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# Homogenization at different linear scales and bounded martingales

Kévin Santugini-Repiquet

September 9, 2011

## Abstract

In this short paper, we look at two scale limits of sequences with varying homogenization periods, each period being a multiple of the previous one. We establish that, up to a measure preserving rearrangement, these two scale limits form a martingale which is bounded: the rearranged two scale limits themselves converge both strongly in  $L^2$  and almost everywhere when the period tend to  $+\infty$ . This limit contains all the information present in all the two scale limits in the sequence.

## 1 Introduction

Homogenization is used to study the solutions of equations when there are multiple scales of interest, usually a microscopic one and a macroscopic one. In particular, one may study the solutions  $u_\varepsilon$  of a PDE with quasi  $\varepsilon$ -periodic coefficient study their behavior as the small period  $\varepsilon$  tend to 0. Two scale convergence, introduced by M. Neuss Radu [5], and G. Allaire [1] is suited to study this particular subset of homogenization problems called periodic homogenization. It can also be used in the presence of periodic holes in the geometry, see [2, 3] or to homogenize multilayers [8, 7].

Intuitively, two scale convergence introduces the concept of two scale limit  $u_0$  which is a function of both a macroscopic variable  $\mathbf{x}$  and a microscopic  $p$ -periodic variable  $\mathbf{y}$  such that, in some “meaning”,  $\mathbf{x} \mapsto u_0(\mathbf{x}, \mathbf{x}/\varepsilon)$  is a good approximation of  $u_\varepsilon$ .

As indicated by its name, two scale convergence captures the behavior at two scales: the macroscopic one and the  $p\varepsilon$ -periodic one. However, two scale convergence does not capture all phenomena that happens at a scale linear in  $\varepsilon$  but only those whose length scale is  $p\varepsilon/m$  where  $m$  is an integer. The

two scale limit of a sequence depends, not only on the asymptotic scale, but also on the precise value of the chosen period. For example, any phenomena happening at the length scale of  $2\varepsilon$  will not be fully apparent in the two scale limit computed with period  $\varepsilon$ . The two scale limit computed with period  $2\varepsilon$  will contain no less —and might actually contain more— information than the two scale limit computed with period  $\varepsilon$ . For example, the homogenization of  $\sin(2\pi x/\varepsilon) + \sin(\pi x/\varepsilon)$  gives a two scale limit of  $u_0 : (x, y) \mapsto \sin(2\pi y)$  if computed with the homogenization period  $\varepsilon$ , *i.e.* when  $p = 1$ , and  $u_0 : (x, y) \mapsto \sin(2\pi y) + \sin(\pi y)$  if computed with the homogenization period  $2\varepsilon$ , *i.e.* when  $p = 2$ . Furthermore, if we choose  $p = 1/2$ , then the two-scale limit is none other than the null function. Worse, the scale factor  $p$  could be irrational. The choice of the scale factor  $p$  used in the homogenization process is therefore of utmost importance in two scale convergence. Using a badly chosen scale factor  $p$  may and will often cause a huge loss of information. At worst, we recover no more information than the one obtained by the standard weak  $L^2$  limit: if  $p\varepsilon$  is the correct choice of homogenization period, the two scale limit computed with period  $\lambda p\varepsilon$  where  $\lambda$  is an irrational number should, intuitively, carry no information about what happens at scale  $p\varepsilon$ .

Fortunately, there is usually a natural choice of period: the coefficients of the PDE are often chosen quasi-periodic in  $\varepsilon$ . The most natural choice is to choose  $p = 1$ , *i.e.* to consider the correct microscopic scale for  $u_\varepsilon$  is  $\varepsilon$  itself. If there are two important periods to consider  $\varepsilon_1$  and  $\varepsilon_2$ , the intuitive solution is to choose a period that is a multiple of both. However, this can only be done if the ratio  $\varepsilon_1/\varepsilon_2$  is a rational number.

To increase the information obtained by two scale convergence, we may also consider an homogenization period of  $m\varepsilon$  instead of  $\varepsilon$ . This is usually not apparent in homogenization without holes or with an acceptable sequences of holes[3], because usually in these cases, the sequence of solutions, after extension to the domain without holes,  $(u_\varepsilon)_\varepsilon$  is bounded in  $H^1$  and the two scale limit does not depend on the fast variable, see [5, 1]. However, if we homogenize the solutions of PDE on domains with periodic non acceptable holes such as multilayers, then the sequence of solutions, after extension to the domain without holes, is no longer bounded in  $H^1$ . And the two scale limit might no longer be independent of the fast variable. Nevertheless, it often only takes a finite number —one per connected component of the periodic cube with holes— of values when the fast variable varies, see for example [8, Theorem 6.1] for such a two scale limit. In that case, homogenizing with period  $m\varepsilon$  with  $m \geq 2$  might give a more precise result than homogenizing with period  $\varepsilon$ .

In this paper, we look at two scale limits of sequences for different choices of the period. In particular, we consider a sequence of periods  $(p_n)_{n \in \mathbb{N}}$  such

that for all integers  $k$ ,  $p_{n+1}/p_n$  is a positive integer and we study the two scale limit of  $(u_\varepsilon)_\varepsilon$  computed with the homogenization period  $p_n\varepsilon$ . This two scale limit, denoted  $u_{0,p_n}$  is  $p_n$ -periodic in each of its fast variable. Since  $p_{n+1}$  is always a multiple of  $p_n$ , one can always recover the two scale limit  $u_{0,p_n}$  from the two scale limit  $u_{0,p_{n+1}}$ . Actually, if  $p_{n+1} = m_n p_n$  and we place ourselves in dimension  $d \geq 1$ , we have

$$u_{0,p_{n+1}}(\mathbf{x}, \mathbf{y}) = \frac{1}{m_n^d} \sum_{\alpha \in \llbracket 0, m_n - 1 \rrbracket^d} u_{0,m_n p_n}(\mathbf{x}, \mathbf{y} + \alpha).$$

The sequence of two scale limits  $(u_{0,p_n})_{n \in \mathbb{N}}$  yields increasing information on the asymptotic behavior of  $(u_\varepsilon)_\varepsilon$ . A natural question is to wonder whether the two scale limits  $u_{0,p_n}$  themselves converge whenever  $n$  tends to  $+\infty$ . *I.E.* does there exist a function that carry the information of all the  $p_n$ -two scale limits? The goal of our paper is to answer this question. The answer is positive. We show in this paper that the sequence of two scale limits is, after a measure preserving rearrangement, a bounded martingale in  $L^2$  and therefore converges both strongly in  $L^2$  and almost everywhere.

In §2, we remind the reader about the known theorems of two scale convergence. In §3, we show how the different two scale limits are bound by martingale type equalities and explain how to transform these two scale limits to get a real martingale. This leads to our stating of our main theorem: Theorem 3.5.

## 2 Notations and the classical notion of two scale convergence

First, as in [1], we introduce some notations. In this paper,  $p$  always refer to a scale factor. It remains constant while taking the two scale limit. However, the goal of this paper is to observe the behavior of the two scale limits as  $p$  tend to  $+\infty$ .

By  $\Omega$ , we denote an open domain of  $\mathbb{R}^d$  where  $n \geq 1$ . By  $Y_p$ , we denote the cube  $[0, p]^d$ . By  $L^2_\#(Y_p)$ , we denote the space of measurable functions defined over  $\mathbb{R}^d$ , that are  $p$ -periodic in each variable and that are square integrable over  $Y_p$ . By  $\mathcal{C}_\#(Y_p)$ , we denote the continuous functions defined on  $\mathbb{R}^d$  that are  $p$ -periodic in each variable.

We reproduce the now classical definition of two scale convergence found in [1, 5, 6]. For convenience, we added the scale factor  $p$

**Definition 2.1** (Two scale convergence). Let  $p$  be a positive real. A sequence  $(u_\varepsilon)_{\varepsilon > 0}$  belonging to  $L^2(\Omega)$  is said to  $p$ -two scale converge if there exist  $u_{0,p}$

in  $L^2(\Omega \times Y_p)$  such that:

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_{\varepsilon}(\mathbf{x}) \psi \left( \mathbf{x}, \frac{\mathbf{x}}{\varepsilon} \right) d\mathbf{x} = \frac{1}{p^d} \int_{\Omega} \int_{Y_p} u_{0,p}(\mathbf{x}, \mathbf{y}) \psi(\mathbf{x}, \mathbf{y}) d\mathbf{y} d\mathbf{x}, \quad (2.1)$$

for all  $\psi$  in  $L^2(\Omega; \mathcal{C}_{\#}(Y_p))$ .

G. Allaire, see [1], and M. Neuss Radu, see [5, 6], proved that any sequence of functions bounded in  $L^2$  has a subsequence that two scale converges. Let's reproduce this precise compactness result.

**Theorem 2.2.** *Let  $(u_{\varepsilon})_{\varepsilon > 0}$  be a continuous sequence of functions belonging to  $L^2(\Omega)$ , then there exist  $u_{0,p}$  in  $L^2(\Omega \times (0, 1))$  and a subsequence  $\varepsilon_k$  converging to 0 such that*

$$\lim_{k \rightarrow \infty} \int_{\Omega} u_{\varepsilon_k}(\mathbf{x}) \psi \left( \mathbf{x}, \frac{\mathbf{x}}{\varepsilon_k} \right) d\mathbf{x} = \frac{1}{p^d} \int_{\Omega} \int_{Y_p} u_{0,p}(\mathbf{x}, \mathbf{y}) \psi(\mathbf{x}, \mathbf{y}) d\mathbf{y} d\mathbf{x}, \quad (2.2)$$

for all  $\psi$  in  $L^2(\Omega; \mathcal{C}_{\#}(Y))$ .

*Proof.* See G. Allaire [1] and M. Neuss Radu [5, 6]. The presence of the scale factor  $p$  has no impact on the proof.  $\square$

We also have the classical proposition

**Proposition 2.3.** *Let  $u_{\varepsilon}$   $p$ -two scale converge to  $u_{0,p}$ , then*

$$\frac{1}{p^d} \|u_{0,p}\|_{L^2(\Omega \times Y_p)} \leq \liminf_{\varepsilon \rightarrow 0} \|u_{\varepsilon}\|_{L^2(\Omega)}.$$

*Proof.* See G. Allaire [1] and M. Neuss Radu [5, 6]. The presence of the scale factor  $p$  has no impact on the proof.  $\square$

The next proposition is easy to derive from Theorem 2.2

**Proposition 2.4.** *Let  $(p_n)_{n \in \mathbb{N}}$  be an increasing sequence of positive real numbers. Let  $(u_{\varepsilon})_{\varepsilon > 0}$  be a continuous sequence of function belonging to  $L^2(\Omega)$ , then there exist a subsequence  $\varepsilon_k$  converging to 0, and a sequence of functions  $u_n$  such that for any integer  $n$  the sequence  $u_{\varepsilon_k}$   $p_n$ -two scale converges to  $u_{0,p_n}$ . I.E., such that for any integer  $n$ :*

$$\lim_{k \rightarrow \infty} \int_{\Omega} u_{\varepsilon_k}(\mathbf{x}) \psi \left( \mathbf{x}, \frac{\mathbf{x}}{\varepsilon_k} \right) d\mathbf{x} = \frac{1}{p_n^d} \int_{\Omega} \int_{Y_{p_n}} u_{0,p_n}(\mathbf{x}, \mathbf{y}) \psi(\mathbf{x}, \mathbf{y}) d\mathbf{y} d\mathbf{x},$$

for all  $\psi$  in  $L^2(\Omega; \mathcal{C}_{\#}(Y_{p_n}))$ .

*Proof.* Apply Theorem 2.2 multiple times and proceed via diagonal extraction.  $\square$

Our goal in this paper is to study the limit of  $u_{0,p_n}$  as  $p_n$  tends to  $+\infty$ .

### 3 Rearrangement of two scale limits and bounded martingales

In this section, we always consider a sequence of scale factors  $(p_n)_{n \in \mathbb{N}}$  such that  $p_{n+1}$  is a multiple of  $p_n$  for every nonnegative integer  $n$ . We set for  $n \geq 1$ ,  $m_n := p_n/p_{n-1}$  and for  $n \geq 0$   $M_n := p_n/p_0$ . We always consider a continuous sequence of functions  $(u_\varepsilon)_{\varepsilon > 0}$  bounded in  $L^2(\Omega)$  and a decreasing sequence of positive  $(\varepsilon_k)_{k \in \mathbb{N}}$  such that the sequence  $(u_{\varepsilon_k})_{k \in \mathbb{N}}$   $p_n$ -two scale converges for all integers  $n$ . This is justified by Proposition 2.4.

To choose the sequence  $p_n$ , one may set  $p_0$  first, then set  $p_n = M_n p_0$  and choose the sequence of integers  $M_n$  such that  $M_{n+1}$  is always a multiple of  $M_n$  and such that any integer eventually divide  $M_n$  when  $n$  is large enough. For example, one may set  $M_n = n!$  or  $M_n$  being the smallest multiple of all positive integers smaller than  $n$ .

Our goal is to study the convergence of the two scale limits  $u_{0,p_n}$  when  $k$  goes to infinity. In this section, we proceed as follows: we begin by establishing a useful equality that looks like a martingale equality in §3.1, then we propose a rearrangement of the two scale limits in §3.2 and finally propose another rearrangement of the two scale limits in §3.3 which transform the sequence of two scale limits into a bounded martingale.

#### 3.1 An almost martingale equality

Consider a sequence  $(p_n)_{k \in \mathbb{N}}$  such that  $p_{n+1}/p_n$  is a positive integer for all  $k$ . We begin by deriving the  $p$ -two scale limit from the  $mp$ -two scale limit when  $m$  is an integer.

**Proposition 3.1.** *Let  $m$  be an integer. Let  $p$  be a positive scale factor. Let  $(u_\varepsilon)_{\varepsilon > 0}$  be a sequence of functions belonging to  $L^2(\Omega)$ , and  $p$ -two scale converging to  $u_{0,p}$  and  $mp$  two scale converging to  $u_{0,mp}$ . Then, for almost all  $\mathbf{x}$  in  $\Omega$  and  $\mathbf{y}$  in  $Y_p$ :*

$$u_{0,p}(\mathbf{x}, \mathbf{y}) = \frac{1}{m^d} \sum_{\boldsymbol{\alpha} \in \llbracket 0, m-1 \rrbracket^d} u_{0,mp}(\mathbf{x}, \mathbf{y} + \boldsymbol{\alpha} p).$$

*Proof.* Let  $\phi$  belong to  $C^\infty(\bar{\Omega} \times \mathbb{R}^d)$  be  $p$ -periodic in the last  $d$  variables. Since  $m$  is an integer,  $\phi$  is also  $mp$ -periodic in the last  $d$  variables. We take the limit of  $\int_{\Omega} u_\varepsilon(\mathbf{x}) \phi(\mathbf{x}, \mathbf{x}/\varepsilon) d\mathbf{x}$ , as  $\varepsilon$  tend to 0, in the sense of two scale

convergence for both scale factors  $mp$  and  $p$ :

$$\begin{aligned} \frac{1}{p^d} \int_{\Omega} \int_{Y_p} u_{0,p}(\mathbf{x}, \mathbf{y}) \phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \, d\mathbf{y} &= \\ &= \frac{1}{m^d p^d} \int_{\Omega} \int_{Y_{mp}} \left( \sum_{\boldsymbol{\alpha} \in \llbracket [0, m-1] \rrbracket^d} u_{0,mp}(\mathbf{x}, \mathbf{y} + \boldsymbol{\alpha}) \right) \phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \, d\mathbf{y}, \end{aligned}$$

for all  $\phi$  in  $\mathcal{C}^\infty(\bar{\Omega} \times \mathbb{R}^d)$ . □

This simple but essential proposition provides all we need to show that up to a rearrangement the sequence of two scale limits  $u_{0,p_n}$  is actually a martingale.

We derive the following corollary.

**Corollary 3.2.** *For all  $\mathbf{x}$  in  $\Omega$ ,  $\mathbf{y}$  in  $\mathbb{R}^d$  and  $j, n$  in  $\mathbb{N}$ , we have*

$$u_{0,p_n}(\mathbf{x}, \mathbf{y}) = \left( \frac{p_n}{p_{n+j}} \right)^d \sum_{\boldsymbol{\alpha} \in \llbracket [0, p_{n+j}/p_n - 1] \rrbracket^d} u_{0,p_{n+j}}(\mathbf{x}, \mathbf{y} + \boldsymbol{\alpha} p_j).$$

Ideally, we would like to consider the limit of  $u_{0,p_n}$  as  $n$  tend to  $+\infty$ . Actually, because of this equality, the sequence  $u_{0,p_n}$  is “morally” a martingale for the filtration made of these  $\sigma$ -fields:

$$\mathcal{F}_n = \mathcal{B}(\Omega) \times (\mathcal{B}(\mathbb{R}^n) + p_n \mathbb{Z}^d).$$

Unfortunately, it isn't technically true as all the sets belonging to these  $\sigma$  fields are of infinite measure and the  $u_{0,p_n}$  are all periodic. However, the equalities defining what is a martingale are satisfied if one replaces the standard integral of  $\mathbb{R}^d$  by the limit of the mean over a ball as its radius tends to  $+\infty$ . *I.E.* we have for all  $F$  in  $\mathcal{F}_n$

$$\begin{aligned} \lim_{R \rightarrow +\infty} \frac{1}{|B(\mathbf{0}, R)|} \int_{\Omega} \int_{B(\mathbf{0}, R)} 1\{(\mathbf{x}, \mathbf{y}) \in F\} u_{0,p_n}(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \, d\mathbf{x} &= \\ &= \lim_{R \rightarrow +\infty} \frac{1}{|B(\mathbf{0}, R)|} \int_{\Omega} \int_{B(\mathbf{0}, R)} 1\{(\mathbf{x}, \mathbf{y}) \in F\} u_{0,p_{n+j}}(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \, d\mathbf{x}, \end{aligned}$$

where  $B(\mathbf{0}, R)$  is the open ball of  $\mathbb{R}^d$  centered on 0 and of radius  $R$  and where  $|A|$  is the Lebesgue measure of set  $A$ . Unfortunately, we were unable to derive a direct convergence result using this pseudo martingale equality. To proceed further, we need to rearrange the two scale limits to get true martingales.

### 3.2 Rearrangement of the two scale limits with integers

In this subsection, we rearrange the two scale limits as functions defined over  $\Omega \times \mathbb{Z}^d \times Y_{p_0}$ . While we were unable to obtain a convergence in that case, the ideas of this sections provide insight on the next section in which we prove convergence for another rearrangement. Given the two scale limits  $u_{0,p_n}$ , we define

$$v_{p_n} : \Omega \times \mathbb{Z}^d \times Y_{p_0} \rightarrow \mathbb{R},$$

$$(\mathbf{x}, \boldsymbol{\alpha}, \mathbf{y}) \mapsto u_{0,p_n}(\mathbf{x}, \mathbf{y} + M_n \boldsymbol{\alpha}).$$

where  $M_n = p_n/p_0$ .

This rearrangement of the two scale limit is natural for homogenization with a non admissible set of holes of period  $p_0\varepsilon$ . In that case, if  $u_\varepsilon$  is regular enough, for example  $H^1$  inside each hypercube, then the  $u_{0,p_n}$  depend only on  $\mathbf{x}$  and on the hypercube of dimension  $p_0$  to which  $\mathbf{y}$  belong. In many applications, the  $v_{p_n}$  depend only on  $\mathbf{x}$  and  $\boldsymbol{\alpha}$  but not on  $\mathbf{y}$ .

We have the following proposition

**Proposition 3.3.** *For all  $n$  in  $\mathbb{N}$ , for almost all  $\mathbf{x}$  in  $\Omega$  and all  $\mathbf{y}$  in  $Y_{p_0}$ , the sequence  $(v_{p_n}(\mathbf{x}, \boldsymbol{\alpha}, \mathbf{y}))_{\boldsymbol{\alpha} \in \mathbb{Z}^d}$  is  $p_n/p_0$ -periodic in each direction. Moreover:*

$$v_{p_n}(\mathbf{x}, \mathbf{y})(\mathbf{x}, \boldsymbol{\alpha}, \mathbf{y}) = \left( \frac{M_n}{M_{n+j}} \right)^d \sum_{\boldsymbol{\beta} \in \llbracket 0, M_{n+j}/M_n - 1 \rrbracket^d} v_{p_{n+j}}(\mathbf{x}, \boldsymbol{\alpha} + M_n \boldsymbol{\beta}, \mathbf{y}). \quad (3.1)$$

*Proof.* This is a direct consequence of Proposition 3.1. □

This in turn should encourage us to look at the following problem.

**Problem 3.4.** Let's call sequences  $(w_{n,\boldsymbol{\alpha}})_{n \in \mathbb{N}, \boldsymbol{\alpha} \in \mathbb{Z}^d}$  that are  $M_n$  periodic in each component of  $\boldsymbol{\alpha}$  and that satisfy

$$t_{n,\boldsymbol{\alpha}} = \left( \frac{M_n}{M_{n+j}} \right)^d \sum_{\boldsymbol{\beta} \in \llbracket 0, M_{n+j}/M_n - 1 \rrbracket^d} t_{n+j,\boldsymbol{\alpha} + M_n \boldsymbol{\beta}}.$$

“imbricated  $(M_n)_n$ -periodic  $d$ -dimensional sequences”. Study the convergence of such sequences as  $n$  tend to  $+\infty$ . Under which condition does there exists a sequence  $t_{\infty,\boldsymbol{\alpha}}$  such that for all positive integers  $n$

$$t_{n,\boldsymbol{\alpha}} = \lim_{N \rightarrow +\infty} \left( \frac{1}{N^d} \right)^d \sum_{\boldsymbol{\beta} \in \llbracket 0, N - 1 \rrbracket^d} t_{\infty,\boldsymbol{\alpha} + M_n \boldsymbol{\beta}}.$$



or such that

$$t_{n,\alpha} = \lim_{N \rightarrow +\infty} \left( \frac{1}{2^d N^d} \right)^d \sum_{\beta \in \llbracket -N, N-1 \rrbracket^d} t_{\infty, \alpha + M_n \beta}.$$

or both?

By Proposition 3.3, for almost all  $\mathbf{x}$  in  $\Omega$  and all  $\mathbf{y}$  in  $Y_{p_0}$ , the sequences  $(v_{n,\alpha})_{\alpha \in \mathbb{Z}^d}$  are imbricated  $(p_n)_n$ -periodic  $d$ -dimensional sequences. Solving Problem 3.4 would be the first step in having a very elegant limit to the  $v_{n,\alpha}$  as a function defined on  $\Omega \times \mathbb{Z}^d$ . Unfortunately, we do not have an answer for Problem 3.4. While this sequence is morally a martingale with respect to the filtration made of the  $\sigma$ -fields  $\{\alpha + \frac{p_n}{p_0} \mathbb{Z}^d, \alpha \in \llbracket 0, \frac{p_n}{p_0} - 1 \rrbracket\}$ , it technically is not: we have the same problem we had in the previous section. To conclude with bounded martingales on the convergence, we would need a measure  $\mu$  on  $\mathbb{Z}^d$  such that  $\mu(\mathbb{Z}^d) = 1$ , invariant by translation and such that  $\mu(m\mathbb{Z}^d) = 1/m$  whenever  $m$  is an integer different from 0. Such a measure cannot be  $\sigma$ -additive. If we remove the  $\sigma$  additivity constraint, then  $\mu$  exists: just set

$$\mu(A) = \lim_{N \rightarrow +\infty} \frac{\#(A \cap \llbracket -N, N \rrbracket^d)}{(2N+1)^d}.$$

It is unknown to the author if bounded martingales converge when they are defined on a non  $\sigma$ -additive measure. To avoid that problem, we introduce, in the next section, a different less natural rearrangement for the  $u_{0,p_n}$  for which we finally prove a convergence result.

### 3.3 Continuous rearrangement of two scale limits

In this section, we finally construct a rearrangement that results in a bounded martingale. And since bounded martingales in  $L^2$  converge both strongly in  $L^2$  and almost everywhere, this establishes a convergence result for the  $u_{0,p_n}$  as  $n$  tend to  $+\infty$ .

Set for  $\mathbf{x}$  in  $\Omega$ ,  $\mathbf{y}$  in  $[0, p_0]^d$  and  $\mathbf{y}'$  in  $[0, 1]^d$ . The variable  $\mathbf{y}'$  replaces the variable  $\alpha$  of the previous section. Consider first

$$w_n(\mathbf{x}, \mathbf{y}, \mathbf{y}') = v_{p_n}(\mathbf{x}, \alpha(\mathbf{y}'), \mathbf{y})$$

where  $\alpha(\mathbf{y}')_i = \lfloor M_n \mathbf{y}'_i \rfloor$  for all integers  $i$  in  $\llbracket 1, d \rrbracket$ . We're now going to permute the hypercubes as in Figure 1 where to simplify the drawing, we considered homogenization only on the last component of  $\mathbb{R}^d$ , hence the presence of layers instead of hypercubes. In that figure, we show one step of the rearrangement. As seen in the drawing, each step of the rearrangement is measure preserving,

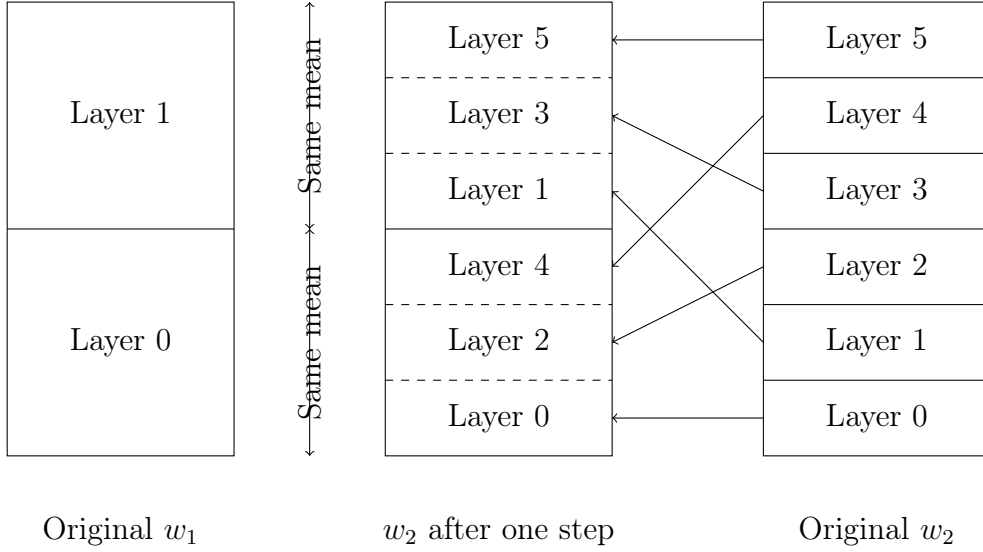


Figure 1: One step of the measure preserving rearrangement  $M_1 = 2$  and  $M_2 = 6$

therefore the full rearrangement is also measure preserving. We need  $n - 1$  such steps to fully rearrange  $w_n$ .

To define rigorously this rearrangement, we begin by defining the function that maps the rearranged layer index onto the unrearranged layer index, see Figure 1:

$$R_{M,m}(i) = M(i \bmod m) + \left\lfloor \frac{i}{m} \right\rfloor. \quad (3.2a)$$

for all  $i$  in  $\llbracket 0, Mm - 1 \rrbracket$ . Then we set the function that maps the rearranged layer onto the unrearranged one:

$$h_{M,m}^* : [0, 1) \rightarrow [0, 1),$$

$$y' \mapsto \frac{R_{M,m}(\lfloor Mmy' \rfloor)}{Mm} + \left( y' - \frac{\lfloor Mmy' \rfloor}{Mm} \right). \quad (3.2b)$$

This represents only one step on the rearrangement on one component. For hypercubes, the permutation is the same but is done componentwise: we set

$$h_{M,m} : (0, 1)^d \rightarrow (0, 1)^d,$$

$$(y'_1, \dots, y'_n) \mapsto (h_{M,m}^*(y'_1), \dots, h_{M,m}^*(y'_n)).$$

And obtain one step of the rearrangement on all  $d$  components. For the

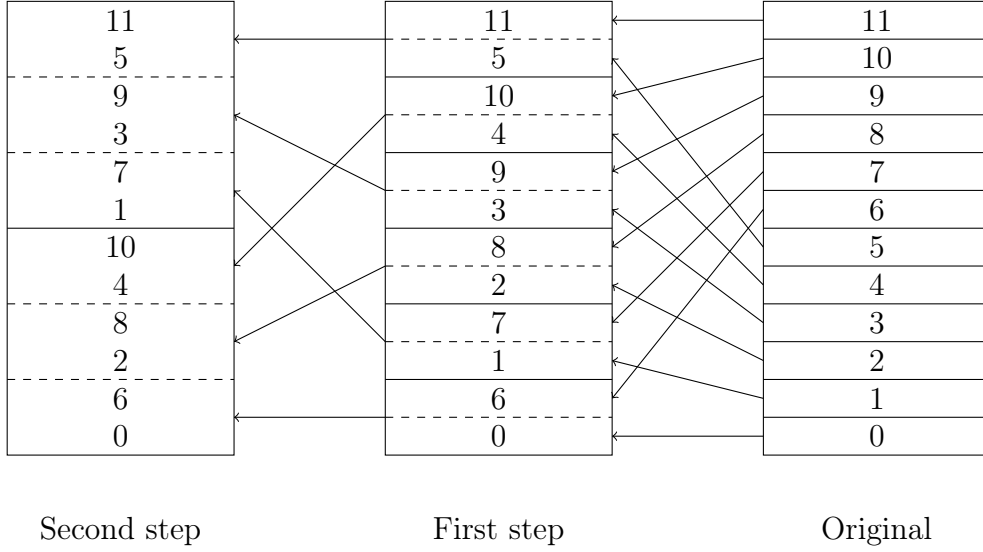


Figure 2: Two steps of the measure preserving rearrangement  $M_1 = 2$ ,  $M_2 = 6$  and  $M_3 = 12$

complete rearrangement on one component, see Figure 2, we set

$$H_n^* := h_{M_{n-1}, m_n}^* \circ \dots \circ h_{M_1, m_2}^* \circ h_{M_0, m_1}^*. \quad (3.2c)$$

To get the complete rearrangement on all components we set

$$\begin{aligned} H_n &: [0, 1]^d \rightarrow [0, 1]^d, \\ (y'_1, \dots, y'_n) &\mapsto (H_n^*(y'_1), \dots, H_n^*(y'_n)). \end{aligned} \quad (3.2d)$$

We also have

$$H_n = h_{M_{n-1}, m_n} \circ \dots \circ h_{M_1, m_2} \circ h_{M_0, m_1}.$$

Finally, we define

$$\tilde{w}_n(\mathbf{x}, \mathbf{y}, \mathbf{y}') := w_n(\mathbf{x}, \mathbf{y}, H_n(\mathbf{y}')). \quad (3.3)$$

This measure preserving rearrangement is purposefully constructed so the  $\tilde{w}_n$  form a martingale for the following filtration of  $\sigma$ -fields  $\mathcal{F}_n = \mathcal{B}(\Omega) \times \mathcal{B}([0, p_0]^d) \times \sigma \left\{ \prod_{i=1}^d \left[ \frac{\beta_i}{M_n}, \frac{\beta_i+1}{M_n} \right), \beta \in \mathbb{Z}^d \right\}$ .

*Remark 1.* The above rearrangement of hypercubes is similar to the one used for computing in place the Discrete Fast Fourier transform: the bit reversal. In the special case where  $M_n = 2^n$ , the rearrangement simply exchanges

layers  $i$ , i.e.  $[i/2^n, (i+1)/2^n)$ , and  $i'$ , i.e.  $[i'/2^n, (i'+1)/2^n)$ , when  $i$  and  $i'$  are bit reversal permutations of each other. I.E. when  $i = \sum_{k=0}^{N-1} b_k 2^k$  and  $i' = \sum_{k=0}^{N-1} b_k 2^{N-1-k}$ .

We now state our complete self contained theorem.

**Theorem 3.5.** *Let  $\Omega$  be an open domain of  $\mathbb{R}^d$  with  $d \geq 1$ . Let  $(u_\varepsilon)_{\varepsilon>0}$  be a bounded continuum of functions belonging to  $L^2(\Omega)$ . Let  $(p_n)$  be an increasing sequence of positive numbers such that for all integer  $n$  the ratio  $p_{n+1}/p_n$  is an integer. Set for all  $n \geq 0$   $M_n := p_n/p_0$  and for all  $n \geq 1$   $m_n = p_n/p_{n-1}$ . Let  $(\varepsilon_k)_{k \in \mathbb{N}}$  be a decreasing sequence of positive real numbers converging to 0 such that the sequence  $(u_{\varepsilon_k})_{k \in \mathbb{N}}$   $p_n$ -two scale converges to  $u_{0,p_n}$  for all non negative integer  $n$ .*

Set

$$\begin{aligned} \tilde{w}_n : \Omega \times [0, p_0]^d \times [0, 1]^d &\rightarrow \mathbb{R} \\ (\mathbf{x}, \mathbf{y}, \mathbf{y}') &\mapsto u_{0,p_n}(\mathbf{x}, p_0 \lfloor M_n H_n(\mathbf{y}') \rfloor + \mathbf{y}). \end{aligned}$$

where  $H_n$  is defined by Equations (3.2).

Then, the sequence  $\tilde{w}_n$  is a bounded martingale in  $L^2$  for the filtration

$$\mathcal{F}_n = \mathcal{B}(\Omega) \times \mathcal{B}([0, p_0]^d) \times \sigma \left\{ \prod_{i=1}^d \left[ \frac{\beta_i}{M_n}, \frac{\beta_i + 1}{M_n} \right), \boldsymbol{\beta} \in \mathbb{Z}^d \right\}. \quad (3.4)$$

And the sequence  $\tilde{w}_n$  converges both strongly in  $L^2$  and almost everywhere to  $\tilde{w}_\infty$ . Moreover,

$$\iiint_A \tilde{w}_n(\mathbf{x}, \mathbf{y}, \mathbf{y}') \, d\mathbf{y}' \, d\mathbf{y} \, d\mathbf{x} = \iiint_A \tilde{w}_\infty(\mathbf{x}, \mathbf{y}, \mathbf{y}') \, d\mathbf{y}' \, d\mathbf{y} \, d\mathbf{x},$$

for all sets  $A$  in  $\mathcal{F}_n$ . I.E.,  $\tilde{w}_n = \mathbb{E}(\tilde{w}_\infty | \mathcal{F}_n)$ .

*Proof.* The  $\tilde{w}_n$  were constructed specifically so as to be a martingale for the filtration (3.4). To prove they are a martingale for the filtration  $(\mathcal{F}_n)_{n \in \mathbb{N}}$ , we only need to prove that for every nonnegative integer  $n$ , we have for almost all  $\mathbf{x}$ , almost all  $\mathbf{y}$  in  $[0, p_0]$  and for all  $\boldsymbol{\beta}$  in  $\llbracket 0, M_n \rrbracket^d$  we have

$$\int_{\prod_{i=1}^d \left[ \frac{\beta_i}{M_n}, \frac{\beta_i + 1}{M_n} \right)} \tilde{w}_{n+1}(\mathbf{x}, \mathbf{y}, \mathbf{y}') \, d\mathbf{y}' = \int_{\prod_{i=1}^d \left[ \frac{\beta_i}{M_n}, \frac{\beta_i + 1}{M_n} \right)} \tilde{w}_n(\mathbf{x}, \mathbf{y}, \mathbf{y}') \, d\mathbf{y}'$$

I.E., we need to show that

$$\int_{\prod_{i=1}^d \left[ \frac{\beta_i}{M_n}, \frac{\beta_i + 1}{M_n} \right)} w_{n+1}(\mathbf{x}, \mathbf{y}, H_{M_n, m_{n+1}} \circ H_n(\mathbf{y}')) \, d\mathbf{y}' = \int_{\prod_{i=1}^d \left[ \frac{\beta_i}{M_n}, \frac{\beta_i + 1}{M_n} \right)} w_n(\mathbf{x}, \mathbf{y}, H_n(\mathbf{y}')) \, d\mathbf{y}'$$

But  $H_n$  maps any hypercube  $[\beta_i/M_n, (\beta_i + 1)/M_n)$  to another hypercube  $[\beta'_i/M_n, (\beta'_i + 1)/M_n)$  and  $H_n$  is measure preserving therefore, we only need to prove that for almost all  $\mathbf{x}$ , almost all  $\mathbf{y}$  in  $[0, p_0]$  and for all  $\beta$  in  $\llbracket 0, M_n \rrbracket^d$

$$\int_{\prod_{i=1}^d [\frac{\beta_i}{M_n}, \frac{\beta_i+1}{M_n})} w_{n+1}(\mathbf{x}, \mathbf{y}, H_{M_n, m_{n+1}}(\mathbf{y}')) \, d\mathbf{y}' = \int_{\prod_{i=1}^d [\frac{\beta'_i}{M_n}, \frac{\beta'_i+1}{M_n})} w_n(\mathbf{x}, \mathbf{y}, \mathbf{y}') \, d\mathbf{y}'$$

is satisfied. But this equality is equivalent to

$$\begin{aligned} \frac{1}{M_n^d} v_{p_n}(\mathbf{x}, \beta, \mathbf{y}) &= \\ &= \frac{1}{M_{n+1}^d} \sum_{\beta' \in \llbracket 0, m_{n+1} \rrbracket^d} v_{p_{n+1}}(\mathbf{x}, R_{M_n, m_{n+1}}(m_{n+1}\beta_i + \beta'_i), \mathbf{y}) = \\ &= \frac{1}{M_{n+1}^d} \sum_{\beta' \in \llbracket 0, m_{n+1} \rrbracket^d} v_{p_{n+1}}(\mathbf{x}, (\beta_i + \beta'_i M_n), \mathbf{y}) \end{aligned}$$

which is true by Proposition 3.3. Therefore, the sequence  $\tilde{w}_n$  is a martingale for the filtration  $(\mathcal{F}_n)_{n \in \mathbb{N}}$ .

This martingale is bounded by Proposition 2.3 in  $L^2$ . It converges both strongly in  $L^2(\Omega \times [0, p_0]^d \times [0, 1]^d)$  and almost everywhere to a function  $\tilde{u}_\infty$ , see [4, Corollary 7.22].  $\square$

## 4 Conclusion

We have proven in this paper that the two scale limits of a given sequence of functions computed for periods that are multiple of the previous ones, form a bounded martingale and thus converge both strongly in  $L^2$  and almost everywhere. From the limit, one can recover any element in the sequence of two scale limits: this limit contains all the information contained in the whole sequence of two scale limits. For a good choice of increasing periods, this limit captures everything that happens at any length scale that is a multiple of  $\varepsilon$ .

Unfortunately, this limit does not capture all phenomena with a period linear in  $\varepsilon$ : it cannot capture phenomena with an irrational scale factor. The construction of the martingale depends on the assumption that  $p_{n+1}$  is always a multiple of  $p_n$ . If there are two interesting scales whose ratio is irrational then no choice of periodic scale carry the information for both scale.

While we were able to conclude using the rearrangement described in §3.3, we feel results on the existence of the limit in the setting of subsection 3.2 would be more satisfying. Solving 3.4 would be a first step to obtain a limit in this setting.

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