Tamari lattices and parking functions: proof of a conjecture of F. Bergeron

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TAMARI LATTICES AND PARKING FUNCTIONS: PROOF OF A CONJECTURE OF F. BERGERON

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Abstract. An m-ballot path of size n is a path on the square grid consisting of north and east unit steps, starting at $(0,0)$, ending at (mn, n), and never going below the line $\{x = my\}$. The set of these paths can be equipped with a lattice structure, called the m-Tamari lattice and denoted by $\mathcal{T}_n^{(m)}$, which generalizes the usual Tamari lattice $\mathcal{T}_n$ obtained when $m = 1$. This lattice was introduced by F. Bergeron in connection with the study of coinvariant spaces. He conjectured several intriguing formulas dealing with the enumeration of intervals in this lattice. One of them states that the number of intervals in $\mathcal{T}_n^{(m)}$ is

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}.$$

This conjecture was proved recently, but in a non-bijective way, while its form strongly suggests a connection with plane trees.

Here, we prove another conjecture of Bergeron, which deals with the number of *labelled* intervals. An interval $[P, Q]$ of $\mathcal{T}_n^{(m)}$ is *labelled* if the north steps of Q are labelled from 1 to n in such a way the labels increase along any sequence of consecutive north steps. We prove that the number of labelled intervals in $\mathcal{T}_n^{(m)}$ is

$$(m+1)^n(mn+1)^{n-2}.$$

The form of these numbers suggests a connection with parking functions, but our proof is non-bijective. It is based on a recursive description of intervals, which translates into a functional equation satisfied by the associated generating function. This equation involves a derivative and a divided difference, taken with respect to two additional variables. Solving this equation is the hardest part of the paper.

Finding a bijective proof remains an open problem.

1. Introduction and main results

An *m-ballot path* of size n is a path on the square grid consisting of north and east unit steps, starting at $(0,0)$, ending at (mn, n), and never going below the line $\{x = my\}$. It is well-known that there are

$$\frac{1}{mn+1}\binom{(m+1)n}{n}$$

such paths [6], and that they are in bijection with $(m+1)$-ary trees with n inner nodes. Bergeron recently defined on the set $\mathcal{T}_n^{(m)}$ of m-ballot paths of size n a partial order. It is convenient to describe it via the associated covering relation, exemplified in Figure 1.

Definition 1. *Let P and Q be two m-ballot paths of size n. Then Q covers P if there exists in P an east step a, followed by a north step b, such that Q is obtained from P by swapping a and S, where S is the shortest factor of P that begins with b and is a (translated) m-ballot path.*

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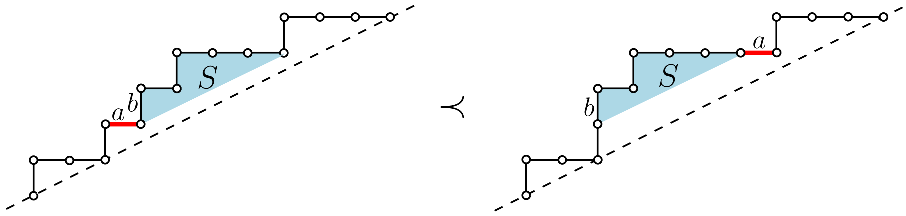

FIGURE 1. The covering relation between m-ballot paths ($m = 2$).

It was shown in [4] that this order endows $\mathcal{T}_n^{(m)}$ with a lattice structure, which is called the *m-Tamari lattice of size n*. When $m = 1$, it coincides with the classical Tamari lattice [2, 7, 15, 16]. Figure 2 shows two of the lattices $\mathcal{T}_n^{(m)}$.

These lattices are conjectured to have deep connections with the ring $\mathrm{DR}_{3,n}$ of polynomials in three sets of variables $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$, $Z = \{z_1, \ldots, z_n\}$, quotiented by the ideal generated by (trivariate) *diagonal invariants*. By diagonal invariants, one means constant term free polynomials that are invariant under the following action of the symmetric group $\mathfrak{S}_n$: for $\sigma \in \mathfrak{S}_n$ and f a polynomial,

$$\sigma(f(X,Y,Z)) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)}, z_{\sigma(1)}, \ldots, z_{\sigma(n)}).$$

We refer to [1, 4] for details about these conjectures, which have striking analogies with the much studied case of *two* sets of variables [10, 11, 13, 14, 17]. In particular, it seems that the role played by ballot paths for two sets of variables (see, *e.g.*, [8, 9, 12]) is played for three sets of variables by *intervals* of ballot paths in the Tamari order.

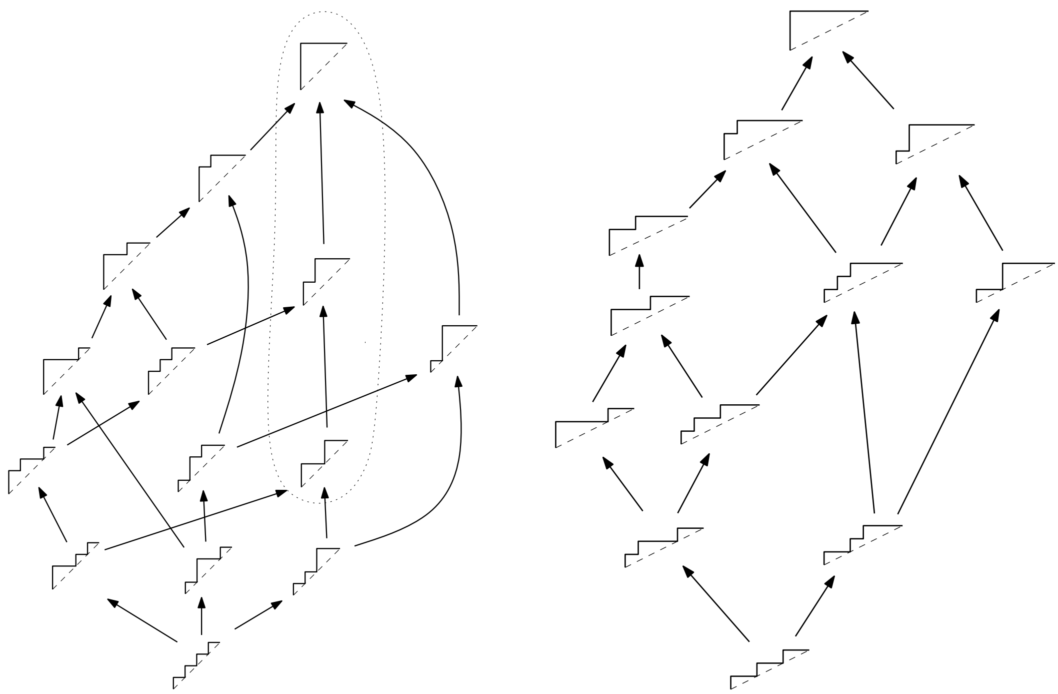

FIGURE 2. The m-Tamari lattice $\mathcal{T}_n^{(m)}$ for $m = 1$ and $n = 4$ (left) and for $m = 2$ and $n = 3$ (right). The three walks surrounded by a line in $\mathcal{T}_4^{(1)}$ form a lattice that is isomorphic to $\mathcal{T}_2^{(2)}$ (see Proposition 4).

For instance, it is conjectured in [1] that the the dimension of a certain polynomial ring related to $\mathrm{DR}_{3,n}$, but involving one more parameter m, is

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}, \tag{1}$$

and that this number counts intervals in the Tamari lattice $\mathcal{T}_n^{(m)}$. The latter statement was proved in [4] (the special case $m=1$ had been proved earlier [5]). The former one is presumably extremely difficult, given the complexity of the corresponding result for two sets of variables [13]. The dimension related result was observed earlier for small values of n by Haiman [14] in the case $m=1$.

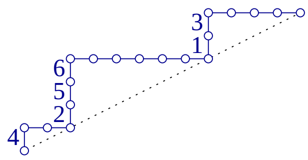

Figure 3. A labelled 2-ballot path.

The aim of this paper is to prove another conjecture of [1], dealing with *labelled* Tamari intervals. Let us say that an m-ballot path of size n is *labelled* if the north steps are labelled from 1 to n, in such a way the labels increase along any sequence of consecutive north steps (Figure 3). The number of labelled m-ballot paths of size n is

$$(mn+1)^{n-1}.$$

Indeed, these paths are in bijection with $(1, m, \ldots, m)$-*parking functions* of size n, in the sense of [19, 20]: the function f associated with a path Q satisfies $f(i)=k$ if the north step of Q labelled i lies at abscissa $k-1$. Now, we say that an m-Tamari interval $[P,Q]$ is labelled, if the upper path Q is labelled. It is conjectured in [1] that the number of labelled m-Tamari intervals of size n is

$$(m+1)^n(mn+1)^{n-2}, \tag{2}$$

and this is what we prove in this paper. It is also conjectured in [1] that this number is the dimension of a certain polynomial ring generalizing $\mathrm{DR}_{3,n}$ (which corresponds to the case $m=1$).

Our proof is, at first blush, analogous to the proof of (1) presented in [4]: we introduce a generating function $F^{(m)}(t;x,y)$ counting labelled intervals according to three parameters; we describe a recursive construction of intervals and translate it into a functional equation defining $F^{(m)}(t;x,y)$; we finally solve this equation, after having partially guessed its solution. However, the labelled case turns out to be significantly more difficult than the unlabelled one. It is not hard to explain the origin of this increased difficulty: for m fixed, the generating function of the numbers (1) is an *algebraic* series, and can be expressed in terms of the series $Z \equiv Z(t)$ satisfying

$$Z=\frac{t}{(1-Z)^{m(m+2)}}.$$

There exists a wealth of tools, both modern or ancient, to handle algebraic series (*e.g.*, factorisation, elimination, Gröbner bases, rational parametrizations when the genus is zero, efficient guessing techniques, all tools made effective in Maple and its packages, like `algcurves` and `gfun`). Such tools play a key role in the proof of (1). But the generating function of the numbers (2) is related to the series Z satisfying

$$Z=te^{m(m+1)Z},$$

which lives in the far less polished world of *differentially algebraic* series, for which much fewer tools are available.

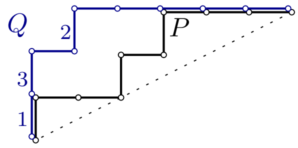

FIGURE 4. A labelled 2-Tamari interval $I = [P,Q]$ of size $|I| = 3$. It has $c(P) = 3$ contacts, and its initial rise is $r(Q) = 2$.

Our main result is actually more general that (2). Indeed, we refine the enumeration by taking into account two more parameters, which we now define. A *contact* of an m-ballot path P is a vertex of P lying on the line $\{x = my\}$. The *initial rise* of a ballot path Q is the length of the initial run of up steps in Q. A *contact* of a Tamari interval $[P,Q]$ is a contact of the *lower* path P, while the *initial rise* of this interval is the initial rise of the *upper* path Q (see Figure 4). We consider the exponential generating function of labelled m-Tamari intervals, counted by the size, the number of contacts and the initial rise. More precisely,

$$F^{(m)}(t;x,y) = \sum_{I=[P,Q]} \frac{t^{|I|}}{|I|!} x^{c(P)} y^{r(Q)}, \tag{3}$$

where the sum runs over all labelled m-Tamari intervals I, $|I|$ denotes the size of I (that is, the number of up steps in P), $c(P)$ the number of contacts of P and $r(Q)$ the initial rise of Q. The main result of this paper is a complicated closed form expression of $F^{(m)}(t;x,y)$, which becomes simple when $y = 1$. In particular, extracting the n^{th} coefficient in $F^{(m)}(t;1,1)$ proves Bergeron's conjecture (2).

Theorem 2. *Let $F^{(m)}(t;x,y) \equiv F(t;x,y)$ be the exponential generating function of labelled m-Tamari intervals, defined by* (3). *Let z and u be two indeterminates, and write*

$$t = ze^{-m(m+1)z} \quad \text{and} \quad x = (1+u)e^{-mzu}. \tag{4}$$

Then $F(t;x,1)$ becomes a series in z with polynomial coefficients in u, and this series has a simple expression:

$$F(t;x,1) = (1+u)e^{(m+1)z-(m-1)zu}\left(1 + \frac{1-e^{mzu}}{u}\right). \tag{5}$$

In particular,

$$F(t;1,1) = (1-mz)e^{(m+1)z},$$

and the number of labelled m-Tamari intervals of size n is

$$n![t^n]F(t;1,1) = (m+1)^n(mn+1)^{n-2}.$$

Our expression of $F^{(m)}(t;x,y)$ is given in Theorem 17. When $m = 1$, it takes a reasonably simple form, which we now present (the case $m = 2$ is also detailed at the end of the paper). Given a Laurent polynomial $P(u)$ in u, we denote by $[u^{\geq}]P(u)$ the *non-negative part of $P(u)$ in u*, defined by

$$[u^{\geq}]P(u) = \sum_{i\geq 0} P_i u^i \quad \text{if} \quad P(u) = \sum_{i\in\mathbb{Z}} P_i u^i.$$

The definition is then extended by linearity to power series whose coefficients are Laurent polynomials in u.

Theorem 3. *Let $F^{(1)}(t;x,y) \equiv F(t;x,y)$ be the generating function of labelled 1-Tamari intervals, defined by* (3). *Let z and u be two indeterminates, and set*

$$t = ze^{-2z} \quad \text{and} \quad x = (1+u)e^{-zu}. \tag{6}$$

Then $F(t;x,y)$ becomes a formal power series in z with polynomial coefficients in u and y, which is given by

$$F(t;x,y) = (1+u)\, e^{2yz}\ [u^{\geq}]\Big(e^{zu(y-1)+zy\bar{u}} - \bar{u}e^{z\bar{u}(y-1)+zyu}\Big), \tag{7}$$

with $\bar{u} = 1/u$. Equivalently,

$$\frac{F(t;x,y)}{(1+u)e^{2yz}} = \sum_{0\leq i\leq j} u^{j-i}\, \frac{z^{i+j}y^i(y-1)^j}{i!j!} - \sum_{0\leq j<i} u^{i-j-1}\, \frac{z^{i+j}y^i(y-1)^j}{i!j!}.$$

It is easily seen that the case $y = 1$ of the above formula reduces to the case $m = 1$ of (5). When $x = 1$, that is, $u = 0$, the double sums in the expression of $F(t;x,y)$ reduce to simple sums, and the generating function of labelled Tamari intervals is expressed in terms of Bessel functions:

$$\frac{F(t;1,y)}{e^{2yz}} = \sum_{i\geq 0} \frac{z^{2i}y^i(y-1)^i}{i!^2} - \sum_{j\geq 0} \frac{z^{2j+1}y^{j+1}(y-1)^j}{(j+1)!j!}.$$

The outline of the paper goes as follows: in Section 2 we derive from a recursive description of labelled Tamari intervals a functional equation satisfied by their generating function. This equation involves a derivative (with respect to y) and a divided difference (with respect to x). We present in Section 3 the principle of the proof, and exemplify it on the case $m = 1$, thus obtaining Theorem 3 above. Section 4 deals with the general case, and proves Theorem 2.

We conclude this introduction with some notation and a few definitions. Let $\mathbb{K}$ be a commutative ring and t an indeterminate. We denote by $\mathbb{K}[t]$ (resp. $\mathbb{K}[[t]]$) the ring of polynomials (resp. formal power series) in t with coefficients in $\mathbb{K}$. If $\mathbb{K}$ is a field, then $\mathbb{K}(t)$ denotes the field of rational functions in t, and $\mathbb{K}((t))$ the field of Laurent series in t (that is, series of the form $\sum_{n\geq n_0} a_n t^n$, with $n_0 \in \mathbb{Z}$). These notations are generalized to polynomials, fractions and series in several indeterminates. We denote by bars the reciprocals of variables: for instance, $\bar{u} = 1/u$, so that $\mathbb{K}[u,\bar{u}]$ is the ring of Laurent polynomials in u with coefficients in $\mathbb{K}$. The coefficient of u^n in a Laurent series $F(u)$ is denoted by $[u^n]F(x)$.

We have defined the non-negative part of a Laurent polynomial $P(u)$ above Theorem 3. We define similarly the positive part of $P(u)$, denoted by $[u^{>}]P(u)$.

The series we handle in this paper involve a main variable t, or z after the change of variables (4), and then additional variables x and y. So they should in principle be denoted $F(t;x,y)$, but we often omit the variable t (or z), to avoid heavy notation and enhance role of the additional variables x and y.

2. A functional equation

The aim of this section is to describe a recursive decomposition of labelled m-Tamari intervals, and to translate it into a functional equation satisfied by the associated generating function. Our description of the decomposition is self-contained, but we refer to [4] for several proofs and details.

2.1. Recursive decomposition of Tamari intervals

We start by modifying the appearance of 1-ballot paths. We apply a 45 degree rotation to 1-ballot paths to transform them into *Dyck paths.* A Dyck path of size n consists of steps $u = (1,1)$ (up steps) and steps $d = (1,-1)$ (down steps), starts at $(0,0)$, ends at $(2n,0)$ and never goes below the x-axis. We say that an up step has *rank* i if it is the i^{th} up step of the path. We often represent Dyck paths by words on the alphabet $\{u,d\}$.

Consider now an m-ballot path of size n, and replace each north step by a sequence of m north steps. This gives a 1-ballot path of size mn, and thus, after a rotation, a Dyck path. In this path, for each $i \in [\![0, n-1]\!]$, the up steps of ranks $mi+1,\ldots,m(i+1)$ are consecutive. We call the Dyck paths satisfying this property *m-Dyck paths*, and say that the up steps of ranks $mi+1,\ldots,m(i+1)$ form a *block*. Clearly, m-Dyck paths of size mn (*i.e.*, having n blocks) are in one-to-one correspondence with m-ballot paths of size n. We often denote by $\mathcal{T}_n$, rather than

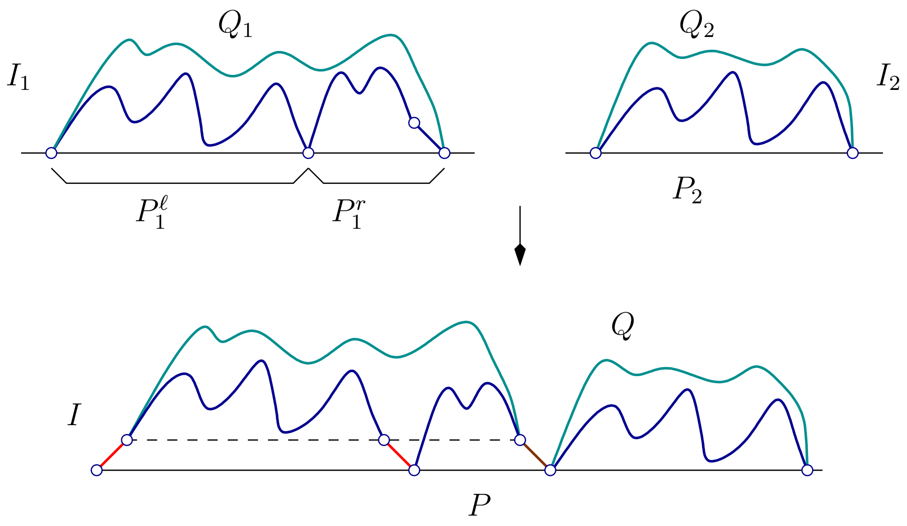

FIGURE 5. The recursive construction of Tamari intervals.

$\mathcal{T}_n^{(1)}$, the usual Tamari lattice of size n. Similarly, the intervals of this lattice are called Tamari intervals, rather than 1-Tamari intervals. As proved in [4], the transformation of m-ballot paths into m-Dyck paths maps $\mathcal{T}_n^{(m)}$ on a sublattice of $\mathcal{T}_{mn}$.

Proposition 4 ([4, Prop. 4])**.** *The set of m-Dyck paths with n blocks is the sublattice of $\mathcal{T}_{nm}$ consisting of the paths that are larger than or equal to $u^m d^m \dots u^m d^m$. It is order isomorphic to $\mathcal{T}_n^{(m)}$.*

We now describe a recursive decomposition of (unlabelled) Tamari intervals, again borrowed from [4]. Thanks to the embedding of $\mathcal{T}_n^{(m)}$ into $\mathcal{T}_{nm}$, it will also enable us to decompose m-Tamari intervals, for any value of m, in the next subsection.

A Tamari interval $I = [P, Q]$ is *pointed* if its lower path P has a distinguished contact (we refer to the introduction for the definition of contacts). Such a contact splits P into two Dyck paths P^ℓ and P^r, respectively located to the left and to the right of the contact. The pointed interval I is *proper* is P^ℓ is not empty, *i.e.*, if the distinguished contact is not $(0, 0)$. We often use the notation $I = [P^\ell P^r, Q]$ to denote a pointed Tamari interval.

Proposition 5. *Let $I_1 = [P_1^\ell P_1^r, Q_1]$ be a pointed Tamari interval, and let $I_2 = [P_2, Q_2]$ be a Tamari interval. Construct the Dyck paths*

$$P = uP_1^\ell dP_1^r P_2 \quad \text{and} \quad Q = uQ_1 dQ_2$$

as shown in Figure 5. Then $I = [P, Q]$ is a Tamari interval. Moreover, the mapping $(I_1, I_2) \mapsto I$ is a bijection between pairs (I_1, I_2) formed of a pointed Tamari interval and a Tamari interval, and Tamari intervals I of positive size. Note that I_1 is proper if and only if the initial rise of P is not 1.

Remarks

1. To recover P_1^ℓ, P_1^r, Q_1, P_2 and Q_2 from P and Q, one proceeds as follows: Q_2 is the part of Q that follows the first return of Q to the x-axis; this defines Q_1 unambiguously. The path P_2 is the suffix of P having the same size as Q_2. This defines $P_1 := uP_1^\ell dP_1^r$ unambiguously. Finally P_1^r is the part of P_1 that follows the first return of P_1 to the x-axis, and this defines P_1^ℓ unambiguously.

2. This proposition is obtained by combining Proposition 5 in [4] and the case $m = 1$ of Lemma 9 in [4]. With the notation $(P'; p_1)$ and (Q', q_1) used therein, the above paths P_2 and Q_2

are respectively the parts of P' and Q' that lie to the right of q_1, while $P_1^\ell P_1^r$ and Q_1 are the parts of P' and Q' that lie to the left of q_1. The pointed vertex p_1 is the endpoint of P_1^ℓ. Proposition 5 in [4] guarantees that, if $P \preceq Q$ in the Tamari order, then $P_1^\ell P_1^r \preceq Q_1$ and $P_2 \preceq Q_2$.
3. One can keep track of several parameters in the construction of Proposition 5. For instance, the initial rise of Q equals the initial rise of Q_1 plus one. Also, the number of contacts of P is

$$c(P) = c(P_1^r) + c(P_2). \tag{8}$$

2.2. From the decomposition to a functional equation

We will now establish the following functional equation.

Proposition 6. *For $m \geq 1$, let $F^{(m)}(t; x, y) \equiv F(x, y)$ be the exponential generating function of labelled m-Tamari intervals, defined by* (3). *Then $F(x, 0) = x$ and*

$$\frac{\partial F}{\partial y}(x, y) = tx \left(F(x, 1) \cdot \Delta\right)^{(m)} (F(x, y)), \tag{9}$$

where Δ is the following divided difference operator

$$\Delta S(x) = \frac{S(x) - S(1)}{x - 1},$$

and the power m means that the operator $G(x, y) \mapsto F(x, 1) \cdot \Delta G(x, y)$ is applied m times.

Proof. We constantly use the inclusion $\mathcal{T}_n^{(m)} \subset \mathcal{T}_{nm}$ given by Proposition 4. That is, we identify elements of $\mathcal{T}_n^{(m)}$ with m-Dyck paths having n blocks.

It is obvious that $F(x, 0) = x$, since the interval of size 0 is the only interval of initial rise 0, and has one contact. The functional equation (9) relies on the decomposition of Tamari intervals described in Proposition 5.

We will actually apply this decomposition to a slight generalization of m-Tamari intervals. For $k \geq 0$, a *k-augmented m-Dyck path* is a Dyck path of size $k + mn$ for some integer n, where the first k steps are up steps, and all the other up steps can be partitioned into *blocks* of m consecutive up steps. The k first steps are not considered to be part of a block, even if k is a multiple of m. We denote by $\mathcal{T}^{(m,k)}$ the set of k-augmented m-Dyck paths.

A Tamari interval $I = [P, Q]$ is a *k-augmented m-Tamari interval* if both P and Q belong to $\mathcal{T}^{(m,k)}$. Assume that P and Q contain n blocks. Then I is *labelled* if the blocks of Q are labelled from 1 to n in such a way the labels increase along any sequence of consecutive blocks. Note that labelled 0-augmented m-Tamari intervals coincide with labelled m-Tamari intervals. Generalizing (3), we denote by $F_k(t; x, y) \equiv F_k(x, y)$ the exponential generating function of labelled k-augmented m-Tamari intervals, counted by the number of blocks (variable t), the number of *non-initial* contacts (that is, contacts distinct from $(0, 0)$ — variable x), and the number of blocks contained in the first ascent (variable y).

In what follows, we first obtain an expression of $F_k(x, y)$ in terms of $F(x, y)$:

$$F_k(x, y) = \begin{cases} F(x, y)/x & \text{if } k = 0, \\ (F(x, 1) \cdot \Delta)^{(k)} F(x, y) & \text{otherwise.} \end{cases} \tag{10}$$

We then relate m-augmented m-Tamari intervals to m-Tamari intervals, proving that

$$txF_m(x, y) = \frac{\partial F}{\partial y}(x, y). \tag{11}$$

This identity, combined with the case $k = m$ of (10), gives (9).

We thus need to prove (10) and (11). The case $k = 0$ of (10) is clear, since 0-augmented m-Tamari intervals are just m-Tamari intervals. The factor x arises from the fact that $F_0(x, y)$ only keeps track of non-initial contacts, while $F(x, y)$ counts all of them.

Let us now address the case $k \geq 1$ of (10). Let $I = [P, Q]$ be a labelled k-augmented m-Tamari interval. By Proposition 5, one can decompose I into a pair (I_1, I_2) of Tamari intervals, with $I_1 = [P_1^\ell P_1^r, Q_1]$ and $I_2 = [P_2, Q_2]$ (see Figure 5). Since the up steps of P_2 and Q_2 are not in the first ascent of P and Q, the paths P_2 and Q_2 are actually m-Dyck paths, so that I_2 is an m-Tamari interval. Similarly, I_1 is a pointed $(k-1)$-augmented m-Tamari interval, which is proper if $k > 1$ (Proposition 5).

The blocks of Q_1 and Q_2 inherit a labelling from Q. We normalise these labellings in the usual way: if P_1 has n_1 blocks and Q_2 has n_2 blocks, we relabel the blocks of Q_1 (resp. Q_2) with $1, \dots, n_1$ (resp. $1, \dots, n_2$) while preserving the relative order of the labels occurring in Q_1 (resp. Q_2).

Conversely, consider a pair (I_1, I_2), where $I_1 = [P_1^\ell P_1^r, Q_1]$ is a labelled pointed $(k-1)$-augmented m-Tamari interval and I_2 is a labelled m-Tamari interval. If $k > 1$, assume moreover that I_1 is proper. If Q_1 and Q_2 have respectively n_1 and n_2 blocks, one can reconstruct from (I_1, I_2) exactly $\frac{(n_1+n_2)!}{n_1!n_2!}$ different labelled k-augmented m-Tamari intervals $I = [P, Q]$, having $n_1 + n_2$ blocks. By (8), the number of non-initial contacts in I is the number of non-initial contacts in P_1^r, plus the number of contacts in P_2. The number of blocks in the first ascent of Q is the number of blocks in the first ascent of Q_1.

The exponential generating function of labelled m-Tamari intervals I_2, counted by the size and the number of contacts, is $F(x, 1)$. Let $F_{k-1}^\circ(t; x, y) \equiv F_{k-1}^\circ(x, y)$ be the exponential generating function of labelled *proper* pointed $(k-1)$-augmented m-Tamari intervals $I_1 = [P_1^\ell P_1^r, Q_1]$, counted by the size, the number of non-initial contacts in P_1^r and the number of blocks in the first ascent of Q_1. Note that the generating function of labelled *non-proper* pointed 0-augmented m-Tamari intervals is simply $F_0(x, y)$. The above construction then implies that

$$F_k(x,y) = F(x,1)\left(F_{k-1}^\circ(x,y) + F_0(x,y)1_{k=1}\right). \tag{12}$$

We claim that

$$F_{k-1}^\circ(x,y) = \Delta F_{k-1}(x,y). \tag{13}$$

It follows from (12–13) that

$$F_k(x,y) = \begin{cases} F(x,1)\cdot\Delta\big(xF_0(x,y)\big) & \text{if } k=1, \\ F(x,1)\cdot\Delta F_{k-1}(x,y) & \text{otherwise.}\end{cases}$$

Now a simple induction on $k > 0$, combined with the case $k = 0$ of (10) and, proves the case $k > 0$ of (10). So (10) will be proved if we establish (13). Write

$$F_{k-1}(x,y) = \sum_{i\ge 0} F_{k-1,i}(y)x^i,$$

so that $F_{k-1,i}(y)$ counts labelled $(k-1)$-augmented m-Tamari intervals having i non-initial contacts. By pointing a non-initial contact, such an interval gives rise to i labelled proper pointed $(k-1)$-augmented m-Tamari intervals $[P_1^\ell P_1^r, Q_1]$, having respectively $0, 1, \dots, i-1$ non-initial contacts in P_1^r. Hence

$$\begin{aligned} F_{k-1}^\circ(x,y) &= \sum_{i\ge 0} F_{k-1,i}(y)\left(1 + x + \cdots + x^{i-1}\right), \\ &= \sum_{i\ge 0} F_{k-1,i}(y)\frac{x^i-1}{x-1} \\ &= \frac{F_{k-1}(x,y) - F_{k-1}(1,y)}{x-1} \\ &= \Delta F_{k-1}(x,y). \end{aligned}$$

This coincides with (13).

We finally want to prove (11), and this will complete the proof of Proposition 6. A labelled m-augmented m-Tamari interval $I = [P, Q]$ having $n-1$ blocks gets in $F_m(x, y)$ a weight

$$\frac{t^{n-1}}{(n-1)!}\,x^{c(P)-1}y^{r(Q)-1} = \frac{n}{tx}\left(\frac{t^n}{n!}\,x^{c(P)}y^{r(Q)-1}\right),$$

where $r(Q)$ is the initial rise of Q, divided by m. Let us interpret the factor n as the choice of a label $i \in [\![1, n]\!]$ assigned to the first m steps of P, while the labels of the blocks, which were $1, \dots, n-1$, are redistributed so as to avoid i. The above identity shows that $txF_m(x, y)$ counts (by the number of blocks, the number of contacts, and the initial rise minus one), m-Tamari intervals $[P, Q]$ in which the blocks are labelled in such a way the labels increase along sequences of consecutive blocks, except that the first block of the first ascent may have a larger label than the second block of the first ascent. Such intervals are obtained from usual labelled m-Tamari intervals by choosing a block in the first ascent and exchanging its label with the label of the very first block. In terms of power series, choosing a block of the first ascent boils down to differentiating with respect to y (this also decreases by 1 the exponent of y), and we thus obtain (11). ∎

3. Principle of the proof, and the case $m = 1$

3.1. Principle of the proof

Let us consider the functional equation (9), together with the initial condition $F(t; x, 0) = x$. Perform the change of variables (4), and denote $G(z; u, y) \equiv G(u, y) = F(t; x, y)$. Then $G(u, y)$ is a series in z with coefficients in $\mathbb{Q}[u, y]$, satisfying

$$\frac{\partial G}{\partial y}(u, y) = z(1+u)e^{-mzu-m(m+1)z} \left(\frac{uG(u,1)}{(1+u)e^{-mzu}-1} \, \Delta_u \right)^{(m)} G(u, y), \tag{14}$$

with $\Delta_u H(u) = \frac{H(u)-H(0)}{u}$, and the initial condition

$$G(u, 0) = (1+u)e^{-mzu}. \tag{15}$$

Observe that this pair of equations defines $G(z; u, y) \equiv G(u, y)$ uniquely as a formal power series in z. Indeed, the coefficient of z^n in G can be computed inductively from these equations (one first determines the coefficient of z^n in $\frac{\partial G}{\partial y}$, which can be expressed, thanks to (14), in terms of the coefficients of z^i in G for $i < n$. Then the coefficient of z^n in G is obtained by integration with respect to y, using the initial condition (15)). Hence, if we exhibit a series $\tilde{G}(z; u, y)$ that satisfies both equations, then $\tilde{G}(z; u, y) = G(z; u, y)$. We are going to construct such a series.

Let

$$G_1(z; u) \equiv G_1(u) = (1+u)e^{(m+1)z-(m-1)zu} \left(1 + \frac{1-e^{mzu}}{u} \right). \tag{16}$$

Then $G_1(u)$ is a series in z with polynomial coefficients in u, which, as we will see, coincides with $G(u, 1)$. Consider now the following equation, obtained from (14) by replacing $G(u, 1)$ by its conjectured value $G_1(u)$:

$$\frac{\partial \tilde{G}}{\partial y}(z; u, y) = z(1+u)e^{-mzu-m(m+1)z} \left(\frac{uG_1(u)}{(1+u)e^{-mzu}-1} \, \Delta_u \right)^{(m)} \tilde{G}(z; u, y), \tag{17}$$

with the initial condition

$$\tilde{G}(z; u, 0) = (1+u)e^{-mzu}. \tag{18}$$

Eq. (17) can be rewritten as

$$\frac{\partial \tilde{G}}{\partial y}(z; u, y) = z(1+u)e^{-mzu} \Lambda^{(m)} \tilde{G}(z; u, y) \tag{19}$$

where Λ is the operator defined by

$$\Lambda(H)(u) = \frac{H(u)-H(0)}{A(u)} \tag{20}$$

with

$$A(u) = \frac{u}{1+u} e^{-zu}, \tag{21}$$

and $\Lambda^{(m)}$ denotes the m-th iterate of Λ. Again, it is not hard to see that (19) and the initial condition (18) define a unique series in z, denoted $\tilde{G}(z;u,y) \equiv \tilde{G}(u,y)$. The coefficients of this series lie in $\mathbb{Q}[u,y]$. The principle of our proof can be described as follows.

If we prove that $\tilde{G}(u,1) = G_1(u)$, then the equation (17) *satisfied by $\tilde{G}$ coincides with the equation* (14) *that defines G, and thus $\tilde{G}(u,y) = G(u,y)$. In particular, $G_1(z;u) = \tilde{G}(z;u,1) = G(z;u,1) = F(t;x,1)$, and Theorem* 2 *is proved.*

3.2. The case $m = 1$

Take $m = 1$. In this subsection, we describe the three steps that, starting from (19), prove that $\tilde{G}(u,1) = G_1(u)$. In passing, we establish the expression (7) of $F(t;x,1)$ (equivalently, of $\tilde{G}(z;u,1)$) given in Theorem 3.

3.2.1. ***A homogeneous differential equation and its solution.*** When $m = 1$, the equation (19) defining $\tilde{G}(z;u,y) \equiv \tilde{G}(u,y)$ reads

$$\frac{\partial \tilde{G}}{\partial y}(u,y) = z(1+u)(1+\bar{u})(\tilde{G}(u,y) - \tilde{G}(0,y)), \tag{22}$$

where $\bar{u} = 1/u$, with the initial condition

$$\tilde{G}(u,0) = (1+u)e^{-zu}. \tag{23}$$

These equations imply that $\tilde{G}(-1,y) = 0$. The coefficient of $\tilde{G}(u,y)$ in the right-hand side of (22) is symmetric in u and $\bar{u}$. We are going to exploit this symmetry to eliminate the term $\tilde{G}(0,y)$. Replacing u by $\bar{u}$ in (22) gives

$$\frac{\partial \tilde{G}}{\partial y}(\bar{u},y) = z(1+u)(1+\bar{u})(\tilde{G}(\bar{u},y) - \tilde{G}(0,y)),$$

so that

$$\frac{\partial}{\partial y}\left(\tilde{G}(u,y) - \tilde{G}(\bar{u},y)\right) = z(1+u)(1+\bar{u})\left(\tilde{G}(u,y) - \tilde{G}(\bar{u},y)\right).$$

This is a homogeneous linear differential equation satisfied by $\tilde{G}(u,y) - \tilde{G}(\bar{u},y)$. It is readily solved, and the initial condition (23) yields

$$\tilde{G}(u,y) - \tilde{G}(\bar{u},y) = (1+u)e^{yz(1+u)(1+\bar{u})}\left(e^{-zu} - \bar{u}e^{-z\bar{u}}\right). \tag{24}$$

3.2.2. ***Reconstruction of $\tilde{G}(u,y)$.*** Recall that $\tilde{G}(u,y) \equiv \tilde{G}(z;u,y)$ is a series in z with polynomial coefficients in u and y. Hence, by extracting from the above equation the positive part in u (as defined at the end of Section 1), we obtain

$$\tilde{G}(u,y) - \tilde{G}(0,y) = [u^>]\left((1+u)e^{yz(1+u)(1+\bar{u})}\left(e^{-zu} - \bar{u}e^{-z\bar{u}}\right)\right).$$

For any Laurent polynomial P, we have

$$[u^>](1+u)P(u) = (1+u)[u^>]P(u) + u[u^0]P(u). \tag{25}$$

Hence

$$\begin{aligned}\tilde{G}(u,y) - \tilde{G}(0,y) = (1+u)[u^>]&\left(e^{yz(1+u)(1+\bar{u})}\left(e^{-zu} - \bar{u}e^{-z\bar{u}}\right)\right)\\ &+ u[u^0]\left(e^{yz(1+u)(1+\bar{u})}\left(e^{-zu} - \bar{u}e^{-z\bar{u}}\right)\right).\end{aligned}$$

Setting $u = -1$ in this equation gives, since $\tilde{G}(-1,y) = 0$,

$$-\tilde{G}(0,y) = -[u^0]\left(e^{yz(1+u)(1+\bar{u})}\left(e^{-zu} - \bar{u}e^{-z\bar{u}}\right)\right),$$

so that finally,

$$\begin{aligned}\tilde{G}(u,y) &= (1+u)[u^{>}]\left(e^{yz(1+u)(1+\bar{u})}\left(e^{-zu}-\bar{u}e^{-z\bar{u}}\right)\right)\\ &\qquad\qquad +(1+u)[u^{0}]\left(e^{yz(1+u)(1+\bar{u})}\left(e^{-zu}-\bar{u}e^{-z\bar{u}}\right)\right)\\ &= (1+u)[u^{\geq}]\left(e^{yz(1+u)(1+\bar{u})}\left(e^{-zu}-\bar{u}e^{-z\bar{u}}\right)\right). \qquad (26)\end{aligned}$$

As explained in Section 3.1, $\tilde{G}(u,y)=G(u,y)$ will be proved if we establish that $\tilde{G}(u,1)=G_1(u)$. This is the final step of our proof.

3.2.3. ***The case*** $y=1$**.** Equation (26) completely describes the solution of (22). It remains to check that $\tilde{G}(u,1)=G_1(u)$, that is

$$\tilde{G}(u,1)=(1+u)e^{2z}\left(1+\frac{1-e^{zu}}{u}\right). \qquad (27)$$

Let us set $y=1$ in (26). We find

$$\begin{aligned}\tilde{G}(u,1) &= (1+u)[u^{\geq}]\left(e^{z(2+\bar{u})}-\bar{u}e^{z(2+u)}\right)\\ &= (1+u)e^{2z}\left(1-\frac{e^{zu}-1}{u}\right),\end{aligned}$$

which coincides with (27). Hence $\tilde{G}(z;u,y)=G(z;u,y)=F(t;x,y)$ (with the change of variables (4)), and Theorem 3 is proved.

3.2.4. ***The trivariate series.*** We have now proved that $\tilde{G}(u,y)=G(u,y)$, so that $F(x,y)=\tilde{G}(u,y)$ after the change of variables (6). The expression (7) of $F(x,y)$ given in Theorem 3 now follows from (26).

4. Solution of the functional equation: the general case

We now adapt to the general case the solution described for $m=1$ in Section 3.2.

4.1. A homogeneous differential equation and its solution

Let us return to the equation (19) satisfied by $\tilde{G}(u,y)$. The coefficient of $\tilde{G}(u,y)$ in the right-hand side of this equation is $zv(u)$, where

$$v(u)=(1+u)e^{-mzu}A(u)^{-m}=(1+u)^{m+1}\bar{u}^{m}.$$

In the case $m=1$, this (Laurent) polynomial was $(1+u)(1+\bar{u})$, and took the same value for u and $\bar{u}$. We are again interested in the series u_i such that $v(u_i)=v(u)$.

Lemma 7. *Denote $v\equiv v(u)=(1+u)^{m+1}u^{-m}$, and consider the following polynomial equation in U:*

$$(1+U)^{m+1}=U^{m}v. \qquad (28)$$

This equation has no double root. We denote its $m+1$ roots by $u_0=u,u_1,\ldots,u_m$.

Proof. A double root of (28) would also satisfy

$$(m+1)(1+u)^{m}=mu^{m-1}v,$$

and this is easily shown to be impossible. ∎

Remark. One can of course express the u_i's as Puiseux series in u (see [18, Ch. 6]), but this will not be needed here, and we will think of them as abstract elements of an algebraic extension of $\mathbb{Q}(u)$. In fact, all the series in z that involve the u_i's in this paper have coefficients that are *symmetric rational functions* of the u_i's, and hence, rational functions of v. At some point, we will have to prove that a symmetric polynomial in the u_i's (and thus a polynomial in v) vanishes at $v=0$, that is, at $u=-1$, and we will consider series expansions of the u_i's around $u=-1$.

Proposition 8. *Denote $v = (1+u)^{m+1}u^{-m}$, and let the series u_i be defined as above. Denote $A_i = A(u_i)$, where $A(u)$ is given by* (21). *Then*

$$\sum_{i=0}^{m} \frac{\tilde{G}(u_i, y)}{\prod_{j\neq i}(A_i - A_j)} = ve^{zvy}. \tag{29}$$

By $\prod_{j\neq i}(A_i - A_j)$ we mean $\prod_{0\leq j\leq m, j\neq i}(A_i - A_j)$ but we prefer the shorter notation when the bounds on j are clear. Observe that the A_i's are distinct since the u_i's are distinct (the coefficient of z^0 in $A(u)$ is $1/(1+\bar{u})$). Note also that when $m = 1$, then $u_0 = u$, $u_1 = \bar{u}$, and (29) coincides with (24). In order to prove the proposition, we need the following two lemmas.

Lemma 9. *Let $x_0, x_1, \ldots, x_m$ be $m+1$ variables. Then*

$$\sum_{i=0}^{m} \frac{x_i{}^m}{\prod_{j\neq i}(x_i - x_j)} = 1 \tag{30}$$

and

$$\sum_{i=0}^{m} \frac{1/x_i}{\prod_{j\neq i}(x_i - x_j)} = (-1)^m \prod_{i=0}^{m} \frac{1}{x_i}. \tag{31}$$

Moreover, for any polynomial Q of degree less than m,

$$\sum_{i=0}^{m} \frac{Q(x_i)}{\prod_{j\neq i}(x_i - x_j)} = 0. \tag{32}$$

Proof. By the Lagrange interpolation, any polynomial R of degree at most m satisfies:

$$R(X) = \sum_{i=0}^{m} R(x_i) \prod_{j\neq i} \frac{X - x_j}{x_i - x_j}.$$

Equations (31) and (32) follow by evaluating this equation at $X = 0$, respectively with $R(X) = 1$ and $R(X) = XQ(X)$. Equation (30) is obtained by taking $R(X) = X^m$ and extracting the leading coefficient. ■

Lemma 10. *Let $H(u) \equiv H(z; u) \in \mathbb{K}(u)[[z]]$ be a formal power series in z whose coefficients are rational functions in u over some field $\mathbb{K}$ of characteristic* 0*. Assume that these coefficients have no pole at $u = 0$. Then there exists a sequence $g_0(z), g_1(z), \ldots$ of formal power series in z such that for every $k \geq 0$ one has:*

$$\Lambda^{(k)} H(u) = \frac{1}{A(u)^k} \left(H(u) - \sum_{j=0}^{k-1} g_j(z) A(u)^j \right). \tag{33}$$

where Λ is the operator defined by (20)*.*

Proof. We denote by $\mathcal{L}$ the subring of $\mathbb{K}(u)[[z]]$ formed by formal power series whose coefficients have no pole at $u = 0$. By assumption, $H(u) \in \mathcal{L}$. We use the notation $O(u^k)$ to denote an element of $\mathbb{K}(u)[[z]]$ of the form $u^k J(z; u)$ with $J(z; u) \in \mathcal{L}$.

First, note that $A(u) = ue^{-zu}/(1+u)$ belongs to $\mathbb{K}(u)[[z]]$. Moreover,

$$A(u) = u + O(u^2). \tag{34}$$

We will first prove that there exists a sequence of formal power series $(g_j)_{j\geq 0} \in \mathbb{K}[[z]]^{\mathbb{N}}$ such that for all $\ell \geq 0$,

$$H(u) = \sum_{j=0}^{\ell-1} g_j(z) A(u)^j + O(u^\ell). \tag{35}$$

We will then prove that these series g_j satisfy (33). In order to prove (35), we proceed by induction on $\ell \geq 0$. The identity holds for $\ell = 0$ since $H(u) \in \mathcal{L}$. Assume it holds for some $\ell \geq 0$, *i.e.*, that there exists series $g_0, \dots, g_{\ell-1}$ in $\mathbb{K}[[z]]$ and $J(u) \in \mathcal{L}$ such that

$$H(u) = \sum_{j=0}^{\ell-1} g_j(z) A(u)^j + u^\ell J(u).$$

By (34) and by induction on r, we have $u^r = A(u)^r + O(u^{r+1})$ for all $r \geq 0$. Using this identity with $r = \ell$, and rewriting $J(u) = J(0) + O(u)$, we obtain $u^\ell J(u) = J(0)A(u)^\ell + O(u^{\ell+1})$, so that:

$$H(u) = \sum_{j=0}^{\ell} g_j(z) A(u)^j + O(u^{\ell+1}),$$

with $g_\ell(z) := J(0)$. Thus (35) holds for $\ell + 1$.

Let us now prove (33). Again, we proceed by induction on $k \geq 0$. The identity clearly holds for $k = 0$. Assume it holds for some $k \geq 0$. In (33), replace $H(u)$ by its expression (35) obtained with $\ell = k+1$, and let u tend to 0: this shows that $g_k(z)$ is in fact $\Lambda^{(k)} H(0)$. From the definition of Λ one then obtains

$$\Lambda^{(k+1)} H(u) = \frac{\Lambda^{(k)} H(u) - g_k(z)}{A(u)} = \frac{1}{A(u)^{k+1}} \left(H(u) - \sum_{j=0}^{k} g_j(z) A(u)^j \right).$$

Thus (33) holds for $k + 1$. ∎

Proof of Proposition 8. Thanks to Lemma 10 (applied with $\mathbb{K} = \mathbb{Q}(y)$), we can rewrite (19) as

$$\frac{\partial \tilde{G}}{\partial y}(u, y) = zv\tilde{G}(u, y) - zv \sum_{j=0}^{m-1} g_j(y) A(u)^j, \tag{36}$$

with $v = (1 + u)^{m+1} \bar{u}^m$, and for all $j \geq 0$ the series $g_j(y) \equiv g_j(z; y)$ belongs to $\mathbb{K}(y)[[z]]$ (one can actually show that $g_j(y) \in \mathbb{K}[y][[z]]$ but we will not need this). As was done in Section 3.2.1, we are going to use the fact that $v(u_i) = v$ for all $i \in [\![0, m]\!]$ to eliminate the m unknown series $g_j(y)$. For $0 \leq i \leq m$, the substitution $u \mapsto u_i$ in (36) gives:

$$\frac{\partial \tilde{G}}{\partial y}(u_i, y) = zv\tilde{G}(u_i, y) - zv \sum_{j=0}^{m-1} g_j(y) A_i^j = zv\tilde{G}(u_i, y) - zvQ(A_i) \tag{37}$$

where $Q(X) = \sum_{j=0}^{m-1} g_j(y) X^j$ is a polynomial in X of degree less than m. Consider the linear combination

$$L(u, y) := \sum_{i=0}^{m} \frac{\tilde{G}(u_i, y)}{\prod_{j \neq i} (A_i - A_j)}. \tag{38}$$

Then by (37),

$$\begin{aligned} \frac{\partial L}{\partial y}(u, y) &= zvL(u, y) - zv \sum_{i=0}^{m} \frac{Q(A_i)}{\prod_{j \neq i} (A_i - A_j)} \\ &= zvL(u, y) \qquad \text{by (32).} \end{aligned}$$

This homogeneous linear differential equation is readily solved:

$$L(u, y) = L(u, 0) e^{zvy}.$$

Recall the expression (38) of L in terms of $\tilde{G}$. The initial condition (18) can be rewritten $\tilde{G}(u,0) = vA(u)^m$, which yields

$$\begin{aligned} L(u,0) &= v\sum_{i=0}^{m} \frac{{A_i}^m}{\prod_{j\neq i}(A_i - A_j)} \\ &= v \end{aligned}$$

by (30). Hence $L(u,y) = ve^{zyv}$, and the proposition is proved. ■

4.2. Reconstruction of $\tilde{G}(u,y)$

We are now going to prove that (29), together with the condition $\tilde{G}(-1,y) = 0$ derived from (19), characterizes the series $\tilde{G}(u,y)$. We will actually obtain a (complicated) expression for this series, generalizing (26).

We first introduce some notation. Consider a formal power series in z, denoted $H(z;u) \equiv H(u)$, having coefficients in $\mathbb{K}[u]$ for some field $\mathbb{K}$ of characteristic 0 (for instance $\mathbb{Q}(y)$). We define a series H_k in z whose coefficients are symmetric functions of $k+1$ variables $x_0, \ldots, x_k$:

$$H_k(x_0,\ldots,x_k) = \sum_{i=0}^{k} \frac{H(x_i)}{\prod\limits_{0\le j\le k, j\neq i} (A(x_i) - A(x_j))},$$

where, as above, A is defined by (21).

Lemma 11. *The series $H_k(x_0,\ldots,x_k)$ has coefficients in $\mathbb{K}[x_0,\ldots,x_k]$. If, moreover, $H(-1) = 0$, then the coefficients of H_k are multiples of $(1+x_0)\cdots(1+x_k)$.*

Proof. Observe that

$$\frac{1}{A(x_i) - A(x_j)} = \frac{1}{x_i - x_j} B(x_i, x_j), \tag{39}$$

where $B(x_i,x_j)$ is a series in z with *polynomial* coefficients in x_i and x_j. Hence

$$H_k(x_0,\ldots,x_k) \prod_{0\le i<j\le k} (x_i - x_j)$$

has polynomial coefficients in the x_i's. But these polynomials are anti-symmetric in the x_i's (since H_k is symmetric), hence they must be multiples of the Vandermonde $\prod_{i<j}(x_i - x_j)$. Hence $H_k(x_0,\ldots,x_k)$ has polynomial coefficients.

A stronger property than (39) actually holds, namely:

$$\frac{1}{A(x_i) - A(x_j)} = \frac{(1+x_i)(1+x_j)}{x_i - x_j} C(x_i, x_j),$$

where $C(x_i,x_j)$ is a series in z with polynomial coefficients in x_i and x_j. Hence, if $H(x) = (1+x)K(x)$,

$$H_k(x_0,\ldots,x_k) = \sum_{i=0}^{k} K(x_i)(1+x_i)^{k+1} \prod_{j\neq i} \frac{(1+x_j)C(x_i,x_j)}{x_i - x_j}.$$

Setting $x_0 = -1$ shows that $H_k(-1,x_1,\ldots,x_k) = 0$, so that $H_k(x_0,\ldots,x_k)$ is a multiple of $(1+x_0)$. By symmetry, it is also a multiple of all $(1+x_i)$, for $1 \le i \le k$. ■

Our treatment of (29) actually applies to equations with an arbitrary right-hand side. We consider a formal power series series $H(z;u) \equiv H(u)$ with coefficients in $\mathbb{K}[u]$, satisfying $H(-1) = 0$ and

$$\sum_{i=0}^{m} \frac{H(u_i)}{\prod_{j\neq i}(A_i - A_j)} = \Phi_m(v),$$

for some series $\Phi_m(v) \equiv \Phi_m(z;v)$ with coefficients in $v\mathbb{K}[v]$, where $v = (1+u)^{m+1}\bar{u}^m$. Using the above notation, this equation can be rewritten as

$$H_m(u_0, \dots, u_m) = \Phi_m(v).$$

We will give an explicit expression of $H(u)$ involving two standard families of symmetric functions [18, Chap. 7], namely the homogeneous functions h_λ and the monomial functions m_λ. We denote by $\ell(\lambda)$ the number of parts in a partition λ, and by $\mathfrak{S}_{m+1}$ the symmetric group on $\{0, 1, \dots, m\}$.

Proposition 12. *Let $H(z;u) \equiv H(u)$ be a power series in z with coefficients in $\mathbb{K}[u]$, satisfying $H(-1) = 0$ and*

$$H_m(u_0, \dots, u_m) = \Phi_m(v), \tag{40}$$

where $\Phi_m(v) \equiv \Phi_m(z;v)$ is a series in z with coefficients in $v\mathbb{K}[v]$.

There exists a unique sequence $\Phi_0, \dots, \Phi_m$ of series in z with coefficients in $v\mathbb{K}[v]$ such that for $0 \le k \le m$, and for all permutation $\sigma \in \mathfrak{S}_{m+1}$,

$$H_k(u_{\sigma(0)}, \dots, u_{\sigma(k)}) = \sum_{j=k}^{m} \Phi_j(v) h_{j-k}(A_{\sigma(0)}, \dots, A_{\sigma(k)}). \tag{41}$$

In particular, $H(u) \equiv H_0(u)$ is completely determined:

$$H(u) = \sum_{j=0}^{m} \Phi_j(v) A(u)^j. \tag{42}$$

The series $\Phi_k(v) \equiv \Phi_k(z;v)$ can be computed by a descending induction on k as follows. Let us denote by $\Phi_{k-1}^{>}(u)$ the positive part in u of $\Phi_{k-1}(v)$, that is

$$\Phi_{k-1}^{>}(u) := [u^{>}]\Phi_{k-1}(\bar{u}^m(1+u)^{m+1}).$$

Then for $1 \le k \le m$, this series can be expressed in terms of $\Phi_k, \dots, \Phi_m$:

$$\Phi_{k-1}^{>}(u) \quad = \quad -\frac{1}{\binom{m}{k}}[u^{>}] \left(\sum_{j=k}^{m} \Phi_j(v) \sum_{\lambda \vdash j-k+1} \binom{m-\ell(\lambda)}{k-\ell(\lambda)} m_\lambda(A_1, \dots, A_m) \right), \tag{43}$$

and $\Phi_{k-1}(v)$ can be expressed in terms of $\Phi_{k-1}^{>}$:

$$\Phi_{k-1}(v) = \sum_{i=0}^{m} \left(\Phi_{k-1}^{>}(u_i) - \Phi_{k-1}^{>}(-1) \right). \tag{44}$$

We first establish three lemmas dealing with the symmetric functions of the series u_i defined in Lemma 7.

Lemma 13. *The elementary symmetric functions of $u_0 = u, u_1, \dots, u_m$ are*

$$e_j(u_0, u_1, \dots, u_m) = (-1)^j \binom{m+1}{j} + v 1_{j=1}$$

with $v = u^{-m}(1+u)^m$.

The elementary symmetric functions of $u_1, \dots, u_m$ are

$$e_{m-j}(u_1, \dots, u_m) = (-1)^{m-j-1} \sum_{p=0}^{j} \binom{m+1}{p} u^{p-j-1}.$$

In particular, they are polynomials in $1/u$, and so is any symmetric polynomial in $u_1, \dots, u_m$.

Finally,

$$\prod_{i=0}^{m} (1+u_i) = v.$$

Proof. The symmetric functions of the roots of a polynomial can be read from the coefficients of this polynomial. Hence the first result follows directly from the equation satisfied by the u_i's, for $0 \le i \le m$, namely

$$(1+u_i)^{m+1} = v u_i^m.$$

For the second one, we need to find the equation satisfied by $u_1, \ldots, u_m$, which is

$$0 = \frac{(1+u_i)^{m+1}u^m - (1+u)^{m+1}u_i^m}{u_i - u} = u^m u_i^m - \sum_{j=0}^{m-1} u_i^j u^{m-j-1} \sum_{p=0}^{j} \binom{m+1}{p} u^p.$$

The second result follows.

The third one is obtained by evaluating at $X = -1$ the identity

$$\prod_{i=0}^{m} (X - u_i) = (1+X)^{m+1} - vX^m.$$

■

Lemma 14. *Denote $v = \bar{u}^m(1+u)^{m+1}$. Let P be a polynomial. Then $P(v)$ is a Laurent polynomial in u. Let $P^>(u)$ denote its positive part:*

$$P^>(u) := [u^>]P(v).$$

Then

$$P(v) = P(0) + \sum_{i=0}^{m} (P^>(u_i) - P^>(-1)). \tag{45}$$

Proof. The right-hand side of (45) is a symmetric polynomial of $u_0, \ldots, u_m$, and thus, by the first part of Lemma 13, a polynomial in v. Denote it by $\tilde{P}(v)$. The second part of Lemma 13 implies that the positive part of $\tilde{P}(v)$ in u is $P^>(u_0) = P^>(u)$. That is, $P(v)$ and $\tilde{P}(v)$ have the same positive part in u. In other words, the polynomial $Q := P - \tilde{P}$ is such that $Q(v)$ is a Laurent polynomial in u of non-positive degree. But since $v = (1+u)^{m+1}\bar{u}^m$, the degree in u of $Q(v)$ coincides with the degree of Q, and so Q must be a constant. Finally, by setting $u = -1$ in $\tilde{P}(v)$, we see that $\tilde{P}(0) = P(0)$ (because $u_i = -1$ for all i when $u = -1$, as follows for instance from Lemma 13). Hence $Q = 0$ and the lemma is proved. ■

Lemma 15. *Let $0 \le k \le m$, and let $R(x_0, \ldots, x_k)$ be a rational function in $k+1$ variables $x_0, \ldots, x_k$, such that for any permutation $\sigma \in \mathfrak{S}_{m+1}$,*

$$R(u_0, \ldots, u_k) = R(u_{\sigma(0)}, \ldots, u_{\sigma(k)}).$$

Then there exists a rational fraction in v equal to $R(u_0, \ldots, u_k)$.

Proof. Let $\tilde{R}$ be the following rational function in $x_0, \ldots, x_m$:

$$\tilde{R}(x_0, \ldots, x_m) = \frac{1}{(m+1)!} \sum_{\sigma \in \mathfrak{S}_{m+1}} R(x_{\sigma(0)}, \ldots, x_{\sigma(k)}).$$

Then $\tilde{R}$ is a symmetric function of $x_0, \ldots, x_m$, and hence a rational function in the elementary symmetric functions $e_j(x_0, \ldots, x_m)$, say $S(e_1(x_0, \ldots, x_m), \ldots, e_{m+1}(x_0, \ldots, x_m))$. By assumption,

$$\tilde{R}(u_0, \ldots, u_m) = S(e_1(u_0, \ldots, u_m), \ldots, e_{m+1}(u_0, \ldots, u_m)) = R(u_0, \ldots, u_k).$$

Since S is a rational function, it follows from the first part of Lemma 13 that $R(u_0, \ldots, u_k)$ can be written as a rational function in v. ■

Proof of Proposition 12. We prove (41) by descending induction on k. For $k = m$, (41) holds by assumption when σ is the identity, and actually for any σ as $H_m(x_0, \ldots, x_m)$ is a symmetric function of the x_i's. Let us assume (41) holds for some $k > 0$, and prove it for $k-1$.

Observe that

$$(A(x_{k-1}) - A(x_k))H_k(x_0, \ldots, x_k) = H_{k-1}(x_0, \ldots, x_{k-2}, x_{k-1}) - H_{k-1}(x_0, \ldots, x_{k-2}, x_k).$$

This is easily proved by collecting the coefficient of $H(x_i)$, for all $i \in [\![0, k]\!]$, in both sides of the equation. We also have, for any indeterminates $a_0, \ldots, a_m$,

$$(a_{k-1} - a_k)h_{j-k}(a_0, \ldots, a_k) = h_{j-k+1}(a_0, \ldots, a_{k-2}, a_{k-1}) - h_{j-k+1}(a_0, \ldots, a_{k-2}, a_k).$$

Hence, multiplying (41) by $(A_{\sigma(k-1)} - A_{\sigma(k)})$ gives

$$H_{k-1}(u_{\sigma(0)}, \ldots, u_{\sigma(k-2)}, u_{\sigma(k-1)}) - \sum_{j=k}^{m} \Phi_j(v) h_{j-k+1}(A_{\sigma(0)}, \ldots, A_{\sigma(k-2)}, A_{\sigma(k-1)}) =$$
$$H_{k-1}(u_{\sigma(0)}, \ldots, u_{\sigma(k-2)}, u_{\sigma(k)}) - \sum_{j=k}^{m} \Phi_j(v) h_{j-k+1}(A_{\sigma(0)}, \ldots, A_{\sigma(k-2)}, A_{\sigma(k)}),$$

for all $\sigma \in \mathfrak{S}_{m+1}$. This implies that the series

$$H_{k-1}(x_0, \ldots, x_{k-1}) - \sum_{j=k}^{m} \Phi_j(v) h_{j-k+1}(A(x_0), \ldots, A(x_{k-1}))$$

takes the same value at all points $(u_{\sigma(0)}, \ldots, u_{\sigma(k-1)})$, with $\sigma \in \mathfrak{S}_{m+1}$. Hence, by Lemma 15, there exists a series in z with *rational* coefficients in v, denoted $\Phi_{k-1}(v)$, such that for all $\sigma \in \mathfrak{S}_{m+1}$,

$$H_{k-1}(u_{\sigma(0)}, \ldots, u_{\sigma(k-1)}) - \sum_{j=k}^{m} \Phi_j(v) h_{j-k+1}(A_{\sigma(0)}, \ldots, A_{\sigma(k-1)}) = \Phi_{k-1}(v). \tag{46}$$

This is exactly (41) with k replaced by $k-1$.

The next point we will prove is that the coefficients of Φ_{k-1} belong to $v\mathbb{K}[v]$. In order to do so, we symmetrize (46) over $u_0, \ldots, u_m$. For any subset $V = \{i_1, \ldots, i_k\}$ of $\{0, \ldots, m\}$, of cardinality k, we denote $u_V = (u_{i_1}, \ldots, u_{i_k})$ and similarly $A_V = (A_{i_1}, \ldots, A_{i_k})$. By (46),

$$\binom{m+1}{k} \Phi_{k-1}(v) = \sum_{V \subset \{0,\ldots,m\}, |V|=k} H_{k-1}(u_V) - \sum_{j=k}^{m} \left(\Phi_j(v) \sum_{V \subset \{0,\ldots,m\}, |V|=k} h_{j-k+1}(A_V) \right). \tag{47}$$

We will prove that both sums in the right-hand side of this equation are series in z with coefficients in $v\mathbb{K}[v]$.

Denote $x_V = (x_{i_1}, \ldots, x_{i_k})$. Observe that

$$\sum_{V \subset \{0,\ldots,m\}, |V|=k} H_{k-1}(x_V)$$

is a series in z with polynomial coefficients in $x_0, \ldots, x_m$, which is symmetric in these variables (Lemma 11). By Lemma 13, the first sum in (47) is thus a series in z with *polynomial* coefficients in v. We still need to prove that this series even vanishes at $v = 0$, that is, at $u = -1$. But this follows from Lemma 11, since $u_i = -1$ for all i when $u = -1$.

Let us now consider the second sum in (47), and more specifically the term

$$\Phi_j(v) \sum_{V \subset \{0,\ldots,m\}, |V|=k} h_{j-k+1}(A_V). \tag{48}$$

Recall that

$$A_i = \frac{u_i}{1+u_i} e^{-zu_i}.$$

But by Lemma 13,

$$\frac{1}{1+u_i} = \frac{1}{v} \prod_{0 \le j \ne i \le m} (1 + u_j).$$

Hence (48) can be written as a series in z with coefficients in $\mathbb{K}[1/v, u_0, \ldots, u_m]$, symmetric in $u_0, \ldots, u_m$. By the first part of Lemma 13, these coefficients belong to $\mathbb{K}[v, 1/v]$. We want to prove that they actually belong to $v\mathbb{K}[v]$, that is, that they are not singular at $v = 0$ (equivalently, at $u = -1$) and even vanish at this point. >From the equation $(1 + u_i)^{m+1} = vu_i^m$, it follows that we can label $u_1, \dots, u_m$ in such a way

$$1 + u_i = \xi^i(1 + u) + o(1 + u),$$

where ξ is a primitive $(m + 1)^{\text{st}}$ root of unity. Since $\Phi_j(v)$ is a multiple of $v = \bar{u}^m(1 + u)^{m+1}$, and the symmetric function h_{j-k+1} has degree $j - k + 1 \le m$, it follows that the series (48) is not singular at $u = -1$, and even vanishes at this point. Hence its coefficients belong to $v\mathbb{K}[v]$.

We finally want to obtain an explicit expression of $\Phi_{k-1}(v)$. Lemma 14, together with $\Phi_{k-1}(0) = 0$, establishes (44). To express $\Phi^{>}_{k-1}(u)$, we now symmetrize (46) over $u_1, \dots, u_m$. With the above notation,

$$\binom{m}{k}\Phi_{k-1}(v) = \sum_{V\subset\{1,\dots,m\},|V|=k} H_{k-1}(u_V) - \sum_{j=k}^{m}\left(\Phi_j(v) \sum_{V\subset\{1,\dots,m\},|V|=k} h_{j-k+1}(A_V)\right). \quad (49)$$

As above,

$$\sum_{V\subset\{1,\dots,m\},|V|=k} H_{k-1}(x_V)$$

is a series in z with polynomial coefficients in $x_1, \dots, x_m$, which is symmetric in these variables. By the second part of Lemma 13, the first sum in (49) is thus a series in z with polynomial coefficients in $1/u$. Since $\Phi_{k-1}(v)$ has coefficients in $\mathbb{K}[v]$, and hence in $\mathbb{K}[u, 1/u]$, the second sum in (49) is also a zeries in z with coefficients in $\mathbb{K}[u, 1/u]$. We can now extract from (49) the positive part in u, and this gives

$$\binom{m}{k}\Phi^{>}_{k-1}(u) = -[u^{>}]\left(\sum_{j=k}^{m}\left(\Phi_j(v) \sum_{V\subset\{1,\dots,m\},|V|=k} h_{j-k+1}(A_V)\right)\right).$$

One easily checks that, for indeterminates $a_1, \dots, a_m$,

$$\sum_{V\subset\{1,\dots,m\},|V|=k} h_{j-k+1}(a_V) = \sum_{\lambda\vdash j-k+1}\binom{m-\ell(\lambda)}{k-\ell(\lambda)} m_\lambda(a_1, \dots, a_m),$$

so that the above expression of $\Phi^{>}_{k-1}(u)$ coincides with (43). ■

4.3. The case $y = 1$

As explained in Section 3.1, Theorem 2 will be proved if we establish $\tilde{G}(u, 1) = G_1(u)$, where

$$G_1(u) = (1 + u)e^{(m+1)z-(m-1)zu}\left(1 + \frac{1 - e^{mzu}}{u}\right).$$

A natural attempt would be to set $y = 1$ in the expression of $\tilde{G}(u, y)$ that can be derived from Proposition 12, as we did when $m = 1$ in Section 3.2.3. However, we have not been able to do so, and will proceed differently.

We have proved in Proposition 8 that the series $\tilde{G}(u, y)$ satisfies (40) with $\Phi_m(v) = ve^{zvy}$. In particular, $\tilde{G}(u, 1)$ satisfies (40) with $\Phi_m(v) = ve^{zv}$. By Proposition 12, this equation, together with the initial condition $\tilde{G}(-1, 1) = 0$, characterizes $\tilde{G}(u, 1)$. It is clear that $G_1(-1) = 0$. Hence it suffices to prove is the following proposition.

Proposition 16. *The series $G_1(u)$ satisfies* (40) *with $\Phi_m(v) = ve^{zv}$.*

Proof. Note that $G_1(u) = e^{(m+1)z}\left(vA(u)^{m-1} - \frac{1}{A(u)}\right)$. Using Lemma 9 with $x_i = A_i$, it follows that

$$\begin{aligned}\sum_{i=0}^{m} \frac{G_1(u_i)}{\prod_{j\neq i}(A_i - A_j)} &= 0 + (-1)^{m+1} e^{(m+1)z} \prod_{i=0}^{m} \frac{1}{A_i} \qquad \text{(by (31) and (32))}\\ &= (-1)^{m+1} e^{(m+1)z+z\sum_i u_i} \prod_{i=0}^{m} \frac{(1+u_i)}{u_i}\\ &= ve^{zv}\end{aligned}$$

by Lemma 13. ∎

4.4. The trivariate series

We have now proved that $\tilde{G}(u,y) = G(u,y)$, so that $F(x,y) = \tilde{G}(u,y)$ after the change of variables (4). As shown in Proposition 8, the series $\tilde{G}(u,y)$ satisfies (40) with $\Phi_m(v) = ve^{zvy}$. Hence Proposition 12 gives an explicit, although complicated, expression of the trivariate series $F(t,x,y)$.

Theorem 17. *Let $F^{(m)}(t;x,y) \equiv F(t;x,y)$ be the exponential generating function of labelled m-Tamari intervals, defined by (3). Let z and u be two indeterminates, and write*

$$t = ze^{-m(m+1)z} \quad \text{and} \quad x = (1+u)e^{-mzu}.$$

Then $F(t;x,y)$ becomes a series in z with polynomial coefficients in u and y, and this series can be computed by an iterative extraction of positive parts. More precisely,

$$F(t;x,y) = \sum_{k=0}^{m} \Phi_k(v) A(u)^k,$$

where $v = u^{-m}(1+u)^{m+1}$, $A(u)$ is defined by (21), and $\Phi_k(v) \equiv \Phi_k(z;v)$ is a series in z with polynomial coefficients in v. This series can be computed by a descending induction on k as follows. First, $\Phi_m(v) = ve^{zyv}$. Then for $k \le m$,

$$\Phi_{k-1}(v) = \sum_{i=0}^{m} \left(\Phi_{k-1}^{>}(u_i) - \Phi_{k-1}^{>}(-1)\right)$$

where

$$\begin{aligned}\Phi_{k-1}^{>}(u) &= [u^{>}]\Phi_{k-1}(v)\\ &= -\frac{1}{\binom{m}{k}}[u^{>}]\left(\sum_{j=k}^{m} \Phi_j(v) \sum_{\lambda \vdash j-k+1} \binom{m-\ell(\lambda)}{k-\ell(\lambda)} m_\lambda(A(u_1),\ldots,A(u_m))\right),\end{aligned}$$

and $u_0 = u, u_1, \ldots, u_m$ are the $m+1$ roots of the equation $(1+u_i)^{m+1} = u_i^m v$.

Remark and examples
The case $k = 1$ of the above identity gives

$$[u^{>}]\Phi_0(v) = -\frac{1}{m}[u^{>}]\left(\sum_{j=1}^{m} \Phi_j(v) \sum_{i=1}^{m} A(u_i)^j\right). \tag{50}$$

Recall that $F(t;x,y) = \tilde{G}(z;u,y)$ has polynomial coefficients in u and y. Hence

$$\begin{aligned} F(t;x,y) &= F(t;1,y) + [u^>]\left(\sum_{k=0}^{m} \Phi_k(v) A(u)^k\right) && \text{(since } u=0 \text{ when } x=1\text{)} \\ &= F(t;1,y) + [u^>]\left(\sum_{k=1}^{m} \Phi_k(v)\left(A(u)^k - \frac{1}{m}\sum_{i=1}^{m} A(u_i)^k\right)\right) && \text{(by (50))} \\ &= (1+u)[u^\geq]\left(\sum_{k=1}^{m} \frac{\Phi_k(v)}{1+u}\left(A(u)^k - \frac{1}{m}\sum_{i=1}^{m} A(u_i)^k\right)\right) \end{aligned}$$

by (25), and given that $F(t;x,y)=0$ when $u=-1$.

- When $m=1$, this is the expression (7) of $F^{(1)}(t;x,y)$ (recall that $\Phi_m = ve^{zyv}$).
- When $m=2$, the generating function of labelled 2-Tamari intervals satisfies

$$\frac{F^{(2)}(t;x,y)}{1+u} = [u^\geq]\left(\frac{\Phi_1(v)}{1+u}\left(A(u) - \frac{A(u_1)}{2} - \frac{A(u_2)}{2}\right) + (1+\bar{u})^2 e^{zyv}\left(A(u)^2 - \frac{A(u_1)^2}{2} - \frac{A(u_2)^2}{2}\right)\right),$$

where

$$A(u) = \frac{u}{1+u}e^{-zu}, \qquad u_{1,2} = \frac{1+3u \pm (1+u)\sqrt{1+4u}}{2u^2},$$

and

$$\Phi_1(v) = \Phi_1^>(u) + \Phi_1^>(u_1) + \Phi_1^>(u_2) - 3\Phi_1^>(-1),$$

with

$$\Phi_1^>(u) = -[u^>]\left((1+u)^3\bar{u}^2 e^{zy(1+u)^3\bar{u}^2}\left(A(u_1)+A(u_2)\right)\right).$$

This expression has been checked with Maple, after computing the first coefficients of $F(t;x,y)$ from the functional equation (9).

5. Final comments

A constructive proof? Our proof would not have been possible without a preliminary task consisting in *guessing* the expression of $F(t;x,1)$. This turned out to be difficult, in particular because the standard tools like the Maple package Gfun can only guess D-finite generating functions, while the generating function of the numbers (2) is not D-finite. More precisely, the expression of $F(t;x,1)$ *becomes D-finite* after the change of variables (4), but what is hard to guess is this change of variables. A constructive proof of our result would be most welcome.

A q-analogue of the functional equation. As described in the introduction, the numbers (2) are conjectured to give the dimension of certain polynomial rings generalizing $\mathrm{DR}_{3,n}$. These rings are tri-graded (with respect to the sets of variables $\{x_i\}$, $\{y_i\}$ and $\{z_i\}$), and it is conjectured [1] that the dimension of the homogeneous component in the x_i's of degree k is the number of labelled intervals $[P,Q]$ in $\mathcal{T}_n^{(m)}$ such that the longest chain from P to Q, in the Tamari order, has length k. One can recycle the recursive description of intervals described in Section 2 to generalize the functional equation of Proposition 6, taking into account (with a new variable q) this distance. Eq. (9) remains valid, upon defining the operator Δ by

$$\Delta S(x) = \frac{S(qx) - S(1)}{qx-1}.$$

The coefficient of t^n in the series $F(t,q;x,y)$ does not seem to factor, even when $x=y=1$. The coefficients of the bivariate series $F(t,q;1,1)$ have large prime factors.

Further developments. There is a natural action of the symmetric group $\mathfrak{S}_n$ on labelled m-Tamari intervals of size n: it consists in permuting the labels according to the permutation one considers, and then to rearrange the labels in each sequence of consecutive steps so that they increase. The dimension of this representation of $\mathfrak{S}_n$ is the number (2) of labelled m-Tamari intervals of size n. Bergeron and Préville Ratelle have a refined conjecture that gives the *character* of this representation[1]. We have very recently proved this conjecture [3].

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