β-expansions for cubic Pisot numbers

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Abstract. Real numbers can be represented in an arbitrary base $\beta > 1$ using the transformation $T_\beta : x \to \beta x \pmod 1$ of the unit interval; any real number $x \in [0, 1]$ is then expanded into $d_\beta(x) = (x_i)_{i\geq 1}$ where $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$.
The closure of the set of the expansions of real numbers of $[0, 1[$ is a subshift of $\{a \in \mathbb{N} \mid a < \beta\}^{\mathbb{N}}$, called the beta-shift. This dynamical system is characterized by the beta-expansion of 1; in particular, it is of finite type if and only if $d_\beta(1)$ is finite; β is then called a simple beta-number.
We first compute the beta-expansion of 1 for any cubic Pisot number. Next we show that cubic simple beta-numbers are Pisot numbers.

Introduction

Representations of real numbers in an arbitrary base $\beta > 1$, called beta-expansions, have been introduced by Rényi ([14]). They arise from the orbits of the piecewise-monotone transformation of the unit interval : $T_\beta : x \to \beta x \pmod 1$. Such transformations were extensively studied in ergodic theory ([13]).

More precisely, any real number $x \in [0, 1]$ is expanded into $d_\beta(x) = (x_i)_{i\geq 1}$ where $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$. The nonnegative integers d_i are elements of the digit alphabet $A = \{a \in \mathbb{N} \mid a < \beta\}$. These representations generalize standard representations in an integral base to a real base; indeed the beta-expansion of any real number of $[0, 1[$ can equivalently be obtained by the greedy algorithm. Only the beta-expansion of 1 differs.

Properties of beta-expansions are strongly related to symbolic dynamics ([4]). The closure of the set of infinite sequences, appearing as beta-expansions of numbers of the interval $[0, 1[$, is a dynamical system, that is, a closed shift-invariant subset of $A^{\mathbb{N}}$, called the *beta-shift*.

An important property of the beta-shift is that its nature is entirely determined, in a combinatorial manner, by the beta-expansion of 1: the beta-shift is sofic, that is to say the set of its finite factors is recognized by a finite automaton, if and only the beta-expansion of 1 is eventually periodic ([3]); it is of finite type, that is to say the set of its finite factors is defined by forbidding a finite set of words, if and only if the beta-expansion of 1 is finite ([12]).

When the beta-expansion of 1 is eventually periodic, β is called a *beta-number* and when the beta-expansion of 1 is finite, β is said to be a *simple beta-number*.

The eventually periodic beta-expansions were extensively studied by Bertrand ([3]) and by Schmidt ([15]). In particular, it is known that Pisot numbers are beta-numbers. Concerning Salem numbers, we only know that if β is a Salem number of degree 4, then the beta-expansion of 1 is eventually periodic ([5]). It is conjectured that Salem numbers of degree 6 are still beta-numbers, but not all Salem numbers of degree 8 ([7]).

The domain of the Galois conjugates of all beta-numbers was also investigated independently by Solomyak ([16]) and by Flatto, Lagarias and Poonen ([8]).

For a general presentation of the beta-shift one can refer to [9].

In the following, we summarize properties of beta-numbers. We compute the beta-expansion of 1 for any cubic Pisot number and we establish a characterization of cubic simple beta-numbers, showing that they are Pisot numbers.

A very close problem, seen from the point of view of numeration systems, was studied by Akiyama ([1]). He showed that in the cubic case, the real numbers of the set $\mathbb{N}[\beta^{-1}]$ have a finite beta-expansion if and only β is a Pisot unit and 1 has a finite beta-expansion. This finiteness problem is equivalent to a problem of fractal tiling generated by Pisot numbers.

1 Beta-numbers

Real numbers can be represented in an arbitrary base $\beta > 1$ using the transformation $T_\beta : x \to \beta x \pmod 1$ of the unit interval; any real number $x \in [0, 1]$ is then expanded into $d_\beta(x) = (x_i)_{i \geq 1}$ where $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$. When a beta-expansion is of the form uv^ω, the expansion is said to be *eventually periodic*. If a representation ends with infinitely many zeros, like $u0^\omega$, it is said to be *finite* and the ending zeros are omitted.

Let us denote by S_β the closure of all beta-expansions of real numbers of $[0, 1[$ and by σ the (one-sided) shift defined by $\sigma((x_i)_{i \geq 1}) = (x_{i+1})_{i \geq 1}$. The set S_β endowed with the shift is called the beta-shift, it is a subshift of $A^{\mathbb{N}}$, A being the digit set, *i.e.*, $A = \{a \in \mathbb{N} \mid a < \beta\}$.

An important property ([13]) of the beta-shift S_β is that its nature is entirely determined by $d_\beta(1)$ the beta-expansion of 1. Indeed, setting $d^*(1) = d_\beta(1)$ if $d_\beta(1)$ is infinite and $d^*(1) = (d_1 d_2 \dots d_{n-1}(d_n - 1))^\omega$ if $d_\beta(1) = d_1 d_2 \dots d_{n-1} d_n$, a sequence x of nonnegative integers belongs to S_β if and only if it satisfies the following lexicographical order conditions: $\forall p \geq 0, \quad \sigma^p(x) \leq d^*(1)$.

Recall that the beta-expansion of 1 also can be characterized ([13]) by lexicographical order conditions: let $d = (d_i)_{i \geq 1}$ be a sequence of nonnegative integers different from 10^ω, such that $\sum_{i \geq 1} d_i \beta^{-i} = 1$, with $d_1 \geq 1$ and for $i \geq 2$, $d_i \leq d_1$, then d is the beta-expansion of 1 if and only if for all $p \geq 1$, $\sigma^p(d) < d$.

We recall that an algebraic integer β strictly greater than 1 is called a *Perron number* if all its Galois conjugates have modulus strictly less than β, a *Pisot number* if all its Galois conjugates have modulus strictly less than 1, and a *Salem number* if all its conjugates are less than 1 in modulus and at least one conjugate has modulus 1.

Let β be a beta-number. Denote by $d_\beta(1) = d_1 \ldots d_n (d_{n+1} \ldots d_{n+p})^\omega$, where n and p are chosen minimal, the beta-expansion of 1. Then the adjacency matrix $\mathcal{M}_\beta$ of the finite automaton recognizing the set of its finite factors (Fig.1) is a *primitive* (*i.e.*, its associated graph is strongly connected and the lengths of its cycles are relatively prime) nonnegative integral matrix whose spectral radius is β; so, from the Perron-Frobenius theorem, β is a Perron number.

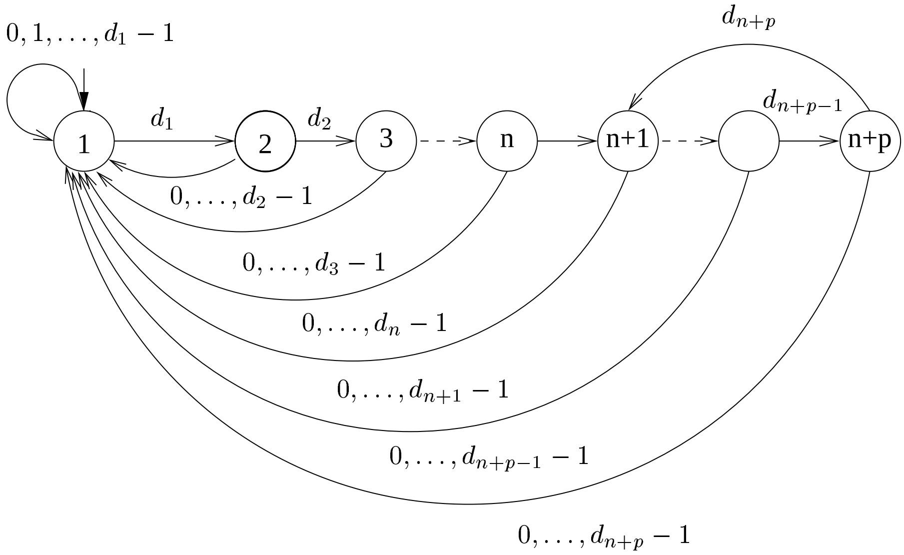

Fig. 1. Automaton recognizing the set of the finite factors of S_β

The characteristic polynomial of $\mathcal{M}_\beta$

$$P(X) = X^{n+p} - \sum_{i=1}^{n+p} d_i X^{n+p-i} - X^n + \sum_{i=1}^{n} d_i X^{n-i}$$

is called, following the terminology introduced by Hollander ([11]), the associated *beta-polynomial*.

As P is a multiple of the minimal polynomial M_β of β, $P(0) = d_{n+p} - d_n$ is a multiple of $|M_\beta(0)| = |\prod \beta_i|$, where β_i runs over the set of algebraic conjugates of β. So, we get that $|\prod \beta_i|$ has to be smaller than $\lfloor \beta \rfloor$.

As a consequence, in the quadratic case, the only beta-numbers are the Pisot numbers. Conversely, it is known that if β is a Pisot number then β is a beta-number ([2]). An important gap remains between Pisot and Perron numbers.

Example 1. The quadratic number $\beta = (1 + \sqrt{13})/2$ is not a beta-number since $M_\beta(X) = X^2 - X - 3$ and $M_\beta(0) > \lfloor \beta \rfloor$.

Let β be the Pisot number $(3+\sqrt{5})/2$, then β is a beta-number and $d_\beta = 21^\omega$.

Let β be the golden ratio $(1+\sqrt{5})/2$, then β is a simple beta-number and $d_\beta(1) = 11$.

On the other hand, the domain of the Galois conjugates of beta-numbers was studied by Solomyak ([16]) and independently by Flatto, Lagarias and Poonen ([8]). They showed in particular that if the beta-expansion of 1 is eventually periodic then the Galois conjugates of β have modulus less than the golden ratio $(1+\sqrt{5})/2$. It was already known (see [9]) that β cannot have a Galois conjugate greater than 1.

Solomyak ([16]) proved that the topological closure of conjugates of beta-numbers and the one of conjugates of simple beta-numbers are the same. However, there is an important difference between the conjugates of beta-numbers and the ones of simple beta numbers: if β is a simple beta-number then β has no algebraic conjugate that is a nonnegative real number.

Indeed, let β be a simple beta-number and set $d_\beta(1) = d_1 \dots d_n$. Consider

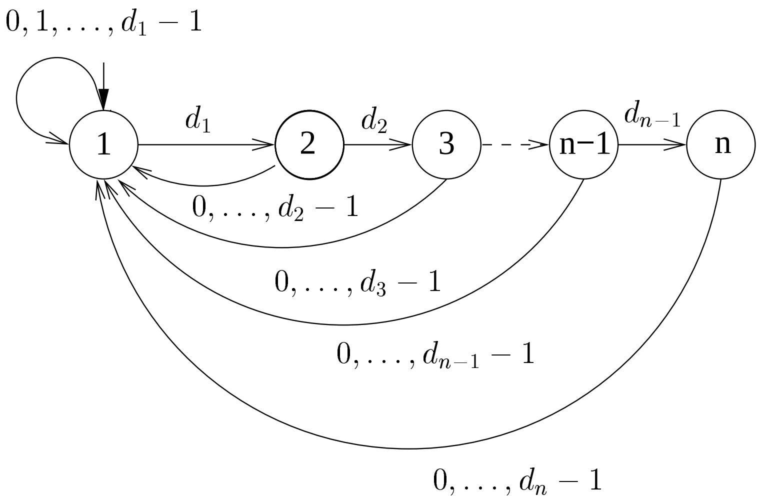

Fig. 2. Automaton recognizing the set of the finite factors of S_β

the finite automaton recognizing the set of the finite factors of the associated beta-shift (Fig. 2). Let $\mathcal{M}_\beta$ be the transition matrix of this automaton. The characteristic polynomial of $\mathcal{M}_\beta$, which is called the associated *beta-polynomial*,

$$P(X) = X^n - \sum_{i=1}^{n} d_i X^{n-i}$$

has only one positive real root.

Example 2. Salem numbers are roots of reciprocal polynomials. Thus if β is a Salem number, $1/\beta > 0$ is a Galois conjugate of β, and so β is not a simple beta-number.

The previous conditions are sufficient for a quadratic algebraic integer to be a simple beta-number.

Proposition 1. *[10] The simple beta-numbers of degree 2 are exactly the quadratic Pisot numbers without a positive real Galois conjugate. They are the positive roots of the polynomials*

$$X^2 - aX - b \quad \text{with} \quad a \geq b \geq 1,$$

The beta-expansion of 1 is then $d_\beta(1) = ab$.

Example 3. The minimal polynomial of $(1+\sqrt{5})/2$ is $X^2 - X - 1$, $(1+\sqrt{5})/2$ is a simple beta-number and $d_\beta(1) = 11$.

The minimal polynomial of $(3+\sqrt{5})/2$ is $X^2 - 3X + 1$, therefore $(3+\sqrt{5})/2$ is not a simple beta-number.

2 Beta-expansions of 1 for cubic Pisot numbers

Let us recall the characterization of cubic Pisot numbers due to Akiyama ([1])

Theorem 1 (Akiyama [1]). *Let $\beta > 1$ be a cubic number and let*

$$M_\beta(x) = X^3 - aX^2 - bX - c$$

be its minimal polynomial.

Then β is a Pisot number if and only if it both inequalities

$$|b-1| < a + c \quad \text{and} \quad (c^2 - b) < sgn(c)(1 + ac)$$

hold.

Remark 1. Note that a must be a nonnegative integer.

The following theorem gives the β-expansion of 1 for any cubic Pisot number.

Theorem 2. *Let β be a cubic Pisot number and let*

$$M_\beta(x) = X^3 - aX^2 - bX - c$$

be its minimal polynomial. Then the beta-expansion of 1 is

- *Case 1 : When $b \geq a$, then $d_\beta(1) = (a+1)(b-1-a)(a+c-b)(b-c)c$.*
- *Case 2: When $0 \leq b \leq a$, if $c > 0$, $d_\beta(1) = abc$, otherwise,*

$$d_\beta(1) = a[(b-1)(c+a)]^\omega.$$

– *Case 3: When* $-a < b < 0$, *if* $b + c \geq 0$, *then* $d_\beta(1) = (a-1)(a+b)(b+c)c$, *otherwise* $d_\beta(1) = (a-1)(a+b-1)(a+b+c-1)^\omega$

– *Case 4: When* $b \leq -a$, *let* k *be the integer of* $\{2, 3, \ldots, a-2\}$ *such that, denoting* $e_k = 1 - a + (a-2)/k$, $e_k \leq b + c < e_{k-1}$.

- *If* $b(k-1) + c(k-2) \leq (k-2) - (k-1)a$, $d_\beta(1) = d_1 \ldots d_{2k+2}$ *with*

$$
\begin{aligned}
&d_1 = a - 2,\\
&d_{k+2-i} = -(k+3-i) + a(k+2-i) + b(k+1-i) + c(k-i), 3 \leq i \leq k\\
&d_k = -k + ak + b(k-1) + c(k-2)\\
&d_{k+1} = -(k-1) + ak + bk + c(k-1)\\
&d_{k+2} = -(k-2) + a(k-1) + bk + ck\\
&d_{2k+2-i} = -(i-2) + a(i-1) + bi + c(i+1) \quad k \geq 3, 2 \leq i \leq (k-1)\\
&d_{2k+1} = b + 2c \quad \text{and} \quad d_{2k+2} = c.
\end{aligned}
$$

- *If* $b(k-1) + c(k-2) > (k-2) - (k-1)a$, *let* m *be the integer defined by* $m = \min\{i \in \mathbb{N}$ *such that* $(i+1)b + ic > i - (i+1)a\}$.

  When $m = 1$, $d_\beta(1) = (a-2)(2a+b-2)(2a+2b+c-2)(2a+2b+2c-2)^\omega$.

  When $m > 1$, $d_\beta(1) = d_1 d_2 \ldots d_{m+2} d_{m+3}^\omega$, *with*

$$
\begin{aligned}
&d_1 = a - 2, \quad d_2 = 2a + b - 3,\\
&d_{m+3-i} = 2a + b - 3 + (m+1-i)(a+b+c-1) \quad m \geq 3, 3 \leq i \leq m,\\
&d_{m+1} = 2a + b - 2 + (m-1)(a+b+c-1),\\
&d_{m+2} = a + b - 1 + m(a+b+c-1),\\
&d_{m+3} = (m+1)(a+b+c-1).
\end{aligned}
$$

Example 4. When $a \geq b \geq 0$ and $c > 0$, we obtain the only beta-expansion of 1 of length 3.

The smallest Pisot number has $M_\beta = X^3 - X - 1$ as minimal polynomial, it is a simple beta-number and $d_\beta(1) = 10001$.

The positive root β of $M_\beta = X^3 - 3X^2 + 2X - 2$ is a simple beta-number and $d_\beta(1) = 2102$.

The case where $b \leq -a$ shows that from a cubic simple beta-number, we can obtain an arbitrary long beta-expansion of 1. For any integer k greater than or equal to 2, the real root β of the irreducible polynomial $X^3 - (k+2)X^2 + 2kX - k$, is a simple beta number whose integer part is equal to k, and the beta-expansion of 1 has length $2k+2$. For $k = 2$, we get $d_\beta(1) = 221002$; for $k = 3$, we get $d_\beta(1) = 31310203$.

Example 5. The greatest positive root β of $M_\beta = X^3 - 2X^2 - X + 1$ is a beta-number and $d_\beta(1) = 2(01)^\omega$.

If β is the positive root of $X^3 - 5X^2 + 3X - 2$, then $d_\beta(1) = 413^\omega$. When β is the greatest positive root of $X^3 - 5X^2 + X + 2$, then $d_\beta(1) = 431^\omega$.

For any integer k greater than or equal to 3, the real root β of the irreducible polynomial $X^3 - (k+2)X^2 + (2k-1)X - (k-1)$, is a beta number whose integer part is equal to k, and the beta-expansion of 1 is eventually periodic of period

1, the length of its preperiod k. For $k = 3$, we get $d_\beta(1) = 3302^\omega$; for $k = 4$, we get $d_\beta(1) = 42403^\omega$.

Proof. It is known that Pisot numbers are beta-numbers, thus, for any cubic Pisot number β, the beta-expansion of 1 is finite or eventually periodic. In any case, we first compute the associated beta-polynomial P. Next we prove that the sequence $d = (d_i)_{i\geq 1}$ of nonnegative integers obtained from the beta-polynomial satisfy lexicographical order conditions: for all $p \geq 1$, $\sigma^p(d) < d$.

First of all, we recall that, from Theorem 1, a cubic number β, greater than 1 and having

$$M_\beta(X) = X^3 - aX^2 - bX - c$$

as minimal polynomial, is a cubic Pisot number if and only if it both

$$|b-1| < a + c \quad \text{and} \quad (c^2 - b) < sgn(c)(1 + ac)$$

hold.

Denote by Q the *complementary factor* of the beta-polynomial P defined by $P(X) = M_\beta(X)Q(X)$. As we shall see in what follows, the value of Q depends upon the value of the coefficients of M_β.

Case 1: When $b > a$, as β is a Pisot number, from Theorem 1, c is a positive integer. In this case, the complementary factor is $Q(X) = X^2 - X + 1$ and $d_\beta(1) = (a+1)(b-1-a)(a+c-b)(b-c)c$.

Indeed, as $(c^2 - b) < sgn(c)(1 + ac)$ and $c > 0$, we get $c \leq a + 1$. As $|b - 1| < a + c$, we get $b - 1 - a \leq a$ and $0 \leq a - b + c$. From $b > a$, we get that $0 \leq b - a - 1$ and, as $c \leq a + 1$, that $a - b + c \leq a$. Finally as $0 \leq a - b + c \leq a$, we obtain $0 \leq b - c \leq a$.

Case 2: When $0 \leq b \leq a$, the complementary factor is then $Q(X) = 1$ and the associated beta-polynomial is equal to the minimal polynomial.

If $c > 0$, then $d_\beta(1) = abc$. Indeed, as $(c^2 - b) < sgn(c)(1 + ac)$, we get $c \leq a$.

If $c < 0$, then $d_\beta(1) = a[(b-1)(a+c)]^\omega$. As $|b-1| < a+c$, we get $b - 1 \leq a - 2$. As $(c^2 - b) < sgn(c)(1 + ac)$, we get that $c \geq -a$ and, consequently, $0 \leq c + a \leq a - 1$.

Case 3: When $-a < b < 0$, if $b + c \geq 0$ then the complementary factor is $Q(X) = X + 1$ and $d_\beta(1) = (a-1)(a+b)(b+c)c$. Indeed, as $-a < b < 0$, we obtain $1 \leq a + b \leq a - 1$. Since $b + c \geq 0$, c is a positive integer. From $(c^2 - b) < sgn(c)(1 + ac)$, we get that $c \leq a - 1$ and $b + c \leq a - 2$.

If $b + c < 0$, then $Q(X) = 1$ and $d_\beta(1) = (a-1)(a+b-1)(a+b+c-1)^\omega$. As $-a < b < 0$, we get $0 \leq a + b - 1 \leq a - 2$. From $|b - 1| < a + c$, we get that $1 \leq a + b + c - 1$ and as $b + c < 0$, we obtain $a + b + c - 1 \leq a - 2$.

Case 4: First of all, since $|b-1| < a+c$, we get $-a+2 \leq b+c$. Moreover as $b \leq -a$, we get $c \geq 2$ and as $(c^2 - b) < sgn(c)(1+ac)$, we obtain $c \leq a-2$, thus $b + c \leq -2$. So, there exists an integer k in $\{2, 3, \ldots, a-2\}$, such that, denoting $e_k = 1 - a + (a-2)/k$, $e_k \leq b + c < e_{k-1}$.

When $b(k-1) + c(k-2) \leq (k-2) - (k-1)a$, the complementary factor is

$$Q(X) = \frac{(X^k - 1)(X^{k+1} - 1)}{(X-1)^2}$$

and $d_\beta(1) = d_1 \dots d_{2k+2}$ with

$$\begin{aligned}
&d_1 = a-2,\\
&d_{k+2-i} = -(k+3-i) + a(k+2-i) + b(k+1-i) + c(k-i), k \geq 3, 3 \leq i \leq k\\
&d_k = -k + ak + b(k-1) + c(k-2)\\
&d_{k+1} = -(k-1) + ak + bk + c(k-1)\\
&d_{k+2} = -(k-2) + a(k-1) + bk + ck\\
&d_{2k+2-i} = -(i-2) + a(i-1) + bi + c(i+1) \quad k \geq 3, 2 \leq i \leq (k-1)\\
&d_{2k+1} = b + 2c \quad \text{and} \quad d_{2k+2} = c.
\end{aligned}$$

We now verify that the lexicographical order conditions on $d_\beta(1)$ are satisfied.

As $2 \leq c \leq a-2$ and $b+c \leq -2$, we get $d_{2k+1} \leq a-4$. From $e_k \leq b+c$ and $b(k-1)+c(k-2) \leq (k-2)-(k-1)a$, we get $d_{2k+1} \geq 0$.

For $k \leq 3$ and $2 \leq i \leq k-1$, $d_{2k+2-i} = -(i-2)+a(i-1)+bi+c(i+1)$. As $b+c < e_i$, we get $d_{2k+2-i} < c$. As $-a+2 \leq b+c$ and $b+2c \geq 0$, we get $d_{2k+2-i} \geq i$.

As $e_k \leq b+c$, we obtain $d_{k+2} \geq 0$. Since $c \leq a-2$, $d_{k+1} > d_{k+2}$ and since $b+c \leq -2$, $d_k > d_{k+1}$. Moreover from $b(k-1)+c(k-2) \leq (k-2)-(k-1)a$, we get $d_k \leq a-2$.

For $k \leq 3$, as $|b-1| < a+c$, we obtain $d_2 < \cdots < d_{k-1}$. As $b+c < e_{k-1}$ and $b+2c \leq 0$, we get $d_{k-1} < a-2$. Moreover from $c \leq a-2$ and $a+b+c-1 > 0$, we get that $d_2 = 2a+b-3$ is nonnegative.

All d_i's are smaller than d_1, only d_{2k+2} and d_k can be equal to d_1. Therefore we have to verify that $d_2 \geq d_{k+1}$ when $k \geq 3$ (otherwise $d_2 = d_k$ and $d_k > d_{k+1}$). If $d_k = a-2$, then $b+c = e_k$, and $d_{k+1} = a-c-1$. As $a+b+c-1 > 0$, we obtain $d_{k+1} \leq d_2$. In case of equality, if $k=3$, then $d_3 = d_k$ and $d_k > d_{k+2}$, otherwise $d_3 > d_2$ and $d_{k+1} > d_{k+2}$, therefore $d_3 > d_{k+2}$.

So lexicographical order conditions are satisfied and $d_1 \dots d_{2k+2}$ is the beta-expansion of 1.

When $b(k-1)+c(k-2) > (k-2)-(k-1)a$, as $b \leq -a$, we get $k \geq 3$. Let m be the integer defined by $m = \min\{i \in \mathbb{N} \text{ such that } (i+1)b+ic > i-(i+1)a\}$. Note that by definition of m, $m \leq k-2$ and since $b \leq -a$, $m \geq 1$. In this case, the complementary factor is

$$Q(X) = \sum_{i=0}^{m} X^i.$$

The beta-expansion of 1 is then eventually periodic with period 1, the length of the preperiod is $m+2$.

When $m=1$, $P(X) = X^4 - (a-1)X^3 - (a+b)X^2 - (b+c)X - c$ and

$$d_\beta(1) = (a-2)(2a+b-2)(2a+2b+c-2)(2a+2b+2c-2)^\omega.$$

Here $d_3 = d_{m+2} = a+b-1+m(a+b+c-1)$ and $d_4 = d_{m+3} = (m+1)(a+b+c-1)$.

When $m > 1$,

$$\begin{aligned}
P(X) = {} & X^{m+3} - (a-1)X^{m+2} - (a+b-1)X^{m+1} - \textstyle\sum_{i=3}^{m}(a+b+c-1)X^i\\
& -(a+b+c)X^2 - (b+c)X - c
\end{aligned}$$

and $d_\beta(1) = d_1 d_2 \dots d_{m+2} d_{m+3}^\omega$, with

$$\begin{aligned}
&d_1 = a-2, \quad d_2 = 2a+b-3,\\
&d_{m+3-i} = 2a+b-3+(m+1-i)(a+b+c-1) \quad m \geq 3, 3 \leq i \leq m,\\
&d_{m+1} = 2a+b-2+(m-1)(a+b+c-1),\\
&d_{m+2} = a+b-1+m(a+b+c-1),\\
&d_{m+3} = (m+1)(a+b+c-1).
\end{aligned}$$

In both cases, $d_1 = a-2$. Since $b(k-1)+c(k-2) > (k-2)-(k-1)a$ and $c \leq a-2$, we get $-2a+3 \leq b$. Moreover as $b \leq -a$, $1 \leq d_2 \leq a-2$ when $m=1$, and $0 \leq d_2 \leq a-3$ otherwise. By definition of m, $(m+1)b+mc > m-(m+1)a$, thus $d_{m+2} \geq 0$ and $d_{m+3} \geq c$. Since $e_k \leq b+c < e_{k-1}$ and $m \leq k-2$, we obtain $d_{m+3} \leq a-3$ and $d_{m+2} \leq a-c-3$.

When $m > 1$, since $mb+(m-1)c \leq (m-1)-ma$, we get $d_{m+1} \leq a-2$. As $0 \leq 2a+b-2$ and $a+b+c-1 > 0$, $d_{m+1} > 0$. Moreover as $a+b+c-1 > 0$, one has $d_2 < d_3 < \dots < d_{m+1}$. Note that, when $m \geq 3$, $d_2 \neq a-2$.

We now study the cases where d_i is not strictly smaller than d_1. When $m=1$, only d_2 may be equal to $a-2$, then $b=-a$ and $d_3 = c-2$, thus $d_3 < d_2$. When $m > 1$, only d_{m+1} may be equal to $a-2$, then $mb = -ma-(m-1)c+(m-1)$, and thus $d_2 - d_{m+2} = a-1-c$ is a positive integer.

We have proved that the lexicographical order conditions on $d_\beta(1)$:

$$d_1 d_2 \dots d_{m+3}^\omega >_{lex} d_i d_{i+1} \dots d_{m+3}^\omega \quad \text{for } 2 \leq i \leq m+3,$$

are satisfied, showing in this way that the announced beta-expansions of 1 are right.

Remark 2. The polynomials Q that appear in the cubic case are cyclotomic. In the general case, Q can be noncyclotomic and even nonreciprocal ([6]).

3 Cubic simple beta-numbers

In the following, we establish that cubic simple beta-numbers are Pisot numbers. Next we give necessary and sufficient conditions on the coefficients of the minimal polynomial of β for β to be a simple beta-number.

Theorem 3. *If β is a cubic simple beta-number then β is a Pisot number.*

Remark 3. This is no longer true for simple beta-numbers of degree 4. For example, the positive root of $X^4 - 3X^3 - 2X^2 - 3$ is a simple beta-number, but is not a Pisot number.

Proof. Let β be a cubic simple beta-number and let

$$M_\beta(X) = X^3 - aX^2 - bX - c$$

be its minimal polynomial. Then β has no positive real algebraic conjugate and c is a positive integer smaller than $\lfloor \beta \rfloor$.

The condition on the product c of the roots of the polynomial M_β, *i.e.*, $|c| \leq \lfloor\beta\rfloor$, directly implies, when the Galois conjugates of β are not real numbers, that β is a Pisot number.

The only other case is the case where both Galois conjugates γ_1 and γ_2 of β are negative real numbers. We then assume that β is a cubic simple beta-number that is not a Pisot number, and show that these hypotheses are contradictory. Let γ_1 and γ_2 be the Galois conjugates of β. As $0 < c \leq \lfloor\beta\rfloor$, if one of the γ_i's is smaller than -1 the other one is greater than -1. Moreover, as the modulus of a Galois conjugate of a beta-number is smaller than the golden ratio, one can suppose, for example, that

$$-\frac{1+\sqrt{5}}{2} < \gamma_2 < -1 < \gamma_1 < 0 < \beta$$

Consequently, $M_\beta(-1) > 0$, in other words, $b > a + c + 1$. Note that here $a \in \{\lfloor\beta\rfloor - 2, \lfloor\beta\rfloor - 1\}$.

As β is a simple beta-number, $d_\beta(1) = d_1 d_2 \ldots d_n$. Denote by P the associated β-polynomial:

$$P(X) = X^n - \sum_{i=1}^{n} d_i X^{n-i}$$

and denote by $Q = \sum_{i\geq 0} q_i X^i$ the quotient of the division upon the increasing powers of P by M_β. In other words,

$$P(X) = M_\beta(X)Q(X)$$

We shall show, by induction, that $q_0 \geq 1$, and that for all $i \geq 0$, $|q_{i+1}| > |q_i|$ with $sgn(q_{i+1}) = -sgn(q_i)$. We shall conclude from the growth of the moduli of its coefficients that Q is an infinite series, and thus that $d_\beta(1)$ is not finite.

In what follows, we mainly use the fact that the d_i's are nonnegative integers smaller than $\lfloor\beta\rfloor$ and the inequality $b \geq a + c + 2$.

First of all, as $d_n = q_0 c$ and d_n and c are positive integers, $q_0 \geq 1$. Since $d_{n-1} = q_0 b + q_1 c$ and $q_0 \geq 1$, $d_{n-1} \geq q_0 a + 2q_0 + (q_0 + q_1)c$. When $a = \lfloor\beta\rfloor - 1$, we directly get from $d_{n-1} \leq \lfloor\beta\rfloor$, that $q_1 < -q_0$. When $a = \lfloor\beta\rfloor - 2$, the lexicographical order conditions on $d_\beta(1)$ imply that

$$d_{n-1} d_n < d_1 d_2 \ldots d_n.$$

By definition of beta-expansions, $d_1 = \lfloor\beta\rfloor$ and here $d_2 < d_n$. Indeed as

$$\gamma_2 = \frac{1}{2}\left(a - \beta + \sqrt{(a-\beta)^2 - \frac{4c}{\beta}}\right),$$

and $\gamma_2 > -(1+\sqrt{5})/2$, we get that

$$c > \frac{\sqrt{5}-1}{2}\beta + \frac{1+\sqrt{5}}{2}\beta\{\beta\},$$

and in particular, that $c > \beta/2$, consequently $d_n = c$ and that $\beta\{\beta\} < c$. Thus $d_2 = \lfloor\beta\{\beta\}\rfloor$ is strictly smaller than d_n. Therefore the previous lexicographical order condition implies that $d_{n-1} < \lfloor\beta\rfloor$. So, as $d_{n-1} \geq \lfloor\beta\rfloor + (q_0+q_1)c$, $q_1 < -q_0$.

As $d_{n-2} = q_0a+q_1b+q_2c$ and $q_1 < -q_0 < 0$, $d_{n-2} \leq (q_1+q_0)a+2q_1+(q_1+q_2)c$, that is $d_{n-2} < -\lfloor\beta\rfloor + (q_1+q_2)c$, so $q_2 > -q_1$.

For all positive integers i, $d_{n-(2i+1)} = -q_{2i-2} + q_{2i-1}a + q_{2i}b + q_{2i+1}c$. From $q_{2i} > 0$, we get $d_{n-(2i+1)} \geq (q_{2i-1}+q_{2i})a + q_{2i} + (q_{2i} - q_{2i-2}) + (q_{2i} + q_{2i+1})c$. From $(q_{2i-1}+q_{2i}) \geq 1$, $q_{2i} > 2i$ and $(q_{2i} - q_{2i-2}) > 1$, we obtain $d_{n-(2i+1)} > \lfloor\beta\rfloor + (q_{2i}+q_{2i+1})c$, and thus $q_{2i+1} < -q_{2i}$.

For all positive integers i, $d_{n-(2i+2)} = -q_{2i-1} + q_{2i}a + q_{2i+1}b + q_{2i+2}c$. From $q_{2i+1} < 0$, we get $d_{n-(2i+1)} \leq (q_{2i}+q_{2i+1})a + q_{2i+1} + (q_{2i+1} - q_{2i-1}) + (q_{2i+1} + q_{2i+2})c$. As $(q_{2i}+q_{2i+1}) \leq -1$, $q_{2i+1} < -(2i+1)$ and $(q_{2i+1} - q_{2i-1}) < -1$, we get $d_{n-(2i+2)} < -\lfloor\beta\rfloor + (q_{2i+1}+q_{2i+2})c$, thus $q_{2i+2} > -q_{2i+1}$.

So Q is an infinite series; consequently if β is not a Pisot number, $d_\beta(1)$ is not finite.

As a consequence of Theorems 2 and 3, we obtain the above characterization of cubic simple beta-numbers.

Proposition 2. *Let β be a cubic Pisot number and let*

$$M_\beta(x) = X^3 - aX^2 - bX - c$$

be its minimal polynomial.

Then β is a simple beta-number if and only it satisfies one of the following conditions:

- *Case 1: $b \geq 0$ and $c > 0$*
- *Case 2: $-a < b < 0$ and $b + c \geq 0$*
- *Case 3: $b \leq -a$ and $b(k-1) + c(k-2) \leq (k-2) - (k-1)a$, where k is the integer in $\{2, 3, \dots, a-2\}$ such that, denoting $e_k = 1 - a + (a-2)/k$, $e_k \leq b + c < e_{k-1}$.*

The problem of finding such a characterization remains open for simple beta-numbers of higher degree.

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