Variable preference relations: existence of maximal elements

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July 16, 2010

Abstract

We consider variable preference relations, called also reference dependent preference relations which are typical in the study of dynamic models in economic theories. We introduce a new concept of weak consistency, a generalization of acyclicity, as an immediate regret condition for variable preferences. The main result to establish is on an existence criterion for maximal elements of a space equipped with a weak consistent variable preference relation. It is expressed via preference completeness condition which is equivalent to existence of aspiration points. As applications we show that a number of results known in the recent literature on maximum principles on a space with or without topological structure can be obtained from the unifying approach of this paper.

Keywords: Variable preference relation, improving path, maximal point, maximum principle.

JEL Classification: C60, C62.

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1 Introduction

Preference is one of the fundamental concepts in economics and social sciences. It refers to a way to put available alternatives in a certain order according to their degree of utility or satisfaction and aims at answering the question "stay or move?". In equilibrium theory a static point of view is dominant, that is, one is looking for situations in which all agents prefer to stay rather then to move; while in innovation theory (Schumpeter 1943 [22]) a dynamic point of view is more important, that is, one is interested in knowing when the agents prefer moving from their current situations than staying there. These points of views are dual, once a preference is given. In the classical models preference relations are a priori determined and constant. They are transitive, and sometimes complete, often defined by utility functions, which greatly facilitate applications of mathematical tools. However, in the real world preference relations vary with the context (experience, characters, emotions, mental state, social links, embeddedness...) and have evolutionary nature, generating dynamic processes of actions. This is why models with changing preferences appeal much attention of researchers for several decades (see Basmann 1956 [4], Bleichbrodt 2007[5], Koszegi and Rabin 2006 [16], Tversky and Kahneman 1991 [27] among many others).

In the present study we consider preference relations, not necessarily being transitive or complete, or given by utility functions, but varying from state to state. Our overall concern is the question under which circumstances maximal elements exist for a variable preference relation without transitivity. Maximal elements play a central role in many economic models, including global maximum of a utility function and Nash equilibrium of a noncooperative game or equilibrium of generalized game (Debreu 1959 [9]). Existence of maximal elements has been extensively studied for constant preference relations and literature on it is abundant. However, to our knowledge there exists few papers dealing with existence of maximal elements in the case of variable preference relations. Koszegi 2010 [15], Koszegi-Rabin 2006 [16] and some others study a personal equilibrium of consumption in which a particular variable preference is used via variable utility functions (see also Gul-Pesendorfer, 2006 [13]); Soubeyran 2009 [23] considers a worthwhile to change function, which generates a variable preference relation to model a number of problems of social science such as habits, routines, behavioral traps etc. in terms of maximal elements. To establish existence criteria for maximal elements of a variable preference relation we exploit the concept of order-completeness borrowed from vector optimization which can be interpreted as a hypothesis on aspiration points recently introduced in the theory of change (see Soubeyran 2009 [23]) or Brezis-Browder's inductivity hypothesis (Brezis and Browder 1976 [8]) when the preference relation is determined by a utility function. Similar to the case of multi-criteria optimization given in Luc [18] and recently in Flores-Bazan et al. [11] the method of order-complete sets is very useful in unifying results on existence of maximal elements in various contexts, including (a) existence of maximal elements in social choice theory with non-transitive and incomplete preferences (Bergstrom 1975 [6] for acyclic relations, Tian and Zhou 1993 [25] for transfer upper continuous preferences, Zuanon 2009 [32] for weakly tc-upper semicontinuous acyclic relations, Alcantud 2002 [1] for upper continuous preference on " $\succ$ "-upper compact sets, Andrikopoulos and Zacharias 2009 [3] for consistent upper tc-S- semicontinuous preferences); (b) maximum ordering principles in partially ordered spaces without compactness (Altman 1982 [2], Turinici 1984 [28], Szaz 2007 [24], Zeidler 1986 [30], Zhu and Li 2007 [31] in the context of partial ordering); and (c) efficiency conditions in the framework of generalized multi-criteria optimization (Flores-Bazan et al. 2008 [11]).

The paper is structured as follows. In Section 2 basic concepts and elementary properties of variable preference relations are given. We introduce the concept of weak consistency which generalizes well-known concepts of acyclicity, path consistency and consistency by Suzumura. We establish equivalence between acyclicity of a variable preference relation and its transitive closure being a partial order. Section 3 deals with a central concept of our paper: ex-ante maximal points and their existence. We prove a criterion for existence of maximal elements in a space equipped with a variable preference relation. It is based on the idea of order-complete sets from vector optimization and the weak consistency introduced in Section 2. Section 4 is devoted to applications of the main existence result of Section 3 to several models of recent literature. Related notions such as ex-ante ideal elements, ex-post ideal elements and ex-post maximal elements are also briefly discussed. Applications of our approach to behavioral traps and equilibrium theory will be addressed in a forthcoming work.

2 Basic concepts and elementary properties

Let X be a set of actions or choices of an agent or a group of agents, or a set of states of an economic system. A preference relation on X that evolves from state to state is called a variable preference relation. Reference dependent, variable, dynamic and evolutionary preferences have the same meaning in our context. They all describe a system in which at a given state $x \in X$ an agent has a set of criteria that forms a preference relation allowing him to choose a new state to move to. At this new state, again, he has a set of criteria that may differ from the one he has had at x and generates a new preference relation for the next move. Examples of variable preference relations are found in multicriteria decision making, consumption theory, psychology and other areas of social sciences (see Giraud 2004 [12]), Kahneman and Tversky 1979 [14], Lindblom 1959 [17] and references given therein).

In this section we develop fundamental concepts related to variable preference relations and some of their elementary properties. Let us first recall basic definitions for constant preference relations. A preference relation P on X is a relation linking pairs of elements of X, that is, P is a binary relation on X

and defined by a subset $\mathcal{R}$ of the product space $X \times X$ as follows: for $x, y \in X$, one has xPy (we say y is preferred to x) if and only if $(x, y) \in \mathcal{R}$. A preference relation P is said to be reflexive if xPx for all $x \in X$; it is irreflexive if xPx is true for no $x \in X$; it is antisymmetric if xPy and yPx implies $x = y$; and it is transitive if xPy and yPz imply xPz. A reflexive, antisymmetric and transitive preference relation is called a partial order. Sometimes irreflexive partial orders are also considered. Reflexivity of a preference relation occurs in a system when an agent is happy with his position, that is, according to his criteria, the current position is acceptable for his next move. Irreflexivity occurs when an agent is unhappy with his position and definitely wishes to move to another position at the next step. If a preference relation P is not reflexive, mathematically one may generate an associated reflexive preference relation by adding the diagonal of the space $X \times X$ to the set $\mathcal{R}$. Similarly, if a preference relation is not irreflexive, by deleting the diagonal from the set $\mathcal{R}$ one obtains an irreflexive preference relation. In general, assuming a preference relation reflexive is widely accepted by researchers and practitioners. Moreover, if P is reflexive, a strict preference relation associated to P is defined to be an irreflexive preference relation P' smaller than P in the sense that $xP'y$ implies xPy (or equivalently the set $\mathcal{R}'$ defining P' is a subset of $\mathcal{R}$). When a reflexive preference relation P is transitive, one defines equivalent classes on X as follows: $x' \in [x]$ if and only if xPx' and $x'Px$. The preference relation induced by P on equivalent classes is denoted $\bar{P}$ and defined by

$$[x]\bar{P}[y] \text{ if and only if } x'Py' \text{ for some } x' \in [x], y' \in [y].$$

Then it is clear that $\bar{P}$ is reflexive, antisymmetric and transitive on the set $[X]$ of all equivalent classes of X. For a reflexive preference relation P on X, we distinguish two kinds of strict preference relations associated with P:

$$xP'y \qquad \text{if and only if } xPy \text{ and not } yPx; \tag{1}$$
$$xP''y \qquad \text{if and only if } xPy \text{ and } x \neq y. \tag{2}$$

Here is a link between these strict preference relations.

Lemma 1 *If P is a reflexive preference relation on X, then*

$$xP'y \textit{ implies } xP''y.$$

Conversely, if P is transitive, then for the induced preference relation $\bar{P}$ on the equivalent classes on X one has

$$[x]\bar{P}''[y] \textit{ implies } [x]\bar{P}'[y].$$

Consequently the two strict preference relations associated with P coincide.

Proof. The first part of the lemma is clear. For the second assertion, assume that $[x]\bar{P}''[y]$. If (1) failed for the induced preference relation, we would have xPy and yPx implying $[x] = [y]$, which shows that (2) does not hold for $\overline{P}$. ■

Throughout this section we will assume that the space X is equipped with a variable preference relation $\{" \leq_x " : x \in X\}$ and that $" \leq_x "$ is reflexive. Given an element $x \in X$, the following sets are of particular interest at the individual or collective levels, ex-ante (before moving) and ex-post (after moving) levels :
i) The "ex-ante" dominant and dominated sets at x,

$$\begin{aligned} S_+(x) &= \{y \in X : x \leq_x y\} \\ S_-(x) &= \{y \in X : y \leq_x x\}. \end{aligned}$$

Thus, $S_+(x)$ can be viewed as the set of states that dominates x by the criteria at x, that is, states to which from his own point of view, being at the state x, an agent or a group of agents wish to move; while $S_-(x)$ contains all states that are dominated by x, that is, states to which, being at x and according to their own judgements, an agent or a group of agents do not want to move.
ii) The "ex-post" dominant and dominated sets at x,

$$\begin{aligned} F_+(x) &= \{y \in X : x \leq_y y\} \\ F_-(x) &= \{y \in X : y \leq_y x\}. \end{aligned}$$

The terminology "ex-post" refers to the fact that, starting from x and moving to y, an agent or a group of agents prefer staying at $y \in F_+(x)$ to coming back to x, this being judged by criteria at y after having moved from x.

From now on we shall fix the strict preference relation $" <_x "$ associated to $" \leq_x "$ by relation (1), that is $x <_x y$ if and only if $x \leq_x y$ and $y \not\leq_y x$ (the negation of $y \leq_y x$). The ex-ante strict dominant and dominated sets at x are defined in a similar manner with strict preference relation $" <_x "$ instead of $" \leq_x "$ and denoted respectively $S_>(x)$ and $S_<(x)$:

$$\begin{aligned} S_>(x) &= \{y \in X : x <_x y\} \\ S_<(x) &= \{y \in X : y <_x x\}. \end{aligned}$$

Likewise, the sets of ex-post strict dominant and dominated sets at x are denoted $F_>(x)$ and $F_<(x)$. An evident relationship between elements of the above sets is the following:

$$\begin{aligned} &y \in S_+(x) \text{ (resp. } y \in S_>(x)) &\text{if and only if} \quad &x \in F_-(y) \text{ (resp. } x \in F_<(y)) \\ &y \in S_-(x) \text{ (resp. } y \in S_<(x)) &\text{if and only if} \quad &x \in F_+(y) \text{ (resp. } x \in F_>(y)). \end{aligned} \tag{3}$$

In general, when the variable preference relation is not constant, the ex-ante and ex-post dominant/ dominated sets are distinct. Let us now define paths of acceptable changes or improving paths. Given two states x and y of X, an upward path from x to y is a finite sequence of elements $x_1, ..., x_n \in X$ such that

$$x = x_1 \leq_{x_1} x_2 \leq_{x_2} x_3 ..... \leq_{x_{n-2}} x_{n-1} \leq_{x_{n-1}} x_n = y. \tag{4}$$

An upward path from x to y means that an agent, being at the state x_k, prefers to move to x_{k+1} along this path. A downward path from x to y is a sequence of elements $x_1, ..., x_n \in X$ such that

$$x = x_1 \leq_{x_2} x_2 \leq_{x_3} x_3 ... \leq_{x_{n-1}} x_{n-1} \leq_{x_n} x_n = y.$$

This means that once arrived at the state x_k an agent prefers staying at this state to coming back to the previous state x_{k-1}. When $n = 2$ the above paths are called direct; otherwise they are called indirect (because y is preferred to x not directly, but through intermediate states x_i for $1 < i < n$).

Since variable preference relations are generally not transitive, upward paths and downward paths provide a way to obtain transitive preference relations on X. Let us define the upper transitive closure and the lower transitive closure of the variable preference relation " $\leq_x$ " as follows:

- $x \leq^u y$ if and only if there is an upward path from x to y;
- $x \leq^\ell y$ if and only if there is a downward path from x to y.

Upper and lower transitive closures being transitive preference relations, we may define equivalent classes on X. The induced preference relations are denoted respectively " $\preceq^u$ " and " $\preceq^\ell$ ". In view of Lemma 1 there is no distinction between strict preference relations by (1) and (2), so we denote them by " $\prec^u$ " and " $\prec^\ell$ ". Below is a relationship between a variable preference relation and its transitive closure.

Proposition 2 *Assume that $x \leq^u y$ is given by (4). Then*

$$\begin{array}{ll} y \not\leq^u x & \text{implies } x_{k+1} \not\leq_{x_{k+1}} x_k \text{ for some } k \in \{1, ..., n-1\} \\ y \neq x & \text{implies } x_{k+1} \neq x_k \text{ for some } k \in \{1, ..., n-1\}. \end{array}$$

Moreover, if $[x_{k+1}] \not\preceq^u [x_k]$ for some $k \in \{1, ..., n-1\}$, then $[y] \not\preceq^u [x]$.

Proof. The first relation follows from the fact that if $x_{k+1} \leq_{x_{k+1}} x_k$ for all $k \in \{1, ..., n-1\}$, then $y \leq^u x$ which contradicts the hypothesis. The second relation is evident. For the last assertion, suppose to the contrary that $[y] \preceq^u [x]$. There is an upward path from y to x. Then the upward path

$$x_{k+1} \leq_{x_{k+1}} x_{k+2} ... \leq_{x_n} y \leq^u x = x_1 \leq_{x_1} x_1 ... \leq_{x_{k-1}} x_k$$

yields $x_{k+1} \leq^u x_k$, implying $[x_{k+1}] = [x_k]$. This contradiction completes the proof. ∎

Related to paths of acceptable changes we present some basic concepts which are already known for constant preference relations. The concept of weak consistency seems to be new and generalizes acyclicity.

Transitivity. The variable preference relation " $\leq_x$ " is said to be transitive if $x \leq^u y$ implies $x \leq_x y$ for all $x, y \in X$. In other words if two states are joint by an indirect path, then they can be joint by a direct path.
Antisymmetry. The variable preference relation " $\leq_x$ " is said to be antisymmetric if $x \leq_x y$ and $y \leq_y x$ imply $x = y$. We notice that given $x_0 \in X$, as an individual preference relation on X, $(\leq_{x_0})$ need not be transitive or antisymmetric even if the variable preference relation is such.

Acyclicity. The variable preference relation " $\leq_x$ " is said to be acyclic if for every upward path from x to itself, all intermediate elements coincide with x. Equivalently, if there are finite numbers of elements $x_1, ..., x_n \in X$ such that an agent moves from $x = x_1$ to x_n through $x_2, ..., x_{n-1}$ and at least one intermediate state x_k is different from x, then $x \neq x_n$.
Path consistency. The variable preference relation " $\leq_x$ " is said to be path consistent if for every $x \in X$ there is no upward path from x to itself $x = x_1 \leq_{x_1} x_2 \leq_{x_2} x_3 ..... \leq_{x_{n-2}} x_{n-1} \leq_{x_{n-1}} x_n = x$ in which at least one intermediate preference is strict in the sense of (1), that is $x_{k+1} \not\leq_{x_{k+1}} x_k$ for some $k \in \{1, ..., n-1\}$. We see that acyclicity implies path consistency, but the converse is not true in general. Actually, acyclicity can be seen as path consistency when (2) is used instead of (1): there is no path from x to itself, say

$$x = x_1 \leq_{x_1} x_2 \leq_{x_2} x_3 ..... \leq_{x_{n-2}} x_{n-1} \leq_{x_{n-1}} x_n = x$$

such that at least one intermediate preference is strict in the sense that $x_k \neq x_{k+1}$ for some k.
Suzumura's consistency. The variable preference relation " $\leq_x$ " is said to be consistent in Suzumura's sense if whenever there is an upward path from x to $y \neq x$, either there is a direct path from x to y, or no direct path from y to x. In other words, for every $x, y \in X, x \neq y$ and $x_1, ..., x_n \in X$ such that $x = x_1 \leq_{x_1} x_2 \leq_{x_2} x_3 ..... \leq_{x_{n-2}} x_{n-1} \leq_{x_{n-1}} x_n = y$ (the agent wants to pass indirectly from x to y), then either $x \leq_x y$ (the agent wants to pass directly from x to y) or $y \not\leq_y x$ (the agent does not want to pass directly from y to x). This definition is an adaptation of consistency by Bossert and Suzumura 2007 [7] for usual preference relations.
Weak consistency. The variable preference relation " $\leq_x$ " is said to be weak consistent if whenever there is a direct path from x to y and an indirect path from y to x, there is a direct path from y to x. In other words,

$$x \leq_x y \leq_y x_1 \leq_{x_1} x_2 \leq_{x_2} x_3 ..... \leq_{x_{n-2}} x_{n-1} \leq_{x_{n-1}} x_n = x \text{ implies } y \leq_y x.$$

This condition can be interpreted as an immediate "regret" of having moved from x to y after a long (indirect) way from y to x. Here is a relationship between the notions of consistency given above.

Proposition 3 *Every path consistent preference relation is Suzumura consistent, and every Suzumura consistent preference relation is weak consistent.*

Proof. Assume that $x = x_1 \leq_{x_1} x_2 \leq_{x_2} x_3 ..... \leq_{x_{n-2}} x_{n-1} \leq_{x_{n-1}} x_n = y$. If $y \leq_y x$, then we have an upward path $x_1, ..., x_n \in X$ such that $x = x_1 \leq_{x_1} x_2 \leq_{x_2} x_3 ..... \leq_{x_{n-2}} x_{n-1} \leq_{x_{n-1}} x_n = y \leq_y x$. If the variable preference relation is path consistent, then all intermediate relations are not strict in the sense of (1), that is, $x_{k+1} \leq_{x_{k+1}} x_k$ for $k = 1, ..., n-1$ and $x \leq_x y$. By this, the preference relation is Suzumura consistent. Now, let $x \leq_x y \leq_y = x_1 \leq_{x_1} x_2 \leq_{x_2} x_3 ..... \leq_{x_{n-2}} x_{n-1} \leq_{x_{n-1}} x_n = x$ be an upward path from x to itself. If the variable preference relation is Suzumura consistent, then one must have $y \leq_y x$, because $x \not\leq_x y$ is not true. ■

It can be seen without difficulty that the converse of Proposition 3 is not true. The utility of acyclicity and consistency is seen from the next result, see also Proposition 6.

Proposition 4 *Given a variable preference relation " $\leq_x$ " on X, the induced preference relation " $\preceq^u$ " is a partial order on equivalent classes of X. Moreover, the transitive closure " $\leq^u$ " is a partial order on X if and only if the variable preference relation is acyclic.*

Proof. We first observe that being reflexive and transitive the transitive closure " $\leq^u$ " defines equivalent classes on X. And the induced preference relation " $\preceq^u$ " is antisymmetric on equivalent classes, hence it is a partial order. Now assume that " $\leq^u$ " is a partial order on X. Let $x = x_1 \leq_{x_1} x_2 \leq_{x_2} x_3 ..... \leq_{x_{n-2}} x_{n-1} \leq_{x_{n-1}} x_n = x$ be an upward path from x to itself. Then for $k = 2, ..., n-1$ one has $x \leq^u x_k$ and $x_k \leq^u x$. By antisymmetry $x = x_k$. Hence the variable preference relation is acyclic. Conversely, assume the preference relation is acyclic. Then $x \leq^u y$ and $y \leq^u x$ imply $x \leq^u y \leq^u x$. Consequently, y is an intermediate element of an upward path from x to itself. By acyclicity, all intermediate elements of that path coincide with x, in particular $y = x$. The proof is complete. ■

In a recent work by Andrikopoulos and Zacharias 2009 [3] the authors have found some sufficient conditions under which a constant preference relation P is acyclic (respectively Suzumura's consistent) if and only if there there exist two functions $G : X \to [0, 1]$ and $C : X \times X \to [0, 1]$ such that for every $x, y \in X$, $xPy \iff G(y) - G(x) > C(x, y)$ (respectively either $G(y) - G(x) > C(x, y)$ or $G(y) = G(x)$.)

3 Ex-ante maximal points

As we have already seen in the previous sections, variable preference relations describe economies in movement. A move from a state x to a state y can be targeted or not. In a move without target the agent has no goal to orient his action, he just wishes to improve his situation as in muddling through process. By changing from $x \in X$ to $y \in X$, he passes from an ex-ante perception to an ex-post perception. In an ex-ante perception the agent starts from doing x and chooses to carry out a new action y, driven by an ex-ante motivation to change: $x \leq_x y$. From his initial point of view at x, he prefers moving to y than staying at x. In an ex-post perception, once arrived at y (having done the new action y), the agent compares, from his new point of view at y, his new situation y with the old one x. Either he does not regret, from his new point of view at y, to be at y because $x \leq_y y$, or he regrets his move because $y \leq_y x$. Given $x \in X$, the ex-ante dominant set $S_+(x)$ is exactly the ex-ante preference change set at x. It contains all possible improving moves from x according to the point of view at x. In a targeted move, a state x being given, the agent starts from a state y with $y \leq_y x$. He sets x as his target to move to. The ex-post dominated set $F_-(x)$ at x contains all states y at which the agent does not want to stay and from which he wishes to move to the target x. Then an important question is to know whether a state exists such that once arrived there, the agent stops his move, finding no better place to go. This is the concept of maximal points that we are going to develop now.

Throughout this section the space X is equipped with a reflexive variable preference relation " $\leq_x$ ". Below we define maximal points with respect to the variable preference relation, and give some characterizations via dominant and dominated sets. Then we introduce the concept of preference-complete sets borrowed from vector optimization (Luc 1989 [19], see also Application 2, Section 4) and prove a general criterion for existence of maximal points. A particular variant of this criterion is proposed when the space is equipped with a topology which turns to be quite useful in applications.

Definition 5 *An element x_* is said to be an ex-ante maximal point of X if there is no $y \in X$ such that the direct path from x_* to y is possible, but the direct path from y to x_* is impossible. It is said to be an ex-ante strict maximal point or a rest point of X if there is no point $y \neq x_*$ such that the direct path from x_* to y is possible, that is, $x_* \leq_{x_*} y$.*

As we have already discussed the terminology "ex-ante" comes from the fact that an agent or a group of agents starts from x_* and considers, before moving, the opportunity to move from x_* to some y. It is clear that every rest point (cycle of order zero) is an ex-ante maximal element (cycle of order zero and one). The converse is generally not true except for the case of antisymmetric preference relations. The definition above is applied to any binary relation on X. Let P be a reflexive and transitive preference relation on X, we say that x_* is P-maximal point of X if there is no $y \in X$ such that xPy and not yPx; and it is a P-rest point if there is no $y \in X$ such that xPy and $x \neq y$. In other words, x_* is a P-maximal point if there is no $y \in X$ such that $x_*P'y$, and it is a P-rest point if there is no $y \in X$ such that $x_*P''y$ in the sense of (1) and (2) respectively. Again we notice that when P is antisymmetric, these two notions coincide (see Duggan 2007 [10]).

Proposition 6 *Let x_* be given. The following assertions hold.*

1) There is equivalence between:

   (a) x_* is an ex-ante maximal element;

   (b) $S_>(x_*) = \emptyset$;

   (c) $x_* \notin \cup_{y\in X} F_<(y)$.

2) There is equivalence between:

   (d) x_* is a rest point;

   (e) $S_+(x_*) = \{x_*\}$;

   (f) $x_* \notin \cup_{y\in X, y\neq x_*} F_-(y)$.

3) If x_* is $\leq^u$-rest point, then it is a rest point.

4) If the variable preference is weak consistent and if an equivalent class $[x_*]$ is $\preceq^u$-maximal on $[X]$, then x_* (and any element of this class) is ex-ante maximal on X.

Proof. Equivalence between (a) and (b) follows from the definition. Equivalence between (b) and (c) follows from the relation (3). The proof of the second assertion uses the same argument. For the third assertion we assume that x_* is a $\leq^u$-rest point. If for some $y \in X$ one has $x_* \leq_{x_*} y$, then $x_* \leq^u y$. By the hypothesis $x_* = y$. Hence x_* is a rest point of X. To prove the last assertion, assume that the class $[x_*]$ is $\preceq^u$-maximal. Suppose that $x_* \leq_{x_*} y$ for some $y \in X$. Then $[x_*] \preceq^u [y]$. By $\preceq^u$-maximality, we have $[y] \preceq^u [x_*]$, which implies $y \leq^u x_*$. Hence there is an upward path from y to x_*:

$$y = x_1 \leq_{x_1} x_2 ... x_{n-1} \leq_{x_{n-1}} x_n = x_*.$$

Completing this path by adding $x_* \leq_{x_*} y$ to the first term of the path we obtain an upward path from x_* to itself:

$$x_* \leq_{x_*} y = x_1 \leq_{x_1} x_2 ... x_{n-1} \leq_{x_{n-1}} x_n = x_*.$$

In view of weak consistency we deduce $y \leq_y x_*$. This proves that x_* is an ex-ante maximal element of X. ■

Maximal elements include the case of personal habits, organizational routines, behavioral traps, personal equilibrium (Koszegi and Rabin 2006 [16]), Nash equilibrium, generalized Nash equilibrium, evolutionary stable equilibrium (Thomas 1985 [26]), norms, conventions (Young 1993 [29]), innovation frontier, Pareto solutions, bargaining solutions, focal points (Schelling 1960 [21]), ideal points, cycling points (Duggan 2007 [10]), recurrent points or recurrent sets, drops, top cycle set, untrapped set etc.

Existence of maximal elements without topology. We recall some standard definitions related to monotone nets. A net $\{y_\alpha\}_{\alpha \in I}$, where I is a directed index set, is said to be (upper) strictly increasing if $y_\alpha <^u y_\beta$ whenever $\alpha < \beta$. It is said to be dominated by a net $\{x_\beta\}_{\beta \in I'}$ if $I \subseteq I'$ and for every $\alpha \in I$ there is some $\beta \in I'$ such that $y_\alpha \leq^u x_\beta$. Given a point $a \in X$, the upper section of X at a is denoted by $S(a) := \{x \in X : a \leq^u x\}$.

Definition 7 *A subset of X is said to be preference-complete (or P-complete for short) if it has no covering of the type $\{X \setminus S(x_\alpha) : \alpha \in I\}$ with $\{x_\alpha\}_\alpha$ a strictly increasing net in that subset.*

Notice that in the above definition no topological structure is necessary on the space X. However, in most applications X is topological and we shall explicitly assume it when necessary. Sufficient conditions for P-completeness will be given later when we consider specific models (see Corollaries 10-15). For the moment let us present the main existence result of this section. There are several ways to prove it under acceptance of Zorn's lemma (see also Flores-Bazan et al. 2008 [11] for constant preferences). The approach below is much inspired by the one of Luc 1989 [18] for efficient points in vector optimization and based on Proposition 6.

Theorem 8 *Assume that the variable preference relation* "$\leq_x$" *on* X *is weakly consistent. Then* X *has ex-ante maximal points if and only if there is some point* $a \in X$ *such that the upper section of* X *at* a *is P-complete.*

The proof of this theorem as well as its applications (Section 4) are given in Appendix. We underline that the merit of the above theorem resides in the sufficient part: if X admits a P-complete section, then it has maximal element. It is to note that when X has maximal elements, it may have sections that are not P-complete. As we'll see later the concept of P-completeness is quite convenient in deriving different kinds of existing conditions for maximal elements when the preference relation is constant. Let us interpret P-completeness in terms of aspiration states. Given a strictly increasing net $\{x_\alpha, \alpha \in I\}$ in X, the family $\{X \setminus S(x_\alpha) : \alpha \in I\}$ is a covering of X if and only if for each $x \in X$ there is some index $\alpha \in I$ such that $x \notin S(x_\alpha)$. Consequently, P-completeness of X is equivalent to the fact that for each strictly increasing net as above, there is a point $x_* \in X$ such that $x_* \in S(x_\alpha)$ for all $\alpha \in I$. This point x_* can be considered as an "aspiration state" of the net. When I is the set of the natural numbers $\mathbb{N}$, a sequence $\{x_n : n = 1, 2, ...\}$ with $x_n \leq_{x_n} x_{n+1}$ is called an improving process. If x_* is an aspiration state of the improving process $\{x_n : n = 1, 2, ...\}$, then it belongs to the ex-ante dominant set $S_+(x_n)$ for all $n \geq 1$. Hence along the improving process if the agent discovers an aspiration state, he moves directly to it without passing through the whole process, economizing time, money and energy. If in addition the aspiration state x_* is ex-ante strict maximal, then it is called an aspiration driven equilibrium of the improving process. Thus, an aspiration driven equilibrium of the process $\{x_n : n = 1, 2, ...\}$ satisfies two conditions: a) $x_* \in S_+(x_n)$ for all $n \geq 1$, and b) $S_+(x_*) = \{x_*\}$. In the framework of partially ordered spaces the hypothesis of existence of aspiration points has already been given by Brezis and Browder under the name of inductivity hypothesis in their famous principle in Brezis and Browder 1976 [8]. In fact, suppose that X is equipped with a partial order $(\leq)$. The inductivity hypothesis is satisfied if a) every improving sequence $\{x_n\}_{n\geq 1}$ with $x_n \leq x_{n+1}$ has an upper bound, and b) there exists a potential function $\Phi : X \to \mathbb{R}$ which is bounded above and increasing, i.e. $\overline{\Phi} = \sup\{\Phi(x), x \in X\} < +\infty$ and $x \leq y \Longrightarrow \Phi(y) \geq \Phi(x)$. It is clear that for every $x \in X$, there exists $x_* \in X$ such that $x \leq x_*$ and $\Phi(y) = \Phi(x_*)$ for all y such that $x_* \leq y$. Moreover, if in addition $x \leq y$ and $x \neq y$ imply $\Phi(x) < \Phi(y)$, then $S_+(x_*) = \{x_*\}$, which means that x_* is an aspiration driven equilibrium. The value $\Phi(x_*)$ is called an improving goal payoff.

Before presenting a variant of Theorem 8 under the presence of a topology on X, we stress once more the fact that Theorem 8 is valid for reflexive variable preferences which are not necessarily transitive, or antisymmetric. This is an important point because it allows to study a lot of anomalies raised by Kahneman and Tversky 1979 in [14] and their followers (we'll address this issue and applications of variable preferences in another study).

Existence of maximal elements with topology. Throughout this part we assume that X is a topological space. The concepts of upper closedness and upper compactness to introduce below are essential for the study of variable preferences in topological spaces.

Definition 9 *The variable preference relation " $\leq_x$ " on X is said to be upper closed if for every convergent strictly increasing net $\{y_\alpha\}_{\alpha\in I}$ with limit y, either $S_>(y)$ is empty, or there is some $z \in X$ such that $y_\alpha <^u z$ for all $\alpha \in I$. The set X is said to be upper compact if every upper strictly increasing net in X is dominated by a strictly increasing net that has a convergent subnet.*

Compact sets are first evident examples of upper compact sets because every net of a compact set has a convergent subnet. The converse is not true in general. For instance the set of all negative numbers together with 0 is not compact, but upper compact, the topology and order being usual. As to upper closedness of variable preference relations, an immediate sufficient condition is that all upper sections are closed, that is, for every $a \in X$, the upper section $\{x \in X : a \leq^u x\}$ is closed. In fact, let $\{x_i\}_{i\in I}$ be a strictly increasing net converging to x in X. Let $i, j \in I$ with $j > i$. Then $x_j \leq^u x$ and $x_i <^u x_j$. Hence $x_i <^u x$ for every $i \in I$. A condition more general then closedness of upper sections is known as transfer upper continuity and will be given in Application 3. Of course, an upper closed preference relation may have non-closed upper sections. For instance the preference relation " $\preceq$ " on $[0, 2]$ defined by

$$x \preceq_x y \text{ if and only if } x \leq y \text{ and } y \neq 1,$$

is upper closed, but the upper section at 0 is not closed. The result below is an useful consequence of Theorem 8. It makes use of upper compactness and upper closedness, and is quite ready for applications.

Corollary 10 *Let X be a topological space. Assume that the variable preference relation " $\leq_x$ " is upper closed, weakly consistent and that for some point $a \in X$, the section $S(a)$ is upper compact. Then $S(a)$ admits a P-complete subsection and consequently X has ex-ante maximal points.*

Proof. Suppose to the contrary that $S(a)$ has no P-complete subsections. In particular there is a strictly increasing net $\{x_i\}_{i\in I}$ in $S(a)$ such that the family $\{S(a) \setminus S(x_i) : i \in I\}$ is a covering of $S(a)$. In view of upper compactness hypothesis there is a strictly increasing net in $S(a)$ that dominates $\{x_i\}_{i\in I}$ in $S(a)$ and admits a convergent subnet $\{y_j\}_{j\in J}$. It can be seen that the family $\{S(a) \setminus S(y_j) : j \in J\}$ is a covering of $S(a)$ too. Let y be the limit of the subnet $\{y_j\}_{j\in J}$. Since $S(a)$ has no P-complete subsection, y cannot be a maximal point, hence there is some $z \in S(a)$ such that $y_j \leq^u z$ for all $j \in J$ because the variable preference " $\leq_x$ " is upper closed. Consequently, $S(a)$ is not covered by the family $\{S(a) \setminus S(y_j) : j \in J\}$, a contradiction. The proof is complete. ■

4 Applications

In this section we show that a number of existing results for maximal elements from the literature under divers contexts can be derived from the criterion we've developed in the previous section. By our knowledge all of them deal with constant preference relations and many of them assume that the relation under consideration is a partial order.

Application 1: Generalization of the Brezis-Browder principle. Let X be a nonempty set (for us an action set) with a reflexive and transitive preference relation " $\leq$ ". It is said to be countably inductive (CIO set for short) if every nondecreasing sequence has an upper bound. Let W be a nonempty set (for us a payoff or value set) with a partial order " $\preceq$ ". It is said to be totally ordered upper separable if for every totally ordered nonempty set $M \subseteq W$, there is a nondecreasing sequence $\{v_n\}_{n\geq 1} \subset M$ such that every $v \in M$ is dominated by some v_n. In term of satisficing, this assumption means that one can always build a satisficing scale in any totally ordered nonempty payoff (utility) subspace $M \subset W$.

Corollary 11 *Assume that X is a CIO set, Y is a totally ordered upper separable set and $G : X \to Y$ is a nondecreasing map. Then for every $x_0 \in X$, there is some x_* with $x_0 \leq x_*$ such that $G(x) = G(x_*)$ for all x with $x_* \leq x$.*

The above corollary is the ordering principle by Zhu and Li in a recent work [31] which generalizes the well-known Brezis-Browder principle as we have already said.

Application 2: General multicriteria optimization. Let X be a nonempty set equipped with a partial order " $\leq$ " and let A be a nonempty subset of X. In multicriteria or vector optimization we are interested in finding maximal elements of A when X is a vector space and in most cases the order is generated by a convex cone. In a recent work [11] Flores-Bazan et al. have studied this question in a space without linear structure, generalizing the existing results in vector optimization. Their main result (Theorem 3.1) states that the set A has a maximal element if it is order-totally-complete in the sense that it has no covering of the form $X \setminus S_+(x), x \in D$, where D is a totally ordered subset of A. It is clear that an order-totally-complete set is P-complete. Hence Theorem 3.1 (see also Theorem 4.3) of Flores-Bazan et al. 2008 [11]) can be derived from Theorem 8 by a direct application. Note that in [11] (Definition 2.2) the authors use nonincreasing nets to define order-complete sets, while in [18] and in Definition 5, we use strictly increasing nets. Working with equivalent classes one may prove that a P-complete set is order-totally-complete, and so, applying Theorem 3.1 of [11] to the transitive closure " $\leq^u$ " is also an alternative proof of Theorem 8. Of course, the weak consistency must be then taken into account to produce a desired result. The proof given in Appendix details the argument by Zorn's lemma as exploited in [11] and [19].

Application 3: Transfer upper-continuous preference relation. We say that a variable preference relation " $\leq_x$ " is transfer upper-continuous if $x <_x y$ implies the existence of some $y' \in X$ and a neighborhood $U(x)$ in X such that $z <_z y'$ for all $z \in U(x)$. The concept of transfer upper-continuity was introduced in Tian and Zhou 1995 [25] for a fixed binary relation on X which generalizes upper continuity of binary relations ($x <_x y$ implies $z <_z y$ for all z in some neighborhood of x). The corollary below is a generalization of the main results of Bergstrom 1975 [6] and Tian and Zhou 1995 [25].

Corollary 12 *Assume that X is a compact set of a topological space and that the variable preference relation is acyclic and transfer continuous. Then X is upper compact, and the variable preference relation is upper closed and weakly consistent. Consequently X has ex-ante maximal points.*

It is to note that transfer continuity implies upper closedness. The converse is not true in general. Here is an example. Consider $X = A \cup B$ where

$$\begin{aligned} A &= \{\frac{2^n - 1}{2^n} : n = 1, 2, ...\} \\ B &= \{\frac{3^n - 1}{3^n} : n = 1, 2, ...\} \cup \{1; 2\}. \end{aligned}$$

A binary relation " $\preceq$ " is defined as follows: $x \preceq y$ if and only if $x \leq y$ (in the usual sense) and $x, y \in B$. It is evident that the preference relation generated by that binary relation is upper closed. It is not transfer continuous under the usual topology of the real numbers because $2 > 1$ and in every neighborhood of 1 there are elements of A which are not comparable with 2.

Application 4: Weakly tc-upper semicontinuous preference relation. We say that the variable preference relation " $\leq_x$ " is weakly tc-upper semicontinuous if it is acyclic and for $x <_x y$ there is an open set $U(x, y)$ satisfying

(i) $x \in U(x, y)$ and $y \notin U(x, y)$;

(ii) $F_-(z) \subseteq U(x, y)$ for every $z \in U(x, y)$;

(iii) $U(x, y) \subset U(z, w)$ and inclusion is strict if $x <_x y$, $z <_z w$ and $y \in U(z, w)$.

The concept of weak tc-upper semicontinuity was introduced in Zuanon 2009 [32] for binary relations which generalizes transfer continuity. As we shall see in the proof of the next corollary that this generalization can be reduced to the case of transfer continuity.

Corollary 13 *Assume that X is a compact set of a topological space and that the variable preference relation "$\leq_x$" is weakly tc-upper semicontinuous. Then X has ex-ante maximal points.*

Application 5: Upper continuous preference relation on "$\prec$"-upper compact sets. Let Y be a topological space and let "$\preceq$" be a partial order on Y. We say that Y is "$\preceq$"-upper compact if every covering of Y by a family of open free disposal sets of Y admits a finite subcovering. Recall that a set $D \subseteq Y$ is free disposal if $x \prec y, y \in D$ implies $x \in D$. The terminology "free disposal" is taken from Debreu 1959 [9] in which the usual Pareto partial order of $\mathbb{R}^n$ is considered. Free disposal sets are called lower comprehensive in Alcantud 2002 [1]. The concept of "$\prec$"-upper compactness was introduced in Alcantud 2002 [1] which generalizes compactness.

Corollary 14 *Assume that the variable preference relation "$\leq_x$" is acyclic such that the sets $\{y \in X : x <^u y\}, x \in X$ are open and that X is "$<^u$"-upper compact. Then X is P-complete and consequently it has ex-ante maximal points.*

The above corollary is essentially Theorem 3 of Alcantud 2002 [1] when the preference relation is constant.

Application 6: Consistent upper tc-S-semicontinuous preference relation. A (Suzumura) consistent preference relation "$\leq_x$" is called upper tc-S-semicontinuous (in the sense of Andrikopoulos and Zacharias 2009 [3]) if for every $x \in X$, the set $\{y \in X : y <^u x\}$ is open. We derive the following result of Andrikopoulos and Zacharias 2009 [3] (Theorem 3).

Corollary 15 *Assume that the variable preference relation "$\leq_x$" is consistent (in Suzumura's sense) and upper tc-S-semicontinuous and X is "$<^u$"-upper compact. Then X has ex-ante maximal points.*

Application 7: Maximum principle. Another generalization of the Brezis-Browder principle given by Szaz 2007 [24] (see also a less general version by Altman 1982 [2], Turinici 1984 [28] and Zeidler 1986 [30]) can also be obtained from Theorem 8. Let X be a nonempty set with a binary relation "$\leq$". Let Δ be a function on $X \times X$ with values in the extended real line $\overline{\mathbb{R}}$. Define a function γ on X by

$$\gamma(x) = \sup_{x \leq y} \Delta(x, y).$$

Here is a generalization of the maximum principle in Szaz 2007 [24].

Corollary 16 *Assume that there are a point $a \in X$ and a function $\Delta : X \times X \to \overline{\mathbb{R}}$ such that*

(i) *γ is decreasing;*

(ii) *$\gamma(x) > -\infty$ for all $x \in X$;*

(iii) $\gamma(a) < \infty$;

(iv) *Every increasing sequence* $\{x_n\}_{n\geq 1}$ *in* $S(a)$ *with* $x_1 = a$ *is bounded above and* $\liminf_{n\to\infty} \Delta(x_n, x_{n+1}) \leq \alpha$;

(v) $\Delta(x, y) > \alpha$ *for every* $x < y$.

Then X *has an ex-ante maximal point in* $S(a)$.

5 Related notions

Other concepts of maximality with respect to a variable preference relation is related to the best action of an agent or a group of agents when he compares ex-ante, before moving from x_*, the current action x_* with all others, and the concept of ex-post maximal points which is a counterpart of ex-ante maximal points when downward paths are used.

Definition 17 *An element* x_* *is said to be an ex-ante ideal point if* $y \leq_{x_*} x_*$ *for every* $y \in X$.

It is clear that an ideal point is maximal, but the converse is not true in general. For instance the segment joining the points $(1, 0)$ and $(0, 1)$ in $\mathbb{R}^2$ has no ideal points, but every point of it is maximal, the preference relation being the usual Pareto order in $\mathbb{R}^2$. An ideal point, if it exists, is not necessarily unique, except for the case when the preference relation is antisymmetric. It is easy to check that $x_* \in X$ is ex-ante ideal if and only if

$$S_-(x_*) = X,$$

or equivalently

$$x_* \in \cap_{y\in X} F_+(y). \tag{5}$$

We shall make use of the last condition to find sufficient conditions of existence of ex-ante ideal points. To this end let us recall the following result of [20]: *If a family* $\{A_i : i \in I\}$ *of subsets of a compact space is intersectionally closed and has the finite intersection property, then the intersection of its members is nonempty.* The family $\{A_i : i \in I\}$ of nonempty subsets in a topological space is intersectionally closed if $\mathrm{cl}(\cap_{i\in I} A_i) = \cap_{i\in I} \mathrm{cl}(A_i)$; and it has the finite intersection property if the intersection of any finite number of elements of the family is nonempty. Several sufficient conditions for intersectional closedness are found in Luc et al. 2010 [20].

Theorem 18 *Assume that* X *is compact and the following conditions hold:*

(i) *For a finite set* $D \subseteq X$, *there is a point* $x_D \in X$ *such that* $y \leq_{x_D} x_D$ *for every* $y \in D$;

(ii) *The family* $\{F_+(x) : x \in X\}$ *is intersectionally closed.*

Then X has ex-ante ideal points.

Proof. In view of (i), the family $\{F_+(x) : x \in X\}$ has the finite intersection property, and by (ii) it is intersectionally closed. According to the finite intersection principle, there is some $x_* \in F_+(x)$ for all $x \in X$. By (5) x_* is an ex-ante ideal point of X. ■

As a consequence of the above theorem we deduce an existence condition for quasi-convex preference relations. Assume that X is a convex subset of a vector space. The variable preference relation " $\leq_x$ " is quasi-convex if $x \leq_y y$ and $x \leq_z z$ imply $x \leq_u u$ for every $u = ty + (1-t)z, t \in [0,1]$.

Corollary 19 *Assume that X is a nonempty convex and compact subset of a Banach space, and that the following conditions hold:*

(i) For every $x \in X$ the set $F_+(x)$ is convex and closed;

(ii) The variable preference relation is quasi-convex.

Then X has ex-ante ideal points.

Proof. Since the set $F_+(x), x \in X$ are closed, Condition (ii) of Theorem 18 is satisfied. Moreover, (ii) implies that the family $\{F_+(x) : x \in X\}$ has the finite intersection property. It remains to apply Theorem 18 to complete the proof. ■

Ex-post maximal points. In this last part we consider a move from x to y, not from the "ex-ante" point of view, before moving from the origin x, but, "ex-post", after moving, from the point of view of the arrival point y. As before, " $\leq_x$ " is a reflexive variable preference relation on X.

Definition 20 *An element x_* is said to be ex-post maximal if there is no other element $y \in X$ such that $x_* <_y y$; and it is said to be ex-post ideal if $y \leq_y x_*$ for all $y \in X$.*

Then, an element x_* is ex-post maximal if there is no point y such that, once arrived at y, from the point of view of y, starting from x_*, an agent or a group of agents strictly prefer to stay at y than to come back to x_*. An element x_* is ex-post ideal if once moved to any y, and based on the judgement at y, an agent or a group of agents regret that move. We can see also that x_* is ex-post maximal if and only if

$$F_>(x_*) = \emptyset; \tag{6}$$

and it is ex-post ideal if and only if

$$F_-(x_*) = X, \tag{7}$$

or equivalently

$$x_* \in \cap_{x \in X} S_+(x). \tag{8}$$

Notice also that when the variable preference relation " $\leq_x$ " is constant in the sense that for every $x, y \in X$ the preference relations " $\leq_x$ " and " $\leq_y$ " coincide, there is no distinction between ex-ante and ex-post maximalities. It is important to note that existence conditions for ex-post maximal points follow the same pattern of ex-ante maximal points. Therefore we do not further go into details of these conditions. Let us give just one counterpart of Theorem 18 as an example.

Theorem 21 *Assume that X is a nonempty compact set and that the following conditions hold.*

(i) For a finite set $D \subseteq X$, there is a point $x_D \in X$ such that $x \leq_x x_D$ for every $x \in D$;

(ii) The family $\{S_+(x) : x \in X\}$ is intersectionally closed.

Then X has ex-post ideal points.

Proof. Apply the argument of Theorem 18 and the relation (8). ∎

We close up this section by observing that under (ii) of Theorem 21 Condition (i) is equivalent to the existence of ex-post ideal points. Moreover, the compactness of X can be relaxed by compactness of an upper section $S_+(x_0)$ for some $x_0 \in X$. This is because when the family of $S_+(x)$ is intersectionally closed, then the family of $S_+(x) \cap S_+(x_0)$ is also intersectionally closed. It is clear that the above observation is valid for ex-ante ideal points as well.

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6 Appendix

Proof of Theorem 8. The "only if " part is evident, by taking a among maximal points. To prove the "if" part we first consider the case where the upper transitive closure " $\leq^u$ " is a partial order on X. As we have seen in Proposition 4 this is the case when the variable preference is acyclic. We wish to prove that the set X has ex-ante maximal points. Suppose to the contrary that it is not the case. Then the set $\mathcal{P}$ of all strictly increasing nets in $S(a)$ is nonempty. Introduce a partial order on $\mathcal{P}$ by inclusion, i.e. for $a, b \in \mathcal{P}$ one writes $a \preceq b$ if and only if $a \subseteq b$ as sets. It is easy to see that this order is a partial order on $\mathcal{P}$ and one may apply Zorn's lemma to obtain a maximal element of $\mathcal{P}$. We give here a detailed proof of application of Zorn's lemma. First we prove that $\mathcal{P}$ satisfies the hypothesis of Zorn's lemma : every chain $D = \{a_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{P}$ has an upper bound. Indeed, denote by $\mathcal{B}$ the family of all finite subsets of Λ. For each $B \in \mathcal{B}$ we set $a_B := \bigcup_{\lambda \in B} a_\lambda$. It is evident that $a_B \in \mathcal{P}$. Now we put $a_0 = \cup\{a_B : B \in \mathcal{B}\}$. We show that a_0 is a strictly increasing net, which means that it belongs to $\mathcal{P}$. Indeed, let x and y be two distinct elements of a_0. There are two indices λ and λ' from Λ such that x is an element of a_λ and y is an element of $a_{\lambda'}$. Since D is a chain, we may assume that a_λ is included in $a_{\lambda'}$. Thus, both x and y are elements of the strictly increasing net $a_{\lambda'}$, and so either $x <^u y$ or $y <^u x$. This proves that a_0 is strictly increasing. Moreover, it is evident that $a \preceq a_0$ for all $a \in D$. Hence a_0 is an upper bound of X. According to Zorn's lemma there is a maximal element a_*, say $a_* = \{x_i\}_{i\in I} \in \mathcal{P}$. We claim that the family $\{x \in S(a) : x_i \not\leq^u x\}_{i\in I}$ is a covering of $S(a)$. Indeed, since a_* is maximal, there is no $x \in S(a)$ such that $x_i \leq^u x$ for all $i \in I$. In other words, for every $x \in S(a)$ there is some $i \in I$ such that $x_i \not\leq^u x$. Thus, the above mention family of sets is a covering of $S(a)$ and we arrive at a contradiction with the P-completeness hypothesis. By this X has an ex-ante maximal point. We now proceed to the case where the upper transitive closure " $\leq^u$ " is not a partial order. Consider the induced preference " $\preceq^u$ " on equivalent classes of X. It is a partial order on $[X]$. Denote by $\hat{S}(a)$ the section of $[X]$ at $[a]$ with respect to the induced order:

$$\hat{S}(a) = \{[x] \in [X] : [a] \preceq^u [x]\}.$$

It can be seen that $[x] \in \hat{S}(a)$ if and only if $x \in S(a)$. We show that $\hat{S}(a)$ is P-complete with respect to the induced preference. Indeed, let $\{[x_\alpha]\}_{\alpha\in I}$ be a strictly increasing net with respect to the induced preference in $\hat{S}(a)$. Then $\{x_\alpha\}_{\alpha\in I}$ is a strictly increasing net with respect to " $\leq^u$ " in $S(a)$. If $\hat{S}(a)$ were covered by the family $\{[X] \setminus \hat{S}(x_\alpha) : \alpha \in I\}$, then for every $x \in S(a)$ there would exist some $i \in I$ such that $[x] \in [X] \setminus \hat{S}(x_i)$. Then $[x] \notin \hat{S}(x_i)$, which implies $[x_i] \not\preceq^u [x]$, and so $x_i \not\leq^u x$. We deduce that $x \in X \setminus S(x_i)$, which contradicts the P-completeness of $S(a)$. Now we apply the first part of the proof to the induced preference to obtain a $\preceq^u$-maximal element $[x_*]$ of $[X]$. Since the variable preference is weakly consistent, in view of Proposition 6, x_* is an ex-ante maximal element of X as well. The proof is complete.

Proof of Corollary 11. By working on equivalent classes if necessary one may assume without loss of generality that X is partially ordered. Now define a new preference relation " $\lhd$ " by $x \lhd x'$ if and only if either $x = x'$, or $x < x'$ and $G(x) < G(x')$. It is clear that this relation is a partial order. Consider a section $S(x_0)$ of X at x_0 (with respect to the new relation). We show that it is P-complete. Indeed, let $\{x_i\}_{i\in I}$ be a strictly increasing net in $S(x_0)$, then $\{G(x_i)\}_{i\in I}$ is an increasing net, hence, by choosing a totally ordered subnet if necessary, we may assume it is a totally ordered set in Y. By hypothesis there is a sequence $\{x_n\}_{n\geq 1}$ from the above net such that for each $i \in I$ there is some $n(i)$ with $G(x_i) \leq G(x_{n(i)})$. We may assume $\{G(x_n)\}_{n\geq 0}$ is increasing, which implies that $\{x_n\}_{n\geq 0}$ is strictly increasing with respect to " $\leq$ ". By the CIO assumption, there is an upper bound of $\{x_n\}_{n\geq 0}$ which is also an upper bound for the net $\{x_i\}_{i\in I}$. Then that upper bound belongs to $S(x_i)$ for all $i \in I$ and the family $\{X \setminus S(x_i) : i \in I\}$ does not cover $S(x_0)$. Thus $S(x_0)$ is P-complete.

According to Theorem 8 there is a maximal point $\overline{x}$ in $S(x_0)$ for the relation " $\lhd$ " which is a point we look for.

Proof of Corollary 12. Since the preference relation is acyclic, it is weakly consistent. The upper compactness of X follows from its compactness as we have already mentioned. It remains to prove the upper closedness of the variable preference relation. Let $\{x_\alpha\}_{\alpha\in I}$ be a convergent strictly increasing net in X. Let x be its limit. If $S_>(x)$ is empty, we are done. If $S_>(x)$ is nonempty, say $x <_x y$ for some $y \in X$, then by transfer continuity, there is some $y' \in X$ and a neighborhood U such that $z <_z y'$, $z \in U$. In particular, there is $\alpha_0 \in I$ such that $x_\alpha <_{x_\alpha} y'$ for all $\alpha > \alpha_0$. This implies that $x_\alpha <^u y'$ for all $\alpha \in I$ (choose $\beta \in I$ with $\alpha < \beta$ and $\alpha_0 < \beta$, then $x_\alpha <^u x_\beta$ and $x_\beta <_{x_\beta} y'$ implying $x_\alpha <^u y'$). Thus, all the assumptions of Corollary 10 hold and so X has maximal points.

Proof of Corollary 13. Let $\{U_\lambda : \lambda \in \Lambda\}$ denote the family of all open sets in the definition of weak tc-upper semicontinuity. Define a new preference relation on X by $x \prec y$ if there is some U_λ such that $x \in U_\lambda$ and $y \notin U_\lambda$. It is easy to see that this new preference added by the diagonal of the product space $X \times X$ is a partial order on X which is upper continuous, hence transfer continuous. In view of Corollary 12 the set X has a maximal element with respect to that new order. It is clear that that maximal element is also maximal with respect to the given variable preference relation on X.

Proof of Corollary 14. Since the preference is acyclic, it is weakly consistent as well. We prove that X is P-complete. Indeed, let $\{x_\alpha\}_{\alpha\in I}$ be a strictly increasing net in X. Consider the family $A_\alpha := X \setminus cl(S(x_\alpha)), \alpha \in I$, where $cl(.)$ stands for the closure. It is clear that every member of this family is open and free disposal. If the family $\{X \setminus S(x_\alpha) : \alpha \in I\}$ were a covering of X, then so would the family $\{A_\alpha : \alpha \in I\}$ be, and by hypothesis, it would admit a finite covering, say $A_{\alpha_i}, i = 1, ..., n$. Take any $\beta > \alpha_i, i = 1, ..., n$. Then $x_{\alpha_i} <^u x_\beta, i = 1, ..., n$. This implies that $x_\beta \notin A_{\alpha_i}$ for all $i = 1, ..., n$ and we arrive at a contradiction. It remains to apply Theorem 8 to complete the proof.

Proof of Corollary 15. Denote by $[X]$ the space of equivalent classes of elements of X with respect to the preference " $\leq_x$ "; and equip $[X]$ with the quotient topology and the quotient preference. It is known that when the preference is consistent, the quotient preference is a partial order. Moreover, with respect to the quotient topology, the quotient preference is continuous and $[X]$ is upper compact. Applying Corollary 13 to the space $[X]$ we obtain a maximal class, say $[x]$. Then any element of this class is an ex-ante maximal point of X

with respect to the given variable preference (see Proposition 6).

Proof of Corollary 16. Consider the section $S(a)$. It follows from (i)-(iii) that $\gamma(x)$ is finite for every $x \in S(a)$. It is also clear from (i) that the transitive closure of the binary relation " $\leq$ " is path consistent. Thus, by working on equivalent classes if necessary one may assume without loss of generality that the transitive closure " $\leq^u$ " is a partial order on $S(a)$. We show that the section $S(a)$ is P-complete. Indeed, let $\{x_i\}_{i \in I}$ be a strictly increasing net in $S(a)$, then one may choose an increasing (with respect to the transitive closure) sequence $\{x_{i(n)}\}_{n \geq 1}$ from that net such that

$$\lim_{n \to \infty} \gamma(x_{i(n)}) = \lim_i \gamma(x_i).$$

Here the limit may take the value $-\infty$. Then for every $i \in I$, there is some $i(n)$ such that $x_i \leq x_{i(n)}$. Indeed, given $i \in I$, by the choice of the sequence $\{x_{i(n)}\}_{n \geq 1}$ there is $i(n)$ such that $\gamma(x_{i(n)} \leq \gamma(x_i)$. The decreasingness of γ implies that $x_i \leq x_{i(n)}$ as requested. Then we deduce that if $S(a)$ is covered by the family $\{x \in S(a) : x_i \not\leq^u x\}, i \in I$, then it is covered by the family $\{x \in S(a) : x_{i(n)} \not\leq^u x\}, n = 1, 2, ..$ too. Notice that the sequence $\{x_{i(n)}\}_{n \geq 1}$ is increasing with respect to the binary relation " $\leq$ ". In view of (iv), it has an upper bound, say $b \in S(a)$ such that $x_{i(n)} \leq b$ for every $n \geq 1$. But then b does not belong to $\{x \in S(a) : x_{i(n)} \not\leq^u x\}, n = 1, 2, ...$ Consequently $S(a)$ is P-complete and by Theorem 8, the set $S(a)$ has a maximal point which is also a maximal point of X.