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# Non quadratic local risk-minimization for hedging contingent claims in the presence of transaction costs

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**Summary.** Following up on the new criterion introduced in Abergel and Millot [1] for hedging contingent claims in incomplete markets, we extend the approach to the case where there are transaction costs on the stock component. The local risk is a convex function of the local costs process. We derive corresponding optimal strategies in both discrete time and continuous time settings. Finally we give an application of our hedging method in the stochastic volatility case as well as in the jump diffusion case.

## 1 Motivations

While the non-quadratic local risk-minimization method is flexible in that it allows to choose a function  $f$  to assess risk, one of its strengths is also to enable the consideration of market imperfections, namely liquidity costs.

In the quadratic framework introduced by Schweizer [17], introduction of bid/ask spread into the picture has been tackled in a discrete time setting by Lambertson, Pham and Schweizer [14] and its extension to continuous time does not seem obvious. On the other hand, taking liquidity costs into account while using self-financing strategies was nicely undertaken by Cetin, Jarrow and Protter [5] through the introduction of a stochastic supply curve.

We propose to use their idea to model liquidity costs in the recently introduced non-quadratic local risk-minimization framework [1] and derive the equations driving the optimal hedge together with the theoretical portfolio value. Numerical implementation should demonstrate the robustness of this approach even with the presence of market imperfections and help choose the best function for assessing risk.

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## 2 Discrete time setting

To better motivate our continuous time presentation we start by investigating the discrete time setting and study existence and uniqueness of solutions to the minimization problem. We first place ourselves in a multi-period model where the evolution of the stock price is given by a strictly positive semi-martingale  $S_k$ , ( $k = 0, \dots, T$ ) on some probability space  $(\Omega, \mathcal{F}, P)$ . Let  $\mathcal{F}_k$  denote the  $\sigma$ -field of events which are observable up to and including time  $k$ . We assume that  $S_k$  is adapted and square-integrable and that the conditional variance of its returns  $\mathbb{E}((S_{k+1} - S_k)^2 | \mathcal{F}_k) - \mathbb{E}(S_{k+1} - S_k | \mathcal{F}_k)^2$  is strictly positive  $P$  almost surely. In order to avoid complicated notations, we work with zero interest rates but this is not a loss of generality as it basically amounts to having discounted stock prices with the existence of a riskless asset.

A contingent claim is described by a square-integrable random variable  $H \in L^2(P)$  that will be assumed of the following form  $H = \delta^H S_T + \beta^H$  with  $\delta^H$  and  $\beta^H$  being  $\mathcal{F}_T$ -measurable random variables. Thus we are considering European type options that can have a cash settled component and a deliverable one.

For example, a call option with strike  $K$  would correspond to  $H = (S_T - K)^+ = 1_{S_T \geq K} S_T - K 1_{S_T \geq K}$ , so that  $\delta^H = 1_{S_T \geq K}$  and  $\beta^H = -K 1_{S_T \geq K}$ .

### 2.1 Trading strategies

A trading strategy  $\Phi$  is given by two stochastic processes  $\delta_k$ , ( $k = 0, \dots, T$ ) and  $\beta_k$ , ( $k = 0, \dots, T$ ).  $\delta_k$  is the amount of stock held in period  $k$ , ( $= [t_k, t_{k+1})$ ) and has to be fixed at the beginning of that period, *i.e.* we assume that  $\delta_k$  is  $\mathcal{F}_k$ -measurable ( $k = 0, \dots, T$ ).

The amount  $\beta_k$  held in the market account in period  $k$  is fixed at the beginning of this period too, *i.e.* we assume that  $\beta_k$  is  $\mathcal{F}_k$ -measurable ( $k = 0, \dots, T$ ).

We further assume that both  $\delta$  and  $\beta$  are in  $L^2(P)$ .

For such a trading strategy, the theoretical value of the portfolio at time  $k$  is given by

$$V_k = \delta_k S_k + \beta_k, \quad (k = 1, \dots, T)$$

and so it is its value just after applying the strategy. We admit only strategies such that each  $V_k$  is square-integrable and such that the contingent claim  $H$  is produced in the end, *i.e.* we require  $V_T = H$ , which can always be met with our measurability requirements upon taking  $\delta_T = \delta_T^H$  and  $\beta_T = \beta_T^H$ .

### 2.2 Costs and risk processes

#### Liquidity costs

The costs  $\Delta C_k$  incurred at time  $t_k$ ,  $k > 0$  from applying the strategy  $\Phi$ , so from changing the stock amount that we hold from  $\delta_{k-1}$  to  $\delta_k$  and from

changing the amount invested in the money market account from  $\beta_{k-1}$  to  $\beta_k$ , are given in the presence of liquidity costs on the stock by

$$\Delta C_k(\Phi) = \mathcal{L}((\delta_k - \delta_{k-1}), S_k, t_k) + (\beta_k - \beta_{k-1}) \forall k \in \{k = 1, \dots, T\}$$

where the function  $\mathcal{L}$  gives the costs of adjusting the stock part and accounts for the liquidity effect:

- If  $(\delta_k - \delta_{k-1}) > 0$ , meaning that we have to buy more stocks, we might not necessarily be able to do so at the theoretical price  $S_k$  but rather at a higher price, so that the bigger the quantity to acquire the greater the marginal costs.
- If on the contrary  $(\delta_k - \delta_{k-1}) < 0$ , meaning that we have to sell more stocks, we might not necessarily be able to do so at the theoretical price  $S_k$  but rather at a lower price, so that the bigger the quantity to sell the greater the marginal costs (costs are negative in this case, so that they are smaller in absolute value).

### Assumptions on Liquidity Costs

As a consequence of the liquidity effect observed on real markets and described above, it is legitimate to assume that  $\mathcal{L} : (\mathbb{R}, \mathbb{R}_+, \mathbb{R}_+) \rightarrow \mathbb{R}$  is a strictly increasing and convex function of its first variable, with  $\mathcal{L}(0, \dots) = 0$  and that it is differentiable with respect to its first variable, with  $\frac{\partial \mathcal{L}}{\partial x}(0, S, \cdot) = S$ . As a matter of fact we don't take into account bid/ask spread at this level. We also neglect the transaction's impact on the price process meaning that there is no feedback effect no matter the quantity. This amounts to assuming that the period of trading is much greater than the relaxation time of the market impact function.

If there exists an adapted function  $g$ , *i.e.*  $g = g(x, t, \omega)$  with  $\omega \in \mathcal{F}_k$ , such that the liquidity costs can be written as  $\mathcal{L}((\delta_k - \delta_{k-1}), S_k, t_k) = (\delta_k - \delta_{k-1})g((\delta_k - \delta_{k-1}), t_k)$ , then  $g$  is called the supply curve. We refer to Cetin et al [5] for more details on the self financing approach in case there is a supply curve. In our case, we will assume that there exists an increasing density function  $l : (\mathbb{R}, \mathbb{R}_+, \mathbb{R}_+) \rightarrow \mathbb{R}$ ,  $l \in \mathcal{C}^1$  which represents the price to pay for buying a marginal amount of stock so that  $\mathcal{L}$  takes the following form:

$$\mathcal{L}(\Delta\delta_k, S_k, t_k) = \int_0^{\Delta\delta_k} l(x, S_k, t_k) dx$$

with then  $l(0, S_k, t_k)$ , the marginal costs for entering a transaction whatever its sign being equal to  $S_k$  in the absence of bid/ask spread. It corresponds to smoothing the orderbook profile which gives the quantity available for a given price. In order to make calculus in continuous time more

tractable whilst not narrowing the scope of the paper we shall assume that the marginal costs can be written as a stationary function times the theoretical spot price  $S$ , *i.e.*  $l(x, S, t) = l(x)S_t$ . We now note that with these assumptions on the liquidity costs function together with the convexity of the risk function  $f$  we cannot be sure that  $(x, y) \mapsto f(\mathcal{L}(x)S + y)$  is a convex function, unlike the case of “infinite” liquidity.

We then define the local risk  $\Delta R_k$  at time  $k$  associated with the costs incurred at time  $k + 1$ :

$$\Delta R_k(\Phi) = \mathbb{E}(f(\Delta C_{k+1})|\mathcal{F}_k)$$

or with obvious notation

$$\Delta R_k(\Phi) = \mathbb{E}_k(f(\Delta C_{k+1}))$$

Our objective is now to find trading strategies that will sequentially minimize the risk process. We state the corresponding minimization program in the next section.

### 2.3 Locally risk-minimizing strategies

The sequential minimization program runs backward in time, *i.e.* given  $(\delta_{k+1}^*, \delta_{k+2}^*, \dots, \delta_T^*)$  and  $(\beta_{k+1}^*, \beta_{k+2}^*, \dots, \beta_T^*)$  (or equivalently  $(V_{k+1}^*, V_{k+2}^*, \dots, V_T^*)$ ), we look for  $\delta_k^*$  and  $\beta_k^*$  (or  $V_k^*$ ) such that  $\Delta R_k$  is minimized.

**Problem (\*)** Given a contingent claim  $H$ , find  $\Phi^*$ , admissible strategy such that

$$\begin{aligned} \forall k \in (0, \dots, T-1), \Delta R_k(\Phi) &\geq \Delta R_k(\Phi^*) \forall \Phi \text{ admissible,} \\ &\text{with } \delta_{k+1} = \delta_{k+1}^* \text{ and } \beta_{k+1} = \beta_{k+1}^* \end{aligned}$$

Given the conditions imposed on  $f$ ,  $\mathcal{L}$ ,  $h$ ,  $S_k$  and  $\beta_k$  (or  $V_k$ ) we have only the existence of the optimal strategy which solves the first-order optimality equations

$$\begin{aligned} &\begin{cases} \mathbb{E}_k(f'(\Delta C_{k+1}(\Phi^*))) &= 0 \\ \mathbb{E}_k(f'(\Delta C_{k+1}(\Phi^*)) \mathcal{L}'((\delta_{k+1} - \delta_k), S_{k+1}, t_{k+1})) &= 0 \end{cases} \\ &\Leftrightarrow \begin{cases} \mathbb{E}_k(f'(\Delta C_{k+1}(\Phi^*))) &= 0 \\ \mathbb{E}_k(f'(\Delta C_{k+1}(\Phi^*)) l(\delta_{k+1} - \delta_k) S_{k+1}) &= 0 \end{cases} \quad (1) \end{aligned}$$

where we have used the notation  $\mathcal{L}'$  for the partial derivatives of  $\mathcal{L}(x, y, z)$  with respect to its first variable.

**Theorem 1.** *Problem (\*) has at least one solution  $\Phi^*$  whose components  $\delta^*$  and  $\beta^*$  solve the set of equations (1).*

*Proof* Let  $h(x, y, \omega) \equiv E_k(f(\mathcal{L}((U - x), S, t_{k+1}) + (V - y)))(\omega)$  with  $U$ ,  $V$  and  $S \in \mathcal{L}^2(P)$ . We first observe that because of our assumptions on the liquidity costs, for a fixed  $\omega$ ,  $h$  is a continuous and differentiable function of  $(x, y)$  so that it reaches a minimum  $(x^*, y^*)$  only if  $(x^*, y^*)$  is a critical point of  $h$ , i.e.  $\nabla h(x^*, y^*) = 0$ . Secondly we have  $\lim_{\|(x, y)\| \rightarrow \infty} h(x, y, \omega) = +\infty$ ,  $P$ -a.e. so that  $h$  has a global minimum  $P$ -almost surely. Finally we show that  $(x^*, y^*)$  is  $\mathcal{F}_k$ -measurable: let  $D_n = \{j2^{-n} | j \in \mathbb{Z}\}$  be the set of dyadic rational of order  $n$ , we define

$$(x_n(\omega), y_n(\omega)) = \inf\{(x, y) \in D_n \times D_n, h(x, y, \omega) \leq h(x', y', \omega) \forall (x', y') \in D_n \times D_n\}$$

Since  $\omega \mapsto h(x, y, \omega)$  is  $\mathcal{F}_k$ -measurable,  $(x_n, y_n)$  is also  $\mathcal{F}_k$ -measurable. As  $(x_n, y_n)$  is bounded in  $n$   $P$ -a.e. and  $h$  is continuous in  $(x, y)$ ,  $(\tilde{x}, \tilde{y}) = \liminf_{n \rightarrow \infty} (x_n, y_n)$  is a  $\mathcal{F}_k$ -measurable minimizer of  $h$ , satisfying  $\nabla h(\tilde{x}, \tilde{y}) = 0$ .

The set of equations (1) is equivalent for the process  $(C_k^f)_k$  with  $C_k^f = \sum_{i=1}^k f'(\Delta C_i)$ ,  $C_0^f = 0$ , and which we will refer to as the  $f$ -costs process, to being a martingale strongly orthogonal to the martingale part of the process  $(S_k^S)_k$  with  $S_k^S = \sum_{i=1}^k (l(\Delta \delta_i) S_i - l(0) S_{i-1}) = \sum_{i=1}^k (l(\Delta \delta_i) S_i - S_{i-1})$ ,  $S_0^S = S_0$ , which we will refer to as the supply price process.

We name this property pseudo-optimality as was done in Abergel and Millot[1] after Schweizer[17] and it will be in this extended case also the main ingredient of the extensions to the continuous time setting. We also note that in the original case of “infinite” liquidity, so with  $l(\cdot) = 1$ , the supply price process is just the stock price  $S$ , as is expected.

### 3 Continuous time setting

Now let  $(\Omega, \mathcal{F}, P)$  be a probability space with a filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$  satisfying the usual conditions of right-continuity and completeness.  $T \in \mathbb{R}^+$  denotes a fixed and finite time horizon. Furthermore, we assume that  $\mathcal{F}_0$  is trivial and that  $\mathcal{F}_T = \mathcal{F}$ . Let  $S = (S_t)_{0 \leq t \leq T}$  be a semimartingale with a decomposition

$$S = S_0 + M + A$$

such that  $M = (M_t)_{0 \leq t \leq T}$  is a square-integrable martingale with  $M_0 = 0$  and  $A = (A_t)_{0 \leq t \leq T}$  is a continuous and adapted process of finite variation  $|A|$  with  $A_0 = 0$ . Throughout the article, we use a right-continuous version of  $S$ .

### 3.1 Trading strategies

A trading strategy  $\Phi$  is a pair of processes  $\delta = (\delta_t)_{0 \leq t \leq T}$ ,  $\beta = (\beta_t)_{0 \leq t \leq T}$  satisfying the following conditions

$$\begin{cases} \delta \text{ is càdlàg and adapted} \\ \beta \text{ is càdlàg and adapted} \end{cases}$$

An option is again described by a square-integrable random variable  $H \in L^2(P)$ , with  $H = \delta^H S_T + \beta^H$ ,  $\delta^H$  and  $\beta^H$  being  $\mathcal{F}_T$ -measurable random variables.

So as to make precise what strategies can be considered, we use the same classical notations and definitions regarding quadratic variations and covariations as in Abergel and Millot [1].

Likewise we introduce the same restrictions on our strategies so that the optimality conditions are well-defined. We shall therefore concentrate on strategies which are  $H$ -admissible in the sense that

$$\begin{cases} \delta_T = \delta^H \text{ } P - a.s. \\ \beta_T = \beta^H \text{ } P - a.s. \\ \delta \text{ has finite and integrable quadratic variation} \\ \beta \text{ has finite and integrable quadratic variation} \\ \delta \text{ and } \beta \text{ have finite and integrable quadratic covariation} \end{cases}$$

#### Local risk-minimization

We briefly recap the definition used to extend the idea of local risk-minimization in discrete time to the continuous time framework.

#### Small perturbations

**Definition** A small perturbation is a bounded admissible<sup>3</sup> strategy  $\phi = (\beta, \delta)$  such that  $\beta_T = 0$  and  $\delta_T = 0$ .

#### Local risk along a partition

We start with an  $H$ -admissible strategy  $\Phi$  and we want to study the increase of risk when the strategy is perturbed at some discrete times. To do so, given a partition  $\tau$  of  $[0, T]$ , where  $\tau = \{0 = t_0, t_1, \dots, t_k = T\}$ , and a small perturbation  $\Delta$ , we define a process  $r_f^\tau$  the following way

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<sup>3</sup> Admissible means that it satisfies the same regularity requirements as an  $H$ -admissible strategy with different terminal conditions.

$$r_f^\tau[\Phi, \phi](t, \omega) = \sum_{t_i, t_{i+1} \in \tau} \frac{\Delta R_{t_i}(\Phi + \phi)|_{[t_i, t_{i+1})}(\omega) - \Delta R_{t_i}(\Phi)(\omega)}{t_{i+1} - t_i} 1_{[t_i, t_{i+1})}(t)$$

with  $\Delta R_{t_i}(\Phi) = \mathbb{E}(f(\Delta C_{t_{i+1}})|\mathcal{F}_{t_i})$ .

Now we can define the local risk-minimization in the same way as we did for the discrete time setting:

**Definition** An  $H$ -admissible strategy  $\Phi$  is called locally risk-minimizing for the option  $H$  if for every small perturbation  $\phi$  and every increasing sequence of partitions  $(\tau_n)_{n \in \mathbb{N}}$  tending to the identity, we have

$$\liminf_{n \rightarrow \infty} r^{\tau_n}[\Phi, \phi] \geq 0 \quad \mathcal{P} - a.e.$$

### 3.2 The $f$ -costs process

Now we proceed with defining the process  $f$ -costs process which will allow us to characterize locally risk-minimizing strategies by analogy with discrete time.

For a replicating trading strategy  $\Phi$  we define the  $f$ -costs process  $C_t(\Phi)$  as the following limit, whenever it exists

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{l_n} f'(\mathcal{L}(\delta^{\tau_k^n} - \delta^{\tau_{k-1}^n}, S^{\tau_k^n}) + \beta^{\tau_k^n} - \beta^{\tau_{k-1}^n})$$

where convergence happens in ucp topology, for any sequences  $\mathcal{P}_n$  of Riemann partitions of  $[0, T]$  of length  $l_n$ . We used the notation  $X^T$  for the process stopped at  $T$ .

We now focus on a  $H$ -admissible strategy  $\Phi$  and state a theorem relative to the existence of the  $f$ -costs process.

**Theorem 2.** *The  $f$ -costs process of a  $H$ -admissible strategy  $\Phi$  is well defined and is given by the following formula*

$$\begin{aligned} C_t(\Phi) = & f''(0) \left( V_t - V_0 - \int_{0+}^t \delta_s - dS_s + \frac{1}{2} l'(0) \int_{0+}^t S_{s-} d[\delta, \delta]_s^c \right) \\ & + \frac{f^{(3)}(0)}{2} \left( [\beta, \beta]_t^c + 2 \int_{0+}^t S_{s-} d[\beta, \delta]_s^c + \int_{0+}^t S_{s-}^2 d[\delta, \delta]_t^c \right) \\ & + \sum_{0 < s \leq t} f'(\Delta \beta_s + \mathcal{L}(\Delta \delta_s, S_s)) - f''(0)(\Delta \beta_s + \Delta \delta_s S_s) \end{aligned} \quad (2)$$

with notation  $[X, Y]^c$  standing for the continuous part of the (càdlàg) quadratic covariation process.

*Proof* The proof relies on the same ingredients as in theorem 2 of Abergel and Millot[1] where the basic case without transaction costs is thoroughly studied, effectively following the lines of Itô formula's proof for general semimartingales in Protter[16]. The changes are in the application of the Taylor's theorem with remainder term which we apply first on  $f'$  as a function of  $\mathcal{L}(x)S + y$  and then on  $\mathcal{L}$  itself.

**Corollary 1.** *The  $f$ -costs process of a  $H$ -admissible strategy  $\Phi$  can also be expressed in terms of the portfolio value  $V$*

$$\begin{aligned}
C_t(\Phi) = & f''(0) \left( V_t - V_0 - \int_{0+}^t \delta_{s-} dS_s \right) \\
& + f''(0)l'(0) \left( \frac{1}{2} \int_{0+}^t S_{s-} d[\delta, \delta]_s^c \right) \\
& + \frac{f^{(3)}(0)}{2} \left( [V, V]_t^c - 2 \int_{0+}^t \delta_{s-} d[V, S]_s^c + \int_{0+}^t \delta_{s-}^2 d[S, S]_t^c \right) \\
& + \sum_{0 < s \leq t} f'(\Delta V_s - \delta_{s-} \Delta S_s + \mathcal{L}(\Delta \delta_s, S_s) - \Delta \delta_s S_s) \\
& - \sum_{0 < s \leq t} f''(0)(\Delta V_s - \delta_{s-} \Delta S_s) \tag{3}
\end{aligned}$$

*Proof* The proof follows easily from applying the definition of  $V$  in the formula (2) and straightforward calculations using quadratic variation properties.

The additional term in the expression of  $C_t(\Phi)$  due to the finite liquidity is  $f''(0)l'(0) \left( \frac{1}{2} \int_{0+}^t S_{s-} d[\delta, \delta]_s^c \right)$  and it is non-decreasing given the convexity of both  $f$  and  $\mathcal{L}$ .

### 3.3 The supply price process

We define the the supply price process required to characterize locally risk-minimizing strategies by analogy with discrete time.

For a replicating trading strategy  $\Phi$  we define the supply price process  $S_t^S(\Phi)$  as the following limit, whenever it exists

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{l_n} \left( l(\delta^{\tau_k^n} - \delta^{\tau_{k-1}^n}) S^{\tau_k^n} - S^{\tau_{k-1}^n} \right)$$

where convergence happens in ucp topology, for any sequences  $\mathcal{P}_n$  of Riemann partitions of  $[0, T]$  of length  $l_n$ .

For  $H$ -admissible strategy  $\Phi$  we have a similar theorem relative to the existence of the supply price process as for the  $f$ -costs.

**Theorem 3.** *The supply price process of  $S^S$  a  $H$ -admissible strategy  $\Phi$  is well defined and is given by the following formula*

$$\begin{aligned}
 S_t^S(\Phi) &= S_t + l'(0) \left( \delta_t S_t - \delta_0 S_0 - \int_{0+}^t \delta_{s-} dS_s \right) + \frac{1}{2} l''(0) \int_{0+}^t S_{s-} d[\delta, \delta]_s^c \\
 &\quad + \sum_{0 < s \leq t} (l((\Delta \delta_s) - 1) S_s - l'(0) \Delta \delta_s S_s) \tag{4}
 \end{aligned}$$

*Proof* The proof follows the same lines as theorem 2 so we don't detail it here.

## 4 Application to stochastic volatility models

In this section we will assume further hypotheses on the trading strategies so that we can derive explicit formula for the  $f$ -cost process and completely characterize the the pseudo-optimal strategies for the local risk-minimization. We model the evolution of  $S$  through a set of SDE with stochastic volatility

$$dS_t = \mu_t dt + \sigma_t dW_t^1 \tag{5}$$

$$d\sigma_t = \gamma_t dt + \Sigma_t dW_t^2 \tag{6}$$

with smooth  $\mu_t$ ,  $\gamma_t$  and  $\Sigma_t$ ,  $W^1$  and  $W^2$  standard Wiener processes under  $\mathcal{P}$  with constant instantaneous correlation  $\rho$ , *i.e.*  $d \langle W^1, W^2 \rangle_t = \rho dt$ .

We shall also assume that appropriate conditions hold for the functions  $\mu_t$ ,  $\gamma_t$  and  $\Sigma_t$  so that the system of SDE (5, 6) admits a unique strong continuous solution for  $S$  and  $\sigma$ , with  $S > 0$  and  $\sigma > 0$ . With these diffusion assumptions we will now place ourselves in a Markovian framework and look for the optimal strategy  $\Phi$  as a smooth function of the state variables

$$\delta_t = \delta(t, S_t, \sigma_t)$$

$$V_t = V(t, S_t, \sigma_t)$$

### 4.1 PDE formulation

We first derive a PDE formulation. For that purpose we firstly express the costs process as a function of the diffusion parameters and the strategy

$$\begin{aligned}
 C_t(\Phi) &= \int_0^t \left[ f''(0) \left( \frac{\partial V}{\partial u} + \frac{\partial V}{\partial S} \mu_u + \frac{\partial V}{\partial \sigma} \gamma_u + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma} \rho \sigma_u \Sigma_u - \delta_u \mu_u \right) \right. \\
 &\quad + f''(0) l'(0) \frac{S_u}{2} \left( \left( \frac{\partial \delta}{\partial S} \right)^2 \sigma_u^2 + \left( \frac{\partial \delta}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial \delta}{\partial S} \frac{\partial \delta}{\partial \sigma} \rho \sigma_u \Sigma_u \right) \\
 &\quad + \frac{f^{(3)}(0)}{2} \left( \left( \frac{\partial V}{\partial S} \right)^2 \sigma_u^2 + \left( \frac{\partial V}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial V}{\partial S} \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) \\
 &\quad - f^{(3)}(0) \delta_u \left( \frac{\partial V}{\partial S} \sigma_u^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) + \left. \frac{f^{(3)}(0)}{2} \delta_u^2 \sigma_u^2 \right] du \\
 &\quad + \int_0^t f''(0) \left( \frac{\partial V}{\partial S} - \delta_u \right) \sigma_u dW_u^1 + \int_0^t f''(0) \frac{\partial V}{\partial \sigma} \Sigma_u dW_u^2
 \end{aligned}$$

which we have obtained from equation (3).  
Secondly we express the supply price process

$$S_t^S(\Phi) = S_t + l'(0) \left( \delta_t S_t - \delta_0 S_0 - \int_0^t \delta_u \mu_u du - \int_0^t \delta_u \sigma_u dW_u^1 \right) \\ + \frac{1}{2} l''(0) \int_0^t \left( \left( \frac{\partial \delta}{\partial S} \right)^2 \sigma_u^2 + \left( \frac{\partial \delta}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial \delta}{\partial S} \frac{\partial \delta}{\partial \sigma} \rho \sigma_u \Sigma_u \right) du$$

which we have obtained from equation (4). Now, applying to the strategy  $\Phi$  the first pseudo-optimality criterion, *i.e.* that  $C$  must be martingale under the measure  $P$ , we find a first fully non-linear PDE satisfied by the strategy  $(V, \delta)$

$$f''(0) \left( \frac{\partial V}{\partial u} + \frac{\partial V}{\partial S} \mu_u + \frac{\partial V}{\partial \sigma} \gamma_u + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma} \rho \sigma_u \Sigma_u - \delta_u \mu_u \right) \\ + f''(0) l'(0) \frac{S_u}{2} \left( \left( \frac{\partial \delta}{\partial S} \right)^2 \sigma_u^2 + \left( \frac{\partial \delta}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial \delta}{\partial S} \frac{\partial \delta}{\partial \sigma} \rho \sigma_u \Sigma_u \right) \\ + \frac{f^{(3)}(0)}{2} \left( \left( \frac{\partial V}{\partial S} \right)^2 \sigma_u^2 + \left( \frac{\partial V}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial V}{\partial S} \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) \\ - f^{(3)}(0) \delta_u \left( \frac{\partial V}{\partial S} \sigma_u^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) + \frac{f^{(3)}(0)}{2} \delta_u^2 \sigma_u^2 = 0$$

with terminal condition corresponding to  $V_T = \delta^H S_T + \beta^H$ .

In order to apply to the strategy  $\Phi$  the second pseudo-optimality criterion, *i.e.* that the martingale  $C$  must be orthogonal to the martingale part of the supply price process  $S^S$ , we first identify its martingale part

$$S_t^S(\Phi) - \mathbb{E}(S_t^S(\Phi)) = \int_0^t \left( 1 + l'(0) S \frac{\partial \delta}{\partial S} \right) \sigma_u dW_u^1 + \int_0^t l'(0) S \frac{\partial \delta}{\partial \sigma} \Sigma_u dW_u^2$$

so that the second PDE satisfied by the strategy  $(V, \delta)$  is

$$\left( \frac{\partial V}{\partial S} - \delta \right) \left( 1 + l'(0) S \frac{\partial \delta}{\partial S} \right) \sigma^2 + \frac{\partial V}{\partial \sigma} \left( 1 + l'(0) S \frac{\partial \delta}{\partial S} \right) \rho \sigma \Sigma + \\ \left( \frac{\partial V}{\partial S} - \delta \right) \frac{\partial \delta}{\partial \sigma} l'(0) S \rho \sigma \Sigma + \frac{\partial V}{\partial \sigma} \frac{\partial \delta}{\partial \sigma} l'(0) S \Sigma^2 = 0$$

With some rearrangements, the pseudo-optimal strategy  $\Phi$  finally solves the following system of PDEs

$$\left\{ \begin{array}{l} \frac{\partial V}{\partial u} + \frac{\partial V}{\partial S} \mu + \frac{\partial V}{\partial \sigma} \gamma + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma^2 + \frac{\partial^2 V}{\partial S \partial \sigma} \rho \sigma \Sigma = \\ \delta \mu + \alpha \left( \left( \frac{\partial V}{\partial S} \sigma + \frac{\partial V}{\partial \sigma} \rho \Sigma - \delta \sigma \right)^2 + (1 - \rho^2) \left( \frac{\partial V}{\partial \sigma} \right)^2 \Sigma^2 \right) \\ \quad + l'(0) \frac{S}{2} \left( \left( \frac{\partial \delta}{\partial S} \sigma + \frac{\partial \delta}{\partial \sigma} \rho \Sigma \right)^2 + (1 - \rho^2) \left( \frac{\partial \delta}{\partial \sigma} \right)^2 \Sigma^2 \right) \\ \left( \frac{\partial V}{\partial S} - \delta \right) \left( 1 + l'(0) S \frac{\partial \delta}{\partial S} \right) \sigma^2 + \frac{\partial V}{\partial \sigma} \left( 1 + l'(0) S \frac{\partial \delta}{\partial S} \right) \rho \sigma \Sigma \\ \quad + \left( \frac{\partial V}{\partial S} - \delta \right) \frac{\partial \delta}{\partial \sigma} l'(0) S \rho \sigma \Sigma + \frac{\partial V}{\partial \sigma} \frac{\partial \delta}{\partial \sigma} l'(0) S \Sigma^2 = 0 \end{array} \right. \quad (7)$$

with  $V_T = \delta^H S_T + \beta^H$ .

### Complete markets case

The case of complete markets allows us to recover an extension of the Black and Scholes PDE ([4], [15]) which does not depend on the choice for the function  $f$ . Indeed, by taking  $\Sigma$  the volatility of volatility equal to zero, the equation for the hedge ratio  $\delta$  reduces to

$$\left( \frac{\partial V}{\partial S} - \delta \right) \left( 1 + l'(0) S \frac{\partial \delta}{\partial S} \right) = 0$$

We can rule out the second factor  $1 + l'(0) S \frac{\partial \delta}{\partial S}$  being zero since it yields an unbounded hedge ratio when  $S$  approaches to zero. Therefore we are left with the following system

$$\delta = \frac{\partial V}{\partial S} \quad (8)$$

$$\frac{\partial V}{\partial u} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 \left( 1 + l'(0) S \frac{\partial^2 V}{\partial S^2} \right) = 0 \quad (9)$$

Equation (8) gives the perfect hedging strategy in that context, provided that the PDE (9) has a solution, which is expected when the contingent claim has a convex payoff. One can verify that in this case the  $f$ -cost process would be identically zero, which amounts to having a self-financing strategy incorporating liquidity costs that perfectly replicates the contingent claim  $H$ . Another remark is that the PDE (9) shows that the value of the portfolio being an increasing function of the volatility for a convex payoff, in the presence of liquidity costs, is increased proportionally to the slope of the marginal costs and to the  $\Gamma = \frac{\partial^2 V}{\partial S^2}$  of the option.

## 4.2 The minimization problem

Despite the fact that in discrete time a pseudo-optimal strategy, satisfying the set of equations (1), might not be optimal, in continuous time, when working with continuous path processes we have a correspondence between the two concepts. As a matter of fact we show that a strategy solving system of equations (7) is locally risk-minimizing for the function  $f$ .

Given the smoothness of the risk function  $f$  and the liquidity costs function  $\mathcal{L}$  we can rewrite the process  $r_f^\tau$  by using a Taylor development around the non-perturbed strategy  $\Phi$ . Let us fix  $t \in [0, T]$ , because of the definition of the process  $r_f^\tau[\Phi, \phi]$  and as we work with increasing sequences of partitions, we may assume that  $t$  is one of the  $t_{i(n)}^n$  (we will thereafter drop the superscript  $n$  and simply write  $t_i$  instead), we have:

$$\begin{aligned} r_f^\tau[\Phi, \phi](t, \omega) &= \frac{\Delta R_{t_i}(\Phi + \phi|_{[t_i, t_{i+1}]}) (\omega) - \Delta R_{t_i}(\Phi) (\omega)}{t_{i+1} - t_i} \\ &= \frac{\mathbb{E}_{t_i} (f(\Delta C_{t_{i+1}}(\Phi + \phi|_{[t_i, t_{i+1}]})) (\omega) - \mathbb{E}_{t_i} (f(\Delta C_{t_{i+1}}(\Phi))) (\omega)}{t_{i+1} - t_i} \end{aligned}$$

Applying Taylor's formula with remainder term to  $g : (x, y) \mapsto f(\mathcal{L}(x) + y)$  in the expectation, we have that

$$\begin{aligned} f(\Delta C_{t_{i+1}}(\Phi + \phi|_{[t_i, t_{i+1}]})) &= \\ f(\Delta C_{t_{i+1}}(\Phi)) - \beta_{t_i} f'(\Delta C_{t_{i+1}}(\Phi)) - \delta_{t_i} \mathcal{L}'(\Phi) f'(\Delta C_{t_{i+1}}(\Phi)) &+ \frac{1}{2} \delta_{t_i}^2 h(\tilde{\phi}) \\ &+ \frac{1}{2} (\beta_{t_i} + \delta_{t_i} \mathcal{L}'(\tilde{\phi}))^2 g(\tilde{\phi}) \end{aligned}$$

where  $g(\tilde{\phi}) = f''(\Delta C_{t_{i+1}}(\tilde{\phi}))$  and  $h(\tilde{\phi}) = \mathcal{L}''(\tilde{\phi}) f'(\Delta C_{t_{i+1}}(\tilde{\phi}))$  with  $\tilde{\phi} = (\tilde{\beta}, \tilde{\delta})$  such that  $|\tilde{\beta}| \leq \beta$  and  $|\tilde{\delta}| \leq \delta$ . With the assumptions on  $f$ , namely  $f'(0) = 0$  and  $f''(0) > 0$ , the remainder term  $\delta_{t_i}^2 h(\tilde{\phi}) + (\beta_{t_i} + \delta_{t_i} \mathcal{L}'(\tilde{\phi}))^2 g(\tilde{\phi})$  will remain strictly positive in a neighborhood of  $t_i$  for  $\delta_{t_i}$  and  $\beta_{t_i}$  small enough.

Rearranging and simplifying we get

$$\begin{aligned} r_f^\tau[\Phi, \phi](t, \omega) &= \beta_{t_i} \frac{\mathbb{E}_{t_i} (f'(\Delta C_{t_{i+1}}(\Phi))) (\omega)}{t_{i+1} - t_i} + \delta_{t_i} \frac{\mathbb{E}_{t_i} (\mathcal{L}'(\Phi) f'(\Delta C_{t_{i+1}}(\Phi))) (\omega)}{t_{i+1} - t_i} \\ &+ \frac{1}{2} \frac{\mathbb{E}_{t_i} (\delta_{t_i}^2 h(\tilde{\phi})) (\omega)}{t_{i+1} - t_i} \\ &+ \frac{1}{2} \frac{\mathbb{E}_{t_i} ((\beta_{t_i} + \delta_{t_i} \mathcal{L}'(\tilde{\phi}))^2 g(\tilde{\phi})) (\omega)}{t_{i+1} - t_i} \end{aligned}$$

Because we work with Itô processes, the following stands:

$$\lim_{t_{i+1} \rightarrow t_i} \frac{\mathbb{E}_{t_i} (f'(\Delta C_{t_{i+1}}(\Phi))) (\omega)}{t_{i+1} - t_i} = \Lambda (f' \circ \Delta C)_{t_i}$$

$$\lim_{t_{i+1} \rightarrow t_i} \frac{\mathbb{E}_{t_i} (\mathcal{L}'(\Phi) f'(\Delta C_{t_{i+1}}(\Phi))) (\omega)}{t_{i+1} - t_i} = \Lambda (\mathcal{L}' \cdot f' \circ \Delta C)_{t_i}$$

and

$$\lim_{t_{i+1} \rightarrow t_i} \frac{\mathbb{E}_{t_i} (h(\tilde{\phi})) (\omega)}{t_{i+1} - t_i} = \Lambda h_{t_i}$$

$$\lim_{t_{i+1} \rightarrow t_i} \frac{\mathbb{E}_{t_i} (g(\tilde{\phi})) (\omega)}{t_{i+1} - t_i} = \Lambda g_{t_i}$$

$$\lim_{t_{i+1} \rightarrow t_i} \frac{\mathbb{E}_{t_i} (\mathcal{L}' g(\tilde{\phi})) (\omega)}{t_{i+1} - t_i} = \Lambda (\mathcal{L}' \cdot g)_{t_i}$$

$$\lim_{t_{i+1} \rightarrow t_i} \frac{\mathbb{E}_{t_i} (\mathcal{L}'^2 g(\tilde{\phi})) (\omega)}{t_{i+1} - t_i} = \Lambda (\mathcal{L}'^2 \cdot g)_{t_i}$$

where  $\Lambda$  is the infinitesimal generator associated with the diffusion:

$$\Lambda h = \frac{\partial h}{\partial S} \mu + \frac{\partial h}{\partial \sigma} \gamma + \frac{1}{2} \frac{\partial^2 h}{\partial S^2} \sigma^2 + \frac{1}{2} \frac{\partial^2 h}{\partial \sigma^2} \Sigma^2 + \frac{\partial^2 h}{\partial S \partial \sigma} \rho \sigma \Sigma$$

Finally the process  $r_f^\tau$  evaluated in  $t$  is worth

$$r_f^\tau[\phi, \Delta](t, \omega) = \beta_t \Lambda (f' \circ \Delta C)_t + \delta_t \Lambda (\mathcal{L}' \cdot f' \circ \Delta C)_t$$

$$+ \frac{1}{2} (\beta_t^2 \Lambda g_t + 2\beta_t \delta_t \Lambda (\mathcal{L}' \cdot g)_t + \delta_t^2 \Lambda (\mathcal{L}'^2 \cdot g + h)_t)$$

Now we first take the component  $\delta$  of the perturbation equal to zero, that is we perturb only  $\beta$ , so that we have the following first condition for the strategy  $\phi$  to be locally risk-minimizing :

$$\beta_t \Lambda (f' \circ \Delta C)_t + \frac{1}{2} \beta_t^2 \Lambda g_t \geq 0 \quad P - a.e. \quad \forall \beta_t$$

As a consequence we must have  $\Lambda (f' \circ \Delta C)_t = 0$ .

Likewise we take the component  $\beta$  equal to zero and we get the following second condition for the strategy  $\phi$  to be locally risk-minimizing :

$$\delta_t \Lambda (\mathcal{L}' \cdot f' \circ \Delta C)_t + \frac{1}{2} \delta_t^2 \Lambda (\mathcal{L}'^2 \cdot g + h)_t \geq 0 \quad P - a.e. \quad \forall \delta_t$$

Therefore we must have  $\Lambda(\mathcal{L}' \cdot f' \circ \Delta C)_t = 0$ . But we observe that

$$\begin{cases} \Lambda(f' \circ \Delta C)_t = 0 \\ \Lambda(\mathcal{L}' \cdot f' \circ \Delta C)_t = 0 \end{cases} \Leftrightarrow \begin{cases} f''(0) \left( \frac{\partial V}{\partial u} + \frac{\partial V}{\partial S} \mu_u + \frac{\partial V}{\partial \sigma} \gamma_u + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma} \rho \sigma_u \Sigma_u - \delta_u \mu_u \right) \\ \quad + f''(0) l'(0) \frac{S_u}{2} \left( \left( \frac{\partial \delta}{\partial S} \right)^2 \sigma_u^2 + \left( \frac{\partial \delta}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial \delta}{\partial S} \frac{\partial \delta}{\partial \sigma} \rho \sigma_u \Sigma_u \right) \\ \quad + \frac{f^{(3)}(0)}{2} \left( \left( \frac{\partial V}{\partial S} \right)^2 \sigma_u^2 + \left( \frac{\partial V}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial V}{\partial S} \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) \\ \quad - f^{(3)}(0) \delta_u \left( \frac{\partial V}{\partial S} \sigma_u^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) + \frac{f^{(3)}(0)}{2} \delta_u^2 \sigma_u^2 = 0 \\ \left( \frac{\partial V}{\partial S} - \delta \right) \left( 1 + l'(0) S \frac{\partial \delta}{\partial S} \right) \sigma^2 + \frac{\partial V}{\partial \sigma} \left( 1 + l'(0) S \frac{\partial \delta}{\partial \sigma} \right) \rho \sigma \Sigma \\ \quad + \left( \frac{\partial V}{\partial S} - \delta \right) \frac{\partial \delta}{\partial \sigma} l'(0) S \rho \sigma \Sigma + \frac{\partial V}{\partial \sigma} \frac{\partial \delta}{\partial \sigma} l'(0) S \Sigma^2 = 0 \end{cases}$$

Finally, as announced in the beginning of this section, we see that in this context of stochastic volatility model, the optimal strategies with respect to the local risk-minimization problem are the same as the pseudo-optimal strategies. Like in the original article with “infinite” liquidity, the only requirement to obtain this result is the existence of the infinitesimal generator and its expression in terms of the parameters of the SDE driving the process so as to identify the sets of two equations, which therefore allows again for a straightforward generalization to Itô processes.

## 5 Application to stochastic volatility/jump diffusion models

In this section, we want to provide an example of a situation where the non-quadratic risk definitely implies a different hedging strategy, not only through the Taylor expansion around zero of the risk function. We therefore model the evolution of  $S$  through an SDE with stochastic volatility and Poisson jumps in the vein of the Bates model [3]

$$\begin{aligned} dS_t &= \mu_t dt + \sigma_t dW_t^1 + k dN_t \\ d\sigma_t &= \gamma_t dt + \Sigma_t dW_t^2 \end{aligned}$$

with  $W^1$  and  $W^2$  Wiener processes under  $\mathcal{P}$  and  $d \langle W^1, W^2 \rangle_t = \rho dt$ ,  $N_t$  a Poisson process of intensity  $\lambda$  and the amplitude of the jumps  $k$  having probability distribution  $K$ . We also assume that  $W_t$ ,  $N_t$  and  $k$  are independent. As in the case of stochastic volatility we shall assume that appropriate conditions hold on the adapted processes  $\mu$ ,  $\sigma$ ,  $\gamma$  and  $\Sigma$  so that the set of SDE has a unique strong solution.

With these assumptions we will again place ourselves in a Markovian framework and look for the optimal strategy  $\Phi$  as a smooth function of the state variables

$$\begin{aligned}\delta_t &= \delta(t, S_t, \sigma_t) \\ V_t &= V(t, S_t, \sigma_t)\end{aligned}$$

### 5.1 PIDE formulation

We first derive a PIDE formulation. For that purpose we express the costs process as a function of the model parameters and the strategy

$$\begin{aligned}C_t(\Phi) &= \int_0^t \left( f''(0) \left( \frac{\partial V}{\partial u} + \frac{\partial V}{\partial S} \mu_u + \frac{\partial V}{\partial \sigma} \gamma_u + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma} \rho \sigma_u \Sigma_u - \delta_u \mu_u \right) \right. \\ &+ \frac{f^{(3)}(0)}{2} \left( \left( \frac{\partial V}{\partial S} \right)^2 \sigma_u^2 + \left( \frac{\partial V}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial V}{\partial S} \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) \\ &+ \frac{l'(0) S_u}{2} \left( \left( \frac{\partial \delta}{\partial S} \right)^2 \sigma_u^2 + \left( \frac{\partial \delta}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial \delta}{\partial S} \frac{\partial \delta}{\partial \sigma} \rho \sigma_u \Sigma_u \right) \\ &- f^{(3)}(0) \delta_{u-} \left( \frac{\partial V}{\partial S} \sigma_u^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) + \frac{f^{(3)}(0)}{2} \delta_{u-}^2 \sigma_u^2 \Big) du \\ &+ \int_0^t f''(0) \left( \frac{\partial V}{\partial S} - \delta_{u-} \right) \sigma_u dW_u^1 \\ &+ \int_0^t f''(0) \frac{\partial V}{\partial \sigma} \Sigma_u dW_u^2 \\ &+ \int_0^t \int_{\mathbb{R}} f'(\Delta V_u - \delta_{u-} \Delta S_u + \mathcal{L}(\Delta \delta_u, S_u) - \Delta \delta_u S_u) K(k) dk dN_u\end{aligned}$$

which we have again obtained from equation (3), and with  $\Delta V_u$  the jump in  $V$  when there is a jump  $\Delta S_u$  of size  $k$  on  $S$  at time  $u$  being equal to  $V(u-, S_{u-} + k, \sigma_{u-}) - V(u-, S_{u-}, \sigma_{u-})$  and likewise for  $\Delta \delta_u$ .

Now, applying to the strategy  $\Phi$  the first pseudo-optimality criterion, *i.e.* that  $C$  must be martingale under the measure  $P$ , we find the PIDE satisfied by the portfolio value  $V$

$$\begin{aligned}f''(0) \left( \frac{\partial V}{\partial u} + \frac{\partial V}{\partial S} \mu_u + \frac{\partial V}{\partial \sigma} \gamma_u + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_u^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma_u^2 + \frac{\partial^2 V}{\partial S \partial \sigma} \rho \sigma_u \Sigma_u - \delta_{u-} \mu_u \right) \\ + \frac{f^{(3)}(0)}{2} \left( \left( \frac{\partial V}{\partial S} \right)^2 \sigma_u^2 + \left( \frac{\partial V}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial V}{\partial S} \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) \\ + \frac{l'(0) S_u}{2} \left( \left( \frac{\partial \delta}{\partial S} \right)^2 \sigma_u^2 + \left( \frac{\partial \delta}{\partial \sigma} \right)^2 \Sigma_u^2 + 2 \frac{\partial \delta}{\partial S} \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) \\ - f^{(3)}(0) \delta_{u-} \left( \frac{\partial V}{\partial S} \sigma_u^2 + \frac{\partial V}{\partial \sigma} \rho \sigma_u \Sigma_u \right) + \frac{f^{(3)}(0)}{2} \delta_{u-}^2 \sigma_u^2 \\ + \int_{\mathbb{R}} f'(\Delta V_u - \delta_{u-} \Delta S_u) K(k) dk \lambda_u = 0\end{aligned}$$

with the terminal conditions corresponding to  $V_T = \delta^H S_T + \beta^H$ .

In order to apply to the strategy  $\Phi$  the second pseudo-optimality criterion, *i.e.* that the martingale  $C$  must be orthogonal to the martingale part of the supply price process  $S^S$ , we first identify its martingale part

$$\begin{aligned} S_t^S(\Phi) - \mathbb{E}(S_t^S(\Phi)) &= \int_0^t \left(1 + l'(0)S \frac{\partial \delta}{\partial S}\right) \sigma_u dW_u^1 + \int_0^t l'(0)S \frac{\partial \delta}{\partial \sigma} \Sigma_u dW_u^2 \\ &\quad + \int_0^t \int_{\mathbb{R}} ((l(\Delta\delta_u) - 1)S_u + k) K(k) dk d\tilde{N}_u \end{aligned}$$

with  $\tilde{N}$  the compensated Poisson process of  $N$ .

Therefore the second PDE satisfied by the strategy  $(V, \delta)$  is

$$\begin{aligned} &\left(\frac{\partial V}{\partial S} - \delta\right) \left(1 + l'(0)S \frac{\partial \delta}{\partial S}\right) \sigma^2 + \frac{\partial V}{\partial \sigma} \left(1 + l'(0)S \frac{\partial \delta}{\partial S}\right) \rho \sigma \Sigma \\ &\quad + \left(\frac{\partial V}{\partial S} - \delta\right) \frac{\partial \delta}{\partial \sigma} l'(0)S \rho \sigma \Sigma + \frac{\partial V}{\partial \sigma} \frac{\partial \delta}{\partial \sigma} l'(0)S \Sigma^2 \\ &+ \int_{\mathbb{R}} f'(\Delta V_u - \delta_u - \Delta S_u + \mathcal{L}(\Delta\delta_u, S_u) - \Delta\delta_u S_u) ((l(\Delta\delta_u) - 1)S_u + k) K(k) dk \lambda_u = 0 \end{aligned}$$

with the terminal conditions corresponding to  $V_T = \delta^H S_T + \beta^H$ .

Contrarily to the stochastic volatility case, where only the local behaviour of the risk and liquidity costs functions  $f$  and  $\mathcal{L}$  in 0 mattered, the optimal strategy in the jump-diffusion model requires the knowledge of both functions on their whole support.

## Conclusion

In this article we have extended the new methodology which we introduced in Abergel and Millot [1] in order to hedge contingent claims in incomplete markets. As was expected this methodology not only gives more flexibility in the choice of the function used to assess the risk of the strategy but it also allows to include part of the transaction costs, namely liquidity costs, in the picture. So it is especially suited for traders hedging large book or if trading occurs on illiquid markets. It also makes the link with the recent research on the orderbook modelling and on the market impact by using a liquidity costs function seen as the smoothing of the orderbook profile. Further extensions will include the study of existence of solutions to the system of PDEs and linking the fully non-linear PDE obtained with a second-order FBSDE to allow for a more tractable numerical scheme.

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