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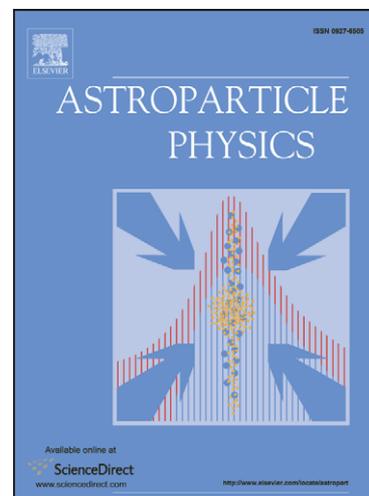
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Diffusive shock acceleration in supernova remnants: on the validity of the Bohm limit

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Abstract

A lot of theoretical effort has been achieved in cosmic ray scattering theory during the previous years. On the other hand several authors employed the so-called Bohm limit to replace cosmic ray diffusion coefficients in the diffusive transport equation for describing particle acceleration in supernova remnants and in other environments. In this article it is shown how the Bohm limit can be derived from cosmic ray diffusion theory by assuming a strong turbulent magnetic field component. Also a test-particle code is employed to explore the regimes for which the Bohm limit is valid and for which not. These results are important to distinguish between the parameter regimes for which the Bohm limit is valid and for which traditional diffusion theories have to be applied.

Key words: Interplanetary shocks, Supernova remnants, Cosmic rays
PACS: 96.50.Fm, 98.38.Mz, 98.70.Sa

1 Introduction

Shocks are common in diverse astrophysical environments on scales ranging from cometary bow shocks and interplanetary shocks to supernova remnants and AGN's (active galactic nuclei). The mechanism which is responsible for the observed power-law spectra of charged particles (cosmic rays) is diffusive shock acceleration (Axford et al. 1977, Bell 1978a, Bell 1978b, Blandford & Ostriker 1978, Krymsky 1977). Observed cosmic rays having energies until about $10^9 eV$ can be explained by assuming acceleration in interplanetary shocks (Zank et al. 2000, Li et al. 2003, Li et al. 2005, Zank et al. 2006) and particles having energies until about $10^{15} eV$ get their high energy due to acceleration in supernova remnants (Völk et al. 1988, Lucek & Bell 2000, Berezhko & Ellison 1999, Berezhko & Völk 2000, Berezhko & Völk 2007). More recently it has been shown that such acceleration processes in AGN's can also lead to

particle energies above $10^{20} eV$ (Honda & Honda 2004). Therefore, the detailed understanding of diffusive shock acceleration is important for solving one of the most important puzzles of astroparticle physics, namely the origin of high energetic cosmic rays. Recently the Pierre Auger experiment found evidence that ultra high energetic particles ($\sim 10^{18} eV$) originate from AGN's (The Pierre Auger Collaboration 2007) indicating that the process of diffusive shock acceleration is one of the most important processes in nature.

One part of the puzzle which has to be solved in diffusive shock acceleration is the interaction between charged particles and stochastic magnetic fields described by the diffusion tensor κ_{ij} . The theoretical determination of this tensor is a fundamental problem of astrophysics and astroparticle physics since this tensor controls cosmic ray propagation in the solar system (see, e.g., Shalchi et al. 2006), in the interstellar medium (see, e.g., Shalchi & Schlickeiser 2005) and most probably also in other environments such as the solar corona (Fletcher 1997, Gkioulidou et al. 2007). However, the knowledge of the diffusion tensor is also essential for understanding the acceleration of charged cosmic rays in shock waves such as interplanetary shocks (see, e.g., Zank et al. 2006). Honda & Honda (2004) have shown that the form of the diffusion coefficient controls the maximum energy which a particle can get due to shock acceleration in AGN's. Most probably the knowledge of the diffusion coefficient is also important for improving the understanding of particle acceleration at supernova shock waves.

The transport equation which has to be solved for obtaining the cosmic ray spectra emitted by supernova remnants has the form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x_i} \kappa_{ij} \frac{\partial}{\partial x_j} f - w_i \frac{\partial}{\partial x_i} f + \frac{1}{3} \frac{\partial w_i}{\partial x_i} p \left(\frac{\partial f}{\partial p} \right) + \frac{1}{p^2} \frac{\partial}{\partial p} \left(\frac{p^3}{\tau_s} f \right) \quad (1)$$

where the terms on the right hand side describe diffusion, convection due to the mean gas speed w_i , adiabatic expansion/compression, and synchrotron losses, respectively. In Eq. (1) we have used the particle distribution function f , the time t , the particle position \vec{x} , and the particle momentum p . The solution of this equation is determined by the diffusion coefficients κ_{ij} , the mean gas speed w_i , and the (momentum dependent) synchrotron loss time $\tau_s = \tau_s(p)$.

One part of the puzzle which has to be investigated theoretically is the interaction between charged particles and stochastic (or turbulent) magnetic fields which are responsible for the acceleration process. These interaction processes are described by the diffusion coefficients κ_{ij} in the diffusion term in Eq. (1). In most previous articles about diffusive shock acceleration (see, e.g., Duffy 1992, Berezhko & Völk 2007) the so-called Bohm limit was employed to replace the diffusion coefficient in the transport equation. In the Bohm limit it is assumed that the spatial diffusion coefficients in all directions (isotropic scattering is also part of the assumption) are given by (see, e.g., Berezhko &

Völk 2007, Section 2)

$$\kappa^{Bohm} = \frac{vpc}{3|q|B} \quad (2)$$

where we used the particle velocity v , the particle momentum p , the speed of light c , the particle charge q , and the magnetic field strength B .

In Section 2 of the current article we present a detailed description of the cosmic ray diffusion theory. In Section 3 we assume a strong turbulent magnetic field to derive analytically the Bohm limit as a special limit. In Section 4 we employ a test particle code to test the different results which we found theoretically to explore their validity. Furthermore, we try to find out for which parameter regimes Bohm diffusion is valid and for which classical diffusion theory has to be applied. In Section 5 we summarize and discuss our findings.

2 Cosmic ray diffusion theory

2.1 General equations

In cosmic ray diffusion theory we investigate the interaction between charged particles (cosmic rays) and a magnetic field configuration. The total magnetic field is assumed to be the superposition of a mean field \vec{B}_0 and fluctuations $\delta\vec{B}(\vec{x})$. The mean field is usually approximated by a constant field aligned parallel to the z -axis ($\vec{B}_0 = B_0\vec{e}_z$), the turbulent contribution has to be replaced by models.

The starting point for the investigation of cosmic ray scattering is the Newton-Lorentz equation whose component parallel to the mean magnetic field is

$$\dot{\mu} = \frac{\Omega}{v} \left(v_x \frac{\delta B_y}{B_0} - v_y \frac{\delta B_x}{B_0} \right). \quad (3)$$

Here we have used the pitch-angle cosine $\mu = v_{\parallel}/v$ and we have neglected electric fields since they are less important for spatial diffusion except for low energetic particles. Furthermore, we used the particle velocity v and the unperturbed¹ gyrofrequency

$$\Omega = \frac{|q|B_0}{\gamma mc} \quad (4)$$

¹ Unperturbed here means that the turbulent field is assumed to be zero ($\delta B = 0$). In this case the particle motion is a helical motion in the direction perpendicular to the mean magnetic field B_0 and a motion with constant velocity in the parallel direction.

with the Lorentzfactor $\gamma = (1 - v^2/c^2)^{-1/2}$. First, we calculate the so-called pitch-angle Fokker-Planck coefficient $D_{\mu\mu}$ which describes pitch-angle scattering. This coefficient can be calculated by employing the well-established Taylor-Green-Kubo-formulation (Taylor 1922, Green 1952, Kubo 1957)

$$D_{\mu\mu} = \int_0^\infty dt \langle \dot{\mu}(t) \dot{\mu}^*(0) \rangle \quad (5)$$

where the operator $\langle \dots \rangle$ denotes the ensemble average. The parallel spatial diffusion coefficient κ_{\parallel} and the parallel mean free path λ_{\parallel} are related to the pitch-angle Fokker-Planck coefficient $D_{\mu\mu}$ by (see, e.g., Earl 1974)

$$\lambda_{\parallel} = \frac{3}{v} \kappa_{\parallel} = \frac{3v}{8} \int_{-1}^{+1} d\mu \frac{(1 - \mu^2)^2}{D_{\mu\mu}}. \quad (6)$$

In the following paragraphs we compute the parameter $D_{\mu\mu}$ by employing two different approaches.

2.2 General form of the pitch-angle Fokker-Planck coefficient

By combining Eqs. (3) and (5) we can see that the ensemble average operator acts on the particle velocity components v_i as well as on the turbulent fields δB_i . First we investigate the particle velocity components. We can write the velocities in the perpendicular direction as a superposition of a gyromotion v_i^{GR} and a (diffusive) guiding center motion v_i^{GC} . The latter effect is described by the equation of motion of the guiding centers which can easily be derived from the field line equation (here we assume that perpendicular scattering is predominantly caused by field line wandering), and thus,

$$dx_i = \frac{\delta B_i}{B_z} dz \quad \Rightarrow \quad v_i^{GC} = \frac{\delta B_i}{B_z} v_z. \quad (7)$$

By combining Eqs. (3) and (7), and by assuming the form $v_i = v_i^{GR} + v_i^{GC}$, we find

$$\begin{aligned} \dot{\mu} &= \frac{\Omega}{v} \left(v_x^{GR} \frac{\delta B_y}{B_0} + v_z \frac{\delta B_x \delta B_y}{B_0 B_z} - v_y^{GR} \frac{\delta B_x}{B_0} - v_z \frac{\delta B_x \delta B_y}{B_0 B_z} \right) \\ &= \frac{\Omega}{v} \left(v_x^{GR} \frac{\delta B_y}{B_0} - v_y^{GR} \frac{\delta B_x}{B_0} \right). \end{aligned} \quad (8)$$

Note, Eq. (8) no longer depends on the parallel component of the stochastic field δB_z and is still valid for arbitrary turbulence. The gyrorotation can be expressed by

$$\begin{aligned}
v_x^{GR}(t) &= v\sqrt{1-\mu^2}\cos[\Phi_0 - \text{sgn}(q)\Omega t] \\
v_y^{GR}(t) &= v\sqrt{1-\mu^2}\sin[\Phi_0 - \text{sgn}(q)\Omega t]
\end{aligned} \tag{9}$$

with the initial gyrophase Φ_0 . **The function $\text{sgn}(q)$ used here is the signum function which is used to allow the motion of positively as well as negatively charged particles. To proceed we combine Eqs. (8) and (9) and assume axisymmetric turbulence corresponding to $\langle \delta B_x(\vec{x}(t))\delta B_x(\vec{x}(0)) \rangle = \langle \delta B_y(\vec{x}(t))\delta B_y(\vec{x}(0)) \rangle$ and $\langle \delta B_x(\vec{x}(t))\delta B_y(\vec{x}(0)) \rangle = \langle \delta B_y(\vec{x}(t))\delta B_x(\vec{x}(0)) \rangle = 0$. By employing the relation $\cos(x)\cos(y) + \sin(x)\sin(y) = \cos(x-y)$ we derive**

$$D_{\mu\mu} = \frac{\Omega^2(1-\mu^2)}{B_0^2} \int_0^\infty dt \cos(\Omega t) \langle \delta B_y(\vec{x}(t))\delta B_y(\vec{x}(0)) \rangle. \tag{10}$$

To proceed we rewrite the magnetic correlation function as

$$\langle \delta B_y(\vec{x}(t))\delta B_y(\vec{x}(0)) \rangle \approx \int d^3k P_{yy}(\vec{k}) \langle e^{i\vec{k}\cdot\vec{x}} \rangle. \tag{11}$$

Here we have employed a random phase approximation and we have assumed homogenous turbulence. The last step in evaluating this expression is the investigation of the characteristic function $\langle e^{i\vec{k}\cdot\vec{x}} \rangle$. This function is defined as

$$\Gamma(\vec{k}, t) := \langle e^{i\vec{k}\cdot\vec{x}} \rangle = \int d^3x f(\vec{x}, t) e^{i\vec{k}\cdot\vec{x}} \tag{12}$$

with the particle distribution function $f(\vec{x}, t)$. Consequently the pitch-angle Fokker-Planck coefficient becomes

$$D_{\mu\mu} = \frac{\Omega^2(1-\mu^2)}{2B_0^2} \int d^3k P_{yy}(\vec{k}) [R_-(\vec{k}) + R_+(\vec{k})] \tag{13}$$

with the resonance function

$$R_\pm(\vec{k}) = \int_0^\infty dt \Gamma(\vec{k}, t) e^{\pm i\Omega t}. \tag{14}$$

We can compute the resonance function and therewith the pitch-angle Fokker-Planck coefficient by determining the characteristic function $\Gamma(\vec{k}, t)$.

2.3 Example: weak turbulence and quasilinear approximation for slab turbulence

To evaluate the formulas derived in the previous paragraph we have to specify the particle distribution function $f(\vec{x}, t)$. In this paragraph we discuss the quasilinear approximation used previously. In the next section we employ a new, strong turbulence approach.

By considering a very weak turbulent magnetic field component ($\delta B_i \ll B_0$) we can assume unperturbed orbits for the charged particles corresponding to the application of a first order perturbation theory. In the current article we employ the so-called slab model² for magnetic turbulence which is defined as

$$P_{ij}(\vec{k}) = g(k_{\parallel}) \frac{\delta(k_{\perp})}{k_{\perp}} \delta_{ij} \quad \text{for } i, j = x, y \quad (15)$$

with the wave spectrum $g(k_{\parallel})$. In the parallel direction, an unperturbed motion corresponds to a motion with constant velocity, and, thus, $z = v\mu t$ in Eq. (14). With Eq. (15) for the slab model one obtains

$$D_{\mu\mu} = \pi \frac{\Omega^2(1-\mu^2)}{B_0^2} \int_{-\infty}^{+\infty} dk_{\parallel} g(k_{\parallel}) [R_{-1}^{QLT}(k_{\parallel}) + R_{+1}^{QLT}(k_{\parallel})] \quad (16)$$

with the resonance function of quasilinear theory (QLT)

$$R_{\pm}^{QLT}(k_{\parallel}) = \int_0^{\infty} e^{i(v\mu k_{\parallel} \pm \Omega)t} = \pi \delta(v\mu k_{\parallel} \pm \Omega). \quad (17)$$

By employing the quasilinear resonance function and a symmetric wave spectrum ($g(k_{\parallel}) = g(-k_{\parallel})$) we find for the pitch-angle Fokker-Planck coefficient

$$D_{\mu\mu} = 2\pi^2 \frac{\Omega^2(1-\mu^2)}{B_0^2 v |\mu|} g\left(k_{\parallel} = \frac{\Omega}{v |\mu|}\right) \quad (18)$$

corresponding to a gyroresonant interaction. Within this approach there is only interaction between charged particles and turbulence if $\mu R_L = k_{\parallel}$. This approach, however, is only valid for weak turbulence. Even then, quasilinear theory is questionable due to its inability for describing scattering at 90° (corresponding to $\mu = 0$). Therefore several theories have been developed for improving the description of cosmic ray propagation. Some of these theories were developed and presented in the seventies of the last century (see, e.g., Völk 1973, Jones et al. 1973, Owens 1974, Völk 1975, Goldstein 1976, Jones et al. 1978) some of them in more recent years (see, e.g., Shalchi 2005a). In the next paragraph we replace the quasilinear approximation by a strong turbulence approach to derive the Bohm limit.

² In the slab model it is assumed that the stochastic magnetic field depends only on the coordinate parallel to the mean magnetic field $\delta B_i(\vec{x}) = \delta B_i(z)$. Although this model is not very realistic, it is often applied due to mathematical tractability. In **section 3** it is shown that the Bohm limit can be derived without specifying the turbulence geometry. **Note, Eq. (13) and the calculations presented in section 3 are valid for arbitrary turbulence. The only assumptions which have been used are the assumption of axisymmetry and homogeneity.** The slab model is only applied to perform quasilinear calculations and the test-particle simulations.

3 Analytical derivation of the Bohm limit for strong turbulence

The correct equation for calculating the particle distribution function $f(\vec{x}, t)$ and the characteristic function $\Gamma(\vec{k}, t)$ in the phase-space is the Fokker-Planck equation which has (in the two-dimensional phase-space) the form

$$\frac{\partial f}{\partial t} + v_{\mu} \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} \left(D_{\mu\mu} \frac{\partial f}{\partial \mu} \right). \quad (19)$$

This formula can be multiplied by $\exp(ik_{\parallel}z)$ and can then be integrated over z to obtain

$$\frac{\partial \Gamma}{\partial t} - ik_{\parallel} v_{\mu} \Gamma = \frac{\partial}{\partial \mu} \left(D_{\mu\mu} \frac{\partial \Gamma}{\partial \mu} \right). \quad (20)$$

The solution of this equation, which yields the characteristic function, for general $D_{\mu\mu}$ is difficult. A simple approximative approach is discussed in the next paragraph.

3.1 The Fokker-Planck coefficient in the strong turbulence limit

In the previous paragraph we have argued that for strong turbulence the solution of Eq. (20) is the correct replacement for the characteristic function in Eq. (14). However, the general solution is very difficult to derive. Therefore, we employ a further simplification by assuming that pitch-angle scattering becomes isotropic for strong turbulence resulting in the form

$$D_{\mu\mu}(\mu) \approx D \cdot (1 - \mu^2) \quad (21)$$

where we have used the pitch-angle Fokker-Planck coefficient D at 90° . In this case we only have to solve Eq. (20) for $\mu = 0$. In this limit Eq. (20) becomes

$$\frac{\partial \Gamma}{\partial t} = \frac{\partial}{\partial \mu} \left(D_{\mu\mu} \frac{\partial \Gamma}{\partial \mu} \right). \quad (22)$$

For the form of Eq. (21), Eq. (22) becomes

$$\begin{aligned} \frac{\partial \Gamma}{\partial t} &= D \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \Gamma}{\partial \mu} \right] \\ &= D \left[\frac{\partial^2 \Gamma}{\partial \mu^2} - 2\mu \frac{\partial \Gamma}{\partial \mu} \right]. \end{aligned} \quad (23)$$

The differential equation can be solved as shown in Appendix A. As demonstrated there we find for the characteristic function

$$\Gamma(t) = 1 + \sum_{l=1}^{\infty} (2l+1) P_l^2(0) e^{-l(l+1)Dt} \quad (24)$$

where we have used Legendre Polynomials $P_n(x)$. With this form the resonance function for strong turbulence (ST) is

$$\begin{aligned} R_{\pm}^{ST} &= \int_0^{\infty} dt e^{\pm i\Omega t} \\ &+ \sum_{l=1}^{\infty} (2l+1) P_l^2(0) \int_0^{\infty} dt e^{-[l(l+1)D \pm i\Omega]t} \\ &= \pi \delta(\pm\Omega) + \sum_{l=1}^{\infty} P_l^2(0) \frac{2l+1}{l(l+1)D \pm i\Omega}. \end{aligned} \quad (25)$$

To obtain Eq. (25) we have combined Eq. (24) with Eq. (14). Since the parameter Ω cannot be zero, we have $\delta(\pm\Omega) = 0$ and, thus,

$$R_{\pm}^{ST} = \sum_{l=1}^{\infty} (2l+1) P_l^2(0) \frac{l(l+1)D \mp i\Omega}{[l(l+1)D]^2 + \Omega^2}. \quad (26)$$

With this resonance function the Fokker-Planck coefficient at 90° becomes (see Eq. (13))

$$\begin{aligned} D &= \frac{\Omega^2}{2B_0^2} \int d^3k P_{yy}(\vec{k}) \\ &\times 2 \sum_{l=1}^{\infty} (2l+1) P_l^2(0) \frac{l(l+1)D}{[l(l+1)D]^2 + \Omega^2}. \end{aligned} \quad (27)$$

By employing

$$\int d^3k P_{yy}(\vec{k}) = \delta B_y^2 = \frac{1}{2} \delta B^2 \quad (28)$$

we can derive

$$D = \frac{1}{2} \Omega^2 \frac{\delta B^2}{B_0^2} \sum_{l=1}^{\infty} (2l+1) P_l^2(0) \frac{l(l+1)D}{[l(l+1)D]^2 + \Omega^2}. \quad (29)$$

For strong turbulence ($\delta B \gg B_0$) we expect that the Fokker-Planck coefficient D is much larger than the Gyrofrequency of the charged particle $D \gg \Omega$ and, thus, we can achieve a further simplification

$$D^2 = \frac{\xi^2}{4} \Omega^2 \frac{\delta B^2}{B_0^2} \quad (30)$$

with the constant

$$\xi^2 = 2 \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} P_l^2(0). \quad (31)$$

Finally we find for the pitch-angle Fokker-Planck coefficient

$$D_{\mu\mu}(\mu) = \frac{\xi}{2} \Omega \frac{\delta B}{B_0} (1 - \mu^2) \quad (32)$$

in the strong turbulence limit. We have evaluated Eq. (31) numerically to find that $\xi^2 = 0.77$ and therefore $\xi = 0.88$.

3.2 The parallel mean free path

In this paragraph we compute the parallel mean free path by combining Eqs. (6) and (32)

$$\lambda_{\parallel} = \frac{3v}{4\xi\Omega} \frac{B_0}{\delta B} \int_{-1}^{+1} d\mu (1 - \mu^2) = \frac{v}{\xi\Omega} \frac{B_0}{\delta B}. \quad (33)$$

With $\xi^{-1} = 1.14 \approx 1$ we find³

$$\lambda_{\parallel} \approx \frac{v}{\Omega} \frac{B_0}{\delta B}. \quad (34)$$

With Eq. (4) one gets

$$\kappa_{\parallel} = \frac{v}{3} \lambda_{\parallel} = \frac{pvc}{3q\delta B}. \quad (35)$$

By comparing with Eq. (2) we find for strong turbulence ($B = B_0 + \delta B \approx \delta B$) indeed Bohm diffusion.

4 Different transport regimes: quasilinear versus Bohm diffusion

As demonstrated we can derive the Bohm limit by using methods of cosmic ray diffusion theory. In this section we employ a test-particle code to test our theoretical result and to explore the parameter regimes for which we indeed find Bohm diffusion of charged cosmic rays. We also compare these results with the traditional gyroresonance picture of cosmic ray transport obtained by employing quasilinear theory. The quasilinear result for the parallel mean

³ The unperturbed Larmor radius of the charged particle is defined as $R_L = v/\Omega$ and Eq. (33) becomes $\lambda_{\parallel} = R_L B_0 / \delta B$. In the case that the particle motion is still a helical motion with the Larmor radius r_L we have for the gyrofrequency $\omega = |q| \delta B / (mc\gamma)$. In this case the Larmor radius r_L can be related to the unperturbed Larmor radius R_L via $r_L = v/\omega = vB_0 / (\Omega\delta B) = R_L B_0 / \delta B$ and, therefore, $\lambda_{\parallel} = r_L$ for Bohm diffusion.

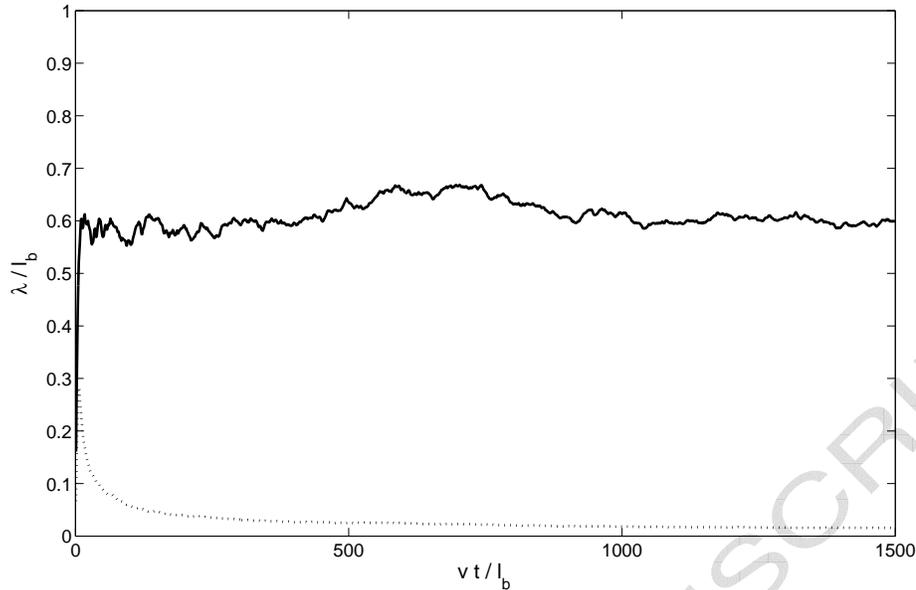


Fig. 1. Running diffusion coefficient in the parallel (solid line) as well as in the perpendicular direction (dotted line). Shown is the mean free path divided by the bendover scale λ/l_b as a function of the dimensionless time $\tau = vt/l_b$.

free path is derived in Appendix B for the same turbulence model used in the test particle simulations.

In the following we employ a test particle code to explore the weak as well as the strong turbulence regime to find out whether we can also find Bohm diffusion in test-particle simulations. We performed the simulations for a Kolmogorov turbulence spectrum ($\nu = 5/6$ in the formula shown in Appendix B). We computed the ratio λ_{\parallel}/l_b as a function of the two independent variables $R = R_L/l_b$ and $B = \delta B/B_0$ (l_b is a characteristic length scale of the turbulence, see Appendix B). In the test-particle simulations we have simulated the mean magnetic field B_0 as well as the stochastic fields δB_i by employing the slab model. Then we have solved the (exact) Newton-Lorentz equation for 1000 charged particles numerically. From the resulting particle trajectories it is a simple matter to compute the mean square deviation $\langle(\Delta x)^2\rangle$ and the (running) diffusion coefficient $\kappa(t) = \langle(\Delta x)^2\rangle/(2t)$ as a function of time in the different directions with respect to the mean magnetic field. Details of the code can be found in Qin et al. (2002a) and Qin et al. (2002b). In Fig. 1 we have shown running diffusion coefficients in the direction parallel and perpendicular with respect to the mean magnetic field obtained by employing computer simulations. From such figures it is a simple matter to obtain the diffusion coefficients. In Figs. 2 and 3 the diffusion coefficients obtained by employing a test-particle code are compared with quasilinear coefficients and Bohm diffusion. In Fig. 2 we have shown the results for $R = R_L/l_b = 1.0$ for different values of the ratio $\delta B/B_0$. In Fig. 3 another case is shown namely

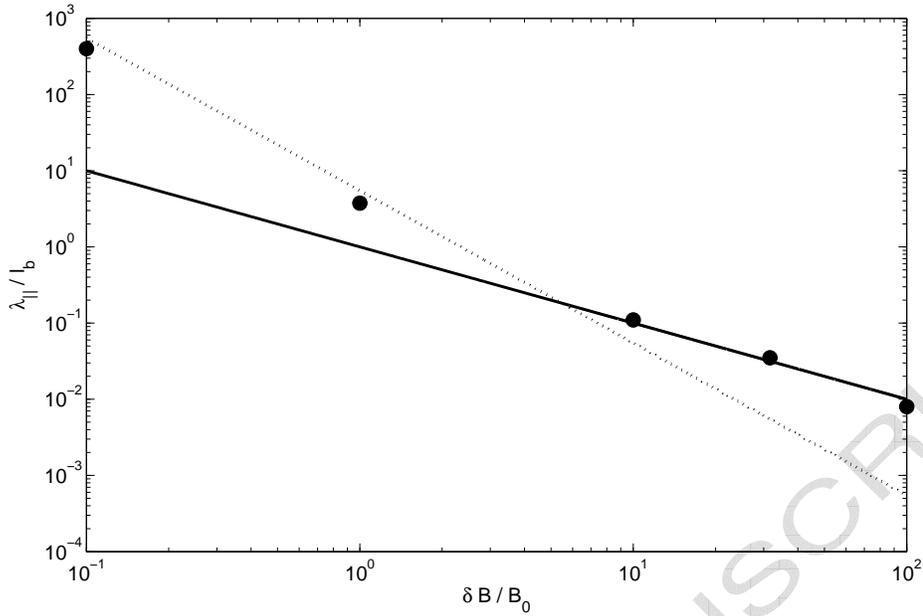


Fig. 2. Numerical test particle simulations (dots) in comparison with quasilinear results (dotted line) and the Bohm limit (solid line). The results shown here are for a Kolmogorov (1941) spectrum and intermediate particle energy $R = R_L/l_b = 1.0$. Clearly we find that quasilinear theory agrees very well with the simulations for weak and intermediate turbulence ($\delta B \leq B_0$). The Bohm limit agrees with the simulations for $\delta B > B_0$.

$R = 0.1$ corresponding to lower particle energies. In this case quasilinear theory does also not agree with the simulations for strong turbulent magnetic fields. It seems that also in this case the simulations approach asymptotically the Bohm-limit.

5 Summary and conclusion

In this article we have investigated charged particle scattering in the direction parallel to a mean magnetic field in the limit of strong turbulence. By employing the method of the nonlinear scattering theory (see Section 2) we derived the Bohm limit analytically (see Section 3). In previous articles about diffusive shock acceleration (e.g., Berezhko & Völk 2007) this limit has been used without being justified. In the current article it is shown that the Bohm limit is indeed the correct limit for strong turbulence ($\delta B \gg B_0$) confirming the previous assumption of bohmian diffusion in the strong turbulence regime.

We also have employed a test particle code to demonstrate that the theoretical results of the current article are correct. It is shown that for weak turbulence ($\delta B \ll B_0$) quasilinear theory is correct whereas for strong turbulence ($\delta B \gg$

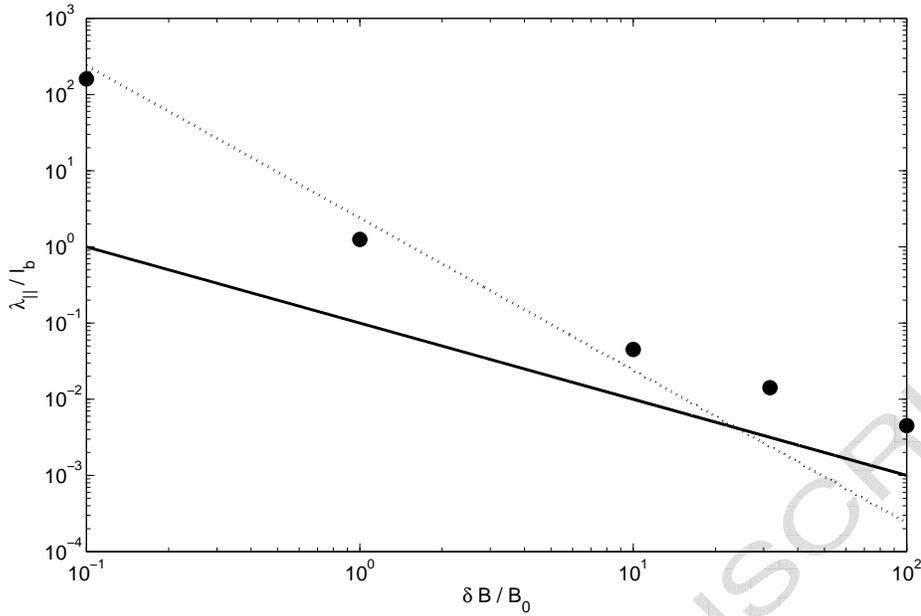


Fig. 3. Numerical test particle simulations (dots) in comparison with quasilinear results (dotted line) and the Bohm limit (solid line). The results shown here are for a Kolmogorov (1941) spectrum and low particle energy $R = R_L/l_b = 0.1$. Clearly we find that quasilinear theory agrees very well with the simulations for weak and intermediate turbulence ($\delta B \leq B_0$). The Bohm limit agrees with the simulations for $\delta B > B_0$.

B_0) we obtain bohmian diffusion. We have also presented a parameter study to find out for which parameter regimes the Bohm limit is valid and for which not. These results could be very important for future studies of diffusive shock acceleration since the choice of the diffusion coefficient (especially the rigidity dependence) determines the acceleration rate as well as the scale of the shock precursor in the upstream region of the shock (see, e.g., Duffy 1992). Honda & Honda (2004) have shown that the form of the diffusion coefficient controls the maximal energy which a particle can get due to acceleration in AGN's. Zank et al. (2006) have shown the the form of the diffusion coefficient is also important for describing particle acceleration at interplanetary shocks. Most probably the knowledge of the diffusion coefficient is also important for improving the understanding of particle acceleration at supernova shock waves.

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A Characteristic function for strong turbulence

In Shalchi (2006) the solution of Eq. (23) is discussed. It is demonstrated there that the solution can be expressed by Legendre polynoms (see Eq. (33) in Shalchi 2006)

$$\Gamma(t) = 1 + \sum_{l=1}^{\infty} \alpha_l P_l(\mu) e^{-l(l+1)Dt}. \quad (\text{A.1})$$

The parameters α_l can be determined from the initial conditions (see Eq. (34) in Shalchi 2006)

$$\alpha_l = \frac{2l+1}{2} \int_{-1}^{+1} d\mu \Gamma(t=0) P_l(\mu). \quad (\text{A.2})$$

By using the definition of the characteristic function we have

$$\Gamma(\vec{k}, \mu, t=0) = \int d^3x f(\vec{x}, \mu, t=0) e^{i\vec{k}\cdot\vec{x}}. \quad (\text{A.3})$$

By assuming sharp initial conditions we can use

$$f(\vec{x}, \mu, t=0) = 2\delta^3(\vec{x} - \vec{x}_0) \delta(\mu - \mu_0) \quad (\text{A.4})$$

and, therefore,

$$\Gamma(\vec{k}, \mu, t=0) = 2\delta(\mu - \mu_0) e^{i\vec{k}\cdot\vec{x}_0}. \quad (\text{A.5})$$

For the initial conditions we can choose $\vec{x}_0 = 0$ due to the assumption of homogenous turbulence. Furthermore, we are only interested in 90° scattering and, thus, we can set $\mu_0 = 0$. As a consequence Eq. (A.5) becomes

$$\Gamma(\vec{k}, \mu, t=0) = 2\delta(\mu). \quad (\text{A.6})$$

Therefore, Eq. (A.2) becomes

$$\alpha_l = (2l+1) \int_{-1}^{+1} d\mu \delta(\mu) P_l(\mu) = (2l+1) P_l(0) \quad (\text{A.7})$$

and we find for the characteristic function at 90°

$$\Gamma(t) = 1 + \sum_{l=1}^{\infty} (2l+1) P_l^2(0) e^{-l(l+1)Dt}. \quad (\text{A.8})$$

In the main part of this article Eq. (A.8) is combined with Eq. (14) to determine the resonance function and therewith the Fokker-Planck coefficient in the strong turbulence limit.

B Quasilinear transport coefficients

In the weak turbulence limit the form of the turbulence wave spectrum $g(k_{\parallel})$ is very important (see, e.g., Eq. (18)). In this part of the appendix we recapitulate the quasilinear calculations presented in Shalchi et al. (2004). In the current article we use the same spectrum as used in Shalchi (2005b), namely a *kappa*-distribution of the form

$$g(k_{\parallel}) = \frac{C(\nu)}{2\pi} l_b \delta B^2 (1 + k_{\parallel}^2 l_b^2)^{-\nu}. \quad (\text{B.1})$$

Here we have used the inertial range spectral index 2ν , the bendover length scale l_b denoting the frequency break between the inertial range and the energy range of the spectrum, the energy of the magnetic fluctuations δB^2 , and the normalization constant

$$C(\nu) = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\nu)}{\Gamma(\nu - 1/2)}. \quad (\text{B.2})$$

For this model spectrum and within quasilinear theory we find for the pitch-angle Fokker-Planck coefficient (see, e.g., Appendix in Shalchi et al. 2004)

$$D_{\mu\mu} = \frac{\pi C(\nu) \nu \delta B^2}{l_b B_0^2} (1 - \mu^2) \mu^{2\nu-1} R^{2\nu-2} (1 + \mu^2 R^2)^{-\nu}, \quad (\text{B.3})$$

and for the parallel mean free path

$$\lambda_{\parallel} = \frac{3l_b}{8\pi C(\nu)} \left(\frac{B_0}{\delta B} \right)^2 R^{2-2\nu} \left[\frac{1}{1-\nu} {}_2F_1(1-\nu, -\nu, 2-\nu, -R^2) - \frac{1}{2-\nu} {}_2F_1(2-\nu, -\nu, 3-\nu, -R^2) \right] \quad (\text{B.4})$$

with the hypergeometric function ${}_2F_1(a, b; c; z)$. In Eqs. (B.3) and (B.4) we used the dimensionless cosmic ray rigidity $R = R_L/l_b$. We expect that these formulas are correct for weak turbulence and for cases for which the 90° scattering problem does not occur. In the current article Eq. (B.4) is only used for a comparison with test particle simulations and the Bohm limit.

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