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# Beta-expansions for cubic Pisot numbers 

Frédérique Bassino<br>I.G.M., Université de Marne La Vallée<br>77454 Marne-la-Vallée Cedex 2. France<br>e-mail: bassino@univ-mlv.fr


#### Abstract

Real numbers can be represented in an arbitrary base $\beta>1$ using the transformation $T_{\beta}: x \rightarrow \beta x(\bmod 1)$ of the unit interval; any real number $x \in[0,1]$ is then expanded into $d_{\beta}(x)=\left(x_{i}\right)_{i \geq 1}$ where $x_{i}=\left\lfloor\beta T_{\beta}^{i-1}(x)\right\rfloor$. The closure of the set of the expansions of real numbers of $[0,1[$ is a subshift of $\{a \in \mathbb{N} \mid a<\beta\}^{\mathbb{N}}$, called the beta-shift. This dynamical system is characterized by the beta-expansion of 1 ; in particular, it is of finite type if and only if $d_{\beta}(1)$ is finite; $\beta$ is then called a simple beta-number. We first compute the beta-expansion of 1 for any cubic Pisot number. Next we show that cubic simple beta-numbers are Pisot numbers.


## Introduction

Representations of real numbers in an arbitrary base $\beta>1$, called beta-expansions, have been introduced by Rényi ([14]). They arise from the orbits of the piecewise-monotone transformation of the unit interval : $T_{\beta}: x \rightarrow \beta x(\bmod 1)$. Such transformations were extensively studied in ergodic theory ([13]).

More precisely, any real number $x \in[0,1]$ is expanded into $d_{\beta}(x)=\left(x_{i}\right)_{i \geq 1}$ where $x_{i}=\left\lfloor\beta T_{\beta}^{i-1}(x)\right\rfloor$. The nonnegative integers $d_{i}$ are elements of the digit alphabet $A=\{a \in \mathbb{N} \mid a<\beta\}$. These representations generalize standard representations in an integral base to a real base; indeed the beta-expansion of any real number of $[0,1[$ can equivalently be obtained by the greedy algorithm. Only the beta-expansion of 1 differs.

Properties of beta-expansions are strongly related to symbolic dynamics ([4]). The closure of the set of infinite sequences, appearing as beta-expansions of numbers of the interval $[0,1[$, is a dynamical system, that is, a closed shiftinvariant subset of $A^{\mathbb{N}}$, called the beta-shift.

An important property of the beta-shift is that its nature is entirely determined, in a combinatorial manner, by the beta-expansion of 1 : the beta-shift is sofic, that is to say the set of its finite factors is recognized by a finite automaton, if and only the beta-expansion of 1 is eventually periodic ([3]); it is of finite type, that is to say the set of its finite factors is defined by forbidding a finite set of words, if and only if the beta-expansion of 1 is finite ([12]).

When the beta-expansion of 1 is eventually periodic, $\beta$ is called a beta-number and when the beta-expansion of 1 is finite, $\beta$ is said to be a simple beta-number.

The eventually periodic beta-expansions were extensively studied by Bertrand ([3]) and by Schmidt ([15]). In particular, it is known that Pisot numbers are beta-numbers. Concerning Salem numbers, we only know that if $\beta$ is a Salem number of degree 4 , then the beta-expansion of 1 is eventually periodic ([5]). It is conjectured that Salem numbers of degree 6 are still beta-numbers, but not all Salem numbers of degree 8 ([7]).

The domain of the Galois conjugates of all beta-numbers was also investigated independently by Solomyak ([16]) and by Flatto, Lagarias and Poonen ([8]).

For a general presentation of the beta-shift one can refer to [9].
In the following, we summarize properties of beta-numbers. We compute the beta-expansion of 1 for any cubic Pisot number and we establish a characterization of cubic simple beta-numbers, showing that they are Pisot numbers.

A very close problem, seen from the point of view of numeration systems, was studied by Akiyama ([1]). He showed that in the cubic case, the real numbers of the set $\mathbb{N}\left[\beta^{-1}\right]$ have a finite beta-expansion if and only $\beta$ is a Pisot unit and 1 has a finite beta-expansion. This finiteness problem is equivalent to a problem of fractal tiling generated by Pisot numbers.

## 1 Beta-numbers

Real numbers can be represented in an arbitrary base $\beta>1$ using the transformation $T_{\beta}: x \rightarrow \beta x(\bmod 1)$ of the unit interval; any real number $x \in[0,1]$ is then expanded into $d_{\beta}(x)=\left(x_{i}\right)_{i \geq 1}$ where $x_{i}=\left\lfloor\beta T_{\beta}^{i-1}(x)\right\rfloor$. When a betaexpansion is of the form $u v^{\omega}$, the expansion is said to be eventually periodic. If a representation ends with infinitely many zeros, like $u 0^{\omega}$, it is said to be finite and the ending zeros are omitted.

Let us denote by $S_{\beta}$ the closure of all beta-expansions of real numbers of [ 0,1 [ and by $\sigma$ the (one-sided) shift defined by $\sigma\left(\left(x_{i}\right)_{i \geq 1}\right)=\left(x_{i+1}\right)_{i \geq 1}$. The set $S_{\beta}$ endowed with the shift is called the beta-shift, it is a subshift of $\bar{A}^{\mathbb{N}}, A$ being the digit set, i.e., $A=\{a \in \mathbb{N} \mid a<\beta\}$.

An important property ([13]) of the beta-shift $S_{\beta}$ is that its nature is entirely determined by $d_{\beta}(1)$ the beta-expansion of 1 . Indeed, setting $d^{*}(1)=d_{\beta}(1)$ if $d_{\beta}(1)$ is infinite and $d^{*}(1)=\left(d_{1} d_{2} \ldots d_{n-1}\left(d_{n}-1\right)\right)^{\omega}$ if $d_{\beta}(1)=d_{1} d_{2} \ldots d_{n-1} d_{n}$, a sequence $x$ of nonnegative integers belongs to $S_{\beta}$ if and only if it satisfies the following lexicographical order conditions: $\forall p \geq 0, \quad \sigma^{p}(x) \leq d^{*}(1)$.

Recall that the beta-expansion of 1 also can be characterized ([13]) by lexicographical order conditions: let $d=\left(d_{i}\right)_{i>1}$ be a sequence of nonnegative integers different from $10^{\omega}$, such that $\sum_{i>1} d_{i} \beta^{-\bar{i}}=1$, with $d_{1} \geq 1$ and for $i \geq 2, d_{i} \leq d_{1}$, then $d$ is the beta-expansion of 1 if and only if for all $p \geq 1, \sigma^{p}(d)<d$.

We recall that an algebraic integer $\beta$ strictly greater than 1 is called a Perron number if all its Galois conjugates have modulus strictly less than $\beta$, a Pisot number if all its Galois conjugates have modulus strictly less than 1, and a Salem number if all its conjugates are less than 1 in modulus and at least one conjugate has modulus 1 .

Let $\beta$ be a beta-number. Denote by $d_{\beta}(1)=d_{1} \ldots d_{n}\left(d_{n+1} \ldots d_{n+p}\right)^{\omega}$, where $n$ and $p$ are chosen minimal, the beta-expansion of 1 . Then the adjacency matrix $\mathcal{M}_{\beta}$ of the finite automaton recognizing the set of its finite factors (Fig.1) is a primitive (i.e., its associated graph is strongly connected and the lengths of its cycles are relatively prime) nonnegative integral matrix whose spectral radius is $\beta$; so, from the Perron-Frobenius theorem, $\beta$ is a Perron number.


Fig. 1. Automaton recognizing the set of the finite factors of $S_{\beta}$

The characteristic polynomial of $\mathcal{M}_{\beta}$

$$
P(X)=X^{n+p}-\sum_{i=1}^{n+p} d_{i} X^{n+p-i}-X^{n}+\sum_{i=1}^{n} d_{i} X^{n-i}
$$

is called, following the terminology introduced by Hollander ([11]), the associated beta-polynomial.

As $P$ is a multiple of the minimal polynomial $M_{\beta}$ of $\beta, P(0)=d_{n+p}-d_{n}$ is a multiple of $\left|M_{\beta}(0)\right|=\left|\prod \beta_{i}\right|$, where $\beta_{i}$ runs over the set of algebraic conjugates of $\beta$. So, we get that $\left|\prod \beta_{i}\right|$ has to be smaller than $\lfloor\beta\rfloor$.

As a consequence, in the quadratic case, the only beta-numbers are the Pisot numbers. Conversely, it is known that if $\beta$ is a Pisot number then $\beta$ is a betanumber ([2]). An important gap remains between Pisot and Perron numbers.

Example 1. The quadratic number $\beta=(1+\sqrt{1} 3) / 2$ is not a beta-number since $M_{\beta}(X)=X^{2}-X-3$ and $M_{\beta}(0)>\lfloor\beta\rfloor$.

Let $\beta$ be the Pisot number $(3+\sqrt{5}) / 2$, then $\beta$ is a beta-number and $d_{\beta}=21^{\omega}$.
Let $\beta$ be the golden ratio $(1+\sqrt{5}) / 2$, then $\beta$ is a simple beta-number and $d_{\beta}(1)=11$.

On the other hand, the domain of the Galois conjugates of beta-numbers was studied by Solomyak ([16]) and independently by Flatto, Lagarias and Poonen ([8]). They showed in particular that if the beta-expansion of 1 is eventually periodic then the Galois conjugates of $\beta$ have modulus less than the golden ratio $(1+\sqrt{5}) / 2$. It was already known (see [9]) that $\beta$ cannot have a Galois conjugate greater than 1 .

Solomyak ([16]) proved that the topological closure of conjugates of betanumbers and the one of conjugates of simple beta-numbers are the same. However, there is an important difference between the conjugates of beta-numbers and the ones of simple beta numbers: if $\beta$ is a simple beta-number then $\beta$ has no algebraic conjugate that is a nonnegative real number.

Indeed, let $\beta$ be a simple beta-number and set $d_{\beta}(1)=d_{1} \ldots d_{n}$. Consider


Fig. 2. Automaton recognizing the set of the finite factors of $S_{\beta}$
the finite automaton recognizing the set of the finite factors of the associated beta-shift (Fig. 2). Let $\mathcal{M}_{\beta}$ be the transition matrix of this automaton. The characteristic polynomial of $\mathcal{M}_{\beta}$, which is called the associated beta-polynomial,

$$
P(X)=X^{n}-\sum_{i=1}^{n} d_{i} X^{n-i}
$$

has only one positive real root.

Example 2. Salem numbers are roots of reciprocal polynomials. Thus if $\beta$ is a Salem number, $1 / \beta>0$ is a Galois conjugate of $\beta$, and so $\beta$ is not a simple beta-number.

The previous conditions are sufficient for a quadratic algebraic integer to be a simple beta-number.

Proposition 1. [10] The simple beta-numbers of degree 2 are exactly the quadratic Pisot numbers without a positive real Galois conjugate. They are the positive roots of the polynomials

$$
X^{2}-a X-b \quad \text { with } \quad a \geq b \geq 1
$$

The beta-expansion of 1 is then $d_{\beta}(1)=a b$.
Example 3. The minimal polynomial of $(1+\sqrt{5}) / 2$ is $X^{2}-X-1,(1+\sqrt{5}) / 2$ is a simple beta-number and $d_{\beta}(1)=11$.

The minimal polynomial of $(3+\sqrt{5}) / 2$ is $X^{2}-3 X+1$, therefore $(3+\sqrt{5}) / 2$ is not a simple beta-number.

## 2 Beta-expansions of 1 for cubic Pisot numbers

Let us recall the characterization of cubic Pisot numbers due to Akiyama ([1])
Theorem 1 (Akiyama [1]). Let $\beta>1$ be a cubic number and let

$$
M_{\beta}(x)=X^{3}-a X^{2}-b X-c
$$

be its minimal polynomial.
Then $\beta$ is a Pisot number if and only if it both inequalities

$$
|b-1|<a+c \quad \text { and } \quad\left(c^{2}-b\right)<\operatorname{sgn}(c)(1+a c)
$$

hold.
Remark 1. Note that $a$ must be a nonnegative integer.
The following theorem gives the $\beta$-expansion of 1 for any cubic Pisot number.
Theorem 2. Let $\beta$ be a cubic Pisot number and let

$$
M_{\beta}(x)=X^{3}-a X^{2}-b X-c
$$

be its minimal polynomial. Then the beta-expansion of 1 is

- Case 1: When $b \geq a$, then $d_{\beta}(1)=(a+1)(b-1-a)(a+c-b)(b-c) c$.
- Case 2: When $0 \leq b \leq a$, if $c>0, d_{\beta}(1)=a b c$, otherwise,

$$
d_{\beta}(1)=a[(b-1)(c+a)]^{\omega} .
$$

- Case 3: When $-a<b<0$, if $b+c \geq 0$, then $d_{\beta}(1)=(a-1)(a+b)(b+c) c$, otherwise $d_{\beta}(1)=(a-1)(a+b-1)(a+b+c-1)^{\omega}$
- Case 4: When $b \leq-a$, let $k$ be the integer of $\{2,3, \ldots, a-2\}$ such that, denoting $e_{k}=1-a+(a-2) / k, e_{k} \leq b+c<e_{k-1}$.
- If $b(k-1)+c(k-2) \leq(k-2)-(k-1) a, d_{\beta}(1)=d_{1} \ldots d_{2 k+2}$ with

$$
\begin{aligned}
& d_{1}=a-2 \\
& d_{k+2-i}=-(k+3-i)+a(k+2-i)+b(k+1-i)+c(k-i), 3 \leq i \leq k \\
& d_{k}=-k+a k+b(k-1)+c(k-2) \\
& d_{k+1}=-(k-1)+a k+b k+c(k-1) \\
& d_{k+2}=-(k-2)+a(k-1)+b k+c k \\
& d_{2 k+2-i}=-(i-2)+a(i-1)+b i+c(i+1) \quad k \geq 3,2 \leq i \leq(k-1) \\
& d_{2 k+1}=b+2 c \quad \text { and } \quad d_{2 k+2}=c .
\end{aligned}
$$

- If $b(k-1)+c(k-2)>(k-2)-(k-1) a$, let $m$ be the integer defined by $m=\min \{i \in \mathbb{N}$ such that $(i+1) b+i c>i-(i+1) a\}$.

When $m=1, d_{\beta}(1)=(a-2)(2 a+b-2)(2 a+2 b+c-2)(2 a+2 b+2 c-2)^{\omega}$.
When $m>1, d_{\beta}(1)=d_{1} d_{2} \ldots d_{m+2} d_{m+3}^{\omega}$, with

$$
\begin{aligned}
& d_{1}=a-2, \quad d_{2}=2 a+b-3 \\
& d_{m+3-i}=2 a+b-3+(m+1-i)(a+b+c-1) \quad m \geq 3,3 \leq i \leq m \\
& d_{m+1}=2 a+b-2+(m-1)(a+b+c-1) \\
& d_{m+2}=a+b-1+m(a+b+c-1) \\
& d_{m+3}=(m+1)(a+b+c-1)
\end{aligned}
$$

Example 4. When $a \geq b \geq 0$ and $c>0$, we obtain the only beta-expansion of 1 of length 3 .

The smallest Pisot number has $M_{\beta}=X^{3}-X-1$ as minimal polynomial, it is a simple beta-number and $d_{\beta}(1)=10001$.

The positive root $\beta$ of $M_{\beta}=X^{3}-3 X^{2}+2 X-2$ is a simple beta-number and $d_{\beta}(1)=2102$.

The case where $b \leq-a$ shows that from a cubic simple beta-number, we can obtain an arbitrary long beta-expansion of 1 . For any integer $k$ greater than or equal to 2 , the real root $\beta$ of the irreducible polynomial $X^{3}-(k+2) X^{2}+2 k X-k$, is a simple beta number whose integer part is equal to $k$, and the beta-expansion of 1 has length $2 k+2$. For $k=2$, we get $d_{\beta}(1)=221002$; for $k=3$, we get $d_{\beta}(1)=31310203$.

Example 5. The greatest positive root $\beta$ of $M_{\beta}=X^{3}-2 X^{2}-X+1$ is a betanumber and $d_{\beta}(1)=2(01)^{\omega}$.

If $\beta$ is the positive root of $X^{3}-5 X^{2}+3 X-2$, then $d_{\beta}(1)=413^{\omega}$. When $\beta$ is the greatest positive root of $X^{3}-5 X^{2}+X+2$, then $d_{\beta}(1)=431^{\omega}$.

For any integer $k$ greater than or equal to 3 , the real root $\beta$ of the irreducible polynomial $X^{3}-(k+2) X^{2}+(2 k-1) X-(k-1)$, is a beta number whose integer part is equal to $k$, and the beta-expansion of 1 is eventually periodic of period

1 , the length of its preperiod $k$. For $k=3$, we get $d_{\beta}(1)=3302^{\omega}$; for $k=4$, we get $d_{\beta}(1)=42403^{\omega}$.
Proof. It is known that Pisot numbers are beta-numbers, thus, for any cubic Pisot number $\beta$, the beta-expansion of 1 is finite or eventually periodic. In any case, we first compute the associated beta-polynomial $P$. Next we prove that the sequence $d=\left(d_{i}\right)_{i \geq 1}$ of nonnegative integers obtained from the beta-polynomial satisfy lexicographical order conditions: for all $p \geq 1, \sigma^{p}(d)<d$.

First of all, we recall that, from Theorem 1, a cubic number $\beta$, greater than 1 and having

$$
M_{\beta}(X)=X^{3}-a X^{2}-b X-c
$$

as minimal polynomial, is a cubic Pisot number if and only if it both

$$
|b-1|<a+c \quad \text { and } \quad\left(c^{2}-b\right)<\operatorname{sgn}(c)(1+a c)
$$

hold.
Denote by $Q$ the complementary factor of the beta-polynomial $P$ defined by $P(X)=M_{\beta}(X) Q(X)$. As we shall see in what follows, the value of $Q$ depends upon the value of the coefficients of $M_{\beta}$.

Case 1: When $b>a$, as $\beta$ is a Pisot number, from Theorem 1, $c$ is a positive integer. In this case, the complementary factor is $Q(X)=X^{2}-X+1$ and $d_{\beta}(1)=(a+1)(b-1-a)(a+c-b)(b-c) c$.

Indeed, as $\left(c^{2}-b\right)<\operatorname{sgn}(c)(1+a c)$ and $c>0$, we get $c \leq a+1$. As $|b-1|<a+c$, we get $b-1-a \leq a$ and $0 \leq a-b+c$. From $b>a$, we get that $0 \leq b-a-1$ and, as $c \leq a+1$, that $a-b+c \leq a$. Finally as $0 \leq a-b+c \leq a$, we obtain $0 \leq b-c \leq a$.

Case 2: When $0 \leq b \leq a$, the complementary factor is then $Q(X)=1$ and the associated beta-polynomial is equal to the minimal polynomial.

If $c>0$, then $d_{\beta}(1)=a b c$. Indeed, as $\left(c^{2}-b\right)<\operatorname{sgn}(c)(1+a c)$, we get $c \leq a$.
If $c<0$, then $d_{\beta}(1)=a[(b-1)(a+c)]^{\omega}$. As $|b-1|<a+c$, we get $b-1 \leq a-2$. As $\left(c^{2}-b\right)<\operatorname{sgn}(c)(1+a c)$, we get that $c \geq-a$ and, consequently, $0 \leq c+a \leq a-1$.

Case 3: When $-a<b<0$, if $b+c \geq 0$ then the complementary factor is $Q(X)=X+1$ and $d_{\beta}(1)=(a-1)(a+b)(b+c) c$. Indeed, as $-a<b<0$, we obtain $1 \leq a+b \leq a-1$. Since $b+c \geq 0, c$ is a positive integer. From $\left(c^{2}-b\right)<\operatorname{sgn}(c)(1+a c)$, we get that $c \leq a-1$ and $b+c \leq a-2$.

If $b+c<0$, then $Q(X)=1$ and $d_{\beta}(1)=(a-1)(a+b-1)(a+b+c-1)^{\omega}$. As $-a<b<0$, we get $0 \leq a+b-1 \leq a-2$. From $|b-1|<a+c$, we get that $1 \leq a+b+c-1$ and as $b+c<0$, we obtain $a+b+c-1 \leq a-2$.

Case 4: First of all, since $|b-1|<a+c$, we get $-a+2 \leq b+c$. Moreover as $b \leq-a$, we get $c \geq 2$ and as $\left(c^{2}-b\right)<\operatorname{sgn}(c)(1+a c)$, we obtain $c \leq a-2$, thus $b+c \leq-2$. So, there exists an integer $k$ in $\{2,3, \ldots, a-2\}$, such that, denoting $e_{k}=1-a+(a-2) / k, e_{k} \leq b+c<e_{k-1}$.

When $b(k-1)+c(k-2) \leq(k-2)-(k-1) a$, the complementary factor is

$$
Q(X)=\frac{\left(X^{k}-1\right)\left(X^{k+1}-1\right)}{(X-1)^{2}}
$$

and $d_{\beta}(1)=d_{1} \ldots d_{2 k+2}$ with

```
\(d_{1}=a-2\),
\(d_{k+2-i}=-(k+3-i)+a(k+2-i)+b(k+1-i)+c(k-i), k \geq 3,3 \leq i \leq k\)
\(d_{k}=-k+a k+b(k-1)+c(k-2)\)
\(d_{k+1}=-(k-1)+a k+b k+c(k-1)\)
\(d_{k+2}=-(k-2)+a(k-1)+b k+c k\)
\(d_{2 k+2-i}=-(i-2)+a(i-1)+b i+c(i+1) \quad k \geq 3,2 \leq i \leq(k-1)\)
\(d_{2 k+1}=b+2 c \quad\) and \(\quad d_{2 k+2}=c\).
```

We now verify that the lexicographical order conditions on $d_{\beta}(1)$ are satisfied.
As $2 \leq c \leq a-2$ and $b+c \leq-2$, we get $d_{2 k+1} \leq a-4$. From $e_{k} \leq b+c$ and $b(k-1)+c(k-2) \leq(k-2)-(k-1) a$, we get $d_{2 k+1} \geq 0$.

For $k \leq 3$ and $2 \leq i \leq k-1, d_{2 k+2-i}=-(i-2)+a(i-1)+b i+c(i+1)$. As $b+c<e_{i}$, we get $d_{2 k+2-i}<c$. As $-a+2 \leq b+c$ and $b+2 c \geq 0$, we get $d_{2 k+2-i} \geq i$.

As $e_{k} \leq b+c$, we obtain $d_{k+2} \geq 0$. Since $c \leq a-2, d_{k+1}>d_{k+2}$ and since $b+c \leq-2, d_{k}>d_{k+1}$. Moreover from $b(k-1)+c(k-2) \leq(k-2)-(k-1) a$, we get $d_{k} \leq a-2$.

For $k \leq 3$, as $|b-1|<a+c$, we obtain $d_{2}<\cdots<d_{k-1}$. As $b+c<e_{k-1}$ and $b+2 c \leq 0$, we get $d_{k-1}<a-2$. Moreover from $c \leq a-2$ and $a+b+c-1>0$, we get that $d_{2}=2 a+b-3$ is nonnegative.

All $d_{i}$ 's are smaller than $d_{1}$, only $d_{2 k+2}$ and $d_{k}$ can be equal to $d_{1}$. Therefore we have to verify that $d_{2} \geq d_{k+1}$ when $k \geq 3$ (otherwise $d_{2}=d_{k}$ and $d_{k}>d_{k+1}$ ). If $d_{k}=a-2$, then $b+c=e_{k}$, and $d_{k+1}=a-c-1$. As $a+b+c-1>0$, we obtain $d_{k+1} \leq d_{2}$. In case of equality, if $k=3$, then $d_{3}=d_{k}$ and $d_{k}>d_{k+2}$, otherwise $d_{3}>d_{2}$ and $d_{k+1}>d_{k+2}$, therefore $d_{3}>d_{k+2}$.

So lexicographical order conditions are satisfied and $d_{1} \ldots d_{2 k+2}$ is the betaexpansion of 1 .

When $b(k-1)+c(k-2)>(k-2)-(k-1) a$, as $b \leq-a$, we get $k \geq 3$. Let $m$ be the integer defined by $m=\min \{i \in \mathbb{N}$ such that $(i+1) b+i c>i-(i+1) a\}$. Note that by definition of $m, m \leq k-2$ and since $b \leq-a, m \geq 1$. In this case, the complementary factor is

$$
Q(X)=\sum_{i=0}^{m} X^{i}
$$

The beta-expansion of 1 is then eventually periodic with period 1 , the length of the preperiod is $m+2$.

When $m=1, P(X)=X^{4}-(a-1) X^{3}-(a+b) X^{2}-(b+c) X-c$ and

$$
d_{\beta}(1)=(a-2)(2 a+b-2)(2 a+2 b+c-2)(2 a+2 b+2 c-2)^{\omega} .
$$

Here $d_{3}=d_{m+2}=a+b-1+m(a+b+c-1)$ and $d_{4}=d_{m+3}=(m+1)(a+b+c-1)$.
When $m>1$,

$$
\begin{aligned}
P(X)= & X^{m+3}-(a-1) X^{m+2}-(a+b-1) X^{m+1}-\sum_{i=3}^{m}(a+b+c-1) X^{i} \\
& -(a+b+c) X^{2}-(b+c) X-c
\end{aligned}
$$

and $d_{\beta}(1)=d_{1} d_{2} \ldots d_{m+2} d_{m+3}^{\omega}$, with

$$
\begin{aligned}
& d_{1}=a-2, \quad d_{2}=2 a+b-3 \\
& d_{m+3-i}=2 a+b-3+(m+1-i)(a+b+c-1) \quad m \geq 3,3 \leq i \leq m, \\
& d_{m+1}=2 a+b-2+(m-1)(a+b+c-1) \\
& d_{m+2}=a+b-1+m(a+b+c-1) \\
& d_{m+3}=(m+1)(a+b+c-1)
\end{aligned}
$$

In both cases, $d_{1}=a-2$. Since $b(k-1)+c(k-2)>(k-2)-(k-1) a$ and $c \leq a-2$, we get $-2 a+3 \leq b$. Moreover as $b \leq-a, 1 \leq d_{2} \leq a-2$ when $m=1$, and $0 \leq d_{2} \leq a-3$ otherwise. By definition of $m,(m+1) b+m c>m-(m+1) a$, thus $d_{m+2} \geq 0$ and $d_{m+3} \geq c$. Since $e_{k} \leq b+c<e_{k-1}$ and $m \leq k-2$, we obtain $d_{m+3} \leq a-3$ and $d_{m+2} \leq a-c-3$.

When $m>1$, since $m b+(m-1) c \leq(m-1)-m a$, we get $d_{m+1} \leq a-2$. As $0 \leq 2 a+b-2$ and $a+b+c-1>0, d_{m+1}>0$. Moreover as $a+b+c-1>0$, one has $d_{2}<d_{3}<\ldots<d_{m+1}$. Note that, when $m \geq 3, d_{2} \neq a-2$.

We now study the cases where $d_{i}$ is not strictly smaller than $d_{1}$. When $m=1$, only $d_{2}$ may be equal to $a-2$, then $b=-a$ and $d_{3}=c-2$, thus $d_{3}<d_{2}$. When $m>1$, only $d_{m+1}$ may be equal to $a-2$, then $m b=-m a-(m-1) c+(m-1)$, and thus $d_{2}-d_{m+2}=a-1-c$ is a positive integer.

We have proved that the lexicographical order conditions on $d_{\beta}(1)$ :

$$
d_{1} d_{2} \ldots d_{m+3}^{\omega}>_{\text {lex }} d_{i} d_{i+1} \ldots d_{m+3}^{\omega} \text { for } 2 \leq i \leq m+3
$$

are satisfied, showing in this way that the announced beta-expansions of 1 are right.

Remark 2. The polynomials $Q$ that appear in the cubic case are cyclotomic. In the general case, $Q$ can be noncyclotomic and even nonreciprocal ([6]).

## 3 Cubic simple beta-numbers

In the following, we establish that cubic simple beta-numbers are Pisot numbers. Next we give necessary and sufficient conditions on the coefficients of the minimal polynomial of $\beta$ for $\beta$ to be a simple beta-number.

Theorem 3. If $\beta$ is a cubic simple beta-number then $\beta$ is a Pisot number.
Remark 3. This is no longer true for simple beta-numbers of degree 4. For example, the positive root of $X^{4}-3 X^{3}-2 X^{2}-3$ is a simple beta-number, but is not a Pisot number.

Proof. Let $\beta$ be a cubic simple beta-number and let

$$
M_{\beta}(X)=X^{3}-a X^{2}-b X-c
$$

be its minimal polynomial. Then $\beta$ has no positive real algebraic conjugate and $c$ is a positive integer smaller than $\lfloor\beta\rfloor$.

The condition on the product $c$ of the roots of the polynomial $M_{\beta}$, i.e., $|c| \leq\lfloor\beta\rfloor$, directly implies, when the Galois conjugates of $\beta$ are not real numbers, that $\beta$ is a Pisot number.

The only other case is the case where both Galois conjugates $\gamma_{1}$ and $\gamma_{2}$ of $\beta$ are negative real numbers. We then assume that $\beta$ is a cubic simple beta-number that is not a Pisot number, and show that these hypotheses are contradictory. Let $\gamma_{1}$ and $\gamma_{2}$ be the Galois conjugates of $\beta$. As $0<c \leq\lfloor\beta\rfloor$, if one of the $\gamma_{i}$ 's is smaller than -1 the other one is greater than -1 . Moreover, as the modulus of a Galois conjugate of a beta-number is smaller than the golden ratio, one can suppose, for example, that

$$
-\frac{1+\sqrt{5}}{2}<\gamma_{2}<-1<\gamma_{1}<0<\beta
$$

Consequently, $M_{\beta}(-1)>0$, in other words, $b>a+c+1$. Note that here $a \in\{\lfloor\beta\rfloor-2,\lfloor\beta\rfloor-1\}$.

As $\beta$ is a simple beta-number, $d_{\beta}(1)=d_{1} d_{2} \ldots d_{n}$. Denote by $P$ the associated $\beta$-polynomial:

$$
P(X)=X^{n}-\sum_{i=1}^{n} d_{i} X^{n-i}
$$

and denote by $Q=\sum_{i \geq 0} q_{i} X^{i}$ the quotient of the division upon the increasing powers of $P$ by $M_{\beta}$. In other words,

$$
P(X)=M_{\beta}(X) Q(X)
$$

We shall show, by induction, that $q_{0} \geq 1$, and that for all $i \geq 0,\left|q_{i+1}\right|>\left|q_{i}\right|$ with $\operatorname{sgn}\left(q_{i+1}\right)=-\operatorname{sgn}\left(q_{i}\right)$. We shall conclude from the growth of the moduli of its coefficients that $Q$ is an infinite series, and thus that $d_{\beta}(1)$ is not finite.

In what follows, we mainly use the fact that the $d_{i}$ 's are nonnegative integers smaller than $\lfloor\beta\rfloor$ and the inequality $b \geq a+c+2$.

First of all, as $d_{n}=q_{0} c$ and $d_{n}$ and $c$ are positive integers, $q_{0} \geq 1$. Since $d_{n-1}=q_{0} b+q_{1} c$ and $q_{0} \geq 1, d_{n-1} \geq q_{0} a+2 q_{0}+\left(q_{0}+q_{1}\right) c$. When $a=\lfloor\beta\rfloor-1$, we directly get from $d_{n-1} \leq\lfloor\beta\rfloor$, that $q_{1}<-q_{0}$. When $a=\lfloor\beta\rfloor-2$, the lexicographical order conditions on $d_{\beta}(1)$ imply that

$$
d_{n-1} d_{n}<d_{1} d_{2} \ldots d_{n}
$$

By definition of beta-expansions, $d_{1}=\lfloor\beta\rfloor$ and here $d_{2}<d_{n}$. Indeed as

$$
\gamma_{2}=\frac{1}{2}\left(a-\beta+\sqrt{(a-\beta)^{2}-\frac{4 c}{\beta}}\right)
$$

and $\gamma_{2}>-(1+\sqrt{5}) / 2$, we get that

$$
c>\frac{\sqrt{5}-1}{2} \beta+\frac{1+\sqrt{5}}{2} \beta\{\beta\},
$$

and in particular, that $c>\beta / 2$, consequently $d_{n}=c$ and that $\beta\{\beta\}<c$. Thus $d_{2}=\lfloor\beta\{\beta\}\rfloor$ is strictly smaller than $d_{n}$. Therefore the previous lexicographical order condition implies that $d_{n-1}<\lfloor\beta\rfloor$. So, as $d_{n-1} \geq\lfloor\beta\rfloor+\left(q_{0}+q_{1}\right) c, q_{1}<-q_{0}$.

As $d_{n-2}=q_{0} a+q_{1} b+q_{2} c$ and $q_{1}<-q_{0}<0, d_{n-2} \leq\left(q_{1}+q_{0}\right) a+2 q_{1}+\left(q_{1}+q_{2}\right) c$, that is $d_{n-2}<-\lfloor\beta\rfloor+\left(q_{1}+q_{2}\right) c$, so $q_{2}>-q_{1}$.

For all positive integers $i, d_{n-(2 i+1)}=-q_{2 i-2}+q_{2 i-1} a+q_{2 i} b+q_{2 i+1} c$. From $q_{2 i}>0$, we get $d_{n-(2 i+1)} \geq\left(q_{2 i-1}+q_{2 i}\right) a+q_{2 i}+\left(q_{2 i}-q_{2 i-2}\right)+\left(q_{2 i}+q_{2 i+1}\right) c$. From $\left(q_{2 i-1}+q_{2 i}\right) \geq 1, q_{2 i}>2 i$ and $\left(q_{2 i}-q_{2 i-2}\right)>1$, we obtain $d_{n-(2 i+1)}>$ $\lfloor\beta\rfloor+\left(q_{2 i}+q_{2 i+1}\right) c$, and thus $q_{2 i+1}<-q_{2 i}$.

For all positive integers $i, d_{n-(2 i+2)}=-q_{2 i-1}+q_{2 i} a+q_{2 i+1} b+q_{2 i+2} c$. From $q_{2 i+1}<0$, we get $d_{n-(2 i+1)} \leq\left(q_{2 i}+q_{2 i+1}\right) a+q_{2 i+1}+\left(q_{2 i+1}-q_{2 i-1}\right)+\left(q_{2 i+1}+\right.$ $\left.q_{2 i+2}\right) c$. As $\left(q_{2 i}+q_{2 i+1}\right) \leq-1, q_{2 i+1}<-(2 i+1)$ and $\left(q_{2 i+1}-q_{2 i-1}\right)<-1$, we get $d_{n-(2 i+2)}<-\lfloor\beta\rfloor+\left(q_{2 i+1}+q_{2 i+2}\right) c$, thus $q_{2 i+2}>-q_{2 i+1}$.

So $Q$ is an infinite series; consequently if $\beta$ is not a Pisot number, $d_{\beta}(1)$ is not finite.

As a consequence of Theorems 2 and 3, we obtain the above characterization of cubic simple beta-numbers.

Proposition 2. Let $\beta$ be a cubic Pisot number and let

$$
M_{\beta}(x)=X^{3}-a X^{2}-b X-c
$$

be its minimal polynomial.
Then $\beta$ is a simple beta-number if and only it satisfies one of the following conditions:

- Case 1: $b \geq 0$ and $c>0$
- Case 2: $-a<b<0$ and $b+c \geq 0$
- Case 3: $b \leq-a$ and $b(k-1)+c(k-2) \leq(k-2)-(k-1) a$, where $k$ is the integer in $\{2,3, \ldots, a-2\}$ such that, denoting $e_{k}=1-a+(a-2) / k$, $e_{k} \leq b+c<e_{k-1}$.

The problem of finding such a characterization remains open for simple betanumbers of higher degree.

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