Stationary free surface viscous flows without surface tension in three dimensions

Frédéric Abergel, Jacques-Herbert Bailly

Stationary free surface viscous flows without surface tension in three dimensions

Frederic Abergel and Jacques-Herbert Bailly

Laboratory of Mathematics Applied to Systems
École Centrale Paris
92290 Châtenay Malabry
France

Abstract

We consider an incompressible, viscous, finite depth fluid flowing down a three dimensional channel. In the absence of surface tension, we prove the existence of a unique stationary solution in weighted Sobolev spaces. The result is based on a thorough study of the linearized problem, particularly the pseudodifferential operator relating the normal velocity of the fluid and the normal component of the associated stress tensor along the free surface, and requires the use of the Nash-Moser Implicit Function Theorem.

Introduction

We consider an incompressible, viscous, finite depth fluid flowing down a three dimensional inclined channel at low Reynolds number. In the case where the bottom of the channel is flat, there exists a well-known stationary solution, usually referred to as the Poiseuille-Nusselt flow [4]. However, the question of finding such a stationary solution for a general bottom profile is not easy to address. First of all, it is expected from previous results going as far back as those in [12], that conditions on the slope of the inclined channel should apply. Such conditions are already present in a two-dimensional version of the current work, see [4]. What is more, standard perturbation methods resting upon the usual Implicit Function Theorem fail to apply, since the linearized operator at the Poiseuille-Nusselt solution is not elliptic when surface tension is neglected. Hence, analysis such as that made in [4] or [17] fails to settle the issue. As a matter of fact, it has been proven, see e.g. [1], that surface tension, when taken into account, plays a particularly important role in the structure of the linearized solution operator, transforming it into an elliptic pseudodifferential operator. In the absence of surface tension, there exist some drastic differences between the two- and three-dimensional cases. Two-dimensional free surface flows are still associated to an elliptic operator, albeit with a purely imaginary principal symbol. However, in the three-dimensional case, the linearized free surface problem is no longer elliptic, and one has to carefully analyze the structure of the associated operator in order to study its invertibility. Moreover, even though the linearized operator may be invertible, one has to overcome a loss of smoothness when inverting it, a phenomenon making the analysis more complicated.
Our aim in this work is to prove the existence of a unique, smooth, stationary free surface for three-dimensional flows close to the Poiseuille-Nusselt flow. The paper is organized as follows: Section 1 contains an exposition of the physical problem and its mathematical formulation, together with the description of the appropriate functional setting. Section 2 introduces the technical tools necessary to compute the derivative of the main operator with respect to the domain. Section 3 is devoted to a detailed study of the linearized

[0]AMS Subject Classification: 35R35; 76D05; 35S05.

operator and its invertibility, whereas Section 4 presents the heavy machinery involved in the Nash-Moser Theorem, and its application to the proof of the main result.

1 Formulation of the problem

1.1 Governing equations

Let Ω_ρ^γ be the unbounded domain

$$\Omega_\rho^\gamma = \{(x,y,z) \in \mathbb{R}^3, -\infty < x < +\infty, -\infty < y < +\infty, \gamma(x,y) < z < z_0 + \rho(x,y)\}$$

in $\mathbb{R}^3$, where ρ is a mapping from $\mathbb{R}^2$ into $\mathbb{R}$ describing the free surface, γ is a mapping from $\mathbb{R}^2$ into $\mathbb{R}$ representing the fixed bottom, while z_0 is the height of the free surface when x, y goes to ∞. Such a representation is legitimate whenever ρ, γ are small enough. Denote by $\partial\Omega_\rho^\gamma$ the boundary of Ω_ρ^γ, so that

$$\partial\Omega_\rho^\gamma = \Gamma_\gamma \cup \Sigma_\rho,$$

Γ_γ being the fixed bottom of the channel and Σ_ρ, the free surface. Let u be the velocity field and p be the pressure. We define the stress tensor

$$\sigma(u) = \nu(\nabla u +^t \nabla u) - pId$$

where $\nu > 0$ is the viscosity of the fluid, and let f be the gravitational field. For simplicity, ν will be set equal to 1 in the rest of the paper.

In the frame shown on Figure 1, f has coordinates $\begin{pmatrix} C_1 \\ 0 \\ -C_2 \end{pmatrix}$ with $C_1 = g\sin\alpha$ and $C_2 = g\cos\alpha$, where α, $0 < \alpha < \frac{\Pi}{2}$, is the angle shown on Figure 1.

The goal of this paper is to study, for a given γ, the existence and uniqueness of a triple $\big(u, p, \rho\big)$ solving the stationary Stokes system

$$-\operatorname{div}\sigma(u) = f \text{ in } \Omega_\rho^\gamma \tag{1.1}$$

$$\operatorname{div} u = 0 \text{ in } \Omega_\rho^\gamma, \tag{1.2}$$

supplemented with the respective fixed- and free-boundary conditions

$$u = \zeta \text{ on } \Gamma_\gamma \qquad\qquad \sigma(u).n = 0 \text{ on } \Sigma_\rho \tag{1.3}$$

as well as the lateral boundary conditions at ∞

$$\lim_{(x,y)\to\infty} u(x,y,z) = \begin{pmatrix} C_1(z_0 z - \frac{1}{2}z^2) \\ 0 \\ 0 \end{pmatrix}. \tag{1.4}$$

Morevoer, a kinematic equation for the free surface should also hold. In the stationary case, this condition is simply

$$u.n = 0 \text{ on } \Sigma_\rho. \tag{1.5}$$

Equations (1.1)-(1.4) will be referred to as the Auxiliary Problem. This system of equations is absolutely standard, and we will not dwell upon the physical derivation of it, nor their interpretation. It is interesting,

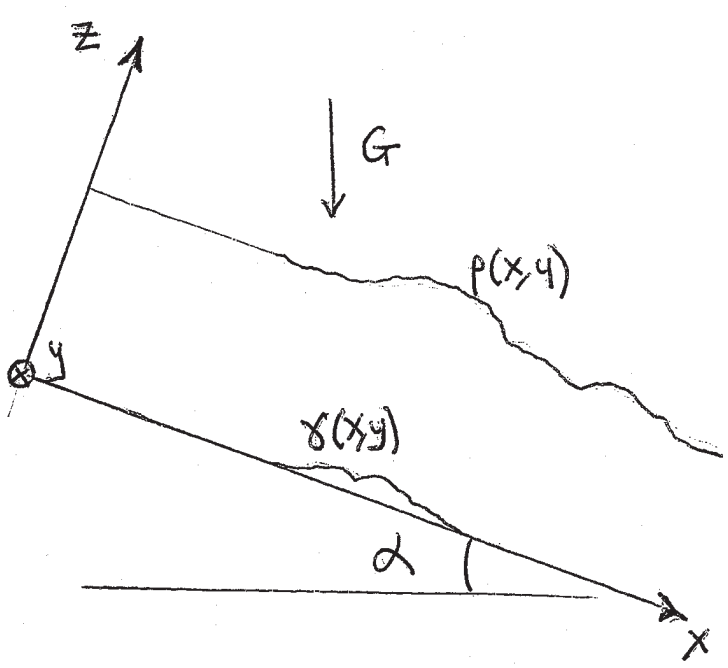

Figure 1: Flow down an inclined plane

and will be useful in the sequel, to fully characterize the Poiseuille-Nusselt flow corresponding to $\rho = \gamma \equiv 0$. The solution to Equations (1.1)-(1.5) is given by the velocity field

$$\begin{pmatrix} U_1(x,y,z) \\ U_2(x,y,z) \\ U_3(x,y,z) \end{pmatrix} = \begin{pmatrix} C_1(z_0 z - \frac{1}{2}z^2) \\ 0 \\ 0 \end{pmatrix}$$

and the pressure field $p(x,y,z) = C_2(z_0 - z)$. In particular, the associated stress tensor has the expression

$$\sigma(u) = (\nabla u +^t \nabla u) - pId = \begin{pmatrix} -C_2(z_0 - z) & 0 & C_1(z_0 - z) \\ 0 & -C_2(z_0 - z) & 0 \\ C_1(z_0 - z) & 0 & -C_2(z_0 - z) \end{pmatrix},$$

and one can easily see that its restriction to the free surface $z = z_0$ has only zero entries. We also note that the normal component of its normal derivative

$$\frac{\partial \sigma}{\partial n}.n.n = \frac{\partial \sigma}{\partial z}.\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is equal to C_2, whereas the tangential part of its normal derivative has coordinates equal to $\begin{pmatrix} -C_1 \\ 0 \\ 0 \end{pmatrix}$.

In particular, C_1 is small when the angle α is, a property that will be crucial in this work.

Before moving on the analysis of the free surface problem, some remarks are in order. Firstly, we consider the Stokes rather than the Navier-Stokes system for the sake of simplicity, and also because, due to the perturbative nature of the argument that will be used, a similar result easily obtains for the Navier-Stokes system. Secondly, the normal stress condition should be $\sigma(u).n = K.n \, \text{on} \, \Sigma_\rho$, expressing the fact that the normal stress has no tangential component, but the constant K can be set equal to 0, a choice that fully

characterizes the pressure field, defined only up to a constant. As regards the boundary condition ζ on the fixed bottom, it will be set equal to 0, which is the case of no-slip boundary condition. One could equally choose ζ to be a small, smooth function rapidly decaying to 0 at ∞.

1.2 Function spaces

Let us now introduce a family of Banach spaces well adapted to the resolution of Equations (1.1)-(1.5). These spaces are weighted Sobolev spaces with an exponential weight function in the (x, y)-plane, and are defined below.
For $\delta \in \mathbb{R}_+^*$, $X = (x, y) \in \mathbb{R}^2$ and $s > 0$, set

$$\Psi_\delta(X) = \exp \sqrt{1 + \delta^2(x^2 + y^2)} \tag{1.6}$$

and define

$$H^s_\delta(\mathbb{R}^D) = \{f \in H^s(\mathbb{R}^D) : \|f(X, z)\Psi(X)\|_{H^s(\mathbb{R}^D)} < +\infty\} \tag{1.7}$$

where $X = (x, y) \in \mathbb{R}^2$, $D \geqslant 2$, $z \in \mathbb{R}^{D-2}$, and $H^s(\mathbb{R}^D)$ stands for the usual Sobolev space. Similarly, one can define the corresponding Hölder spaces

$$C^s_\delta(\mathbb{R}^D) = \{f \in C^s(\mathbb{R}^D) : \sum_{|\alpha| \leqslant s} Sup(\|D^\alpha f\|)\Psi(X) + \sum_{|\alpha|=[s]} Sup([D^\alpha f]) < +\infty\} \tag{1.8}$$

where $[s]$ is the integer part of s and $[f] = \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^{s-[s]}}$. We recall for further use a classical embedding theorem.

Proposition 1.1 *Let $m \in \mathbf{N}$, $\alpha \in]0, 1[$ be such that $s - \frac{D}{2} \geqslant m + \alpha$. Then $\forall f \in H^s_\delta(\mathbb{R}^D), f \in C^{m+\alpha}_\delta(\mathbb{R}^D)$ and there exists a constant $C > 0$ such that $\forall f \in H^s_\delta(\mathbb{R}^D)$, $\|f\|_{C^{m+\alpha}_\delta} \leqslant C\|f\|_{H^s_\delta}$.*

1.3 The main result

Denote by $B^\epsilon(X)$ the ball of radius ϵ centered at the origin in X, when X is a Banach space and $\epsilon > 0$. Our main result can now be stated.

Theorem 1.2 *There exists an $\alpha_0 > 0$ with the property that, whenever the angle α in Figure 1 is less than α_0, there exists $\delta_0 > 0$ such that there holds*

$$\forall \delta \leqslant \delta_0, \exists \epsilon > 0, \forall \gamma \in B^\epsilon(H^{10}_\delta(\mathbf{R^2})), \exists!(u, p, \rho) \text{ solution to Equations } (1.1)\text{-}(1.5).$$

Moreover, $\rho \in H^5_\delta(\mathbf{R^2})$, $(u, \nabla p) \in H^3_\delta(\Omega^\gamma_\rho) \times H^1_\delta(\Omega^\gamma_\rho)$, and the solution depends continuously on the data; that is, there exists a constant $C > 0$ depending only on δ and the physical parameters of the problem such that

$$\|\rho\|_{H^5_\delta} + \|u\|_{H^3_\delta} + \|\nabla p\|_{H^1_\delta} \leqslant C(\|f\|_{H^1_\delta} + \|\gamma\|_{H^{10}_\delta})$$

where f is the right-hand side of Equation(1.1).

The proof of Theorem (1.2) is quite lengthy, and will be presented in the rest of the paper. It relies on a thorough study of the linearization of Equations (1.1)-(1.5), as can be expected from our formulation as a perturbation problem. A similar approach has been used in various works related to free surface stationary flows for the Stokes system. However, to the best of our knowledge, all correct results that are available in the litterature are based either on the assumption of a non-zero surface tension coefficient, or on a two-dimensional formulation, see [1] and [5] for general expositions. Now, in the case considered here, the situation is much more complex, because the standard approach using the Implicit Function Theorem fails as we now explain. In fact, Equation (1.5) can be interpreted as the problem of finding the zero of the

nonlinear mapping G associating, to a candidate free surface ρ, the value of the normal velocity $u.n$ on Σ_ρ, *viz.*

$$G(\gamma, \rho) = u_{|\Sigma_\rho}.n_\rho \tag{1.9}$$

where γ (describing the fixed bottom of the domain) plays the role of a small parameter. Note that G is considered, by composition, as a function with domain $\mathbf{R}^2$ rather than Σ_ρ, so that the function space it lives in will be fixed for all γ, ρ.
It is clear that the classical Poiseuille-Nusselt flow corresponds to the particular, trivial solution $G(0,0) = 0$ and therefore, the use of the Implicit Function Theorem may seem appropriate. However, a major difficulty arises when studying the derivative $\frac{DG}{D\rho}(0,0)$ of G. One can see that it is indeed invertible in some sense, thanks to the fact that the velocity field never vanishes along the flat free surface, but also that, unfortunately, $(\frac{DG}{D\rho}(0,0))^{-1} \circ \frac{DG}{D\rho}(0,0)$ is an unbounded operator. We therefore have to invoke, and verify the cumbersome assumptions of, a Nash-Moser-type iterative scheme to overcome the loss of regularity. In particular, this program requires the study of the invertibility of the linearized operator $\frac{DG}{D\rho}(\gamma, \rho)$ in a full neighbourhood of the origin, and not only at the particular solution $(0,0)$.
In the next section, we introduce the appropriate formalism to study the dependence of the operator G on the candidate free surface, explicitly computing its derivative.

2 The free-surface operator

2.1 Some geometric prerequisites

We briefly recall some classical results pertaining to the parameterization of a free surface Σ_ρ as a graph above a reference manifold, see [5] for a comprehensive treatment.
Let $\bar{\Sigma}$ be a two-dimensional reference manifold. Denote by $\bar{M}(s)$ a generic point on $\bar{\Sigma}$ and by $\bar{n}(s)$ the unit normal vector to $\bar{\Sigma}(s)$ pointing outward, where $s = \big(s_1, s_2\big)$ is a system of local coordinates on $\bar{\Sigma}$. We will use the notation ∂_{s_k} for the tangential derivative operator with respect to the k^{th} local coordinate s_k on $\bar{\Sigma}$. Classical results in Differential Geometry, see e.g. [14], allow us to assert the existence of a neighourhood of $\bar{\Sigma}$ such that, for all points M in this neighbourhood, there exists a unique $d \in \mathbf{R}$, $\bar{M} = \bar{M}(s) \in \bar{\Sigma}$ such that $M = \bar{M}(s) + d\bar{n}(s)$. Here, d stands for the distance of M to $\bar{\Sigma}$ and $\bar{M}(s)$ is the projection of M onto $\bar{\Sigma}$. For $l_0 > 0$ small enough (the size of l_0 depending only on the maximum principal curvature of $\bar{\Sigma}$), this representation defines a C^∞-diffeomorphism from $\bar{\Sigma}\times]-l_0, l_0[$ onto the l_0-neighbourhood of $\bar{\Sigma}$, that is, the set of points who sit at a distance less than l_0 from $\bar{\Sigma}$. Hence, to any smooth, small enough function ρ defined on $\bar{\Sigma}$, one can associate a manifold using the parameterization

$$\Sigma_\rho = \{M \in \mathbf{R}^3 : \exists \bar{M} = \bar{M}(s) \in \bar{\Sigma}, M = \bar{M} + \rho(s)\bar{n}\}. \tag{2.1}$$

For instance, one can obviously represent any point in the three dimensional half-space over the (x, y) plane using tangential and normal coordinates - this property still holds true for small enough perturbations and small enough neighbourhoods of this plane.

2.2 Properties of G

Let us now turn to the study of the nonlinear operator G introduced formally in (1.9). The following standard result yields that G is well-defined for γ, ρ small enough.

Proposition 2.1 *There exists $\delta_{01} > 0$ such that $\forall \delta \leqslant \delta_{01}, \exists \epsilon > 0, \forall s \geqslant 4$, G defined in (1.9) is of class C^2 from $B^\epsilon(H^s_\delta(\mathbf{R^2})^2)$ into $B^\epsilon(H^{s-\frac{3}{2}}_\delta(\mathbf{R}^2))$.*

The proof of Proposition 2.1 is performed in two steps. First, one uses a global change of variables mapping Ω^γ_ρ onto the domain Ω^0_0 corresponding to the unperturbed flow. The well-posedness of the transformed Stokes system in the exponentially weighted Sobolev spaces is ensured by classical results, see e.g. [7][22], and a perturbation argument as in [5]. The only technical point is that one requires the algebra structure

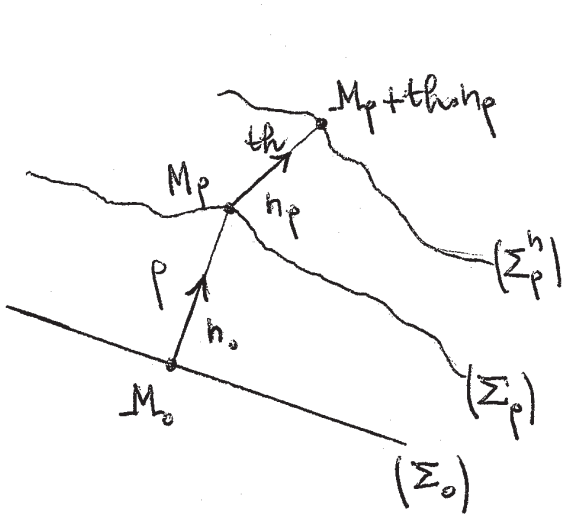

Figure 2: Deformation along the normal field

of the Sobolev spaces under composition and pointwise multiplication. Such a condition is classically known to hold for $s > \frac{D}{2}$, see [23], and must be satisfied by the derivatives of ρ and γ up to the second order with $D = 2$. We choose to impose the slightly more stringent condition $s \geqslant 4$ ensuring that the coefficients of the transformed Stokes system on the fixed domain Ω_0^0 are globally bounded, exponentially decaying Hölder-continuous functions for all exponent α, $0 < \alpha < 1$. Then, a differentiability argument for the solution of a well-posed boundary-value-problem with respect to the coefficients of the equation is used. These arguments are identical to the ones given in great details in [5], Section 2.2 or [4] for the case of Hölder spaces.

2.3 The derivative of G

In the rest of Section 2, as well as in Section 3, we shall make the assumption that the perturbation of the profile γ and the candidate free surface ρ are smooth. This assumption is not a restriction, as will become clear in the use of Zehnder's Implicit Function Theorem, and allow us to state in a more comfortable way the various results regarding the linearization of G. In particular, the use of normal variations of the domain for the explicit computation of the derivative of G will allow for more convenient expressions than those that could be obtained through a general change of variables.
Since G is differentiable in a neighbourhood of the origin, we can compute its partial derivative $\frac{DG}{D\rho}(\rho).h$ as the limit when $t \to 0$ of

$$\frac{1}{t}(G(M_\rho + thn_\rho, \rho + th) - G(M_\rho, \rho)), \tag{2.2}$$

with $M_\rho = M_0 + \rho n_0$, see Figure 2.

The computations we need to perform are well understood and detailed at length in [5], hence they will only be briefly sketched here. We first specialize the geometric description presented in Subsection 2.1 by choosing conformal coordinates on Σ_ρ, see [14]. Denoting by E the coefficient of the first fundamental form on Σ_ρ, the local Frénet frame on the tangent plane to Σ_ρ is given by

$$e_i = \frac{1}{E}\frac{\partial M}{\partial s_i}$$

and there holds $\frac{\partial M}{\partial s_i}.\frac{\partial M}{\partial s_j} = 0$, $i \neq j$, $|\frac{\partial M}{\partial s_i}| = E$, $n = \bigwedge_i e_i$.

The computation in (2.2) is split into three parts. The first part comes from the variation of the normal field, that is $\frac{1}{t}u_{\rho+th}(M_\rho + thn_\rho) \times (n_{\rho+th}(M_\rho + thn_\rho) - n_\rho(M_\rho))$, the limit of which as $t \to 0$ is simply

$$- u.\nabla h \tag{2.3}$$

where the gradient operator ∇ is defined intrinsically on Σ_ρ, and is equal to $\frac{1}{E}(\partial_{s_1}, \partial_{s_2})$ in the chosen coordinates system (s_1, s_2). The second term in (2.2) comes from the variation of the point at which the function u is evaluated,

$$\frac{1}{t}(u_{\rho+th}(M_\rho + thn_\rho) - u_{\rho+th}(M_\rho)).n_\rho(M_\rho)$$

the limit of which is obviously

$$\frac{\partial u}{\partial n_\rho}(M_\rho).n_\rho. \tag{2.4}$$

Finally, the third term in (2.2) comes from the variation of u with respect to the domain. Defining the local derivative of u with respect to normal domain variations as

$$v(M) = \lim_{t \to 0} \frac{u_{\rho+th}(M) - u_\rho(M)}{t}, \tag{2.5}$$

for any point M lying in the interior of the domain Ω_ρ^γ, then the limit of the last term is

$$v(M_\rho).n_\rho \tag{2.6}$$

for a function v that we now characterize. v is the solution of a boundary-value problem akin to the Stokes system with non-homogenous boundary conditions, the derivation of which mimicks that in [5], Section 2.2.2. Consequently, the next result is stated without proof.

Proposition 2.2 *The derivative with respect to normal variations of the domain of the solution u to the Auxiliary Problem* (1.1) *to* (1.4) *is given by the solution (v, q) to the linearized Stokes system*

$$- \operatorname{div} \tau(v) = 0 \operatorname{in} \Omega_\rho^\gamma, \tag{2.7}$$

$$\operatorname{div} v = 0 \operatorname{in} \Omega_\rho^\gamma \tag{2.8}$$

$$v = 0 \operatorname{on} \Gamma_\gamma, \tag{2.9}$$

$$\tau(v).n_\rho = h\frac{\partial \sigma}{\partial n_\rho}.n_\rho + \sigma \nabla h \operatorname{on} \Sigma_\rho. \tag{2.10}$$

The results of this section are summarized in the following proposition.

Proposition 2.3 *The derivative with respect to ρ of the operator G defined in* (1.9) *is given by*

$$\frac{DG}{D\rho}(\rho).h = -u.\nabla h + \frac{\partial u}{\partial n_\rho}(M_\rho).n_\rho h + v.n_\rho, \tag{2.11}$$

where v is characterized in Proposition 2.2.

Note that in the propositions 2.2 and 2.3 above, u is the solution to the Auxiliary System (1.1)-(1.4) in Ω_ρ^γ, and σ is the associated stress tensor.

3 Inverting the linearized operator

In this section, the invertibility of $\frac{DG}{D\rho}(\rho).h$ is addressed. The main result is stated and proven in Theorem 3.5 below, under a main "physical" assumption on the inclination angle α, as well as a smallness assumption on the perturbation of the fixed and free boundaries (we recall the assumption that the perturbation of the profile γ and the candidate free surface ρ are smooth).
From its expression (2.11), one can see that $\frac{DG}{D\rho}(\rho)$ is the sum of a linear first-order differential operator and a non-local term. In the simple case of the Poiseuille-Nusselt flow, one uses the expressions for u and σ in Section 1.1 to obtain that $\frac{\partial\sigma}{\partial n_0}.n_0.n_0 > 0$, $\frac{\partial u}{\partial n_0}.n_0 = 0$ and $\sigma = 0$ on Σ_0. Moreover, the first-order differential operator

$$u_{|\Sigma_0}.\nabla h \equiv C_1 \frac{1}{2} z_0^2 \frac{\partial h}{\partial x}$$

is non-degenerate. These remarks come in handy when studying the invertibility of $\frac{DG}{D\rho}(\rho)$. In fact, one can think of the equation

$$\frac{DG}{D\rho}(\rho)h = f \tag{3.1}$$

as a linear hyperbolic problem given by the vector field $u_{|\Sigma_0}$, perturbed by non-local terms. This hyperbolicity is essential, and typically fails to hold true on a closed surface, say, due to Poincaré's theorem for vector field on the sphere. However, it is not sufficient to solve (3.1), unless one knows how to control the lower-order terms in (3.1). This problem has been tackled in a general setting in [3], where the classical approach of symmetrizers has been extended to a situation very much inspired by the current one. The results in [3] provide a way to control a small 0^{th}-order operator "sandwiched" between a vector field as above, and an invertible operator of order -1. This approach will now be applied to the resolution of (3.1).
The emphasis is then on the non-local term, $v.n_\rho$. In the rest of the article, $\mathbf{P}$ will denote the operator $h \mapsto v.n_\rho$ where v is the solution of Equations (2.7)-(2.10). The operator $\mathbf{P}$ can be decomposed into the form $\mathbf{P} = \mathbf{PS} \circ D + \mathbf{PS} \circ M$, where

$$Dh = \begin{pmatrix} (h\frac{\partial\sigma}{\partial n_\rho}.n_\rho + \sigma\nabla h).t_\rho^1 \\ (h\frac{\partial\sigma}{\partial n_\rho}.n_\rho + \sigma\nabla h).t_\rho^2 \\ 0 \end{pmatrix} \tag{3.2}$$

is a first-order differential operator acting on functions defined on Σ_ρ and

$$Mh = h \begin{pmatrix} 0 \\ 0 \\ \frac{\partial\sigma}{\partial n_\rho}.n_\rho.n_\rho \end{pmatrix} \tag{3.3}$$

is the normal component of the RHS in the linearized problem. As for $\mathbf{PS}$, it is a Poincaré-Steklov-type operator

$$\mathbf{PS}f = w.n_\rho \tag{3.4}$$

for the Stokes system

$$-\operatorname{div}\tau(w) = 0 \text{ in } \Omega_\rho^\gamma \qquad \operatorname{div} w = 0 \text{ in } \Omega_\rho^\gamma \tag{3.5}$$

$$w = 0 \text{ on } \Gamma_\gamma, \tag{3.6}$$

$$\tau(w).n_\rho = f \text{ on } \Sigma_\rho, \tag{3.7}$$

with Neuman boundary conditions on the candidate free surface.
In the remainder of this section, denote by $\mathbf{PS}_n$ the scalar "normal to normal" operator defined as

$$\mathbf{PS}_n h = w.n_\rho \tag{3.8}$$

for a Stokes system similar to (3.5)-(3.7) but with specific right-hand-side

$$-\operatorname{div}\tau(w) = 0 \text{ in } \Omega_\rho^\gamma \qquad \operatorname{div} w = 0 \text{ in } \Omega_\rho^\gamma$$

$$w = 0 \,\text{on}\, \Gamma_\gamma,$$
$$\tau(w).n_\rho.t_\rho^i = 0 \,\text{on}\, \Sigma_\rho\,,\ i = 1, 2,$$
$$\tau(w).n_\rho.n_\rho = h \,\text{on}\, \Sigma_\rho.$$

$\mathbf{PS}_n$ will play a fundamental part in the study of the invertibility of $\mathbf{P}$, as will soon become clear.
One can obviously rewrite $\mathbf{P} = \mathbf{P}_{orth} + \mathbf{P}_{tan}$ as the sum of two operators, $\mathbf{P}_{orth}$ (resp. $\mathbf{P}_{tan}$) mapping the normal (resp. tangential) stress to the normal component of the velocity field.
Regarding $\mathbf{P}_{orth}$, the following result holds true.

Proposition 3.1 *The operator $\mathbf{P}_{orth}$ is a pseudodifferential operator of order -1 whose principal symbol is given by*

$$p(\xi) = \frac{\frac{\partial \sigma}{\partial n_\rho}.n_\rho.n_\rho}{2|\xi|}.$$

Proof: the fact the $\mathbf{P}_{orth}$ is a pseudodifferential operator is straightforward, since it is the composition of two such operators. Its principal symbol is simply the product of that of the operator $\mathbf{PS}_n$ with the function $\frac{\partial \sigma}{\partial n_\rho}.n_\rho.n_\rho$. To compute the principal symbol - defined unambiguously in any system of local coordinates - of $\mathbf{PS}_n$, we use again conformal coordinates on Σ_ρ. Upon referring once more to [5], in particular Section 3.2.3, one is easily convinced that it is sufficient to freeze the coefficients of the equation at $\lambda = 0$, and therefore to focus on the simpler operator $h \to V_3(s_1, s_2, 0)$ where V is the solution to the system:

$$-\triangle V + \nabla Q = 0 \,\text{in}\, \mathbf{R^2} \times \mathbf{R}_*^- \qquad \text{div}\, V = 0 \,\text{in}\, \mathbf{R^2} \times \mathbf{R}_*^- \qquad \lim_{\lambda \to -\infty} V = 0 \tag{3.9}$$

$$\begin{pmatrix} \frac{\partial V_1}{\partial \lambda} + \frac{\partial V_3}{\partial s_1} \\ \frac{\partial V_2}{\partial \lambda} + \frac{\partial V_3}{\partial s_2} \\ 2\frac{\partial V_3}{\partial \lambda} - Q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \text{ on } \{\lambda = 0\}. \tag{3.10}$$

Upon taking the Fourier transform in the s variables, the system

$$\left(-\frac{\partial^2 \hat{V}_1}{\partial \lambda^2} + |\xi^2| \hat{V}_1\right) + i\xi_1 \hat{Q} \ = \ 0, \tag{3.11}$$

$$\left(-\frac{\partial^2 \hat{V}_2}{\partial \lambda^2} + |\xi^2| \hat{V}_2\right) + i\xi_2 \hat{Q} \ = \ 0, \tag{3.12}$$

$$\left(-\frac{\partial^2 \hat{V}_3}{\partial \lambda^2} + |\xi^2| \hat{V}_3\right) + \frac{\partial \hat{Q}}{\partial \lambda} \ = \ 0, \tag{3.13}$$

$$i\xi_1 \hat{V}_1 + i\xi_2 \hat{V}_2 + \frac{\partial \hat{V}_3}{\partial \lambda} = 0, \tag{3.14}$$

obtains. Then, solving for $\hat{Q}$ and combining the equations above, one immediately finds that

$$\hat{Q}(\xi_1, \xi_2 \lambda) = \frac{1}{|\xi|^2} \left(\frac{\partial}{\partial \lambda}(\frac{\partial^2}{\partial \lambda^2} - |\xi|^2)\right) \hat{V}_3 \tag{3.15}$$

and therefore

$$\left(\frac{\partial^2}{\partial \lambda^2} - |\xi|^2\right) \frac{\partial^2 \hat{V}_3}{\partial \lambda^2} = |\xi|^2 \frac{\partial \hat{Q}}{\partial \lambda}. \tag{3.16}$$

On the other hand, there also holds

$$-|\xi|^2 \left(\frac{\partial^2}{\partial \lambda^2} - |\xi|^2\right) \hat{V}_3 = -|\xi|^2 \frac{\partial \hat{Q}}{\partial \lambda}, \tag{3.17}$$

so that we finally obtain the ODE

$$\left(\frac{\partial^2}{\partial \lambda^2} - |\xi|^2\right)^2 \hat{V}_3 = 0 \tag{3.18}$$

for $\hat{V}_3$. Hence, $\hat{V}_3$ has the representation

$$\hat{V}_3(\xi,\lambda) = (A + B\lambda)\exp(-\lambda|\xi|) + (C + D\lambda)\exp(\lambda|\xi|) \tag{3.19}$$

with $A = B = 0$ because of boundedness conditions as $\lambda \to -\infty$. In particular, it must be the case that $\hat{V}_3(\xi, 0) = C$, and we need to compute the value of C. From (3.15), one easily derives that $\hat{Q}(\xi,\lambda) = 2D\exp(\lambda|\xi|)$. The third coordinate of the stress tensor on $\lambda = 0$ is given by $2\frac{\partial \hat{V}_3}{\partial \lambda}(\xi, 0) - Q(\xi, 0)$, which simplifies to $2C|\xi|$. Using the boundary conditions (3.10), it follows that

$$C = \frac{\widehat{h}}{2|\xi|} \tag{3.20}$$

and the proposition is proven.

Next, we state and prove the following important lemma, which will be essential in showing that $\mathbf{P}_{orth}$ is invertible for all small enough ρ.

Lemma 3.2 $\mathbf{PS}_n$ *is a coercive, self-adjoint operator from* $H^{-\frac{1}{2}}(\Sigma_\rho)$ *into* $H^{\frac{1}{2}}(\Sigma_\rho)$. *Therefore, there exists a constant* $C > 0$ *such that, for all* $h \in H^{-\frac{1}{2}}(\Sigma_\rho)$

$$(\mathbf{PS}_n h, h) \geqslant C\|h\|^2.$$

Proof : the self-adjointess of PS is a straightforward consequences of Green's formula, and the coercivity follows easily from Korn's inequality. In fact, setting $v.n = \mathbf{PS}_n h$ and writing, for a test function w,

$$\int_{\Omega_\rho} div\,\tau(v).w + \frac{1}{2}\int_{\Omega_\rho} (\nabla v +^t \nabla v) : (\nabla w +^t \nabla w) = \int_{\Sigma_\rho} (\tau(v).n)w,$$

one obtains, using v itself as a test function, that

$$\frac{1}{2}\int_{\Omega_\rho} |\nabla v +^t \nabla v|^2 = \int_{\Sigma_\rho} (\tau(v).n.n)(w.n),$$

which means that

$$(\mathbf{PS}_n h, h) = \frac{1}{2}\int_{\Omega_\rho} |\nabla v +^t \nabla v|^2$$

and thanks to Korn's inequality $(\mathbf{PS}_n h, h) \geqslant C\|v\|^2_{H^1}$. This inequality implies that $\mathbf{PS}_n$ is coercive, and hence, invertible from $H^{-\frac{1}{2}}(\Sigma_\rho)$ into $H^{\frac{1}{2}}(\Sigma_\rho)$. The lemma then results from the following trace theorem.

Lemma 3.3 *Let* Ω *be an open set in* $\mathbf{R}^3$ *with a* C^2 *boundary. Then for* $v \in H^1(\Omega)$ *such that* $\tau.n$ *is parallel to* n *and* $div\,\tau(v) = 0$, *one can define a normal stress operator* $v \to \tau.n.n \in H^{\frac{-1}{2}}(\partial\Omega)$ *satisfying the continuity property*

$$\|v\|_{H^1(\Omega)} \geqslant C\|\tau.n.n\|_{H^{\frac{-1}{2}}(\partial\Omega)}.$$

The operator thus defined coincides with the usual normal stress operator when v *is* C^1 *up to the boundary of* Ω

The proof of Lemma 3.3 is a direct consequence of [24], Chapter 1, Theorem 1.2 and so is omitted. Upon using Lemma 3.3, one then has

$$(\mathbf{PS}_n h, h) \geqslant C\|v\|^2_{H^1(\Omega_\rho^\gamma)} \geqslant C\|\tau.n.n\|^2_{H^{\frac{-1}{2}}(\partial\Omega_\rho^\gamma)},$$

which ends the proof of Lemma 3.2.
We may now state the main result concerning $\mathbf{P}_{orth}$.

Lemma 3.4 *For all $\delta > 0$, $s > 4$, there exists a neigbourhood of 0 in $C^s_\delta(\mathbb{R}^2)^2$ such that whenever (γ, ρ) lies in this neigbourhood, $\mathbf{P}_{orth}$ has a bounded inverse defined from $H^{\frac{1}{2}}(\Sigma_\rho)$ into $H^{-\frac{1}{2}}(\Sigma_\rho)$.*

Proof: an easy consequence of the invertibility of $\mathbf{PS}_n$ and the fact that the multiplier $\frac{\partial \sigma}{\partial n_\rho}.n_\rho.n_\rho$, being bounded from below by a positive number along the unperturbed free surface corresponding to $\gamma = \rho \equiv 0$, therefore stays so whenever (γ, ρ) is small enough. In fact, $P^{-1}_{orth} f$ is defined, for $f \in H^{\frac{1}{2}}(\Sigma_\rho)$, as the solution $h \in H^{\frac{-1}{2}}(\Sigma_\rho)$ of $\mathbf{P}_{orth} h = f$, that is

$$h = \frac{1}{\frac{\partial \sigma}{\partial n_\rho}.n_\rho.n_\rho} \mathbf{PS}_n^{-1} f$$

where the existence of $\mathbf{PS}_n^{-1}$ has just been established.

Regarding the second term in the decomposition of $\mathbf{P}$, $\mathbf{P}_{tan}$, it is *a priori* a pseudodifferential operator of order $\leqslant 0$, whose most salient feature is that it has a small norm as a bounded operator from $\mathbf{L}^2$ into $\mathbf{L}^2$, say, as long as the angle α is small enough and the perturbation of the fixed profiles and the free surfaces γ, ρ are small enough. As a matter of fact, one can see by inspection of (3.2) that there are two components in the tangential operator Dh. The first one is a 0^{th}-order term parallel to the tangential part of $\frac{\partial \sigma}{\partial n_\rho}.n_\rho$, a quantity equal to the constant vector $\begin{pmatrix} -C_1 \\ 0 \\ 0 \end{pmatrix}$ in the case of the unperturbed Poiseuille-Nusselt flow, and the other is the vector field $\sigma \nabla h$, identically equal to 0 in the unperturbed case. Therefore, one can safely choose an α small enough and a small enough neighbourhood for the perturbation γ, ρ so as to ensure that $\mathbf{P}_{tan}$ has a small norm.
Note that this condition on α is in fact a natural one, as has been already demonstrated in [4][12].
We are now in a position to state the main result of this section, pertaining to the invertibility of the linearized operator defined in (2.11).

Theorem 3.5 *There exists an $\alpha_0 > 0$ and a $\delta_0 > 0$ such that $\forall \alpha < \alpha_0$, $\forall \delta, 0 < \delta < \delta_0$, $\forall s > 4$, there exists a neigbourhood of 0 in $C^s_\delta(\mathbb{R}^2)^2$ with the property that whenever (γ, ρ) lies in this neigbourhood, $\frac{DG}{D\rho}$ defined in (2.11) has a bounded inverse defined from $H^{t+\frac{1}{2}}(\Sigma_\rho)$ into $H^{t-\frac{1}{2}}(\Sigma_\rho)$ for all $t > 0$.*

Proof: this result is essentially that of Proposition 1.1 in [3], with a few minor modifications. In fact, one must check that the assumptions of Proposition 1.1 of [3] hold true for the linearized operator $\frac{DG}{D\rho}$. A careful examination shows that there is a slight difference, namely, that $\mathbf{P}$ is not self-adjoint. However, it has just been proven to be invertible, clearly a sufficient condition in the light of the proof of Proposition 1.1 in [3]. Therefore, so is the linearized operator $\frac{DG}{D\rho}$ in the usual Sobolev spaces.
A straightforward extension of the proof in [3] also yields the next result.

Corollary 3.6 *Under the same set of assumptions as above, there exists a $\delta_{02} > 0$ such that, $\forall \delta, 0 < \delta \leqslant \delta_{02}$, $\frac{DG}{D\rho}$ has a bounded inverse defined from $H^{t+\frac{1}{2}}_\delta(\Sigma_\rho)$ into $H^{t-\frac{1}{2}}_\delta(\Sigma_\rho)$ for all $t > 0$.*

Proof: one uses the changes of right-hand-side and unknown function $f(x) = f_0(x) \exp(-\delta\sqrt{1+|x|^2})$, $h(x) = h_0(x) \exp(-\delta\sqrt{1+|x|^2})$ to obtain a modified operator for which Theorem 3.5 still holds true provided that δ is small enough.

4 Proof of the main result

This final section is devoted to the proof of Theorem 1.2.

4.1 Zehnder's Implicit Function Theorem

We start by presenting the machinery fully developped in [26]. Let $\{X_t\}_{t\geqslant 0}$ be a one-parameter family of Banach spaces, with norm denoted by $|.|_t$, such that for all t, t' with $0 \leqslant t' \leqslant t \leqslant +\infty$

$$X_0 \supseteq X_{t'} \supseteq X_t \supseteq X_\infty \equiv \bigcap_{t\geqslant 0} X_t$$

and

$$|z|_{t'} \leqslant |z|_t.$$

We will work with three such scales of Banach spaces, respectively denoted by $\{X_t\}_{t\geqslant 0}$, $\{Y_t\}_{t\geqslant 0}$ and $\{Z_t\}_{t\geqslant 0}$. The mapping G in (1.9) will be defined from $D \subset Y_0 \times X_0$ into $R \subset Z_0$. There holds that $G(0,0) = 0$, and Zehnder's Implicit Function Theorem provides conditions ensuring that, for γ close enough to 0, there exists a unique $\rho(\gamma)$ solution to

$$G(\gamma, \rho(\gamma)) = 0.$$

We recall that γ is the perturbation of the bottom profile.
For $\epsilon > 0$, let $B_t^\epsilon = \{(\gamma,\rho) \in Y_t \times X_t : |\gamma| < \epsilon, |\rho| < \epsilon\}$. Below are listed the hypotheses that have to be satisfied by G for a suitably chose $\epsilon > 0$.
Hypothesis 1 (smoothness)
The operator $G : B_0^\epsilon \to Z_0$ is at least C^2 with respect to ρ and C^0 with respect to (γ,ρ), and there exists a constant $M_0 > 0$ such that

$$\forall (\gamma,\rho) \in B_0^\epsilon, \sup(|D_\rho G(\gamma,\rho)|_0, |D_\rho^2 G(\gamma,\rho)|_0) \leqslant M_0. \tag{4.1}$$

and

$$\forall (\gamma,\rho),\ (\gamma',\rho) \in B_0^\epsilon, |G(\gamma,\rho) - G(\gamma',\rho)|_0 \leqslant M_0|\gamma - \gamma'|_0. \tag{4.2}$$

Hypothesis 2 (approximate inverse with loss of ν derivatives)
For all (γ,ρ) in B_ν^ϵ, there exists a linear mapping $H(\gamma,\rho) : Z_\nu \to X_0$ such that

$$\forall \phi \in Z_\nu,\ |H(\gamma,\rho)\phi|_0 \leqslant M_0|\phi|_\nu. \tag{4.3}$$

$H(\gamma,\rho)$ is continuous from Z_t into $X_{t-\nu}$ whenever $(\gamma,\rho) \in B_\nu^\epsilon \bigcap (Y_t \times X_t)$ and there holds

$$\forall (\gamma,\rho) \in B_t^K,\ |H(\gamma,\rho)G(\gamma,\rho)|_{t-\nu} \leqslant M_t K. \tag{4.4}$$

Moreover, H is an approximate left inverse of $D_\rho G(\gamma,\rho)$ in the sense that

$$\forall \phi \in Z_\nu,\ |[D_\rho G(\gamma,\rho) \circ H(\gamma,\rho) - Id]\phi|_0 \leqslant N_0|F(\gamma,\rho)|_\nu|\phi_\nu|. \tag{4.5}$$

Hypothesis 3 (order)
$(G,0,0)$ is of order s, where $s > \nu \geqslant 1$ (see Hypothesis 2 above for the definition of ν). More precisely,

$$\forall t \in [1,s],\ G(B_0^\epsilon \bigcap (Y_t \times X_t)) \subset Z_t \tag{4.6}$$

and

$$\forall t \in [1,s],\ \exists M_t > 0, \exists K_0 > 0, \forall (\gamma,\rho) \in B_t^{K_0} \bigcap B_1^\epsilon,\ |G(\gamma,\rho)|_t \leqslant M_t|f|_t. \tag{4.7}$$

When all the hypotheses above are satisfied, the following deep result applies.

Theorem 4.1 *(Zehnder [26]) Let (G,γ_0,ρ_0) be of order s and satisfy Hypotheses 1 to 3 above. Then there exists a neighbourhood $N_\lambda = \{\gamma \in Y_\lambda, |\gamma| < C\}$ with μ, λ in the admissible parameter set*

- $1 < \kappa < 2$,

- $1 < \alpha$,
- $1 \leqslant \nu \leqslant \mu < \lambda < s$,
- $\lambda > max\{2\kappa\nu(2-\kappa)^{-1}, \kappa(\nu + \kappa\mu)\}$,
- $s > max\{\alpha\nu(\alpha-1)^{-1}, \lambda + \alpha\nu(\kappa-1)^{-1}\}$,

and a mapping $\Psi : N_\lambda \to X_\mu$ *such that for all* $\gamma \in N_\lambda$, $\rho_\gamma \equiv \Psi(\gamma)$ *is the unique solution to*

$$G(\gamma, \rho_\gamma) = 0 \tag{4.8}$$

with $|\rho_\gamma|_\mu < C^{-1}|\gamma|_\lambda$. *Furthermore,* Ψ *is continuous whenever* H *is.*

To apply this theorem to the proof of Theorem 1.2, we use the weighted Sobolev spaces introduced in Section 3

$$X_t = H^{4+t}_\delta(\mathbf{R}^2) \qquad Y_t = H^{4+t}_\delta(\mathbf{R}^2) \qquad Z_t = H^{\frac{5}{2}+t}_\delta(\mathbf{R}^2),$$

so that

$$X_\infty = S(\mathbf{R}^2) \qquad Y_\infty = S(\mathbf{R}^2) \times S(\mathbf{R}^2) \qquad Z_\infty = S(\mathbf{R}^2).$$

The choice of the exponent in the Sobolev spaces just introduced is guided by Proposition 2.1, as $\frac{5}{2} = 4 - \frac{3}{2}$. Before proceeding to the verification of Hypotheses 1-3 above, let us explain the use of the Nash-Moser Implicit Function Theorem: its constructive proof, see see e.g. [26], p. 118 to 125, or [6], Chapter III, p. 128, relies on a Newton iteration having the form

$$\rho_{n+1} = \rho_n - (\mathcal{S}_n \circ (\frac{DG}{D\rho}(\rho_n, \gamma))^{-1} \circ G)(\rho_n, \gamma), \tag{4.9}$$

where $\mathcal{S}_n$ is a smoothing operator coping for the loss of derivatives. Combining Proposition 2.1 with Corollary 3.6, one can see that the loss of regularity when applying $\frac{DG}{D\rho}^{-1} \circ G$ is equal to $\frac{5}{2}$, so that $\frac{DG}{D\rho}^{-1} \circ G$ will be bounded and continuous from X_t to $X_{t-\frac{5}{2}}$, see the estimate (4.10) below. We therefore proceed to check the hypotheses of Theorem 4.1 with the parameter ν (the loss in regularity) set equal to $\frac{5}{2}$.
Fix $\delta < \mathrm{Max}(\delta_{01}, \delta_{02})$, δ_{01} (resp. δ_{02}) being introduced in Proposition 2.1 (resp. Corollary 3.6). We also pick s (the order parameter in Hypothesis 3) equal to $+\infty$, meaning that we are interested in smooth solutions only. Hypothesis 1 is an easy consequence of Proposition 2.1, or, more precisely, an extension thereof so as to include the fixed boundary γ as well as the candidate free surface ρ in the differentiability. The proof relies again on a change of variables and results on the smooth dependence of the solution to the Auxiliary Problem with respect to its coefficients, and will be omitted.
As for Hypothesis 3, we remark that $(0,0) \in U_\infty \times X_\infty$ and that we already established as a general result that $G(\gamma, \rho) \in Z_t$ if $(\gamma, \rho) \in Y_t \times X_t$ for $t \in [1, +\infty[$. The last point to check is that for all $t \in [1, +\infty[$, there exists an M_t such that $|G(\gamma, \rho)|_t \leqslant M_t K$ whenever $|\gamma - \gamma_0|_t < K$, $|\rho - \rho_0|_t < K$ in $(Y_t \times X_t) \bigcap B_1$ (here, we have chosen $K < \epsilon$). Such an inequality follows from computations similar to those involved in the proof of Hypothesis 2, albeit with respect to both γ and ρ this time, the key point being (see the remark at the bottom of p. 120 in [26]) that the control of a low-order norm is sufficient to ensure the linear growth of higher-order norms for the operator G. As a consequence, one can even choose $M_t = M_0$ for all t, a remark that will be essential in verifying Hypothesis 2. Regarding this hypothesis, we state and prove a more precise result.

Proposition 4.2 *For all* $(\gamma, \rho) \in Y_\infty \times X_\infty$, *there exists a linear mapping* $\mathbf{H} : Z_{\frac{5}{2}} \to X_0$ *such that* $|\mathbf{H}(\gamma, \rho)\phi|_0 \leqslant M_0 |\phi|_{Z_{\frac{5}{2}}}$. $\mathbf{H}$ *can be extended to a continuous operator from* Z_t *into* $X_{t-\frac{5}{2}}$ *for all* $t \geqslant \frac{5}{2}$.
Moreover, there exists a constant M_t *such that the inequality*

$$|\mathbf{H}(\gamma, \rho)(G(\gamma, \rho))|_{t-\frac{5}{2}} \leqslant M_t K \tag{4.10}$$

holds true for all $(\gamma,\rho) \in (Y_t \times X_t) \bigcap B_1$ *with* $|\gamma - f_0|_t < K$, $|\rho - \rho_0|_t < K$. *Finally,* $\mathbf{H}(\gamma,\rho)$ *is an exact inverse to* $D_\rho G(\gamma,\rho)$, *i.e.*

$$\forall \phi \in Z_{t+\nu},\ \big(D_\rho G(\gamma,\rho) \circ \mathbf{H}(\gamma,\rho)\ -\ Id\big)\,\phi = 0. \tag{4.11}$$

Proof: In Theorem 3.5 and Corollary 3.6, we have established the existence of an exact inverse $\mathbf{H} : Z_{\frac{5}{2}} \to X_0$ to $D_\rho G$, and also that $\mathbf{H}$ is a continuous operator from Z_t into $X_{t-\frac{5}{2}}$ for all $t \geqslant 0$, provided $(\gamma,\rho) \in Y_\infty \times X_\infty$. There still remains to prove that $|\mathbf{H}(\gamma,\rho)(G(\gamma,\rho))|_{t-\frac{5}{2}} \leqslant M_t K$ whenever $(\gamma,\rho) \in (Y_t \times X_t) \bigcap B_1$ with $|\gamma - \gamma_0|_t < K$, $|\rho - \rho_0|_t < K$. This is a consequence of the boundedness of $\mathbf{H}$ and the uniform (in t) estimate on G in Hypothesis 3, see (4.7).
Finally, we must prove that Hypothesis 2 can be replaced by Proposition 4.2. In the original proof by Zehnder ([26], p. 118 to 125), the use of an approximate inverse is made after the smoothing procedure for (γ,ρ). Hence, the estimate 4.5 $|(D_\rho G(\gamma,\rho) \circ \mathbf{H}(\gamma,\rho) - Id)\phi|_0 \leqslant N_0 |F(\gamma,\rho)|_{\frac{5}{2}} |\phi|_{\frac{5}{2}}$ in Hypothesis 2 is only required for the regularized approximating sequence $(\gamma_\tau^n, \rho_\tau^n)$, τ being the smoothing factor. Since this sequence comprises smooth functions, Proposition 4.2 above applies, and actually yields a stronger result that the one required in the original proof by Zehnder. There, the constant N_0 had to be uniform with respect to the smoothing parameter τ, a condition that becomes irrelevant thanks to the exact invertibility of $D_\rho G(\gamma,\rho)$.
This concludes the proof of Theorem 1.2.

References

[1] F. Abergel, *A geometric approach to the study of stationary free surface flow for viscous liquids,* Proc. Royal Soc. Edinburgh, 123A (1993), 209-229.

[2] F. Abergel, *Well-posedness for a Cauchy problem associated to time-dependent free boundaries with non-local leading terms,* Comm. Partial Diff. Equations, 21 (1996), 1307-1379.

[3] F. Abergel, J.-H. Bailly, *Higher order symmetrizers and application to an unusual transport equation,* Comm. Partial Diff. Equations, 24 (1999), 1593-1610.

[4] F. Abergel, J. L. Bona, *A mathematical theory for a viscous, free-surface flow over an inclined plane,* Arch. rational Mech. Anal, 118 (1992), 71-93.

[5] F. Abergel, E. Rouy, *Interfaces stationnaires pour les 'équations de Navier Stokes,* rapport de recherche INRIA 2651 (1995).

[6] S. Alinhac, P. Gerard, "Pseudodifferential Operators and the Nash Moser Theorem," American Mathematical Society, Providence, Rhode Island, 2000.

[7] C. J. Amick, *Steady solutions of the Navier-Stokes equations in unbounded channels and pipes,* Annali della Scuola Normale Superiore di Pisa, Classe di Scienze Serie 4, tome 4, nÂř 3 (1977), 473-513.

[8] J.-H. Bailly, *Existence of classical solutions to first-order parabolic equations describing free boundaries,* Nonlinear Analysis T. M. A., 32, 5 (1998).

[9] J. T. Beale, *The initial value problem for the Navier-Stokes equations with free surface,* Comm. Pure Appl. Math., 34 (1980), 359-392.

[10] J. T. Beale, *Large-time regularity of viscous surface waves,* Arch. rational Mech. Anal, 84 (1984), 307-352.

[11] J. T. Beale, T. Nishida, *Large-time behaviour of viscous surface waves,* Lecture Notes in Num. Appl. Anal., 8 (1985), 1-14.

[12] T. Brooke Benjamin, *Wave formation in laminar flow down an inclined plane,* Journal of Fluid Mechanics, 2 (1957), 554-573.

[13] X. Chen, J. Hong, F. Yi, *Existence, uniqueness and regularity of classical solutions of the Mullins-Sekerka problem,* Comm. Partial Diff. Equations, 21 (1996), 1705-1727.

[14] B. Doubrovine, S. Novikov, A. Fomenko, "Géométrie contemporaine," MIR editions, Moscou, 1982.

[15] H. Fujita-yashima, *Problème de la surface libre de l'équation de Navier-Stokes - cas stationnaire et cas p'ériodique,* Anal. Scuo. Norm. Sup. Pisa, 12 (1985), 532-587.

[16] M. Jean, *Free surface of the steady flow of a newtonian fluid in a finite channel,* Arch. rational Mech. Anal, 62 (1980), 1-52.

[17] S. Nazarov, K. Pileckas, *On noncompact free boundary problems for the plane stationary Navier-Stokes equations,* J. Reine Angew. Math, 438 (1993), 103-141.

[18] V. Pukhnachov, *Stationary viscous flows at zero surface tension,* Acta Appl. Math., 37 (1994), 147-152.

[19] V. Solonnikov, *Solvability of a problem on the motion of a viscous, incompressible fluid bounded by a free surface,* Math USSR Isvestija, 11 (1977), 1323-1358.

[20] V. Solonnikov, *Solvability of a problem on plane motion of a heavy viscous incompressible capillary liquid partially filling a container,* Math USSR Isvestija, 14 (1980), 193-221.

[21] V. Solonnikov, *On the Stokes equations in domains with nonsmooth boundaries and on viscous incompressible flows with a free surface,* Nonlinear Partial Differential Equation and their Applications, III, Collège de france seminar, 1980-1981, eds H. brezis, J.-L. Lions, research Notes in Mathematics, Pitman, 70 (1982), 340-423.

[22] V. Solonnikov, *Solvability of the problem of effluence of a viscous incompressible fluid in an infinite open basin,* Proc. Steklov Inst. Math, 179 (1989), 192-225.

[23] R. S. Strichartz, *A note on Sobolev algebras,* Proc. Amer. Math. Soc., 29, 1 (1971), 205-207.

[24] R. Temam, "Navier-Stokes equations," North Holland, Amsterdam, 1984.

[25] Y. Teramoto, *On the Navier-Stokes flow down an inclined plane,* J. Kyoto Univ., 32, 3, (1992), 593-619.

[26] E. Zehnder, *Generalized implicit function theorem with applications to small divisors problems,* Comm. Pure Appl. Math, 28 (1975), 91-140.