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**Mini-Workshop: Wellposedness and Controllability of
Evolution Equations**

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Abstracts

Fast control cost for heat-like semigroups: Lebeau-Robbiano strategy and Hautus test

LUC MILLER

(joint work with Thomas Duykaerts)

Since the seminal work of Russell and Weiss in [7], resolvent conditions for various notions of admissibility, observability and controllability, and for various notions of linear evolution equations have been studied intensively, sometimes under the name of infinite-dimensional Hautus test, cf. [8, 2]. This talk based on [1] investigates resolvent conditions for null-controllability in arbitrary time: necessary conditions for general semigroups, and sufficient conditions for analytic normal semigroups and semigroups with negative self-adjoint generators.

1. INTRODUCTION

Let $-A$ be the generator of a strongly continuous semigroup on a Hilbert space \mathcal{E} . Let C be a bounded operator from the domain $D(A)$ with the graph norm to another Hilbert space \mathcal{F} . The norms in \mathcal{E} and \mathcal{F} are denoted $\|\cdot\|$. We refer to the monograph [8] for a full account of the control theory of semigroups.

Recall the usual *admissibility* condition (for some time $T > 0$ hence all $T > 0$),

$$(1) \quad \exists K_T > 0, \forall v \in D(A), \quad \int_0^T \|Ce^{-tA}v\|^2 dt \leq K_T \|v\|^2.$$

If C is admissible for A then null-controllability at time T is equivalent to *final-observability* at time T (cf. [8], i.e.

$$(2) \quad \exists \kappa_T > 0, \forall v \in \mathcal{E}, \quad \|e^{-TA}v\|^2 \leq \kappa_T \int_0^T \|Ce^{-tA}v\|^2 dt.$$

The control property investigated here is (2) for all $T > 0$.

1.1. Control cost. The coefficient κ_T in (2) is the *control cost*: it is the ratio of the size of the input over the size of the initial state which the input steers to the zero final state in a lapse of time T . It blows up as $T \rightarrow 0$. E.g. for the heat semigroup on a compact manifold M with Dirichlet boundary conditions observed from a subset Ω : $\kappa_T \leq c_0 \exp(2c/T)$, $T \in (0, 1)$, where c_0 is a positive constant, implies $c \geq d^2/4$ where d is the furthest a point of M can be from Ω , and is implied by $c > 3L^2/4$ where L is the length of the longest generalized geodesic in M which does not intersect Ω ($L < +\infty$ is known as the condition of Bardos-Lebeau-Rauch).

For many evolutions of parabolic type, κ_T is bounded by $c_0 \exp(2c/T^\beta)$ where c , c_0 and β are positive constants. E.g. thermoelastic plates without rotatory inertia, the plate equation with square root damping, diffusions in discontinuous media or in a potential well, diffusions generated by the fractional Laplacian or non-selfadjoint elliptic generators, cf. references in [5].

1.2. **Resolvent conditions.** The resolvent condition: $\exists M > 0$,

$$(3) \quad \|v\|^2 \leq \frac{M}{(\operatorname{Re} \lambda)^2} \|(A - \lambda)v\|^2 + \frac{M}{\operatorname{Re} \lambda} \|Cv\|^2, \quad v \in D(A), \quad \operatorname{Re} \lambda > 0.$$

was introduced in [7] as a necessary condition for exact observability in infinite time of exponentially stable semigroups.

When A is skew-adjoint (equivalently when the semigroup is a unitary group), it was proved in [3] that the following resolvent condition is necessary and sufficient for final-observability (hence exact observability) in some time $T > 0$: $\exists M > 0$,

$$\|v\|^2 \leq M \|(iA - \lambda)v\|^2 + M \|Cv\|^2, \quad v \in D(A), \quad \lambda \in \mathbb{R}.$$

We refer to [6] for more background and references. This result was extended to some more general groups in [2, theorem 1.2].

When $-A$ generates an exponentially stable normal semigroup, [2, theorem 1.3] proves that the resolvent condition (3) is sufficient for the weaker notion:

$$(4) \quad \exists T > 0, \exists \kappa_T > 0, \forall v \in \mathcal{E}, \|e^{-TA}v\|^2 \leq \kappa_T \int_0^\infty \|Ce^{-tA}v\|^2 dt.$$

In this framework (4) implies (2) for *some* time T .

But it seems that resolvent conditions for final-observability for *any* $T > 0$ in (2) has not been investigated yet, although it is quite natural for heat-like semigroups.

2. RESULTS

2.1. **Necessary resolvent conditions for semigroups.** The proof mainly consists in changing i into -1 in [3, lemma 5.2]. Cf. also the proof of [7, theorem 1.2].

Theorem 1. Let $B_T = \sup_{t \in [0, T]} \|e^{-tA}\|$ be the semigroup bound up to time T .

If (1) and (2) hold then : $\forall v \in D(A), \lambda \in \mathbb{C}, \operatorname{Re} \lambda > 0$,

$$\|v\|^2 \leq 2e^{2T \operatorname{Re} \lambda} \left((B_T^2 + 2\kappa_T K_T) \frac{\|(A - \lambda)v\|^2}{(\operatorname{Re} \lambda)^2} + \kappa_T \frac{\|Cv\|^2}{\operatorname{Re} \lambda} \right),$$

Theorem 2. If final-observability (2) holds for all $T \in (0, T_0]$ with the control cost $\kappa_T = c_0 e^{\frac{2c}{T^\beta}}$ for some positive β, c and c_0 then the resolvent condition

$$\|v\|^2 \leq a_0 e^{2a(\operatorname{Re} \lambda)^\alpha} (\|(A - \lambda)v\|^2 + \|Cv\|^2), \quad v \in D(A), \quad \operatorname{Re} \lambda > 0,$$

holds with power $\alpha = \frac{\beta}{\beta+1}$ and rate $a = c^{\frac{1}{\beta+1}} \frac{\beta+1}{\beta^\alpha}$.

It still holds for $\lambda \in \mathbb{C}$ if $\operatorname{Re} \lambda$ is replaced by $\operatorname{Re}_+ \lambda := \max\{\operatorname{Re} \lambda, 0\}$.

2.2. **Sufficient resolvent conditions for an analytic normal semigroup.** The proof is based on the Lebeau-Robbiano strategy of [5]. N.b. (1) is not assumed.

Theorem 3. Assume that $-A$ generates an analytic normal semigroup, hence there exists $\omega \in \mathbb{R}$ and $\theta \in [0, \frac{\pi}{2})$ such that $\sigma(A) \subset \{z \in \mathbb{C} : \arg(z - \omega) \leq \theta\}$.

The resolvent condition with $\alpha \in (0, 1)$, $\omega_0 < \omega$, $\lambda_0 > \omega_0$, positive a_0 and a ,

$$\|v\|^2 \leq \frac{\cos^2 \theta}{(\lambda - \omega_0)^2} \|(A - \lambda)v\|^2 + a_0 e^{2a\lambda^\alpha} \|Cv\|^2, \quad v \in D(A), \quad \lambda \geq \lambda_0,$$

implies final-observability (2) for all time $T > 0$ with the control cost estimate

$$\limsup_{T \rightarrow 0} T^\beta \ln \kappa_T \leq 2a^{\beta+1}(\beta+1)^{\beta(\beta+1)}\beta^{-\beta^2}, \quad \text{where } \beta = \frac{\alpha}{1-\alpha}.$$

2.3. Sufficient resolvent condition for a negative self-adjoint generator.

The proof combines the Lebeau-Robbiano strategy of [5], the control transmutation method of [4] (which deduces the final-observability of the heat-like equation $\dot{v} + Av = 0$ from the exact observability of the wave-like equation $\ddot{w} + Aw = 0$) and results on resolvent conditions from [6].

Theorem 4. *Assume that the positive self-adjoint operator A and the operator C bounded from $D(\sqrt{A})$ with the graph norm to \mathcal{F} satisfy the admissibility and observability conditions with nonnegative powers γ and positive δ , L_* and M_* :*

$$\|Cv\|^2 \leq L_* \lambda^\gamma \left(\frac{1}{\lambda} \|(A - \lambda)v\|^2 + \|v\|^2 \right), \quad v \in D(A), \quad \lambda \geq \inf A,$$

$$\|v\|^2 \leq M_* \lambda^\delta \left(\frac{1}{\lambda} \|(A - \lambda)v\|^2 + \|Cv\|^2 \right), \quad v \in D(A), \quad \lambda \geq \inf A.$$

If $\gamma + \delta < 1$ then final-observability (2) holds for all $T > 0$ with the cost estimate

$$\limsup_{T \rightarrow 0} T^\beta \ln \kappa_T < +\infty, \quad \text{where } \beta = \frac{1 + \gamma + \delta}{1 - \gamma - \delta}.$$

The assumption of the control transmutation method corresponds to $\gamma = \delta = 0$. The Russell-Weiss condition (3) corresponds to $\delta = -1$.

A logarithmic improvement of this theorem is also proved in [1] thanks to this new variant of the Lebeau-Robbiano strategy of [5]:

Theorem 5. *Assume the admissibility condition (1) and that $-A$ generates a normal semigroup. If the logarithmic observability condition on spectral subspaces*

$$\|v\|^2 \leq a_0 e^{2a\lambda/((\log(\log \lambda))^\alpha \log \lambda)} \|Cv\|^2, \quad v \in \mathbf{1}_{\operatorname{Re} A < \lambda} \mathcal{E}, \quad \lambda \geq \lambda_0.$$

holds with $\alpha > 2$, λ_0 , a_0 , a positive then final-observability (2) holds for all $T > 0$.

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