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On representations of cyclotomic Hecke algebras

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Abstract

An approach, based on Jucys–Murphy elements, to the representation theory of cyclotomic Hecke algebras is developed. The maximality (in the cyclotomic Hecke algebra) of the set of the Jucys–Murphy elements is established. A basis of the cyclotomic Hecke algebra is suggested; this basis is used to prove the flatness of the deformation without use of the representation theory.

1 . Introduction

In the paper [3] a Hamiltonian for an open chain model was suggested. This Hamiltonian is written on the universal level, as an element of (a certain localization of) the group ring of an "affine Hecke algebra", see [3] for details; the model induces an integrable system for any representation of the affine Hecke algebra. For the study of the model it is important to understand different aspects of the representation theory of the affine Hecke algebra. In the present article we discuss finite-dimensional representations of the affine Hecke algebra. A finite dimensional representation of the affine Hecke algebra factors through a certain finite-dimensional quotient of the affine Hecke algebra, a "cyclotomic Hecke algebra". The cyclotomic Hecke algebra is a particular example of a Hecke-type deformation which group rings of Coxeter groups and, more generally, of complex reflection groups admit (the usual Hecke algebra is a deformation of the Coxeter group of type A, that is, the symmetric group). The cyclotomic Hecke algebras $H(m,1,n)$ are the Hecke algebras associated to the complex reflection groups of type $G(m,1,n)$; for $m=1$ and $m=2$, these are the Hecke algebras associated to the Coxeter groups of type A and type B. The cyclotomic Hecke algebras have been introduced in [1] wherein the representation theory was established and used to prove that the deformation is flat.

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Section 2 contains necessary definitions. In Section 3 we exhibit a basis of the algebra $H(m,1,n)$, quite different from the basis presented in [1]; we use this basis it to prove the flatness of the deformation without appealing to the representation theory.

In Sections 4–6 an inductive approach to the representation theory of the algebras $H(m,1,n)$ is developed. It uses heavily the Jucys-Murphy elements which form a maximal commutative set in the algebras $H(m;1;n)$. This approach generalizes the approach developed in [8] for the symmetric groups (and generalized to the Brauer algebras and their q-deformations in [5]). Our aim is not the construction of the representation theory, it has been already constructed in [1]. We rather rederive the representation theory directly from the properties of the Jucys-Murphy operators; we encode the representation bases and the Young multi-tableaux by sets of common eigenvalues of the Jucys-Murphy elements. The irreducible representations of $H(m,1,n)$ are constructed with the help of a new associative algebra; the underlying vector space of this algebra is the tensor product of the cyclotomic Hecke algebra with a free associative algebra generated by standard m-tableaux. The proofs of statements are often omitted in the present letter; details of the results and proofs and also the classical limit of the approach will be published in [6].

Throughout the paper the ground field is the field $\mathbb{C}$ of complex numbers.

2. Cyclotomic Hecke algebras and Jucys-Murphy elements

The braid group of type A (or simply the braid group) on n strands is generated by elements $\sigma_1, \ldots, \sigma_{n-1}$ with the relations:

$$\begin{cases} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for all } i = 1,\ldots,n-2\ , \qquad (2.1)\\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for all } i,j \text{ such that } |i-j|>1\ . \qquad (2.2)\end{cases}$$

The braid group of type B (sometimes called affine braid group) is obtained by adding to the generators σ_i the generator τ with the relations:

$$\begin{cases} \tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau\ , & (2.3)\\ \tau\sigma_i = \sigma_i\tau \qquad \text{for } i>1\ . & (2.4)\end{cases}$$

The elements J_i, $i = 1,\ldots,n$, of the braid group of type B defined inductively by the following initial condition and recursion:

$$J_1 = \tau\ , \quad J_{i+1} = \sigma_i J_i \sigma_i\ , \qquad (2.5)$$

are called Jucys-Murphy elements. It is well known that they form a commutative set of elements. In addition, J_i commutes with all σ_k except σ_{i-1} and σ_i,

$$J_i\sigma_k = \sigma_k J_i \quad \text{if } \ k>i \ \text{ and if } \ k<i-1\ . \qquad (2.6)$$

The affine Hecke algebra $\hat{H}_n$ is the quotient of the group algebra of the braid group of type B by:

$$\sigma_i^2 = (q - q^{-1})\sigma_i + 1 \qquad \text{for all } i = 1, \ldots, n-1 \ . \tag{2.7}$$

The usual Hecke algebra H_n is the algebra generated by elements $\sigma_1, \ldots, \sigma_{n-1}$ with the relations (2.1)–(2.2) and (2.7).

The cyclotomic Hecke algebra $H(m, 1, n)$ is the quotient of the affine Hecke algebra $\hat{H}_n$ by

$$(\tau - v_1) \ldots (\tau - v_m) = 0 \ . \tag{2.8}$$

The algebra $H(m, 1, n)$ is a deformation of the group algebra $\mathbb{C}G(m, 1, n)$ of the complex reflection group $G(m, 1, n)$. In particular, $H(1, 1, n)$ is isomorphic to the Hecke algebra of type A and $H(2, 1, n)$ is the Hecke algebra of type B.

The group $G(m, 1, n)$ is isomorphic to $S_n \wr C_m$, the wreath product of the symmetric group S_n with the cyclic group of m elements.

The specialization of $H(m, 1, n)$ is semi-simple if and only if the numerical values of the parameters satisfy (see [2])

$$1 + q^2 + \cdots + q^{2N} \neq 0 \ \text{ for all } \ N : \ N < n \tag{2.9}$$

and

$$q^{2i} v_j - v_k \neq 0 \ \text{ for all } \ i, j, k \ \text{ such that } \ j \neq k \text{ and } -n < i < n \ . \tag{2.10}$$

In the sequel we work either with a generic cyclotomic Hecke algebra (that is, $v_1, \ldots, v_m$ and q are indeterminates) or in the semi-simple situation with an additional requirement:

$$v_j \neq 0 \ , \ j = 1, \ldots, m \ . \tag{2.11}$$

As n varies, the algebras $H(m, 1, n)$ form an ascending chain of algebras :

$$\{e\} \subset H(m, 1, 0) \subset H(m, 1, 1) \subset \cdots \subset H(m, 1, n) \subset \ldots . \tag{2.12}$$

We shall denote by the same symbols J_i the images of the Jucys-Murphy elements in the cyclotomic Hecke algebra. As a by-product of our construction, we shall see that the set of the Jucys-Murphy elements $\{J_1, \ldots, J_n\}$ is maximal commutative in the generic algebra $H(m, 1, n)$; more precisely, the algebra of polynomials in Jucys-Murphy elements coincides with the algebra generated by the union of the centers of $H(m, 1, k)$ for $k = 1, \ldots, n$.

3 . Normal form for $H(m, 1, n)$

We exhibit a basis for the algebra $H(m, 1, n)$. Several known facts about the chain (with respect to n) of the algebras $H(m, 1, n)$ are reestablished with the help of this basis. In particular, we show that $H(m, 1, n)$ is a flat deformation of the group ring of $G(m, 1, n)$. This was proved in [1] with the use of the representation theory.

Let $\tilde{W}$ be the subalgebra of $H(m, 1, n)$ generated by the elements $\tau, \sigma_1, \ldots, \sigma_{n-2}$.

Proposition 1. *If $\tilde{\mathcal{B}}$ is a linear basis of $\tilde{W}$, the elements*

$$\sigma_j^{-1}\sigma_{j-1}^{-1}\dots\sigma_1^{-1}\tau^\alpha\sigma_1\sigma_2\dots\sigma_{n-1}\tilde{b}\ , \tag{3.1}$$

where $j \in \{0,\dots,n-1\}$, $\alpha \in \{0,\dots,m-1\}$ and $\tilde{b}$ is a basis element of $\tilde{\mathcal{B}}$, form a linear basis of $H(m,1,n)$.

Corollary 2. *(i) The algebra $H(m,1,n)$ is a flat deformation of the group ring $\mathbb{C}G(m,1,n)$; in other words, $H(m,1,n)$ is a free $\mathbb{C}[q,q^{-1},v_1,\dots,v_m]$-module of dimension*

$$\dim\big(H(m,1,n)\big) = |G(m,1,n)| = n!m^n\ . \tag{3.2}$$

(ii) Moreover the subalgebra $\tilde{W}$ is isomorphic to $H(m,1,n-1)$.

We can construct recursively a global normal form for elements of $H(m,1,n)$ using now the Proposition 1 and the Corollary 2, statement (ii). Let R_k be the following set of elements:

$$R_k = \{\sigma_j^{-1}\sigma_{j-1}^{-1}\dots\sigma_1^{-1}\tau^\alpha\sigma_1\sigma_2\dots\sigma_{k-1},\ j=0,\dots,k-1,\ \alpha=0,\dots,m-1\}.$$

Corollary 3. *Any element $x \in H(m,1,n)$ can be written uniquely as $x = u_n u_{n-1}\dots u_1$ where $u_k \in R_k$ for $k = 1,\dots,n$.*

4. Spectrum of Jucys–Murphy elements and Young m-tableaux

We begin to develop an approach, based on the Jucys-Murphy elements, to the representation theory of the chain (with respect to n) of the cyclotomic Hecke algebras $H(m,1,n)$. This is a generalization of the approach of [8].

1. The first step consists in construction of all representations of $H(m,1,n)$ verifying two conditions. First, the Jucys-Murphy elements $J_1,\dots,J_n$ are represented by semi-simple (diagonalizable) operators. Second, for every $i = 1,\dots,n-1$ the action of the sub-algebra generated by J_i, J_{i+1} and σ_i is completely reducible. We shall use the name C-representations (C is the first letter in "completely reducible") for these representations. At the end of the construction we shall see that all irreducible representations of $H(m,1,n)$ are C-representations.

Following [8] we denote by $\mathrm{Spec}(J_1,\dots,J_n)$ the set of strings of eigenvalues of the Jucys-Murphy elements in the set of C-representations: $\Lambda = (a_1^{(\Lambda)},\dots,a_n^{(\Lambda)})$ belongs to the set $\mathrm{Spec}(J_1,\dots,J_n)$ if there is a vector e_Λ in the space of some C-representation such that $J_i(e_\Lambda) = a_i^{(\Lambda)}e_\Lambda$ for all $i = 1,\dots,n$. Every C-representation possesses a basis formed by vectors e_Λ (this is a reformulation of the first condition in the definition of C-representations). Since σ_k commutes with J_i for $k > i$ and $k < i-1$, the action of σ_k on a vector e_Λ, $\Lambda \in \mathrm{Spec}(J_1,\dots,J_n)$, is "local" in the sense that $\sigma_k(e_\Lambda)$ is a linear combination of $e_{\Lambda'}$ such that $a_i^{(\Lambda')} = a_i^{(\Lambda)}$ for $i \neq k, k+1$.

2. Affine Hecke algebra $\hat{H}_2$. Consider the affine Hecke algebra $\hat{H}_2$, which is generated by X, Y and σ with the relations:

$$XY = YX, \quad Y = \sigma X \sigma, \quad \sigma^2 = (q - q^{-1})\sigma + 1 \ . \tag{4.1}$$

It is realized by J_i, J_{i+1} and σ_i for all $i = 1, \ldots, n-1$. We reproduce here the result of [4] concerning the classification of irreducible representations with diagonalizable X and Y of $\hat{H}_2$.

There are one-dimensional and two-dimensional irreducible representations.

- The one-dimensional irreducible representations are given by

$$X \mapsto a, \quad Y \mapsto q^{\pm 2} a, \quad \sigma \mapsto \pm q^{\pm 1} \ . \tag{4.2}$$

- The two-dimensional irreducible representations are given by

$$\sigma \mapsto \begin{pmatrix} 0 & 1 \\ 1 & q - q^{-1} \end{pmatrix}, \quad X \mapsto \begin{pmatrix} a & -(q - q^{-1})b \\ 0 & b \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} b & (q - q^{-1})b \\ 0 & a \end{pmatrix},$$

  with $b \neq a$ in order for X and Y to be diagonalizable and with $b \neq q^{\pm 2} a$ to ensure irreducibility. By a change of basis we transform X and Y to a diagonal form:

$$\sigma \mapsto \begin{pmatrix} \frac{(q-q^{-1})b}{b-a} & 1 - \frac{(q-q^{-1})^2 ab}{(b-a)^2} \\ 1 & -\frac{(q-q^{-1})a}{b-a} \end{pmatrix}, \quad X \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}. \tag{4.3}$$

3. We return to strings of eigenvalues of Jucys-Murphy elements.

Proposition 4. *Let $\Lambda = (a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) \in \mathrm{Spec}(J_1, \ldots, J_n)$ and let e_Λ be a corresponding vector. Then:*

(a) We have $a_i \neq a_{i+1}$.

(b) If $a_{i+1} = q^{\pm 2} a_i$ then $\sigma_i(e_\Lambda) = \pm q^{\pm 1} e_\Lambda$.

(c) If $a_{i+1} \neq q^{\pm 2} a_i$ then $\Lambda' = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) \in \mathrm{Spec}(J_1, \ldots, J_n)$; moreover, the vector $\sigma_i(e_\Lambda) - \frac{(q-q^{-1})a_{i+1}}{a_{i+1}-a_i} e_\Lambda$ corresponds to the string Λ' (see (4.3) with $b = a_{i+1}$ and $a = a_i$).

4. Content strings.

Definition 5. *A content string $(a_1, \ldots, a_n)$ is a string of numbers satisfying the following conditions:*

(c1) $a_1 \in \{v_1, \ldots, v_m\}$;

(c2) for all $j > 1$: if $a_j = v_k q^{2z}$ for some k and $z \neq 0$, then $\{v_k q^{2(z-1)}, v_k q^{2(z+1)}\} \cap \{a_1, \ldots, a_{j-1}\}$ is non-empty;

(c3) if $a_i = a_j = v_k q^{2z}$ with $i < j$ for some k and z, then $\{v_k q^{2(z-1)}, v_k q^{2(z+1)}\} \subset \{a_{i+1}, \ldots, a_{j-1}\}$.

The set of content strings of length n we denote by $\mathrm{Cont}(n)$.

Here is the "cyclotomic" analogue of the Theorem 5.1 in [8] and the Proposition 4 in [4].

Proposition 6. *If a string of numbers $(a_1, \ldots, a_n)$ belongs to* $\mathrm{Spec}(J_1, \ldots, J_n)$ *then it belongs to* $\mathrm{Cont}(n)$.

5. Young m-diagrams and m-tableaux. A Young m-diagram, or m-partition, is an m-tuple of Young diagrams $\lambda^{(m)} = (\lambda_1, \ldots, \lambda_m)$. The length of a Young diagram λ is the number of nodes of the diagram and is denoted by $|\lambda|$. By definition the length of an m-tuple $\lambda^{(m)} = (\lambda_1, \ldots, \lambda_m)$ is

$$|\lambda^{(m)}| := |\lambda_1| + \cdots + |\lambda_m| \ . \tag{4.4}$$

Let the length of the m-tuple be n. We place the numbers $1, \ldots, n$ in the nodes of these diagrams in such a way that in every diagram the numbers in the nodes are in ascending order along rows and columns in right and down directions. This is a standard Young m-tableau of shape $\lambda^{(m)}$.

We associate to each node of a Young m-diagram a number (the "content") which is $v_k q^{2(s-r)}$ for the node in the line r and column s in the kth tableau of the m-tuple.

Proposition 7. *There is a bijection between the set of standard Young m-tableaux of length n and the set* $\mathrm{Cont}(n)$.

5 . Construction of representations

We proceed as in [7]. We first define an algebra structure on a tensor product of the algebra $H(m, 1, n)$ with a free associative algebra generated by the standard m-tableaux corresponding to m-partitions of n. Then, by evaluation (with the help of the simplest one-dimensional representation of $H(m, 1, n)$) from the right, we build representations.

Define, for any σ_i among the generators $\sigma_1, \ldots, \sigma_{n-1}$ of $H(m, 1, n)$,

$$\sigma_i(\alpha, \beta) := \sigma_i + (q - q^{-1}) \frac{\beta}{\alpha - \beta} \ . \tag{5.1}$$

This is called the "Baxterization" of the elements σ_i. The parameters α and β are called spectral parameters. We collect the relations for the Baxterized generators σ_i.

Proposition 8. *The following relations hold:*

$$\begin{aligned} &\sigma_i(\alpha,\beta)\sigma_i(\beta,\alpha) = f(\alpha,\beta)f(\beta,\alpha) \qquad \text{with } f(\alpha,\beta) = \tfrac{q\alpha - q^{-1}\beta}{\alpha-\beta}, \\ &\sigma_i(\alpha,\beta)\sigma_{i+1}(\alpha,\gamma)\sigma_i(\beta,\gamma) = \sigma_{i+1}(\beta,\gamma)\sigma_i(\alpha,\gamma)\sigma_{i+1}(\alpha,\beta), \\ &\sigma_i(\alpha,\beta)\sigma_j(\gamma,\delta) = \sigma_j(\gamma,\delta)\sigma_i(\alpha,\beta) \qquad \text{if } |i-j|>1. \end{aligned} \tag{5.2}$$

Let $\lambda^{(m)}$ be an m-partition of length n. Consider a set of free generators labeled by standard m-tableaux of shape $\lambda^{(m)}$; for a standard m-tableau $X_{\lambda^{(m)}}$ we denote by $\mathcal{X}_{\lambda^{(m)}}$ the corresponding free generator and by $c(X_{\lambda^{(m)}}|i)$ the content (see the preceding Section for the definition) of the node carrying the number i.

Proposition 9. *The relations*

$$\Big(\sigma_i + \frac{(q-q^{-1})c(X_{\lambda^{(m)}}|i+1)}{c(X_{\lambda^{(m)}}|i) - c(X_{\lambda^{(m)}}|i+1)}\Big)\cdot \mathcal{X}_{\lambda^{(m)}} = \mathcal{X}_{\lambda^{(m)}}s_i \cdot \Big(\sigma_i + \frac{(q-q^{-1})c(X_{\lambda^{(m)}}|i)}{c(X_{\lambda^{(m)}}|i+1) - c(X_{\lambda^{(m)}}|i)}\Big) \tag{5.3}$$

and

$$\Big(\tau - c(X_{\lambda^{(m)}}|1)\Big)\mathcal{X}_{\lambda^{(m)}} = 0 \tag{5.4}$$

are compatible with the relations for the generators $\tau, \sigma_1, \dots, \sigma_{n-1}$ of the algebra $H(m,1,n)$. The element $\mathcal{X}_{\lambda^{(m)}}s_i$ corresponds to the m-tableau obtained from $X_{\lambda^{(m)}}$ by exchanging the nodes with numbers i and $(i+1)$. The tableau $\mathcal{X}_{\lambda^{(m)}}s_i$ is of the same (as $\mathcal{X}_{\lambda^{(m)}}$) shape $\lambda^{(m)}$. If the resulting m-tableau is not standard, we put $\mathcal{X}_{\lambda^{(m)}}s_i = 0$.

We explain the meaning of the word "compatible" used in the formulation of the Proposition.

Let $\mathcal{F}$ be the free associative algebra generated by $\tilde{\tau}$, $\tilde{\sigma}_1, \dots, \tilde{\sigma}_{n-1}$. The algebra $H(m,1,n)$ is naturally the quotient of $\mathcal{F}$.

Let $\mathbb{C}[\mathcal{X}]$ be the free associative algebra whose generators $\mathcal{X}_{\lambda^{(m)}}$ range over all standard m-tableaux of shape $\lambda^{(m)}$ for all m-partitions $\lambda^{(m)}$ of n.

Consider an algebra structure on the space $\mathbb{C}[\mathcal{X}]\otimes\mathcal{F}$ for which: (i) the map $\iota_1 : x \mapsto x\otimes 1$, $x\in\mathbb{C}[\mathcal{X}]$, is an isomorphism of $\mathbb{C}[\mathcal{X}]$ with its image with respect to ι_1; (ii) the map $\iota_2 : \phi \mapsto 1\otimes\phi$, $\phi\in\mathcal{F}$, is an isomorphism of $\mathcal{F}$ with its image with respect to ι_2; (iii) the formulas (5.3)-(5.4), extended by associativity, provide the rules to rewrite elements of the form $(1\otimes\phi)(x\otimes 1)$, $x\in\mathbb{C}[\mathcal{X}]$, $\phi\in\mathcal{F}$, as elements of $\mathbb{C}[\mathcal{X}]\otimes\mathcal{F}$.

The "compatibility" means that we have an induced structure of an associative algebra on the space $\mathbb{C}[\mathcal{X}]\otimes H(m,1,n)$. More precisely, if we multiply any relation of the cyclotomic Hecke algebra $H(m,1,n)$ (the relation is viewed as an element of the free algebra $\mathcal{F}$) from the right by a generator $\mathcal{X}_{\lambda^{(m)}}$ (this is a combination of the form "relation of $H(m,1,n)$ times $\mathcal{X}_{\lambda^{(m)}}$") and use the "instructions" (5.3)-(5.4) to move all appearing $\mathcal{X}$'s to the left (the free generator

changes but the expression stays always linear in $\mathcal{X}$) then we obtain a combination of terms of the form "$\mathcal{X}$ times relation of $H(m,1,n)$".

Let $|\rangle$ be a "vacuum" - a basic vector of a one-dimensional $H(m,1,n)$-module; for example, $\sigma_i|\rangle = q|\rangle$ for all i and $\tau|\rangle = v_1|\rangle$. Moving, in the expressions $\phi\mathcal{X}_{\lambda^{(m)}}|\rangle$, $\phi \in H(m,1,n)$, the elements $\mathcal{X}$'s to the left and using the module structure, we build, due to the compatibility, a representation of $H(m,1,n)$ on the vector space $V_{\lambda^{(m)}}$ spanned by $\mathcal{X}_{\lambda^{(m)}}|\rangle$. We shall, by a slight abuse of notation, denote the symbol $\mathcal{X}_{\lambda^{(m)}}|\rangle$ again by $\mathcal{X}_{\lambda^{(m)}}$. This procedure leads to the following formulas for the action of the generators $\tau, \sigma_1, \ldots, \sigma_{n-1}$ on the basis vectors $\{\mathcal{X}_{\lambda^{(m)}}\}$ of $V_{\lambda^{(m)}}$:

$$\begin{aligned} \sigma_i \,:\, \mathcal{X}_{\lambda^{(m)}} \quad \mapsto \quad & -(q-q^{-1})\frac{c(X_{\lambda^{(m)}}|i+1)}{c(X_{\lambda^{(m)}}|i)-c(X_{\lambda^{(m)}}|i+1)}\mathcal{X}_{\lambda^{(m)}} \\ & + \frac{qc(X_{\lambda^{(m)}}|i+1)-q^{-1}c(X_{\lambda^{(m)}}|i)}{c(X_{\lambda^{(m)}}|i+1)-c(X_{\lambda^{(m)}}|i)}\mathcal{X}_{\lambda^{(m)}}s_i \end{aligned} \tag{5.5}$$

and

$$\tau \,:\, \mathcal{X}_{\lambda^{(m)}} \mapsto c(X_{\lambda^{(m)}}|1)\mathcal{X}_{\lambda^{(m)}} \ . \tag{5.6}$$

As before, it is assumed here that $\mathcal{X}_{\lambda^{(m)}}s_i = 0$ if $X_{\lambda^{(m)}}s_i$ is not a standard m-tableau.

6 Completeness

We conclude the study of representations of $H(m,1,n)$. First, with the construction of the preceding Section, we complete the description of the set $\mathrm{Spec}(J_1,\ldots,J_n)$ started in Section 4. Then we check that we obtain within this approach all irreducible representations of the algebra $H(m,1,n)$.

We underline that the restrictions (2.9)–(2.11) are essential for the statements below.

Proposition 10. *The set* $\mathrm{Spec}(J_1,\ldots,J_n)$*, the set* $\mathrm{Cont}(n)$ *and the set of standard* m*-tableaux are in bijection.*

Theorem 11. *The representations* $V_{\lambda^{(m)}}$ *(labeled by the* m*-partitions) of the algebra* $H(m,1,n)$ *constructed in the Section 5 are irreducible and pairwise non-isomorphic. Moreover the sum of the squares of the dimensions of the constructed representations equals the dimension of* $H(m,1,n)$*.*

As a by-product we obtain that the algebra $H(m,1,n)$ is semi-simple under the restrictions (2.9)–(2.11).

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