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Stationary free surface viscous flows without surface tension in three dimensions

Frederic Abergel*

Jacques-Herbert Bailly

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Abstract

We consider an incompressible, viscous, free surface flow down a three dimensional channel. In the absence of surface tension, we prove the existence of a unique stationary solution in weighted Sobolev spaces. The result is based on a thorough study of the pseudodifferential operator relating the normal velocity of the fluid and the normal component of the associated stress tensor along the free surface, and requires the use of the Nash-Moser Implicit Function Theorem.

Introduction

We consider an incompressible, viscous, free surface fluid flowing down a three dimensional inclined channel at low Reynolds number. In the case where the bottom of the channel is flat, there exists a well-known stationary solution, usually referred to as the Poiseuille-Nusselt flow [4]. However, the question of finding such a stationary solution for a general profile of the bottom is not easy to address: standard perturbation methods resting upon the usual Implicit Function Theorem fail to apply, since the linearized operator at the Poiseuille-Nusselt solution is not elliptic when surface tension is neglected, and results such as those in [4] or [14] do not hold true. As a matter of fact, it has been proven, see e.g. [1], that the surface tension coefficient, when taken into account, plays a particularly important role in the structure of the linearized solution operator, transforming it into an elliptic second order pseudodifferential operator. In the absence of surface tension, there exist some drastic differences between the two- and three-dimensional cases. Two-dimensional free surface flows are still associated to an elliptic operator, albeit with a purely imaginary principal symbol; however, in the three-dimensional case, the linearized free surface problem is no longer elliptic, and one has to carefully analyze the structure of the associated operator in order to study its invertibility. Moreover, even though the linearized operator may be invertible, one has to overcome a loss of smoothness when inverting it, a phenomenon making the analysis more complicated.

Our aim in this work is to prove the existence of a unique smooth, stationary free surface for three-dimensional flows close to the Poiseuille-Nusselt flow. The paper is organized as follows: Section 1 contains an exposition of the physical problem and its mathematical formulation, together with the description of the appropriate functional setting. Section 2 introduces the technical tools necessary to compute the derivative of the main operator with respect to the domain. Section 3 is devoted to a detailed study of the linearized operator and its invertibility, whereas Section 4 presents the heavy machinery involved in the Nash-Moser Theorem, and its application to the proof of the main result.

*Laboratory of Mathematics Applied to Systems, Ecole Centrale Paris. E-mail: frederic.abergel@ecp.fr

1 Formulation of the problem

1.1 Governing equations

Let Ω_ρ^γ be an unbounded domain of \mathbb{R}^3

$$\Omega_\rho^\gamma = \{(x, y, z) \in \mathbb{R}^3, -\infty < x < +\infty, -\infty < y < +\infty, \gamma(x, y) < z < z_0 + \rho(x, y)\} \quad (1.1)$$

where ρ is a mapping from \mathbb{R}^2 into \mathbb{R} describing the free surface, γ is a mapping from \mathbb{R}^2 into \mathbb{R} representing the fixed bottom, while z_0 is the height of the free surface when x, y goes to ∞ . Such a representation is legitimate whenever ρ, γ are small enough. Denote by $\partial\Omega_\rho^\gamma$ the boundary of Ω_ρ^γ

$$\partial\Omega_\rho^\gamma = \Gamma_\gamma \cup \Sigma_\rho, \quad (1.2)$$

Γ_γ being the fixed bottom of the channel and Σ_ρ , the free surface. Let u be the velocity field and p be the pressure. We define the stress tensor

$$\sigma(u) = \frac{1}{2}(\nabla u + {}^t\nabla u) - pId \quad (1.3)$$

and let f be the gravitational field. In the frame shown on Figure 1.1 below, f has coordinates $\begin{pmatrix} C_1 \\ 0 \\ C_2 \end{pmatrix}$ where

$C_1 = g \sin \alpha$ and $C_2 = g \cos \alpha$, $\alpha, 0 < \alpha < \frac{\pi}{2}$ being the angle shown on Figure 1.1.

The goal of this paper is to study, for a given γ , the existence and uniqueness of a triple (u, p, ρ) solution of the stationary Stokes system

$$-\operatorname{div} \sigma(u) = f \text{ in } \Omega_\rho^\gamma, \operatorname{div} u = 0 \text{ in } \Omega_\rho^\gamma, \quad (1.4)$$

supplemented with the respective fixed- and free-boundary conditions

$$u = \zeta \text{ on } \Gamma_\gamma, \sigma(u).n = 0 \text{ on } \Sigma_\rho \quad (1.5)$$

as well as the lateral boundary conditions at ∞

$$\lim_{(x,y) \rightarrow +\infty} u(x, y, z) = \begin{pmatrix} C_1(z_0 z - \frac{1}{2}z^2) \\ 0 \\ 0 \end{pmatrix}. \quad (1.6)$$

Moreover, a kinematic equation for the free surface should also hold. In the stationary case, this condition simply reads:

$$u.n = 0. \quad (1.7)$$

Equations (1.4)(1.5)(1.6) will be referred to as the auxiliary problem, while (1.7) will be called the main equation. This system of equations is absolutely standard, and we will not dwell upon the physical derivation of it, nor the interpretation of the equations above. It is interesting - and will be useful in the sequel - to fully characterize the Poiseuille-Nusselt flow corresponding to $\rho = \gamma = 0$. The solution to Equations(1.4) to (1.7) is given by the velocity field

$$\begin{pmatrix} U_1(x, y, z) \\ U_2(x, y, z) \\ U_3(x, y, z) \end{pmatrix} = \begin{pmatrix} C_1(z_0 z - \frac{1}{2}z^2) \\ 0 \\ 0 \end{pmatrix} \quad (1.8)$$

and the pressure field $p(x, y, z) = C_2(z_0 - z)$. In particular, the associated stress tensor has the following expression

$$\sigma(u) = \frac{1}{2}(\nabla u + {}^t \nabla u) - pId = \begin{pmatrix} -C_2(z_0 - z) & 0 & \frac{1}{2}C_1(z_0 - z) \\ 0 & -C_2(z_0 - z) & 0 \\ \frac{1}{2}C_1(z_0 - z) & 0 & -C_2(z_0 - z) \end{pmatrix}, \quad (1.9)$$

and one can easily see that its restriction to the free surface $z = z_0$ has only zero entries. We also note that the normal component of its normal derivative

$$\frac{\partial \sigma}{\partial n} \cdot n \cdot n = \frac{\partial \sigma}{\partial z} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (1.10)$$

is equal to C_2 , a strictly positive constant.

Before moving on the analysis of the free surface problem, a couple of remarks are in order: firstly, we use the Stokes rather than the Navier-Stokes system for the sake of simplicity, and also because, due to the perturbative nature of the argument that will be used, a similar result easily obtains for the Navier-Stokes system. Secondly, the normal stress condition should read $\sigma(u) \cdot n = K \cdot n$ on Σ_ρ , but the constant K can be set equal to 0, a choice that fully characterizes the pressure field, defined only up to a constant. As regards the boundary condition ζ on the fixed bottom, it is set to 0 - the case of no-slip boundary condition - although one could choose a small, rapidly decaying at ∞ , function.

1.2 Function spaces

Let us now introduce a family of Banach spaces well adapted to the resolution of Equations (1.4)(1.5)(1.6)(1.7). These spaces are weighted Sobolev spaces with an exponentially growing weight function, and are defined below.

For $\delta = (\delta_1, \delta_2)$ in \mathbb{R}_+^2 and $s > 0$, we set

$$H_\delta^s(\mathbb{R}^3) = \{f \in H^s(\mathbb{R}^3) : \|f(X, z) \exp(\delta_1|x| + \delta_2|y|)\|_{H^s(\mathbb{R}^3)} < +\infty\} \quad (1.11)$$

where $X = (x, y) \in \mathbb{R}^2$ and $H^s(\mathbb{R}^3)$ stands for the usual Sobolev space. Similarly, we define

$$C_\delta^s(\mathbb{R}^3) = \{f \in C^s(\mathbb{R}^3) : \sum_{|\alpha| \leq s} \text{Sup}(|D^\alpha f|) \exp(\delta_1|x| + \delta_2|y|) + \sum_{|\alpha|=s} \text{Sup}([D^\alpha f]) < +\infty\} \quad (1.12)$$

where $[s]$ is the integer part of s and $[f] = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{s - [s]}}$. We recall for further use a classical embedding theorem:

Proposition 1.1 *Let $m \in \mathbf{N}$, $\alpha \in]0, 1[$ be such that $s - \frac{3}{2} \geq m + \alpha$ and $2s > n$. Then $\forall f \in H_\delta^s(\mathbb{R}^3)$, $f \in C_\delta^{m+\alpha}(\mathbb{R}^3)$ and there exists a constant $C > 0$ such that $\forall f \in H_\delta^s(\mathbb{R}^3)$, $\|f\|_{C_\delta^{m+\alpha}} \leq C \|f\|_{H_\delta^s}$.*

1.3 The main result

For the sake of convenience, we shall denote by $B^\epsilon(X)$ the ball of radius ϵ centered at the origin in X , when X is a Banach space and $\epsilon > 0$. Our main result can now be stated:

Theorem 1.2 *There exists $\delta_0 > 0$ such that $\forall \delta \leq \delta_0, \exists \epsilon > 0, \forall \gamma \in B^\epsilon(H_\delta^{10}(\mathbf{R}^2)), \forall \zeta \in B^\epsilon(H_\delta^8(\Gamma_\gamma)), \exists!(u, p, \rho)$ solution of Equations (1.4) to (1.7) with $\rho \in H_\delta^5(\mathbf{R}^2), (u, \nabla p) \in H_\delta^3(\Omega_\rho^\gamma) \times H_\delta^1(\Omega_\rho^\gamma)$.*

Moreover, the solution depends continuously on the data:

$$\|\rho\|_{H_\delta^3} + \|u\|_{H_\delta^3} + \|\nabla p\|_{H_\delta^1} \leq C(\|f\|_{H_\delta^1} + \|\zeta\|_{H_\delta^8} + \|\gamma\|_{H_\delta^{10}}),$$

f being the right-hand side of Equation(1.4).

The proof of 1.2 is quite lengthy, and will be presented in the rest of the paper. It relies on a thorough study of the linearization of Equations (1.4) to (1.7), as can be expected from our formulation as a perturbation problem. A similar approach has been used in various works related to free surface stationary flows for the Stokes system. However, to the best of our knowledge, all rigorous results that are available in the literature are based either on the assumption of a non-zero surface tension coefficient, or on that of a zero surface tension coefficient for two-dimensional flows, see [1][5] for a general exposition. Now, in the case considered here, the situation is much more complex, because it is essentially impossible to use the standard Implicit Function Theorem. In fact, Equation(1.7) can be interpreted as the problem of finding the zero of the nonlinear mapping G associating, to a candidate free surface ρ , the value of the normal velocity $u.n$ on Σ_ρ :

$$G(\rho, \gamma) = u|_{\Sigma_\rho} \cdot n_\rho \quad (1.13)$$

where γ (describing the fixed bottom of the domain) plays the role of a small parameter. It is clear that the classical Poiseuille-Nusselt flow corresponds to a particular, trivial solution

$$G(0, 0) = 0 \quad (1.14)$$

and therefore, the use of the Implicit Function Theorem may seem appropriate. However, a major difficulty must be overcome: when studying the derivative of G , $\frac{DG}{D\rho}(0, 0)$ say, one can see that it is indeed invertible in some sense, thanks to the fact that the velocity field never vanishes along the flat free surface, but also that, unfortunately, $(\frac{DG}{D\rho}(0, 0))^{-1} \circ \frac{DG}{D\rho}(0, 0)$ is an unbounded operator. We therefore have to invoke, and verify the cumbersome assumptions of, a Nash-Moser-type iterative scheme in order to overcome this loss of regularity. In particular, this program requires the study of the invertibility of the linearized operator $\frac{DG}{D\rho}(\rho, \gamma)$ in a full neighbourhood of the flat case - and not only at the particular solution $(0, 0)$.

In the next section, we will introduce the appropriate formalism to study the dependence of the operator G on the candidate free surface, explicitly computing its derivative.

2 The free-surface operator

2.1 Some geometric prerequisites

We briefly recall some classical results pertaining to the parameterization of a free surface Σ_ρ as a graph above a reference manifold, see [5] for a detailed exposition.

Let $\bar{\Sigma}$ be a two-dimensional reference manifold, denote by $\bar{M}(s)$ a generic point on $\bar{\Sigma}$ and by $\bar{n}(s)$ the unit normal vector to $\bar{\Sigma}(s)$ pointing outward, where s is a system of local coordinates on $\bar{\Sigma}$. We will use the notation ∂_{s_k} for the tangential derivative operator with respect to the k^{th} local coordinate s_k on $\bar{\Sigma}$.

In a neighbourhood of $\bar{\Sigma}$, there exists a unique representation of any point M in this neighbourhood under the following form

$$M(s, d) = \bar{M}(s) + d\bar{n}(s). \quad (2.1)$$

For $l_0 > 0$ small enough (the size of l_0 depending only on the maximum principal curvature of $\bar{\Sigma}$), this representation defines a C^∞ -diffeomorphism from $\bar{\Sigma} \times]-l_0, l_0[$ onto the l_0 -neighbourhood of $\bar{\Sigma}$, that is, the set of points who sit at a distance less than l_0 from $\bar{\Sigma}$. Hence, to a given $\rho \in H_\delta^s(\bar{\Sigma}; \mathbf{R})$ in a small enough neighbourhood of the origin, one can associate a manifold using the parameterization defined above

$$\Sigma_\rho = \bar{\Sigma} + \rho\bar{n} \quad (2.2)$$

or, more specifically

$$\Sigma_\rho = \{M \in \mathbf{R}^3 : \exists \bar{M} \in \bar{\Sigma}, M = \bar{M} + \rho(s)\bar{n}\}. \quad (2.3)$$

For instance, one can obviously represent any point in the three dimensional half-space over the (x, y) plane using tangential and normal coordinates - this property still holds true for small enough perturbations of this plane.

2.2 Properties of G

Let us now turn to the study of the nonlinear operator G introduced formally in 1.13. In order to make sure that G is well-defined for γ, ρ small enough, we first recall the

Proposition 2.1 *There exists $\delta_{01} > 0$ such that there holds:*

$\forall \delta \leq \delta_{01}, \exists \epsilon > 0, \forall s \geq 4, G$ defined in 1.13 is of class C^2 from $B^\epsilon(H_\delta^s(\mathbf{R}^2)^2)$ into $B^\epsilon(H_\delta^{s-\frac{3}{2}}(\mathbf{R}))$.

The proof of Proposition 2.1 is performed in two steps: first, one uses a global change of variables mapping Ω_ρ^γ onto the domain Ω_0^0 corresponding to the unperturbed flow. Then, a differentiability argument for the solution of a well-posed boundary value problem w.r. to the coefficients of the equation is used. It is identical to the one given in great details in [5], Section 2.2, to which the interested reader is referred.

2.3 The derivative of G

Since G is differentiable in a neighbourhood of the origin, we can compute its partial derivative $\frac{DG}{D\rho}(\rho).h$ as the limit when $t \rightarrow 0$ of

$$\frac{1}{t}(G(M_\rho + thn_\rho, th + \rho) - G(M_\rho, \rho)), \quad (2.4)$$

with $M_\rho = M_0 + \rho n_0$, see Figure 2.3. Again, the computations we need to perform are classical and detailed at length in [5], and will only be briefly sketched here. We first specialize the geometric description presented in Subsection 2.1 by choosing conformal coordinates on Σ_ρ , see e.g. [11]. Denoting by E the coefficient of the first fundamental form on Σ_ρ , the local frame on the tangent plane to Σ_ρ is given by

$$e_i = \frac{1}{E} \frac{\partial M}{\partial s_i} \quad (2.5)$$

and there holds

$$\frac{\partial M}{\partial s_i} \cdot \frac{\partial M}{\partial s_j} = 0, \quad i, j, \quad \left| \frac{\partial M}{\partial s_i} \right| = E, \quad n = \bigwedge_i e_i. \quad (2.6)$$

The computation in (2.4) are split into three parts. The first part comes from the variation of the normal field, that is $\frac{1}{t}u_{\rho+th}(M_\rho + thn_\rho) \times (n_{\rho+th}(M_\rho + thn_\rho) - n_\rho(M_\rho))$, the limit of which as $t \rightarrow 0$ is simply

$$-u \cdot \nabla h \quad (2.7)$$

where the gradient operator ∇ is defined intrinsically on Σ_ρ , and is equal to $\frac{1}{E}(\partial_{s_1}, \partial_{s_2})$ in the chosen coordinates system (s_1, s_2) . The second term in (2.4) comes from the variation of the point at which the function u is evaluated

$$\frac{1}{t}(u_{\rho+th}(M_\rho + thn_\rho) - u_{\rho+th}(M_\rho)) \cdot n_\rho(M_\rho) \quad (2.8)$$

the limit of which is obviously

$$\frac{\partial u}{\partial n_\rho}(M_\rho) \cdot n_\rho. \quad (2.9)$$

Finally, the third term in (2.4) comes from the variation of u with respect to the domain: defining the local derivative of u with respect to normal domain variations as

$$v(M) = \lim_{t \rightarrow 0} \frac{u_{\rho+th}(M) - u_\rho(M)}{t} \quad (2.10)$$

for any point M lying in the interior of the domain Ω_ρ^γ , then the limit of the last term is

$$v(M_\rho).n_\rho. \quad (2.11)$$

Let us now characterize v : it is the solution of a boundary value problem akin to the Stokes system with non homogenous boundary conditions, the derivation of which follows intimately the lines of [5], Section 2.2.2. We state without proof the

Proposition 2.2 *The derivative with respect to normal variations of the domain of the solution u to the Auxiliary Problem (1.4) to (1.6) is given by (v, q) solution to the following linearized Stokes system*

$$-\operatorname{div} \tau(v) = 0 \text{ in } \Omega_\rho^\gamma, \operatorname{div} v = 0 \text{ in } \Omega_\rho^\gamma \quad (2.12)$$

$$v = 0 \text{ on } \Gamma_\gamma, \quad (2.13)$$

$$\tau(v).n_\rho = h \frac{\partial \sigma}{\partial n_\rho}.n_\rho + \sigma \nabla h \text{ on } \Sigma_\rho. \quad (2.14)$$

We summarize the results of this section in the

Proposition 2.3 *The derivative with respect to ρ of the operator G defined in (1.13) is given by*

$$\frac{DG}{D\rho}(\rho).h = -u.\nabla h + \frac{\partial u}{\partial n_\rho}(M_\rho).n_\rho h + v.n_\rho, \quad (2.15)$$

where v is characterized in Proposition 2.2.

Note that in Propositions 2.4 and 2.2 above, u is the solution to the original Stokes system (1.4) to (1.6) in Ω_ρ^γ , and σ is the associated stress tensor.

3 Inverting the linearized operator

In this section, the invertibility of $\frac{DG}{D\rho}(\rho).h$ is addressed.

We start with some remarks on the Poiseuille-Nusselt flow: from the expressions of u and σ in Section 1.1, one has that $\frac{\partial \sigma}{\partial n}.n.n > 0$ and $\sigma = 0$ on Σ_ρ in the particular case $\gamma = 0$. Moreover, the first-order differential

operator $u.\nabla h$ is non degenerate, since $u = \begin{pmatrix} C_1 \frac{1}{2} z_0^2 \\ 0 \\ 0 \end{pmatrix}$ on Σ_0 . Upon inspecting the terms involved in the

derivative of G in (2.15), it also appears clearly that the only non-trivial term is $v.n_\rho$.

In the rest of the article, we will denote by \mathbf{P} the operator $h \mapsto v.n_\rho$ where v is the solution of Equations (2.12) (2.13) (2.14). \mathbf{P} can be decomposed under the form $\mathbf{P} = \mathbf{PS} \circ D + \mathbf{PS} \circ M$, where

$$Dh = \begin{pmatrix} (h \frac{\partial \sigma}{\partial n_\rho}.n_\rho + \sigma \nabla h).t_1 \\ (h \frac{\partial \sigma}{\partial n_\rho}.n_\rho + \sigma \nabla h).t_2 \\ 0 \end{pmatrix} \quad (3.1)$$

is a first-order differential operator acting on function defined on Σ_ρ ,

$$Mh = h \begin{pmatrix} 0 \\ 0 \\ \frac{\partial \sigma}{\partial n_\rho}.n_\rho.n_\rho \end{pmatrix} \quad (3.2)$$

is the normal component of the RHS in the linearized problem, and \mathbf{PS} is a Poincare-Steklov-type operator

$$\mathbf{PS}f = w.n_\rho \quad (3.3)$$

for the Stokes system with Neuman boundary conditions on the candidate free surface

$$-\operatorname{div} \tau(w) = 0 \text{ in } \Omega_\rho^\gamma, \operatorname{div} w = 0 \text{ in } \Omega_\rho^\gamma \quad (3.4)$$

$$w = 0 \text{ on } \Gamma_\gamma, \quad (3.5)$$

$$\tau(w) \cdot n_\rho = f \text{ on } \Sigma_\rho. \quad (3.6)$$

In other words, one can rewrite $\mathbf{P} = \mathbf{P}_{orth} + \mathbf{P}_{tan}$ as the sum of two operators, \mathbf{P}_{orth} (resp. \mathbf{P}_{tan}) mapping the normal (resp. tangential) stress to the normal component of the velocity field.

Regarding \mathbf{P}_{orth} , the following result holds true.

Proposition 3.1 *For any $s \geq 0$ and $\delta \leq \delta_{01}$, \mathbf{P}_{orth} is a continuous linear operator from $H_\delta^s(\mathbf{R}^2)$ into $H_\delta^{s+1}(\mathbf{R}^2)$. Moreover, \mathbf{P}_{orth} is a pseudodifferential operator of order -1 whose principal symbol is given by*

$$p(\xi) = \frac{\frac{\partial \sigma}{\partial n_\rho} \cdot n_\rho \cdot n_\rho}{|\xi|}.$$

Proof: the fact the \mathbf{P}_{orth} is a pseudodifferential operator is straightforward, as a composition of two such operators. Its principal symbol is simply the product of that of the Poincaré-Steklov operator \mathbf{PS} with the function $\frac{\partial \sigma}{\partial n_\rho} \cdot n_\rho \cdot n_\rho$. In order to compute the principal symbol - defined unambiguously in any system of local coordinates - of \mathbf{PS} , we use again conformal coordinates on Σ_ρ . Upon referring once more to [5], in particular Section 3.2.3, one is easily convinced that it is sufficient to freeze the coefficients of the equation at $\lambda = 0$, and therefore to focus on the simpler operator $h \rightarrow V_3(s_1, s_2, 0)$ where V is the solution to:

$$-\frac{1}{2} \Delta V + \nabla Q = 0 \text{ in } \mathbf{R}^2 \times \mathbf{R}_*^-, \operatorname{div} V = 0 \text{ in } \mathbf{R}^2 \times \mathbf{R}_*^-, \lim_{\lambda \rightarrow -\infty} V = 0, \left(\begin{array}{c} \frac{1}{2} \left(\frac{\partial V_1}{\partial \lambda} + \frac{\partial V_3}{\partial s_1} \right) \\ \frac{1}{2} \left(\frac{\partial V_2}{\partial \lambda} + \frac{\partial V_3}{\partial s_2} \right) \\ \frac{\partial V_3}{\partial \lambda} - Q \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \\ h \end{array} \right) \text{ on } \lambda = 0. \quad (3.7)$$

Upon using the Fourier transform in the s variables, the following system obtains

$$\frac{1}{2} \left(-\frac{\partial^2 \hat{V}_1}{\partial \lambda^2} + |\xi|^2 \hat{V}_1 \right) + i \xi_1 \hat{Q} = 0 \quad (3.8)$$

$$\frac{1}{2} \left(-\frac{\partial^2 \hat{V}_2}{\partial \lambda^2} + |\xi|^2 \hat{V}_2 \right) + i \xi_2 \hat{Q} = 0 \quad (3.9)$$

$$\frac{1}{2} \left(-\frac{\partial^2 \hat{V}_3}{\partial \lambda^2} + |\xi|^2 \hat{V}_3 \right) + \frac{\partial \hat{Q}}{\partial \lambda} = 0 \quad (3.10)$$

$$i \xi_1 \hat{V}_1 + i \xi_1 \hat{V}_1 + \frac{\partial \hat{V}_3}{\partial \lambda} = 0. \quad (3.11)$$

Then, solving for \hat{Q} and combining the equations above, one immediately obtains that

$$\hat{Q}(\xi_1, \xi_2 \lambda) = \frac{1}{2|\xi|^2} \frac{\partial}{\partial \lambda} \left[\frac{\partial^2}{\partial \lambda^2} - |\xi|^2 \right] \hat{V}_3 \quad (3.12)$$

and therefore

$$\left[\frac{\partial^2}{\partial \lambda^2} - |\xi|^2 \right] \frac{\partial^2 \hat{V}_3}{\partial \lambda^2} = 2|\xi|^2 \frac{\partial \hat{Q}}{\partial \lambda}. \quad (3.13)$$

On the other hand, there also holds

$$-|\xi|^2 \left[\frac{\partial^2}{\partial \lambda^2} - |\xi|^2 \right] \hat{V}_3 = -2|\xi|^2 \frac{\partial \hat{Q}}{\partial \lambda}, \quad (3.14)$$

so that we finally obtain the following ODE for \hat{V}_3 :

$$\left[\frac{\partial^2}{\partial \lambda^2} - |\xi|^2\right]^2 \hat{V}_3 = 0. \quad (3.15)$$

Hence, \hat{V}_3 has the following representation

$$\hat{V}_3(\xi, \lambda) = (A + B\lambda) \exp(-\lambda|\xi|) + (C + D\lambda) \exp(\lambda|\xi|) \quad (3.16)$$

with $A = B = 0$ because of boundedness conditions at $\lambda \rightarrow -\infty$. In particular, there holds $\hat{V}_3(\xi, 0) = C$, and we need to compute the value of C . From (3.12), one easily derives that $\hat{Q}(\xi, \lambda) = D \exp(\lambda|\xi|)$. The third coordinate of the stress tensor on $\lambda = 0$ is given by $\frac{\partial \hat{V}_3(\xi, 0)}{\partial \lambda} - Q(\xi, 0)$, which simplifies to $C|\xi|$. Using the boundary conditions for $\tau(V, Q)_3$, it follows that

$$C = \frac{\widehat{U_1 h}}{|\xi|}, \quad (3.17)$$

and the proposition is proven.

Next, we state and prove the following important lemma:

Lemma 3.2 *\mathbf{P}_{orth} is a coercive operator from $H^{-\frac{1}{2}}(\mathbf{R}^2)$ into $H^{\frac{1}{2}}(\mathbf{R}^2)$. Therefore, there exists a constant $C > 0$ such that, for all $h \in H^{-\frac{1}{2}}(\mathbf{R}^2)$, there holds:*

$$(\mathbf{P}_{orth} h, h) \geq C \|h\|^2$$

Proof : a straightforward consequence of Green's formula and Korn's inequality. In fact, setting $v.n = \mathbf{P}Sh$ and writing for a test function w

$$\int_{\Omega_\rho} \operatorname{div} \tau(v).w + \frac{1}{2} \int_{\Omega_\rho} (\nabla v + {}^t \nabla v) : (\nabla w + {}^t \nabla w) = \int_{\Sigma_\rho} (\tau(v).n)w, \text{ one obtains, using } v \text{ itself as a test function}$$

$$\frac{1}{2} \int_{\Omega_\rho} |\nabla v + {}^t \nabla v|^2 = \int_{\Sigma_\rho} (\tau(v).n.n)(w.n), \text{ that is:}$$

$$(\mathbf{P}Sh, h) = \frac{1}{2} \int_{\Omega_\rho} |\nabla v + {}^t \nabla v|^2$$

and thanks to Korn's inequality $(\mathbf{P}Sh, h) \geq C \|v\|_{H^1}^2$. This inequality shows that $\mathbf{P}S$ is a coercive operator, a result easily extended to \mathbf{P}_{orth} because - see Section 1.1 - the multiplier $\frac{\partial \sigma}{\partial n_\rho}.n_\rho.n_\rho$ is a strictly positive function for the unperturbed free surface and therefore, stays so whenever ρ is in a small enough neighbourhood of the origin. The lemma then results from a trace theorem, the proof of which is similar to that provided in [20] and is therefore omitted:

Lemma 3.3 *Let Ω_ρ be an open set in \mathbf{R}^3 with a C^2 boundary. Then for $v \in H^1(\Omega)$ such that $\tau.n$ is parallel to n and $\operatorname{div} \tau(v) = 0$, one can define a normal stress operator $v \rightarrow \tau.n.n \in H^{\frac{-1}{2}}(\partial\Omega_\rho)$ satisfying the following continuity property:*

$$\|v\|_{H^1(\Omega_\rho)} \geq C \|\tau.n.n\|_{H^{\frac{-1}{2}}(\partial\Omega_\rho)}. \text{ This operator coincides with the usual normal stress operator when } v \text{ is } C^1 \text{ up to the boundary of } \Omega_\rho$$

Upon using this result, one then has $(\mathbf{P}_{orth} h, h) \geq C \|v\|_{H^1(\Omega_\rho)}^2 \geq C \|\tau.n.n\|_{H^{\frac{-1}{2}}(\partial\Omega_\rho)}^2$, which ends the proof of Lemma 3.2.

Regarding the second term in the decomposition of \mathbf{P} , namely, \mathbf{P}_{tan} , the truly relevant point is that \mathbf{P}_{tan} is an operator of order 0, with a "small" principal symbol.

We now prove the invertibility of the linearized operator defined in 2.15, essentially rephrasing a more general result due to the authors [3]

Theorem 3.4 (Abergel, Bailly) *Let V be a smooth real-valued vector field in \mathbf{R}^2 , g a smooth real-valued function, and consider the following problem*

$$- \langle V, \nabla h \rangle + L_0(\rho)h + M_{-1,0}h = f \quad (3.18)$$

where

- $V=(V_1, V_2)$
- *there exists a non-zero constant vector field C such that $\|V - C\|_{W^{1,\infty}} \ll 1$*
- L_0 *is a bounded operator of order zero with symbol σ_{L_0} such that $\|\sigma_{L_0}\|_L \ll 1$*
- $\nabla V, \sigma_{L_0}$ *are rapidly decreasing at infinity.*

We assume furthermore that $M_{-1,0}$ is a pseudodifferential operator of order -1 with principal symbol

$$\sigma_{P_{-1}}(x, \xi) = \frac{m(x)}{|\xi|}, \quad (3.19)$$

$m(x)$ being such that $0 < m \leq m(x) \leq M$, and that the following coercivity inequality

$$(M_{-1,0}h, h) \geq C\|\rho\|_{H^{-\frac{1}{2}}(\mathbf{R}^2)} \quad (3.20)$$

holds. Then, for all $t > 0$ and $f \in H^{t+\frac{1}{2}}(\mathbf{R}^2)$, there exists a unique solution $h \in H^{t-\frac{1}{2}}(\mathbf{R}^2)$ to (3.4), and the solution h depends continuously on the data f .

Let us check that the conditions of Theorem 3.4 hold true for the linearized operator $\frac{DG}{D\rho}$. The smoothness of V and g is obvious, because they correspond to smooth data (f, ρ) , and (V_1, V_2) is clearly close to the constant vector field $(C_1 \frac{z_0^2}{2}, 0)$ - the boundary value of the unperturbed Poiseuille-Nusselt velocity field in Ω_0^0 - for (ρ, γ) small enough. The second condition bears on the operator of order 0, L_0 , which is the sum of two parts: one coming from the term $h \mapsto \frac{\partial U_\rho}{\partial n} \cdot n_\rho h$, another coming from the operator $h \mapsto \mathbf{P}_{tan}h$ introduced above. Hence, the principal symbol of L_0 consists in the product of homogenous functions of order 0 in the frequency variable multiplied by $\frac{\partial U_\rho}{\partial n} \cdot n_\rho$, $\frac{\partial \sigma_\rho}{\partial n} \cdot n_\rho \cdot t_i$, $i = 1, 2$ and $\sigma_\rho \cdot t_i \cdot t_j$, and therefore, is a small, rapidly decaying function (see the remarks at the beginning of Section 1.1). Finally, the third term is precisely the operator \mathbf{P}_{orth} , which has just been proven to be coercive.

Now, upon examining the hypotheses of Proposition 1.1 in [3], one can see that there is a slight difference - \mathbf{P}_{orth} is not self-adjoint - but the solution to the equation $\mathbf{P}_{orth}h = f$ is clearly given by $h = \frac{1}{\frac{\partial \sigma}{\partial n_\rho} \cdot n_\rho \cdot n_\rho} \mathbf{P}\mathbf{S}^{-1}h$ and therefore, \mathbf{P}_{orth} is invertible, clearly a sufficient condition in the light of the proof of Proposition 1.1 in [3]. Therefore, so is the linearized operator \mathbf{P} in 2.15 in the usual Sobolev spaces.

A straightforward extension of the proof in [3] also proves the

Corollary 3.5 *Under the same set of assumptions as above, there exists $\delta_{02} > 0$ such that for $\delta \leq \delta_{02}$ and for all $f \in H_\delta^{s+\frac{1}{2}}(\mathbf{R}^2)$, there exists a unique $h \in H_\delta^{s-\frac{1}{2}}(\mathbf{R}^2)$ solution to (3.4).*

Proof: one uses the changes of RHS and unknown function $f(x) = f_0(x) \exp(-\delta|x|)$, $h(x) = h_0(x) \exp(-\delta|x|)$ to obtain a modified equation:

$$- \langle V, \nabla h_0 \rangle + L'_0 h_0 + \mathbf{P}_{orth} h_0 + \exp(\delta|x|) [\mathbf{P}_{orth}, \exp(-\delta|x|)] h_0 = f_0 \quad (3.21)$$

where $L'_0 = L_0 - \langle V, \nabla(\exp(-\delta|x|)) \rangle$ and $[\cdot, \cdot]$ stands for the commutator of two operators. Regarding L'_0 , it can clearly be considered a small perturbation of L_0 for δ small enough. As for $\exp(\delta|x|) [\mathbf{P}_{orth}, \exp(-\delta|x|)]$, it is an operator of order -2 that has a small norm when δ is small enough, so that the inverse of the perturbed operator $\mathbf{P}_{orth} h_0 + \exp(\delta|x|) [\mathbf{P}_{orth}, \exp(-\delta|x|)]$ still exists. This proves Corollary 3.5.

4 Proof of the main result

This final section is devoted to the proof of Theorem 1.2.

4.1 Zehnder's Implicit Function Theorem

We start by presenting the machinery fully developed in [22]. Let then $\{X_t\}_{t \geq 0}$ be a one-parameter family of Banach spaces, with norm denoted by $|\cdot|_t$, such that for all t, t' with $0 \leq t \leq t' \leq +\infty$, there holds:

$$X_0 \supseteq X_{t'} \supseteq X_t \supseteq X_\infty \equiv \bigcap_{t \geq 0} X_t \quad (4.1)$$

and

$$\forall t, t', t' \leq t, |z|_{t'} \leq |z|_t. \quad (4.2)$$

We will work with three such scales, respectively denoted by $\{X_t\}_{t \geq 0}$, $\{Y_t\}_{t \geq 0}$ and $\{Z_t\}_{t \geq 0}$. The mapping G in 1.13 will be defined from $D \subset Y_0 \times X_0$ into $R \subset Z_0$. Since $G(0, 0) = 0$, our purpose is to prove that, for γ close enough to 0, there exists a $\rho(\gamma)$ in a suitable function space such that $G(\gamma, \rho(\gamma)) = 0$.

For the sake of clarity, we introduce the

Definition 4.1.1

$$B_t^\epsilon = \{(f, \rho) \in Y_t \times X_t : |f| < \epsilon, |\rho| < \epsilon\}. \quad (4.3)$$

$\epsilon > 0$ will be determined later. Below are listed the four hypotheses that have to be satisfied by G .

Hypothesis 1 (smoothness w.r. to ρ)

$G : B_0^\epsilon \rightarrow Z_0$ is at least C^2 with respect to ρ and C^0 with respect to (f, ρ) , and there exists a constant M_0 such that

$$\forall (f, \rho) \in B_0^\epsilon, \sup(|D_\rho G(f, \rho)|_0, |D_\rho^2 G(f, \rho)|_0) \leq M_0. \quad (4.4)$$

Hypothesis 2 (uniform lipschitz estimates w.r. f)

For all couple $(f, \rho), (f', \rho)$ in B_0^ϵ , there holds

$$|G(f, \rho) - G(f', \rho)|_0 \leq M_0 |f - f'|_0. \quad (4.5)$$

Hypothesis 3

$(G, 0, 0)$ is of order s , $s > \nu \geq 1$ (see Hypothesis 4 below for the definition of ν), that is

$$\forall t \in [1, s], G(B_0^\epsilon \cap (Y_t \times X_t)) \subset Z_t \quad (4.6)$$

and

$$\forall t \in [1, s], \exists M_t > 0, \exists K_0 > 0, \forall K, 0 < K < K_0, \forall (f, \rho) \in B_t^K \cap B_1^\epsilon, |G(f, \rho)|_t \leq M_t K. \quad (4.7)$$

Hypothesis 4 (approximate inverse with loss of ν derivatives)

For all (f, ρ) in B_ν^ϵ , there exists a linear mapping $H(f, \rho) : Z_\nu \rightarrow X_0$ such that

$$\forall \phi \in Z_\nu, |H(f, \rho)\phi|_0 \leq M_0 |\phi|_\nu. \quad (4.8)$$

$H(f, \rho)$ is continuous from Z_t into $X_{t-\nu}$ whenever $(f, \rho) \in B_\nu^\epsilon \cap (Y_t \times X_t)$ and there holds

$$\forall (f, \rho) \in B_t^K, |H(f, \rho)G(f, \rho)|_{t-\nu} \leq M_t K. \quad (4.9)$$

Moreover, H is an approximate left inverse of $D_\rho G(f, \rho)$ in the sense that

$$\forall \phi \in Z_\nu, |[D_\rho G(f, \rho) \circ H(f, \rho) - Id]\phi|_0 \leq N_0 |F(f, \rho)|_\nu |\phi|_\nu. \quad (4.10)$$

When all four hypotheses above are satisfied, the following deep result applies:

Theorem 4.1 (Zehnder [22]) *Let (G, f_0, ρ_0) be of order s and satisfying Hypotheses 1 to 4 above. Then there exists a neighbourhood $D_\lambda = \{f \in Y_\lambda, |f| < C\}$ with μ, λ in the admissible parameter set defined as*

- $1 < \kappa < 2$
- $1 < \alpha$
- $1 \leq \nu \leq \mu < \lambda < s$
- $\lambda > \max\{2\kappa\nu(2 - \kappa)^{-1}, \kappa(\nu + \kappa\mu)\}$
- $s > \max\{\alpha\nu(\alpha - 1)^{-1}, \lambda + \alpha\nu(\kappa - 1)^{-1}\}$

and a mapping $\Psi : D_\lambda \rightarrow X_\mu$ such that for all $f \in D_\lambda$ there exists a unique $\rho_f \equiv \Psi(f)$ solution to

$$G(f, \rho_f) = 0 \quad (4.11)$$

with $|\rho_f|_\nu < C^{-1}|f|_\lambda$. Furthermore, Ψ is continuous whenever H is.

In order to apply this theorem to the proof of Theorem 1.2, we recast the weighted Sobolev spaces defined in Section 3 into this formalism

Definition 4.1.2 For all $t \in [0, +\infty[$ and $\delta > 0$, we se

$$X_t = H_\delta^{4+t}(\mathbf{R}^2)$$

$$Y_t = H_\delta^{4+t}(\mathbf{R}^2) \times H_\delta^{2+t}(\mathbf{R}^2)$$

$$Z_t = H_\delta^{\frac{5}{2}+t}(\mathbf{R})$$

and their respective intersections

$$X_\infty = S(\mathbf{R}^2)$$

$$Y_\infty = S(\mathbf{R}^2) \times S(\mathbf{R}^2)$$

$$Z_\infty = S(\mathbf{R})$$

and proceed to check all four hypotheses of Theorem 4.1.

Let us fix $\delta < \max(\delta_{01}, \delta_{02})$, δ_{01} (resp. δ_{02}) being introduced in Proposition 2.1 (resp. Corollary 3.5). We also pick s (the order parameter in Hypothesis 3) equal to $+\infty$, meaning by that that we are interested in smooth solutions only. Hypotheses 1 and 2 are easy consequences of Proposition 2.1, or, more precisely, an extension thereof so as to include the fixed boundary γ as well as the candidate free surface ρ in the differentiability. The proof relies again on a change of variables and results on the smooth dependence of the solution to the Auxiliary System w. r. to its coefficients, and will be omitted.

As for Hypothesis 3, we remark that $(0, 0) \in U_\infty \times X_\infty$ and that we already established as a general result that $G(f, \rho) \in Z_t$ if $(f, \rho) \in Y_t \times X_t$ for $t \in [1, +\infty[$. The last point to check is that for all $t \in [1, +\infty[$, there exists an M_t such that $|G(f, \rho)|_t \leq M_t K$ whenever $|f - f_0|_t < K$, $|\rho - \rho_0|_t < K$ in $(Y_t \times X_t) \cap B_1$ (here, we have chosen $K < \epsilon$). Such an inequality follows from computations similar to those involved in the proof of Hypothesis 2, albeit with respect to both f and ρ this time, the key point being (see the remark at the bottom of p. 120 in [22]) that the control of a low order norm is sufficient to ensure the linear growth of higher order norms for the operator G . As a consequence, one can even choose $M_t = M_0$ for all t , a remark that will be essential in verifying Hypothesis 4. Regarding the former, we state and prove the slightly more precise

Proposition 4.2 For all pair $(f, \rho) \in Y_\infty \times X_\infty$ there exists a linear mapping $H : Z_\gamma \rightarrow X_0$ with $\gamma = \frac{5}{2}$ such that $|H(f, \rho)|_0 \leq M_0 |\phi|_{Z_\gamma}$.

When $(f, \rho) \in Y_\infty \times X_\infty$, H is continuous from Z_t into $X_{t-\gamma}$ for all $t \geq 0$.

Moreover, the inequality

$$|H(f, \rho)|_{t-\gamma} \leq M_t K \quad (4.12)$$

holds true for all $(f, \rho) \in (Y_t \times X_t) \cup B_1$ with $|f - f_0|_t < K$, $|\rho - \rho_0|_t < K$. Finally, $H(f, \rho)$ is an exact inverse to $D_\rho G(f, \rho)$, i.e.

$$\forall \phi \in Z_{t+\gamma}, [D_\rho G(f, \rho)(f, \rho) - Id] \phi = 0. \quad (4.13)$$

Proof: we need to show the existence, for (f, ρ) given in $Y_\infty \times X_\infty$ and for any $F \in H_\delta^{t+\frac{1}{2}}(\mathbf{R}^2)$, of a unique $h \in H_\delta^{t-\frac{1}{2}}(\mathbf{R}^2)$ such that $D_\rho G(f, \rho)h = F$. In Theorem 3.4 and corollary 3.5, we have already established the existence of an exact inverse $H : Z_\gamma \rightarrow X_0$ to $D_\rho G$ for $\gamma = \frac{5}{2}$, H being a continuous operator from Z_t into $X_{t-\gamma}$ for all $t \geq 0$, when $(f, \rho) \in Y_\infty \times X_\infty$. There still remains to prove that $|H(f, \rho)[G(f, \rho)]|_{t-\gamma} \leq M_t K$ whenever $(f, \rho) \in (Y_t \times X_t) \cap B_1$ with $|f - f_0|_t < K$, $|\rho - \rho_0|_t < K$. Now, this is a consequence of the continuity of H : $|H(f, \rho)\phi| \leq M_t |\phi|_{t+\gamma}$ and the uniform - in t - estimate on G , see Hypothesis 3: $|G(f, \rho)\phi|_t \leq M_0 K$. The proof of Theorem 1.2 is almost over: we must still prove that Hypothesis 4 can be replaced by Proposition 4.2. In the original proof by Zehnder ([22], p. 118 to 125), the use of an approximate inverse is made after the smoothing procedure for (f, ρ) - the proof itself relies on a Newton iterative method. Hence, the estimate 4.10 in Hypothesis 4 $||[D_\rho G(f, \rho) \circ H(f, \rho) - Id]\phi|_0 \leq N_0 |F(f, \rho)|_\gamma |\phi_\gamma|$ is only required for the regularized approximating sequence (f_τ^n, ρ_τ^n) , τ being the smoothing factor. Since this sequence is made of smooth functions, Proposition 4.2 above applies, and actually yields a stronger result than the one required in the original proof by Zehnder : there, the constant N_0 has to be uniform with respect to the smoothing parameter τ , a condition that becomes irrelevant thanks to the exact invertibility of $D_\rho G(f, \rho)$. This concludes the proof of Theorem 1.2.

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