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► **To cite this version:**

Frédéric Abergel, Jacques-Herbert Bailly. Stationary free surface viscous flows without surface tension in three dimensions. 1999. hal-00621191v1

**HAL Id: hal-00621191**

**<https://hal.science/hal-00621191v1>**

Preprint submitted on 9 Sep 2011 (v1), last revised 14 Jan 2012 (v7)

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# Stationary free surface viscous flows without surface tension in three dimensions

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July 29, 2011

## Abstract

We consider an incompressible, viscous free surface flow down a three dimensional channel. In the absence of surface tension, we prove the existence of a unique stationary flow in weighted Sobolev spaces. The proof relies on a careful study of a particular pseudodifferential operator and the use of a Nash-Moser implicit function theorem.

## Introduction

We consider an incompressible viscous free surface fluid flowing down a three dimensional inclined channel. In the case where the bottom of the channel is flat, there exists a well-known stationary solution, usually referred to as the Poiseuille-Nusselt flow[4]. However, the question of finding such a stationary solution for a general profile of the bottom is not easy to address, as the linearized operator in the neighbourhood of the Poiseuille-Nusselt solution is not elliptic, and results such as those obtained by [4] or [14] do not apply in this situation. As a matter of fact, it has been proven, see e.g. [1] that the surface tension coefficient, when it exists, plays a particularly important role in the structure of the linearized solution operator, transforming it into an elliptic second order pseudodifferential operator. In the general three dimensional case when surface tension is neglected, the linearized free surface problem is no longer elliptic, and one has to carefully analyze the structure of the associated operator in order to study its invertibility. Moreover, even though this linearized operator may be invertible, there is a loss of smoothness when inverting it, which makes the analysis even more complicated.

## 1 Formulation of the problem

### 1.1 Governing equations

Let  $\Omega_\rho^\gamma$  be an unbounded domain of  $\mathbb{R}^3$ , with  $\delta\Omega = \Gamma_\gamma \cup \Sigma_\rho$ ,  $\Gamma_\gamma$  being the fixed bottom of the channel. More precisely we define:

$$\Omega_\rho^\gamma = \{(x, y, z) \in \mathbb{R}^3, -\infty < x < +\infty, -\infty < y < +\infty, \gamma(x, y) < z < z_0 + \rho(x, y)\}$$

where  $\rho$  is a mapping from  $\mathbb{R}^2$  into  $\mathbb{R}$  describing the free surface, while  $z_0$  is the height of the free surface when  $x, y$  goes to  $\infty$ . Similarly,  $\gamma$  is a mapping from  $\mathbb{R}^2$  into  $\mathbb{R}$  representing the fixed bottom.

We now let  $u$  be the velocity field and  $p$  be the pressure in the classical Stokes equations. We denote by

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$\sigma(u) = \frac{1}{2}(\nabla(u) + {}^t \nabla(u)) - pId$  the stress tensor, by  $f$  the gravitational field with coordinates  $\begin{pmatrix} C_1 \\ 0 \\ C_2 \end{pmatrix}$  with  $C_1 = g \sin \alpha$  and  $C_2 = g \cos \alpha$ ,  $\alpha$  being the angle shown on Figure 1.1 and  $g$ , the gravity constant.

In this paper, we shall prove the existence and uniqueness of a triple  $(u, p, \Sigma_\rho)$  solution of the following system of equations:

$$-\operatorname{div} \sigma(u) = f \text{ in } \Omega_\rho^\gamma, \operatorname{div} u = 0 \text{ in } \Omega_\rho^\gamma, u = \zeta \text{ on } \Gamma_\gamma, \sigma(u).n = C.n \text{ on } \Sigma_\rho \quad (1.1)$$

with the conditions at infinity:

$$\lim_{(x,y) \rightarrow +\infty} u(x, y, z) = \begin{pmatrix} C_1(z_0 z - \frac{1}{2} z^2) \\ 0 \\ 0 \end{pmatrix} \quad (1.2)$$

and the main kinematic equation for the free surface itself, which reads in the stationary case:

$$u.n = 0 \quad (1.3)$$

Equations (1.1)(1.2) will be referred to as the Auxiliary problem, while (1.3) will be called the Main Equation. Note that the constant  $C$  involved in the normal stress condition can be set equal to 0 by subtracting off a constant pressure field, and we will assume so in the sequel. The function  $g$  is 0 in the case of the no-slip boundary condition, but can be chosen as a small - in a sense to be made precise later - rapidly decaying function in  $S(\mathbb{R}^2)$ .

## 1.2 Function spaces

We now introduce the family of Banach spaces to which belong the data and unknowns of our problem. For  $\delta = (\delta_1, \delta_2)$  in  $\mathbb{R}_+^2$  and  $s > 0$ , we set:

$$H_\delta^s(\mathbb{R}^3) = \{f \in H^s(\mathbb{R}^3) / \|f(X, z) \exp(\delta_1|x| + \delta_2|y|)\|_{H^s(\mathbb{R}^3)} \leq +\infty\} \quad (1.4)$$

where  $X = (x, y) \in \mathbb{R}^2$  and  $H^s(\mathbb{R}^3)$  is the usual Sobolev space. Similarly, we define:

$$C_\delta^s(\mathbb{R}^3) = \{f \in C^s(\mathbb{R}^3) / \sum_{|\alpha| \leq s} \operatorname{Sup}(|D^\alpha f|) \exp(\delta_1|x| + \delta_2|y|) + \sum_{|\alpha|=s} \operatorname{Sup}(|D^\alpha f|) \leq +\infty\} \quad (1.5)$$

where  $[s]$  is the integer part of  $s$  and  $[f] = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{s - [s]}}$ .

We recall for further use a classical embedding theorem:

**Proposition 1.1** *Let  $f \in H_\delta^s(\mathbb{R}^3)$ , and assume that, for  $m \in \mathbb{N}$ ,  $\alpha \in ]0, 1[$ , there holds:  $s - \frac{3}{2} + \alpha$  and  $2s > n$ , then  $f \in C_\delta^{m+\alpha}(\mathbb{R}^3)$  and there exists a constant  $C > 0$  such that:  $\|f\|_{C_\delta^{m+\alpha}} \leq C \|f\|_{H_\delta^s}$*

### 1.3 The main result

We introduce the following notation : for  $X$  a Banach space, we denote by  $B^\epsilon(X)$  the ball of radius  $\epsilon$  centered at the origin in  $X$ . Our main result can now be stated:

**Theorem 1.2** *There exists  $\delta_0$  small enough such that, for  $\delta \leq \delta_0$  and  $\epsilon$  small enough,  $f$  and  $\zeta$  as in (1.1) with  $\zeta \in B^\epsilon(H_\delta^8(\Gamma))$  and  $\Gamma \in B^\epsilon(H_\delta^{10}(\mathbf{R}^2))$ , there exists a unique triple  $(u, p, \Sigma)$  solution of (1.1)(1.2)(1.3), with  $\Sigma \in H_\delta^5(\mathbf{R}^2)$ ,  $(u, \nabla p) \in H_\delta^3(\Omega)_\delta^1(\Omega)$ .*

*Moreover, the solution depends continuously on the data:*

$$\|u\|_{H_\delta^3} + \|\nabla p\|_{H_\delta^1} \leq C(\|f\| + \|\zeta\|).$$

The proof of 1.2 is quite lengthy, and will be presented in the rest of this article. It relies on a thorough study of the linearization of (1.1)(1.2)(1.3), as can be expected from our formulation as a small perturbation problem. However, and this is where the technical difficulty of this result lies, it is impossible to use a standard Implicit Function Theorem: when considering the nonlinear mapping  $G$  formally defined by

$$G(\Sigma) = u|_\Sigma \cdot n \tag{1.6}$$

we will see that  $DG(\Sigma)^{-1} \circ DG(\Sigma)$  is an unbounded operator. We therefore have to invoke, and verify the cumbersome assumptions of, a Nash-Moser type iterative scheme in order to overcome this loss of regularity, the most difficult task in this program being the study of the invertibility of the linearized operator  $DG(\Sigma)$  in a full neighbourhood of the Poiseuille-Nusselt flow - and not only at this particular solution.

In Section2, we shall define the appropriate formalism in order to study the operator  $G$ . In particular, we will explicitly compute its derivative around the Poiseuille-Nusselt flow. This linearized operator has an unusual and interesting characterization as a first-order transport like- operator with a nonlocal term of order  $-1$  of the type studied in [3], and this nonlocal term will play a major role in the invertibility of the derivative of  $G$ . Section3 contains an exposition of the heavy machinery involved in the Nash-Moser Theorem.

## 2 Kinematic condition

Let us recall a standard parametric representation one can use in order to describe a family of free surfaces  $\Sigma_\rho$  as a graph along the normal above a reference manifold, see eg[5][6] for a detailed exposition.

Let then  $\bar{\Sigma}$  be a 2-dimensional reference manifold, denote by  $\bar{M}(s)$  a generic point on  $\bar{\Sigma}$  and by  $\bar{n}(s)$  the unit outward normal vector to  $\bar{\Sigma}(s)$ . We will use the notation  $\partial_{s_k}$  for the generic tangential derivative operator with respect to the  $k^{th}$  local coordinate on  $\bar{\Sigma}$ .

For a small enough neighbourhood of  $\bar{\Sigma}$  there exists a system of local coordinates and a unique representation, for any point  $M$  in this neighbourhood

$$M(s, d) = \bar{M}(s) + d\bar{n}(s) \tag{2.1}$$

and this representation defines a  $C^\infty$ -diffeomorphism from  $\bar{\Sigma} \times ]-L_0, L_0[$  onto the  $L_0$ -neighbourhood of  $\bar{\Sigma}$ , for  $L_0$  small enough (the size of  $L_0$  depending only on the maximum principal curvature of  $\bar{\Sigma}$ ). Hence, by choosing  $\rho \in H_\delta^5(\bar{\Sigma}; \mathbf{R})$  with a small enough norm, one can represent any manifold close enough to  $\bar{\Sigma}$  using the parameterization just defined:

$$\Sigma_\rho = \bar{\Sigma} + \rho\bar{n} \tag{2.2}$$

or equivalently:

$$\Sigma_\rho = \{M_\rho = \bar{M} + \rho(s)\bar{n}, \bar{M} \in \bar{\Sigma}\} \tag{2.3}$$

For instance, one can trivially represent any point in the three dimensional space over the  $(x, y)$  plane corresponding to the unperturbed Poiseuille-Nusselt flow using its tangential and normal coordinates, and

this property still holds true locally for small enough perturbations of this plane. We can now define precisely the nonlinear operator  $G$  we are interested in:

**Definition 2.0.1** *Let  $G$  be the normal velocity operator*

$$G: \begin{array}{l} S(\mathbf{R}^2) \rightarrow S(\mathbf{R}^2) \\ \rho \mapsto u|_{\Sigma_\rho} \cdot n_\rho \end{array}. \quad (2.4)$$

$G$  well defined in a neighbourhood of the origin in  $H_\delta^s(\bar{\Sigma}; \mathbf{R})$ .

We first state the

**Proposition 2.1** *There exists  $\delta_{01} > 0$  such that, for  $\delta \leq \delta_{01}$ , there exists  $\epsilon > 0$  such that, for every  $s \geq 4$  and  $(\rho, \gamma) \in B^\epsilon(H_\delta^s(\mathbf{R}^2))^2$ , the operator  $G$  defined in 2.0.1 is at least of class  $C^2$  from  $B^\epsilon(H_\delta^s(\mathbf{R}^2))$  into  $B^\epsilon(H_\delta^{s-\frac{3}{2}}(\mathbf{R}))$ .*

The proof of proposition 2.1 is rather classical and rely on usual differentiability arguments for the solution of a well-posed boundary value problem w.r. to the coefficients of the equation. One uses as in [10][3][5] a change of variables - the Hanzawa transform - shrinking  $\Omega_\rho^\gamma$  to the domain corresponding to the Poiseuille-Nusselt flow,  $\Omega_0$  say. More specifically, let  $\zeta \in C^\infty(\mathbf{R})$  be a cut-off function such that for every  $\rho \in H_\delta^s(\Sigma_0)$  and  $\gamma \in H_\delta^s(\Sigma_0)$ , the mapping  $Y_{\rho, \gamma}$  defined by

$$M(x) \mapsto M - [\zeta(\frac{D}{L_0})\rho(S(x)) + \zeta(\frac{D+z_0}{L_0})\gamma(S(x))]n_0(S(x)) \quad (2.5)$$

for  $\rho, \gamma$  small enough is a  $C^2$  diffeomorphism from  $\mathbf{R}^3$  into itself. One can then set

$$M(x) = M_0(S(x)) + D(x)n_0(S(x)) \quad (2.6)$$

leading to

$$Y_{\rho, \gamma}(M(x)) = M_0(S(x)) + [D(x) - \zeta(\frac{D}{L_0})\rho(S(x)) + \zeta(\frac{D+z_0}{L_0})\gamma(S(x))]n_0(S(x)) \quad (2.7)$$

One easily notices that  $Y_{\rho, \gamma}$  maps  $\Omega_\rho^\gamma$  into  $\Omega_0$ ,  $\Sigma_\rho$  into  $\Sigma_0$  and  $\Gamma_\rho$  into  $\Gamma_0$ .

We now consider the change of unknown function:

$$v(\hat{x}) = u(Y_{\rho, \gamma}^{-1}(\hat{x})), q(\hat{x}) = p(Y_{\rho, \gamma}^{-1}(\hat{x})), F(\hat{x}) = f(Y_{\rho, \gamma}^{-1}(\hat{x})) \quad (2.8)$$

and

$$\zeta(\hat{x}) = g(Y_{\rho, \gamma}^{-1}(\hat{x})) \quad (2.9)$$

for  $\hat{x} \in \Omega_0$ . After performing this change of variable, the auxiliary problem(1.1)(1.2) becomes:

$$A(\rho, \gamma)(v, q) = F \text{ in } \Omega_0, B(\rho, \gamma)v = 0 \text{ in } \Omega_0, v = \zeta \text{ on } \Gamma_0, C(\rho, \gamma)(v, q) \cdot n_0 = C \cdot n_0 \text{ on } \Sigma_0 \quad (2.10)$$

with the conditions

$$\lim_{\|(\hat{x}, \hat{y}, \hat{z})\| \rightarrow \infty} v(\hat{x}, \hat{y}, \hat{z}) = \begin{pmatrix} C_1(z_0\hat{z} - \frac{1}{2})\hat{z}^2 \\ 0 \\ 0 \end{pmatrix} \quad (2.11)$$

and

$$\lim_{\|(\hat{x}, \hat{y}, \hat{z})\| \rightarrow \infty} q(\hat{x}, \hat{y}, \hat{z}) = C_2(\hat{z} - z_0) \quad (2.12)$$

Denoting by  $L_\rho^\gamma$  the operator defining the left-hand side of (2.10), one easily checks that  $L_\rho^\gamma$  maps  $H_\delta^s(\Omega_0) \times H_\delta^{s-1}(\Omega_0)$  into  $H_\delta^{s-2}(\Omega_0) \times H_\delta^{s-1}(\Omega_0) \times H_\delta^s(\Gamma_0) \times H_\delta^{s-1}(\Sigma_0)$ . In order to solve (2.10), we first eliminate the

conditions at  $\infty$  by seeking a solution  $(v, q)$  under the following form:

$$v = V + U_0, q = Q + P_0 \quad (2.13)$$

where  $(U_0, P_0)$  is the unperturbed solution corresponding to a flat bottom:

$$A(0, 0)(v, q) = F \text{ in } \Omega_0, B(0, 0)v = 0 \text{ in } \Omega_0, v = 0 \text{ on } \Gamma_0, C(0, 0)(v, q).n_0 = C.n_0 \text{ on } \Sigma_0 \quad (2.14)$$

We now state and prove the following

**Proposition 2.2** *There exists  $\delta_0 > 0$  such that if  $\delta \leq \delta_0$  and  $s \geq 4$ , then for every  $(\rho, \gamma) \in B_\epsilon(H_\delta^s(\mathbf{R}^2)) \times B_\epsilon(H_\delta^s(\mathbf{R}^2))$ , there exists a unique solution  $(v, q)$  to (2.10) with*

$$v = V + U_0, q = Q + P_0 \quad (2.15)$$

$(U_0, P_0)$  as in (2.14). Moreover, there holds:

$$\|V\|_{H_\delta^{s-1}(\Omega_0)} + \|\nabla Q\|_{H_\delta^{s-3}(\Omega_0)} \leq C(\|g\|_{H_\delta^{s-1}(\Omega_0)} + \epsilon_{\rho, \gamma}) \quad (2.16)$$

$\epsilon_{\rho, \gamma}$  being small for  $(\rho, \gamma) \in B_\epsilon(H_\delta^s(\mathbf{R}^2)) \times B_\epsilon(H_\delta^s(\mathbf{R}^2))$ .

The proof of (2.2) relies on a classical contraction argument, as follows: one first rewrites (2.14) under the following form

$$\begin{aligned} A(0, 0)(V, Q) &= [A(0, 0) - A(\rho, \gamma)](v, q) \text{ in } \Omega_0, \\ B(0, 0)V &= [B(0, 0) - B(\rho, \gamma)]v \text{ in } \Omega_0, \\ V &= \zeta \text{ on } \Gamma_0, \\ C(0, 0)(V, Q).n_0 &= [C(0, 0) - C(\rho, \gamma)](v, q).n_0 C.n_0 \text{ on } \Sigma_0 \end{aligned} \quad (2.17)$$

and look for a fixed point of the following operator  $(V, Q) = M_\rho^\gamma(W, R)$  defined as the solution of:

$$\begin{aligned} A(0, 0)(V, Q) &= [A(0, 0) - A(\rho, \gamma)](W + U_0, R + P_0) \text{ in } \Omega_0 \\ B(0, 0)V &= [B(0, 0) - B(\rho, \gamma)](W + U_0) \text{ in } \Omega_0 \\ V &= \zeta \text{ on } \Gamma_0 \\ C(0, 0)(V, Q).n_0 &= [C(0, 0) - C(\rho, \gamma)](W + U_0, R + P_0).n_0 C.n_0 \text{ on } \Sigma_0 \end{aligned} \quad (2.18)$$

At this stage, the main point is to prove that (2.14) is well-posed when the right-hand sides are replaced by small data, that is, that the following problem:

$$\begin{aligned} A(0, 0)(V, Q) &= f_1 \text{ in } \Omega_0 \\ B(0, 0)V &= f_2 \text{ in } \Omega_0 \\ V &= f_3 \text{ on } \Gamma_0 \\ C(0, 0)(V, Q).n_0 &= f_4 \text{ on } \Sigma_0 \end{aligned} \quad (2.19)$$

has a unique solution for small enough quadruple  $(f_1, f_2, f_3, f_4)$  in  $B_\epsilon(H_\delta^{s-3}(\Omega_0) \times H_\delta^{s-2}(\Omega_0) \times H_\delta^{s-1}(\Sigma_0) \times H_\delta^{s-2}(\Sigma_0))$ .

This result follows easily from arguments in [21] for the standard Sobolev spaces, and from the detailed study of the weighted case that we postpone to Section 3, see in particular corollary 3.5. Using this result, one can now show that the mapping  $(W, R) \mapsto (V, Q) = M_\rho^\gamma(W, R)$  is well-defined and is a contraction for  $(\rho, \gamma) \in B_\epsilon(H_\delta^s(\mathbf{R}^2)) \times B_\epsilon(H_\delta^s(\mathbf{R}^2))$  and  $\epsilon$  small enough. Property (2.16) is then a simple consequence of the Lipschitz continuity of the coefficients of the operator in (2.10).

Let us note that the  $C^2$  regularity of the solution operator  $G$  of (1.6) obtains through standard arguments relying on the chain rule, and move on to the computation of the differential of  $G$  with respect to the free surface  $\rho$ .

For  $\rho$  small enough, we compute the derivative  $\frac{DG}{D\rho}(\rho).h$  as the limit when  $t \rightarrow 0$  of

$$\frac{1}{t}(G(M_\rho + thn_\rho, th + \rho) - G(M_\rho, \rho)) \quad (2.20)$$

with - as previously defined -  $M_\rho = M_0 + \rho n_0$ , see figure 2. The computations we need to perform are classical and detailed in [5]. Let us first recall- using coordinates in a local frame - that

$$n_0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (2.21)$$

$$M_\rho \begin{pmatrix} x \\ y \\ z_0 + \rho(x, y) \end{pmatrix} \quad (2.22)$$

$$n_\rho \frac{1}{E} \begin{pmatrix} -\rho_x \\ -\rho_y \\ 1 \end{pmatrix} \quad (2.23)$$

where  $E = \sqrt{1 + |\nabla \rho|^2}$ . Therefore, we have that

$$M_{th} := M_{\rho+thn_\rho} \begin{pmatrix} x - \frac{th\rho_x}{E} \\ y - \frac{th\rho_y}{E} \\ z_0 + \rho(x, y) + \frac{th}{E} \end{pmatrix} \quad (2.24)$$

We now determine the normal vector  $n_{th}$  at  $M_{th}$ . Up to smaller order terms in  $t$ , one can work out the following expressions, see [5] for details:

$$n_{\rho+th}(M_{\rho+th}) = n_\rho(M_\rho) - A \cdot \nabla h + Lh + o(t) \quad (2.25)$$

with

$$A(\rho) = \begin{pmatrix} 1 + (\rho_y)^2 & -\rho_x \rho_y & 0 \\ -\rho_x \rho_y & 1 + (\rho_x)^2 & 0 \\ \rho_x & \rho_y & 0 \end{pmatrix} \quad (2.26)$$

$$L(\rho) := \begin{pmatrix} L_1(\rho) \\ L_2(\rho) \\ L_3(\rho) \end{pmatrix} = \begin{pmatrix} \rho_y \rho_{xy} + \rho_x \rho_{yy} \\ \rho_x \rho_{xy} + \rho_y \rho_{xx} \\ -\rho_{xx} - \rho_{yy} \end{pmatrix} \quad (2.27)$$

We now return to the derivative of  $G$  itself: (2.20) is split into three parts. First, a part coming from the variation of the normal field, that is

$$\frac{1}{t} u_{\rho+th}(M_\rho + thn_\rho) \times (n_{\rho+th}(M_\rho + thn_\rho) - n_\rho(M_\rho))$$

whose limit as  $t \rightarrow 0$  is given using (2.25) by:

$$- \langle u_\rho(M_\rho), A(\rho) \cdot \nabla h \rangle + L_0(\rho) \cdot h \quad (2.29)$$

with

$$L_0(\rho) \cdot h = \langle u_\rho(M_\rho), L(\rho) \rangle \cdot h \quad (2.30)$$

We rewrite (2.29) as

$$- \langle V, \nabla h \rangle + L_0(\rho) \cdot h \quad (2.31)$$

with  $V$  easily obtained using the transpose of  $A(\rho)$ . If we let  $\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}$  be the coordinates of  $u_\rho(M_\rho)$ , then one has:

$$V \begin{pmatrix} U_1(1 + |\nabla\rho|^2) + \rho_x G(\rho) \\ U_2(1 + |\nabla\rho|^2) + \rho_y G(\rho) \end{pmatrix} \quad (2.32)$$

We remark that  $V$  is a small perturbation of  $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$  which is itself close to  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

The second part of (2.20) comes from the variation of the point at which the function  $u$  is evaluated:

$$\frac{1}{t} u_{\rho+th}(M_\rho + thn_\rho) - u_{\rho+th}(M_\rho) \times (n_{\rho+th}(M_\rho + thn_\rho) - n_\rho(M_\rho))$$

the limit of which is obviously

$$\frac{\partial u}{\partial n_\rho}(M_\rho).n_\rho. \quad (2.34)$$

Finally, the third term in (2.20) comes from the variation of  $u$  with respect to the domain, see e.g. [5]: setting

$$v(M) = \lim_{t \rightarrow 0} \frac{u_{\rho+th}(M) - u_\rho(M)}{t}, \quad (2.35)$$

then the limit of the last term is

$$v(M_\rho).n_\rho. \quad (2.36)$$

Finally, the derivative of  $G$  is given by the following formula:

$$\frac{DG}{D\rho}(\rho).h = -\langle V, \nabla h \rangle + L_0 h + v.n_\rho. \quad (2.37)$$

Similarly to  $v$ , we define the local derivatives of the stress tensor and the pressure field

$$v(M) = \lim_{t \rightarrow 0} \frac{\sigma(u_{\rho+th})(M) - \sigma(u_\rho)(M)}{t} \quad (2.38)$$

and

$$q(M) = \lim_{t \rightarrow 0} \frac{p_{\rho+th}(M) - p_\rho(M)}{t} \quad (2.39)$$

One clearly has that

$$\tau(v) = \frac{1}{2} \nabla v + {}^t \nabla v - qI. \quad (2.40)$$

Using standard arguments, one easily obtains the following system of equation for the local derivative  $v$  with respect to the domain:

$$-\operatorname{div} \tau(v) = 0 \text{ in } \Omega_\rho^\gamma, \operatorname{div} v = 0 \text{ in } \Omega_\rho^\gamma, v = 0 \text{ on } \Gamma_\gamma, \tau(v).n_\rho = \Lambda(\rho).\nabla h - \left( \frac{\partial \sigma}{\partial n_\rho}.n_\rho.n_\rho h \right) + M_0(\rho)h \text{ on } \Sigma_\rho, \quad (2.41)$$

where we have set

$$\Lambda(\rho) = (1 + |\nabla\rho|^2)(\sigma - C) \quad (2.42)$$



and

$$M_0(\rho) = (\sigma - C)L_0(\rho). \quad (2.43)$$

Again, one notices that  $\Lambda(\rho)$  is a small perturbation of  $(\sigma - C)$  and also that  $M_0(\rho)$  is an operator of order 0 with a small norm whenever  $\rho \in B_\epsilon$ .

In the rest of the article, we will denote by  $P_{-1}$  the operator  $h.n_\rho$  where  $v$  is the solution of (2.37). We will first focus on a simplified version of it,  $P_{-,01}$  say, defined as  $h.n_\rho$  where  $w$  is a solution to the set of equation (2.44) below:

$$-\operatorname{div} \tau(v) = 0 \text{ in } \Omega_\rho^\gamma, \operatorname{div} v = 0 \text{ in } \Omega_\rho^\gamma, v = 0 \text{ on } \Gamma_\gamma, \tau(v).n_\rho = \Lambda(\rho).\nabla h + h \text{ on } \Sigma_\rho \quad (2.44)$$

The study of the operator  $P_{-,01}$  is key to the proof of the invertibility of the linearized operator  $\frac{DG}{D\rho}$ .

## 2.1 Properties of $P_{-,01}$

We first state the following important technical result:

**Proposition 2.3** *For any  $s \geq 0$  and  $\delta \leq \delta_{01}$ ,  $P_{-,1,0}$  is a continuous linear operator from  $H_\delta^s(\mathbf{R}^2)$  into  $H_\delta^{s+1}(\mathbf{R}^2)$*

Proof: this easy result stems from the obvious linearity of  $P_{-,1,0}$  and the fact that the Stokes problem is well-posed on unbounded domains for small data, see e.g. [19][21]. In fact, the much more accurate result holds true:

**Proposition 2.4**  *$P_{-,1,0}$  is a pseudodifferential operator of order  $-1$  whose principal symbol is given by  $p(\xi) = \frac{1}{|\xi|}$*

Proof: the techniques we use are described with great details in [5], to which we refer the interested reader. Let us however recall the gist of the computation:  $P_{-,1,0}$  is a pseudodifferential operator on  $\Sigma_\rho$  if it is a pseudodifferential operator in  $\mathbf{R}^2$  in any local coordinate system on  $\Sigma_\rho$ . Moreover, its principal symbol is uniquely defined, irrespective of the choice of local coordinates, and can therefore be computed using conformal coordinates on  $\Sigma_\rho$ , see e.g. [11].

Denoting again by  $E$  the coefficient of the first fundamental form on  $\Sigma_\rho$ , the local basis is now given by

$$e_i = \frac{1}{E} \frac{\partial M}{\partial s_i} \quad (2.45)$$

and the following properties are satisfied:

$$\frac{\partial M}{\partial s_i} \cdot \frac{\partial M}{\partial s_j} = 0, \quad i, j, \quad \left| \frac{\partial M}{\partial s_i} \right| = E, \quad n = \bigwedge_i e_i \quad (2.46)$$

For every point  $N$  we set  $N = M(s_1, s_2) + \lambda n_\rho$ ,  $M \in \Sigma_\rho$ . The coefficients in (2.44) now becomes dependent on the space variables, and we write the new system as follows:

$$A(s_1, s_2, \lambda)(W, Q) = 0 \text{ in } \mathbf{R}_2 \times \mathbf{R}_*^-, \quad B(s_1, s_2, \lambda)W = 0 \text{ in } \mathbf{R}_2 \times \mathbf{R}_*^-, \quad \lim_{\lambda \rightarrow -\infty} W(s_1, s_2, \lambda) = 0, \quad (2.47)$$

$$C(s_1, s_2, \lambda)(W, Q) = D(s_1, s_2, \lambda)h \text{ on } \lambda = 0 \quad (2.48)$$

where  $W, Q$  are the unknown functions  $w, q$  after the change of variables (these computations are obviously

similar to those in Section2).

Now, in order to compute the principal symbol, one can see that it is sufficient to consider the problem with the coefficients frozen at  $\lambda = 0$ :

$$A(s_1, s_2, 0)(V, R) = 0 \text{ in } \mathbf{R}_2 \times \mathbf{R}_*^-, B(s_1, s_2, 0)V = 0 \text{ in } \mathbf{R}_2 \times \mathbf{R}_*^-, \lim_{\lambda \rightarrow -\infty} V(s_1, s_2, \lambda) = 0, \quad (2.49)$$

$$C(s_1, s_2, )(V, R) = D(s_1, s_2, 0)h \text{ on } \lambda = 0 \quad (2.50)$$

The proof of this simplification is straightforward : one first forms the difference  $W - V, Q - R$ , a solution to:

$$A(s_1, s_2, 0)(W - V, Q - R) = (A(s_1, s_2, 0) - A(s_1, s_2, \lambda))(W, Q) \text{ in } \mathbf{R}_2 \times \mathbf{R}_*^- \quad (2.51)$$

$$B(s_1, s_2, 0)(W - V) = (B(s_1, s_2, 0) - B(s_1, s_2, \lambda))W \text{ in } \mathbf{R}_2 \times \mathbf{R}_*^-, \lim_{\lambda \rightarrow -\infty} (W - V)(s_1, s_2, \lambda) = 0, \quad (2.52)$$

$$C(s_1, s_2, )(W - V, Q - R) = (C(s_1, s_2, ) - C(s_1, s_2, \lambda))(W - V, Q - R)(W, Q) \text{ on } \lambda = 0 \quad (2.53)$$

and use the Taylor formula to obtain an expansion of  $W - V, Q - R$ :

$$W - V = \lambda Z + H, Q - R = \lambda M + K \quad (2.54)$$

where  $H, K$  are smoother than  $W - V, Q - R$  and the terms coming from  $Z, M$ , being multiplied by  $\lambda$ , do not contribute to the normal trace operator on  $\lambda = 0$ . Therefore, we may legitimately focus on the simpler operator  $h \rightarrow V_3(s_1, s_2, 0)$  where  $V$  is the solution to:

$$-\frac{1}{2}\Delta V + \nabla Q = 0 \text{ in } \mathbf{R}^2 \times \mathbf{R}_*^-, \text{div } V = 0 \text{ in } \mathbf{R}^2 \times \mathbf{R}_*^-, \lim_{\lambda \rightarrow -\infty} V = 0, \tau(V, Q) = (\sigma - C)\nabla h + h \text{ on } \lambda = 0 \quad (2.55)$$

We now use the Fourier transform in the  $s$  variables, and obtain the following:

$$\frac{1}{2}\left(-\frac{\partial^2 \hat{V}_1}{\partial \lambda^2} + |\xi^2 \hat{V}_1\right) + i\xi_1 \hat{Q} = 0 \quad (2.56)$$

$$\frac{1}{2}\left(-\frac{\partial^2 \hat{V}_2}{\partial \lambda^2} + |\xi^2 \hat{V}_2\right) + i\xi_2 \hat{Q} = 0 \quad (2.57)$$

$$\frac{1}{2}\left(-\frac{\partial^2 \hat{V}_3}{\partial \lambda^2} + |\xi^2 \hat{V}_3\right) + \frac{\partial \hat{Q}}{\partial \lambda} = 0 \quad (2.58)$$

$$i\xi_1 \hat{V}_1 + i\xi_1 \hat{V}_1 + \frac{\partial \hat{V}_3}{\partial \lambda}. \quad (2.59)$$

Upon solving for  $\hat{Q}$  and combining the equations above, one immediately obtains that

$$\hat{Q}(\xi_1, \xi_2 \lambda) = \frac{1}{2|x_i|^2} \frac{\partial}{\partial \lambda} \left[ \frac{\partial^2}{\partial \lambda^2} - |\xi|^2 \right] \hat{V}_3. \quad (2.60)$$

It follows that  $[\frac{\partial^2}{\partial \lambda^2} - |\xi|^2] \frac{\partial^2 \hat{V}_3}{\partial \lambda^2} = 2|\xi|^2 \frac{\partial \hat{Q}}{\partial \lambda}$ . On the other hand, there also holds:  $-|\xi|^2 [\frac{\partial^2}{\partial \lambda^2} - |\xi|^2] \hat{V}_3 = -2|\xi|^2 \frac{\partial \hat{Q}}{\partial \lambda}$ , so that we finally obtain the following ODE for  $\hat{V}_3$ :

$$\left[ \frac{\partial^2}{\partial \lambda^2} - |\xi|^2 \right] \hat{V}_3 = 0. \quad (2.61)$$

Such an equation is trivially solved using the following representation:

$$\hat{V}_3(\xi, \lambda) = (A + B\lambda) \exp(-\lambda|\xi|) + (C + D\lambda) \exp(\lambda|\xi|) \quad (2.62)$$

so that  $\hat{V}_3(\xi, 0) = A + C$   
 We can then compute  $\hat{Q}$ :

$$\hat{Q}(\xi, \lambda) = B \exp(-\lambda|\xi|) + D \exp(\lambda|\xi|). \quad (2.63)$$

Upon considering the third coordinate of the stress tensor on  $= 0$ , there follows:

$$\frac{\partial \hat{V}_3}{\partial \lambda}(\xi, 0) = \hat{Q}(\xi, 0) + i(\xi_1 \sigma_{31} + \xi_2 \sigma_{32}) \hat{h} + \hat{h}, \quad (2.64)$$

an expression which can be simplified since one has that  $\sigma_{31} = \sigma_{32} = 0$  on the boundary. Finally, one obtains that:

$$C - A = \frac{\hat{h}}{|\xi|} \quad (2.65)$$

which finally leads to the following equality - given that  $A = 0$  because of the normal velocity boundary condition:

$$\hat{V}_3(\xi, 0) = \frac{1}{|\xi|} \hat{h} \quad (2.66)$$

and the lemma is proven.

Next, we need the following lemma

**Lemma 2.5**  $P_{-1,0}$  is a self-adjoint operator in  $L^2(\mathbf{R}^2)$ .

The proof of the lemma consists in a straightforward use of Green formula.

In fact, one can prove the more important result:

**Lemma 2.6**  $P_{-1,0}$  is a coercive operator from  $H^{-\frac{1}{2}}(\mathbf{R}^2)$  into  $H^{\frac{1}{2}}(\mathbf{R}^2)$ . Therefore, there exists a constant  $C > 0$  such that, for every  $h \in H^{-\frac{1}{2}}(\mathbf{R}^2)$ , there holds:

$$(P_{-1,0}h, h) \geq C \|h\|^2$$

Proof : a straightforward consequence of the Green formula and the Korn inequality. In fact, setting  $v.n = P_{-1,0}h$  and writing for a test function  $w$

$$\int_{\Omega_\rho} \operatorname{div} \tau(v).w + \frac{1}{2} \int_{\Omega_\rho} (\nabla v + {}^t \nabla v) : (\nabla w + {}^t \nabla w) = \int_{\Sigma_\rho} (\tau(v).n)w, \text{ one obtains, using } v \text{ itself as a test function:}$$

$$\frac{1}{2} \int_{\Omega_\rho} |\nabla v + {}^t \nabla v|^2 = \int_{\Sigma_\rho} (\tau(v).n.n)(w.n), \text{ that is:}$$

$$(P_{-1,0}h, h) = \frac{1}{2} \int_{\Omega_\rho} |\nabla v + {}^t \nabla v|^2$$

and, thanks to the Korn inequality:  $(P_{-1,0}h, h) \geq C \|v\|_{H^1}^2$ . The lemma then results from the following technical result, the proof of which is similar to that provided in [20] and is therefore omitted:

**Lemma 2.7** Let  $\Omega_\rho$  be an open set in  $\mathbf{R}^3$  with a  $C^2$  boundary. Then for  $v \in H^1(\Omega)$  such that  $\tau.n$  is parallel to  $n$  and  $\operatorname{div} \tau(v) = 0$ , one can define a normal stress operator  $v \rightarrow \tau.n.n \in H^{-\frac{1}{2}}(\partial\Omega_\rho)$  satisfying the following continuity property:

$$\|v\|_{H^1(\Omega_\rho)} \geq C \|\tau.n.n\|_{H^{-\frac{1}{2}}(\partial\Omega_\rho)}. \text{ This operator coincides with the usual normal stress operator when } v \text{ is } C^1 \text{ up to the boundary of } \Omega_\rho$$

Upon using this result, one then has:

$$(P_{-1,0}h, h) \geq C \|v\|_{H^1(\Omega_\rho)}^2 \geq C \|\tau.n.n\|_{H^{-\frac{1}{2}}(\partial\Omega_\rho)}^2 \text{ which ends the proof of Lemma 2.6.}$$

### 3 The main result

We now prove our main result, namely, the existence of a unique steady solution  $(u_\rho, p_\rho, \Sigma_\rho)$  to the full problem 1.11.21.3 in weighted Sobolev spaces for a large enough Sobolev index  $s$ . As we will be using a Nash Moser iteration technique, we have to make precise our choice of a scale of Banach spaces.

### 3.1 Scale of Banach spaces

In this subsection, we recall the machinery fully developed in [22].

Let then  $\{X_t\}_{t \geq 0}$  be a one-parameter family of Banach spaces, with norm denoted by  $|\cdot|_t$ , such that for every  $t, t'$  with  $0 \leq t \leq t' \leq +\infty$ , there holds:

$$X_0 \supseteq X_{t'} \supseteq X_t \supseteq X_\infty \equiv \bigcap_{t \geq 0} X_t \quad (3.1)$$

and

$$\forall t, t', t' \leq t, |z|_{t'} \leq |z|_t. \quad (3.2)$$

We will work with three such scales, respectively denoted by  $\{X_t\}_{t \geq 0}$ ,  $\{Y_t\}_{t \geq 0}$  and  $\{Z_t\}_{t \geq 0}$ . The mapping  $G$  in 1.6 will be defined from  $D \subset Y_0 \times X_0$  into  $R_0$ . Upon setting

$$f_0 = (\gamma_0, 0), \rho_0 = 0 \quad (3.3)$$

with  $\gamma_0 \equiv 0$  being the flat bottom corresponding to the Poiseuille-Nusselt flow, one clearly has that:

$$G(f_0, 0) = 0. \quad (3.4)$$

Our purpose is now to prove that, for  $f$  close enough to  $f_0$ , there exists a  $\rho_f$  in a suitable function space such that  $G(f, \rho_f) = 0$ .

Below is the set of four hypotheses that have to be satisfied in order for the Nash-Moser Implicit Function Theorem to apply:

*Hypothesis 1*

Let  $B_t = \{(f, \rho) \in Y_t \times X_t / |f - f_0| < \epsilon, |\rho - \rho_0| < \epsilon\}$ ; then  $G : B_0 \rightarrow Z_0$  is at least  $C^2$  with respect to  $\rho$  and  $C^0$  with respect to  $(f, \rho)$ , and there exists a constant  $M_0$  such that:

$$\forall (f, \rho) \in B_t, \sup(|D_\rho G(f, \rho)|_0, |D_\rho^2 G(f, \rho)|_0) \leq M_0.$$

*Hypothesis 2*

For every couple  $(f, \rho), (f', \rho')$  in  $B_0$ , there holds:

$$|G(f, \rho) - G(f', \rho')|_0 \leq M_0 |f - f'|_0$$

*Hypothesis 3*

$(G, f_0, \rho_0)$  is of order  $s$ ,  $s > \nu \geq 1$  (see Hypothesis 4 below for the definition of  $\nu$ ), that is:

$(f_0, \rho_0) \in Y_s \times X_s$ ,  $G(B_0 \cap (Y_t \times X_t)) \subset Z_t$  and for every  $t \in [1, s]$ , for every  $t \in [1, s]$ , there exists  $M_t > 0$  such that  $|G(f, \rho)| \leq M_t K$  whenever  $|f - f_0|_t < K$ ,  $|\rho - \rho_0|_t < K$

*Hypothesis 4* (existence of an approximate inverse)

For every  $(f, \rho)$  in  $B_\nu$ , there exists a linear mapping  $H(f, \rho) : Z_\nu \rightarrow X_0$  such that  $\forall \phi \in Z_\nu$ ,  $|H(f, \rho)\phi|_0 \leq M_0 |\phi|_\nu$  and moreover,  $H(f, \rho)$  is continuous from  $Z_t$  into  $X_{t-\nu}$  whenever  $(f, \rho) \in B_\nu \cap (Y_t \times X_t)$ .

One also has that  $|H(f, \rho)G(f, \rho)|_{t-\nu} \leq M_t K$  for every  $(f, \rho)$  such that  $|f - f_0|_t < K$ ,  $|\rho - \rho_0|_t < K$ .

Finally,  $H$  is an approximate left inverse of  $D_\rho G(f, \rho)$  in the sense that  $|[D_\rho G(f, \rho)(f, \rho) - Id]\phi|_0 \leq M_0 |F(f, \rho)_\nu| |\phi|_\nu$  for every  $\phi \in Z_\nu$ .

When all four hypotheses above are satisfied, the following deep result applies:

**Theorem 3.1** (Zehnder [22]) *Let  $(G, f_0, \rho_0)$  be of order  $s$  and satisfying Hypotheses 1 to 4 above. Then there exists a neighbourhood  $D_\lambda = \{f \in Y_\lambda, |f - f_0| < C\}$  with  $\mu, \lambda$  in the admissible parameter set defined as follows:*

$$1 < \kappa < 2,$$

$$1 < \alpha,$$

$$1 \leq \nu \leq \mu < \lambda < s,$$

$$\lambda > \max\{2\kappa\nu(2-\kappa)^{-1}, \kappa(\nu + \kappa\mu)\},$$

$$s > \max\{\alpha\nu(\alpha-1)^{-1}, \lambda + \alpha\nu(\kappa-1)^{-1}\},$$

and a mapping  $\Psi : D_\lambda \rightarrow X_\mu$  such that for every  $f \in D_\lambda$  there exists a unique  $\rho_f \equiv \Psi(f)$  solution to  $G(f, \rho_f) = 0$  with  $|\rho_f - \rho_0|_\nu < C^{-1}|f - f_0|_\lambda$ .

Finally, one has that  $\Psi$  is continuous whenever  $H$  is.

Our next task is to apply this theorem in the context defined above. To this end, we have to choose the scales of Banach spaces in which we work and check all four hypotheses in that particular case.

Thus, we recast the weighted Sobolev spaces defined in Section 2 into the above formalism:

**Definition 3.1.1** For every  $t \in [0, +\infty[$  and every  $\delta < \delta_0$  we set:

$$X_t = H_\delta^{4+t}(\mathbf{R}^2)$$

$$Y_t = H_\delta^{4+t}(\mathbf{R}^2) \times H_\delta^{2+t}(\mathbf{R}^2)$$

$$Z_t = H_\delta^{\frac{3}{2}+t}(\mathbf{R})$$

and their respective intersections

$$X_\infty = S(\mathbf{R}^2)$$

$$Y_\infty = S(\mathbf{R}^2) \times S(\mathbf{R}^2)$$

$$Z_\infty = S(\mathbf{R})$$

We set:  $f = (\gamma, \zeta)$   $\gamma$  being the function describing the bottom of the domain occupied by the fluid and  $\zeta$ , the value of the solution  $u_\rho$  on  $\Gamma_\gamma$ , and recall for the sake of completeness the

**Definition 3.1.2**  $G(f, \rho) = u_\rho|_{\Sigma_\rho} \cdot n_\rho$ .

## 3.2 Proof of the main result

In this final section, we end the proof of Theorem 1.2. First, let us proceed to check the hypotheses of Theorem 3.1. Given the obvious equality  $f_0 = (\gamma_0, g_0) = (0, 0)$  corresponding to the Poiseuille-Nusselt flow, one concludes trivially that  $G(0, 0) = 0$ .

Let us now check that Hypothesis 1 is fulfilled: first,  $B_0 = \{\rho, |\rho|_{X_0} < \epsilon, |f|_{Y_0} < \epsilon\}$ , so that  $G$  is well defined from  $(Y_0 \times X_0)$  into  $Z_0$ . In fact, we can invoke the computations performed in Section 2 and conclude that  $G(f, \rho) \in H_\delta^{1+t+\frac{1}{2}}(\mathbf{R})$  for every  $t \geq 0$ . The other prerequisites for Hypothesis 1 come directly from Proposition 2.1, in particular the fact that  $M_0$  can be chosen as a uniformly bounded constant.

Regarding Hypothesis 2, it is nothing else than a Lipschitz estimate for  $G$  with respect to the first variable:  $|G(f, \rho) - G(f', \rho')|_0 \leq M_0|f - f'|_0$ .

In order to prove such an inequality, one has to compare the respective solutions  $u_\rho, u_{\rho'}$  corresponding to different  $f, f'$  and hence, defined on different domains. The key tool is once again the Hanzawa transform mapping each domain onto the same, unperturbed one. We first write for  $\rho$  (resp.  $\rho'$ ) the system satisfied by the solution  $v, q$  (resp.  $v', q'$ ) and form the difference, thereby obtaining the following set of equations similar to those introduced in the proof of Proposition 2.2:

$$A(\rho, \gamma)(v - v', q - q') = [A(\rho, \gamma) - A(\rho, \gamma')](v', q') \text{ in } \Omega_0, \quad (3.5)$$

$$B(\rho, \gamma)(v - v') = [B(\rho, \gamma) - B(\rho, \gamma')]v' \text{ in } \Omega_0, v - v' = \zeta - \zeta' \text{ on } \Gamma_0, \quad (3.6)$$

$$C(\rho, \gamma)(v - v', q - q') \cdot n_0 = [C(\rho, \gamma) - C(\rho, \gamma')]v' \cdot n_0 \text{ on } \Sigma_0. \quad (3.7)$$

The key point here is that  $v', q'$  remains in a uniformly bounded ball when  $(\gamma, g) \in B_0$ . One then applies Taylor formula to the RHS of (3.5) and obtain the bounds required for Hypothesis 2.

We now examine Hypothesis 3: first of all, let us notice that  $(f_0, \rho_0) \in U_\infty \times X_\infty$  and that we already established as a general result that  $G(f, \rho) \in Z_t$  if  $(f, \rho) \in Y_t \times X_t$  for  $t \in [1, +\infty[$ . The last point to check is that for every  $t \in [1, +\infty[$ , there exists an  $M_t$  such that  $|G(f, \rho)|_t \leq M_t K$  whenever  $|f - f_0|_t < K, |\rho - \rho_0|_t < K$  in  $(Y_t \times X_t) \cup B_1$  (here, we have chosen  $K < \epsilon$ ).

Such an inequality follows directly from the proof of Hypothesis 2, upon performing similar computations

with respect to both  $f$  and  $\rho$  this time. In fact, one can even choose  $M_t = M_0$  for all  $t$ , a remark that will be used in a crucial way for the verification of Hypothesis 4. In order to proceed to this verification, we state and prove the

**Proposition 3.2** *For every pair  $(f, \rho) \in Y_\infty \times X_\infty$  there exists a linear mapping  $H : Z_\gamma \rightarrow X_0$  with  $\gamma = \frac{5}{2}$  such that  $|H(f, \rho)|_0 \leq M_0 |\phi|_{Z_\gamma}$ .*

*When  $(f, \rho) \in Y_\infty \times X_\infty$ ,  $H$  is continuous from  $Z_t$  into  $X_{t-\gamma}$  for every  $t \geq 0$ .*

*Moreover, the following inequality:*

$$|H(f, \rho)|[G(f, \rho)]|_{t-\gamma} \leq M_t K$$

*holds true for every  $(f, \rho) \in (Y_t \times X_t) \cup B_1$  with  $|f - f_0|_t < K$ ,  $|\rho - \rho_0|_t < K$ .*

*Finally,  $H(f, \rho)$  is an exact inverse to  $D_\rho G(f, \rho)$ , i.e.:  $\forall \phi \in Z_{t+\gamma}$ ,  $[D_\rho G(f, \rho)(f, \rho) - Id] = 0$*

Before proving Proposition 3.2 and showing that it is sufficient to ensure that Hypothesis 4 holds true, let us now explain the way we shall apply Theorem 3.1 to our problem.

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We now proceed to prove Proposition 3.2. Our goal is to show the existence, for  $(f, \rho)$  given in  $Y_\infty \times X_\infty$  and for any  $F \in H_\delta^{t+\frac{1}{2}}(\mathbf{R}^2)$ , of a unique  $h \in H_\delta^{t-\frac{1}{2}}(\mathbf{R}^2)$  such that  $D_\rho G(f, \rho)h = F$ .

Recalling that  $D_\rho G(f, \rho)h = -\langle V, \nabla h \rangle + L_0(\rho)h + P_{-1,0}h$ , we want to invoke a result due to the authors:

**Theorem 3.3** ([3]) *Let  $V$  be a smooth real-valued vector field in  $\mathbf{R}^2$ ,  $g$  a smooth real-valued function, and consider the following problem:*

$$-\langle V, \nabla h \rangle + L_0(\rho)h + P_{-1,0}h = f \quad (3.8)$$

where  $V = (V_1, V_2)$ , there exists a non-zero constant vector field  $C$  such that  $\|V - C\|_{W^{1,\infty}} \ll 1$ ,  $L_0$  is a bounded operator of order zero with symbol  $\sigma_{L_0}$  such that  $\|\sigma_{L_0}\|_L \ll 1$ , and  $\nabla V, \sigma_{L_0}$  are rapidly decreasing at infinity.

We assume furthermore that  $P_{-1,0}$  is a self-adjoint pseudodifferential operator of order  $-1$  satisfying a coercivity inequality  $(P_{-1,0}h, h) \geq C\|\rho\|_{H^{-\frac{1}{2}}(\mathbf{R}^2)}$ . Its principal symbol is given by  $\sigma_{P_{-1}}(x, \xi) = \frac{m(x)}{|\xi|}$ ,  $m(x)$  being such that  $0 < m \leq m(x) \leq M$ .

Then, for every  $t > 0$  and  $f \in H^{t+\frac{1}{2}}(\mathbf{R}^2)$ , there exists a unique solution  $h \in H^{t-\frac{1}{2}}(\mathbf{R}^2)$  to (3.3), and the solution  $h$  depends continuously on the data  $f$ .

We now check that the assumptions of Theorem 3.3 are satisfied in the problem we are considering. The smoothness of  $V$  and  $g$  is obvious, because they correspond to smooth data  $(f, \rho)$ . On the other hand,  $(V_1, V_2)$  is given explicitly as in Section 2

$$V \begin{pmatrix} U_1(1 + |\nabla \rho|^2) + \rho_x G(\rho) \\ U_2(1 + |\nabla \rho|^2) + \rho_y G(\rho) \end{pmatrix},$$

where  $(U_1, U_2)$  is the solution to (1.1)(1.2) for  $\Omega_\rho$  close enough to  $\Omega_0$ . In fact, using results of Section 2, in particular Proposition 2.2, one can show that:

$$V \begin{pmatrix} U_1(1 + \epsilon_1(x_1, x_2)) \\ U_2(1 + \epsilon_2(x_1, x_2)) \end{pmatrix} \quad (3.9)$$

with  $\epsilon_1, \epsilon_2$  close to 0. The second condition has to do with the operator  $L_0$ , which has to be of order 0. This is a consequence of the computations in Section 2. The principal symbol  $\sigma_{L_0}$  of  $L_0$  is given by the first-order derivatives of  $\rho$  and by  $\frac{\partial U_\rho}{\partial n} \cdot n$ , and therefore  $\|\sigma_{L_0}\|_\infty \ll 1$ . Finally, results concerning the operator  $P_{-1,0}$  have been proven for the particular case  $P_{-1,0}h = (\sigma - C)\nabla h + h$ , and a perturbation argument will now give us the desired conclusion for the more general case of  $P_{-1,0}h = (\sigma - C)\nabla h + \frac{\partial \sigma}{\partial n} \cdot n \cdot h$ , allowing us to

apply Theorem 3.3 to obtain the existence of an exact inverse to  $D_\rho G(f, \rho)$ . In fact, there holds:

**Corollary 3.4** *Under the assumptions of Theorem 3.3, and under the following assumptions:  $0 < C_0 \leq |\frac{\partial \sigma}{\partial n} \cdot n \cdot n| \leq C_1$ ,  $\frac{\partial \sigma}{\partial n} \cdot n \cdot n < 0$ , then, for every  $f \in H_\delta^{s+\frac{1}{2}}(\mathbf{R}^2)$ , there exists a unique  $h \in H_\delta^{s-\frac{1}{2}}(\mathbf{R}^2)$  solution to the equation:*

$$- \langle V, \nabla h \rangle + L_0 h + P_{-1} h = f \quad (3.10)$$

Proof: we use, once again, a perturbation argument, and must go back to the proof of the main result in [3]. We shall denote as in [3]  $A = - \langle V, \nabla \cdot \rangle + L_0$ . As for the principal symbol of  $P_{-1}$ , it is computed exactly as in Section 2, and we obtain that  $\sigma(P_{-1})(x, \xi) = \frac{\partial \sigma}{\partial n} \cdot n \cdot n \frac{1}{|\xi|}$

We know already that  $P_{-1}$  is elliptic, as was already the case for  $P_{-1,0}$ . However,  $P_{-1}$  is not necessarily coercive, so we now adapt slightly the proof of Theorem 4.2 in [3] in order to make it work in this particular case. This modification amounts to rewriting the weak formulation (see the proof of Theorem 4.2 in [3]) and using a suitably chosen test function of the form  $\Lambda(x)h$ . This easy extension of [3] is left as an exercise to the reader.

We also state and prove the easy

**Corollary 3.5** *Under the same set of assumptions as above, there exists  $\delta_{02} > 0$  such that for  $\delta \leq \delta_{02}$  and for every  $f \in H_\delta^{s+\frac{1}{2}}(\mathbf{R}^2)$ , there exists a unique  $h \in H_\delta^{s-\frac{1}{2}}(\mathbf{R}^2)$  solution to (3.8).*

Proof: one uses the changes of RHS and unknown function  $f(x) = f_0(x) \exp(-\delta|x|)$ ,  $h(x) = h_0(x) \exp(-\delta|x|)$  to obtain a modified equation:

$$- \langle V, \nabla h_0 \rangle + L'_0 h_0 + P_{-1} h_0 + \exp(\delta|x|)[P_{-1}, \exp(-\delta|x|)]h_0 = f_0 \quad (3.11)$$

where  $L'_0 = L_0 - \langle V, \nabla(\exp(-\delta|x|)) \rangle$  and  $[\cdot, \cdot]$  stands for the commutator of two operators. Regarding  $L'_0$ , it can clearly be considered a small perturbation of  $L_0$  for  $\delta$  small enough. As for  $[P_{-1}, \exp(-\delta|x|)]$ , it is an operator of order  $-2$  that has a small norm when  $\delta$  is small enough. This proves Corollary??.

At this stage, we have established the existence of an exact inverse  $H : Z_\gamma \rightarrow X_0$  to  $D_\rho G$  for  $\gamma = \frac{5}{2}$ ,  $H$  being a continuous operator from  $Z_t$  into  $X_{t-\gamma}$  for every  $t \geq 0$ , when  $(f, \rho) \in Y_\infty \times X_\infty$ .

There still remains to prove that  $|H(f, \rho)[G(f, \rho)]|_{t-\gamma} \leq M_t K$  whenever  $(f, \rho) \in (Y_t \times X_t) \cap B_1$  with  $|f - f_0|_t < K$ ,  $|\rho - \rho_0|_t < K$ . Now, this is a consequence of the continuity of  $H$ :  $|H(f, \rho)\phi| \leq M_t |\phi|_{t+\gamma}$  and the estimate on  $G$  in Hypothesis 3:  $|G(f, \rho)\phi| \leq M_0 K$ .

The proof of Theorem 1.2 is almost over: we must still prove that Hypothesis 4 can be replaced by Proposition 3.2. In the original proof by Zehnder ([22], p. 118 to 125), the use of an approximate inverse is made after the smoothing procedure for  $(f, \rho)$  - the proof itself relying on a Newton iterative method. Hence, the estimate in Hypothesis 4  $|[D_\rho G(f, \rho)(f, \rho) - Id]\phi|_0 \leq M_0 |F(f, \rho)_\gamma|_\gamma$  is in fact only required for the regularized approximating sequence  $(f_\tau^n, \rho_\tau^n)$ ,  $\tau$  being the smoothing factor. The result that we prove in Proposition 3.2 is actually stronger. In the original proof by Zehnder, the constant  $M_0$  has to be uniform with respect to the smoothing parameter  $\tau$ , a condition that becomes irrelevant in our case thanks to the exact invertibility of  $D_\rho G(f, \rho)$  (the interested reader is referred to [22], p. 124-125 for more details). This finally ends the proof of Theorem 1.2.

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