=1}^{k-1} (Coord(\boldsymbol{x_{i+1}}) - Coord(\boldsymbol{x_i})) + Coord(\boldsymbol{x_1}) - Coord(\boldsymbol{x_k}) = \boldsymbol{0}$$

Thus, if the algorithm returns false, each m-loop of X has a null wrapping vector and, according to Th. 25, belongs to the homotopy class of a trivial loop: there is no toric m-loop in X which is therefore not wrapped in $\mathbb{T}^n$.

• If the algorithm returns true, then, there exists $\boldsymbol{x}, \boldsymbol{y} \in X$ and an m-step $\boldsymbol{a}$, such that $\boldsymbol{x} \oplus \boldsymbol{a} = \boldsymbol{y}$ and $Coord(\boldsymbol{y}) - Coord(\boldsymbol{x}) \neq \boldsymbol{a}$.

It is therefore possible to find two sequences γ_x and γ_y of m-adjacent points in X, with $\gamma_x = (\boldsymbol{p} = \boldsymbol{x_1}, \boldsymbol{x_2}, ..., \boldsymbol{x_q} = \boldsymbol{x})$ and $\gamma_y = (\boldsymbol{y} = \boldsymbol{y_t}, ..., \boldsymbol{y_2}, \boldsymbol{y_1} = \boldsymbol{p})$, such that, for all $i \in [1; q-1], \boldsymbol{x_i}$ is the label predecessor of $\boldsymbol{x_{i+1}}$, and for all $i \in [1; t-1], \boldsymbol{y_i}$ is the label predecessor of $\boldsymbol{y_{i+1}}$. Therefore, we can set

$$\begin{cases} .\ \text{for all } i \in [1; q-1], \boldsymbol{u_i} = Coord(\boldsymbol{x_{i+1}}) - Coord(\boldsymbol{x_i}) \\ \quad \text{is an m-step such that } \boldsymbol{x_i} \oplus \boldsymbol{u_i} = \boldsymbol{x_{i+1}} \\ .\ \text{for all } i \in [1; t-1], \boldsymbol{v_i} = Coord(\boldsymbol{y_i}) - Coord(\boldsymbol{y_{i+1}}) \\ \quad \text{is an m-step such that } \boldsymbol{y_{i+1}} \oplus \boldsymbol{v_i} = \boldsymbol{y_i} \end{cases}$$

Let $\mathcal{N}_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{a}} = (\boldsymbol{p}, V)$ be the m-loop such that $V = (\boldsymbol{u_1}, ..., \boldsymbol{u_{q-1}}, \boldsymbol{a}, \boldsymbol{v_{t-1}}, ..., \boldsymbol{v_1})$. The m-loop $\mathcal{N}_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{a}}$ is lying in X and its wrapping vector $\boldsymbol{w}$ is equal to:

$$\boldsymbol{w} = \textstyle\sum_{i=1}^{q-1} \boldsymbol{u_i} + \boldsymbol{a} + \sum_{i=1}^{t-1} \boldsymbol{v_i} = \boldsymbol{a} - (Coord(\boldsymbol{y}) - Coord(\boldsymbol{x})) \neq \boldsymbol{0}$$

Thus, when the algorithm returns true, it is possible to find, inside X, an m-loop with a non-null wrapping vector: by Th. 25, there is a toric m-loop in X which is therefore wrapped in $\mathbb{T}^n$. $\square$

6 Conclusion

In this article, we give a formal definition of loops and homotopy inside discrete toric spaces in order to define various notions such as loop homotopy and the fundamental group. We then propose a linear time algorithm for detecting toric loops in a subset X of $\mathbb{T}^n$: the proof of the algorithm relies on the notions previously given, such as the wrapping vector which, according to Th. 25, completely characterizes toric loops.

In Sec. 1, we have seen that detecting toric loops is important in order to filter grains from a material's sample and perform a fluid flow simulation on the sample. The WSD algorithm proposed in this article detects which subsets of a sample, embedded inside a toric space, will create grains and should be removed. Future works will include analysis of the relationship between other topological characteristics of materials and their physical properties.

7 Annex

Proof. (of Prop. 18) First, let $\mathcal{L} = (\boldsymbol{p}, V)$ be an m-loop of wrapping vector $\boldsymbol{w}= (w_1, ..., w_n)$, with $\boldsymbol{p} = (p_1, ..., p_n)$. As $\mathcal{L}$ is a loop, for all $i \in [1; n]$, $p_i \oplus_{d_i} w_i = p_i$. Hence, for all $i \in [1; n]$, $w_i \equiv 0(mod\ d_i)$.

Let $\boldsymbol{w} = (w_1, ..., w_n)$ be a vector of $\mathbb{Z}^n$ such that for all $i \in [1; n]$, w_i is a multiple of d_i. If we denote by $(\boldsymbol{p}, B_i)$ the i-th basic loop of base point $\boldsymbol{p}$, we see that $(\prod_{i=1}^{n} (\boldsymbol{p}, B_i)^{w_i/d_i})$ is an m-loop whose wrapping vector is equal to $\boldsymbol{w}$. $\square$

Proof. (of Lem. 24) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-null 1-steps. Let i (resp. j) be the index of the non-null coordinate of $\boldsymbol{a}$ (resp $\boldsymbol{b}$). We say that $\boldsymbol{a}$ *is index-smaller than* $\boldsymbol{b}$ if $i < j$.

Let $\mathcal{L} = (\boldsymbol{p}, V)$ be an m-loop of normalized wrapping vector $\boldsymbol{w^*} \in \mathbb{Z}^n$.

- **1 -** The m-loop $\mathcal{L}$ is homotopic to a 1-loop $\mathcal{L}_1 = (\boldsymbol{p}, V_1)$ (see Lem. 28).
- **2 -** By Def. 5 and 7, the 1-loop $\mathcal{L}_1$ is homotopic to a 1-loop $\mathcal{L}_2 = (\boldsymbol{p}, V_2)$, where V_2 contains no null vector.
- **3 -** Let $\mathcal{L}_3 = (\boldsymbol{p}, V_3)$ be such that V_3 is obtained by iteratively permuting all pairs of consecutive 1-steps $(\boldsymbol{v_j}, \boldsymbol{v_{j+1}})$ in V_2 such that $\boldsymbol{v_{j+1}}$ is index-smaller than $\boldsymbol{v_j}$. Thanks to Lem. 29, $\mathcal{L}_3$ is homotopic to $\mathcal{L}_2$.
- **4 -** Consider $\mathcal{L}_4 = (\boldsymbol{p}, V_4)$, where V_4 is obtained by iteratively replacing all pairs of consecutive 1-steps $(\boldsymbol{v_j}, \boldsymbol{v_{j+1}})$ in V_3 such that $\boldsymbol{v_{j+1}} = (-\boldsymbol{v_j})$ by two null vectors, and then removing these two null vectors. The loop $\mathcal{L}_4$ is homotopic to $\mathcal{L}_3$.

The 1-loop $\mathcal{L}_4$ is homotopic to $\mathcal{L}$, it has therefore the same normalized wrapping vector $\boldsymbol{w^*} = (w_1^*, ..., w_n^*)$ (see Prop. 21). By construction, each pair of consecutive 1-steps $(\boldsymbol{v_j}, \boldsymbol{v_{j+1}})$ of V_4 is such that $\boldsymbol{v_j}$ and $\boldsymbol{v_{j+1}}$ are non-null and either $\boldsymbol{v_j} = \boldsymbol{v_{j+1}}$ or $\boldsymbol{v_j}$ is index-smaller than $\boldsymbol{v_{j+1}}$.

Let $\boldsymbol{d} = (d_1, ..., d_n)$ be the size vector of $\mathbb{T}^n$. As the normalized wrapping vector of $\mathcal{L}_4$ is equal to $\boldsymbol{w^*}$, we deduce that the $(d_1.|w_1^*|)$ first elements of V_4 are equal to $(w_1^*/|w_1^*|.\boldsymbol{b_1})$ (see Def. 16). Moreover, the $(d_2.|w_2^*|)$ next elements are equal to $(w_2^*/|w_2^*|.\boldsymbol{b_2})$, etc. Therefore, we have $\mathcal{L}_4 = (\prod_{i=1}^{n} (\boldsymbol{p}, B_i)^{w_i^*})$. $\square$

Lemma 28. *Any m-loop $\mathcal{L} = (\boldsymbol{p}, V)$ is homotopic to a 1-loop.*

Proof. Let us write $V = (\boldsymbol{v_1}, ..., \boldsymbol{v_k})$ and let $j \in [1; n]$ be such that $\boldsymbol{v_j}$ is not a 1-step. The m-loop $\mathcal{L}$ is directly homotopic to $\mathcal{L}_1 = (\boldsymbol{p}, V_1)$, with $V_1 = (\boldsymbol{v_1}, ..., \boldsymbol{v_{j-1}}, \boldsymbol{v_j}, \boldsymbol{0}, \boldsymbol{v_{j+1}}, \ ..., \boldsymbol{v_k})$. As $\boldsymbol{v_j}$ is not a 1-step, there exists an (m-1)-step $\boldsymbol{v_j'}$ and a 1-step $\boldsymbol{v_{j1}}$ such that $\boldsymbol{v_j} = (\boldsymbol{v_{j1}} + \boldsymbol{v_j'})$. The m-loop $\mathcal{L}_1$ is directly homotopic to $\mathcal{L}_2 = (\boldsymbol{p}, V_2)$, with $V_2 = (\boldsymbol{v_1}, ..., \boldsymbol{v_{j-1}}, \boldsymbol{v_{j1}}, \boldsymbol{v_j'}, \boldsymbol{v_{j+1}}, ..., \boldsymbol{v_k})$. By iteration, it is shown that $\mathcal{L}$ is homotopic to a 1-loop. $\square$

Lemma 29. *Let $\mathcal{L}_A = (\boldsymbol{p}, V_A)$ and $\mathcal{L}_B = (\boldsymbol{p}, V_B)$ be two m-loops such that $V_A = (\boldsymbol{v_1}, ..., \boldsymbol{v_{j-1}}, \boldsymbol{v_{j1}}, \boldsymbol{v_{j2}}, \boldsymbol{v_{j+1}}, ..., \boldsymbol{v_k})$ and $V_B = (\boldsymbol{v_1}, ..., \boldsymbol{v_{j-1}}, \boldsymbol{v_{j2}}, \boldsymbol{v_{j1}}, \boldsymbol{v_{j+1}}, ..., \boldsymbol{v_k})$ where $\boldsymbol{v_{j1}}$ and $\boldsymbol{v_{j2}}$ are 1-steps. Then, $\mathcal{L}_A$ and $\mathcal{L}_B$ are homotopic.*

Proof. As $\boldsymbol{v_{j1}}$ and $\boldsymbol{v_{j2}}$ are 1-steps, they have at most one non-null coordinate. If $(\boldsymbol{v_{j1}} - \boldsymbol{v_{j2}})$ is an n-step, the two loops are directly homotopic. If $(\boldsymbol{v_{j1}} - \boldsymbol{v_{j2}})$ is not an n-step, then necessarily $\boldsymbol{v_{j1}} = (-\boldsymbol{v_{j2}})$. Therefore, $\mathcal{L}_A$ is directly homotopic to $\mathcal{L}_C = (\boldsymbol{p}, V_C)$, with $V_C = (\boldsymbol{v_1}, ..., \boldsymbol{v_{j-1}}, \boldsymbol{0}, \boldsymbol{0}, \ \boldsymbol{v_{j+1}}, ..., \boldsymbol{v_k})$. Furthermore, $\mathcal{L}_C$ is also directly homotopic to $\mathcal{L}_B$. $\square$

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