On the computation of some external or partially enclosed natural convection flows

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ON THE COMPUTATION OF SOME EXTERNAL OR PARTIALLY ENCLOSED NATURAL CONVECTION FLOWS

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ABSTRACT

We address the issue of the computation of natural convection in external or partially enclosed domains, such as natural convection flow in vertical open channels or along heated vertical plates which raises the question of the proper choice of artificial boundary conditions and of their numerical implementation. To this aim we perform a singular value decomposition of the discrete Stokes operator that arises from the discretization of the governing Navier-Stokes equations, using the classical staggered grid formulation. We show that some choices of boundary conditions lead to an increase of the kernel of this operator making the solution of the full nonlinear equations indefinite. This arises in particular when Neumann type boundary conditions are imposed on the velocity component normal to the main inlet and outlet boundaries. The existence of one or more velocity-pressure combinations solutions of the homogeneous Stokes operator can be used to derive a superposition algorithm in which a linear combination of these modes is added to a particular solution of the nonlinear equations in order to satisfy given constraints such as one or more pressure conditions for instance. Sample calculations are performed to demonstrate the effectiveness of this methodology, which has many applications in other configurations such as pipe flows with several outlets or free plane jets for instance.

NOMENCLATURE

A	aspect ratio of heated zone
g	gravity acceleration, m/s^2
G	flow rate, m^2/s
ΔT	temperature difference, K
Δt	time step
H	height of computational domain, m
h	heat transfer coefficient, $W/m^2°C$
Pr	Prandtl number ($=\nu/\kappa$)
Ra	Rayleigh number ($=g\beta\Delta TH^3/\nu\kappa$)
Nu	Nusselt number
W	width of computational domain, m
β	coefficient of thermal expansion
ν	kinematic viscosity, m/s^2
κ	thermal diffusivity, m/s^2
x,z	horizontal and vertical coordinates
u,w	horizontal and vertical velocity components
P_m	motion pressure

Subscripts

H	relative to height
w	relative to width

INTRODUCTION

Natural convection flows abound in nature or in industrial configurations (e.g [1]). Such flows can occur either in confined spaces or in unbounded domains, very often also in partially bounded domains, that is in domains connected to the ambient surroundings through one or several apertures. Surprisingly enough external or partially enclosed flows have received much less interest in the computational literature than fully enclosed flows. Since both types of flows are governed by the same partial derivative equations, the explanation has to be found in the difficulty of deriving and imposing appropriate boundary conditions. Whereas for forced convection the imposed flow can be imposed at the entrance boundaries provided those are set sufficiently far upstream, this procedure can no longer be applied for natural convection as the flow magnitude and direction at the inflow boundary are determined by what happens close to the heated source located downstream the entrance. This difficulty has generally led people to consider boundary conditions that exert as less constraints as possible on the flow, in the aim of letting the flow establish itself in magnitude and direction.

In particular, the computation of flows such as developing external natural convection boundary

layers along a heated plate, of channel flows, or flows due to isolated heated bodies necessitates considering artificial boundaries of the computational domain located at some distance away from the heated parts. For such configurations most computations have considered homogeneous Neumann boundary conditions for the component of the velocity normal to the far field boundaries where the flow is drawn inside the computational domain, while considering that the tangential velocity can be assumed negligibly small, i.e. satisfies homogeneous Dirichlet conditions. At the outflow boundaries, homogeneous Neumann boundary or convective boundary conditions are generally used, in the aim of translating the assumption of no longer evolving flow structure, a vague concept.

The apparently simplest of the external flows listed above is the vertical channel, open at both ends, subject to various heating boundary conditions. This configuration is very interesting because, for appropriate values of the governing parameters, it may give rise to the so called chimney effect, an acceleration of the through flow, leading to heat transfer augmentation. Quantifying the chimney effect has been the subject of many works in the past (e.g. [2]). Owing to the geometry, one would like to consider a computational domain restricted to the channel geometry which raises the question of the inlet and outlet boundary conditions. As said above it is impossible to specify the shape of the vertical velocity component w, nor its magnitude i.e. the global flow rate, at entrance, as those are part of the unknowns of the problem. One possibility thus seems to set its vertical derivative to zero, while it seems logical to impose that the flow enters vertically i.e. to specify the horizontal component u to zero. At outlet it seems also logical, at least if the channel is long enough, to impose the same boundary conditions, i.e. vertical derivative of w equal to zero and zero horizontal velocity flow, conditions corresponding to established flow.

Now the question that one may ask is the following.: is this problem well posed, both from a continuous standpoint and after numerical discretization? After recalling the governing equations, we will answer to this question, making use of the singular value decomposition of the matrix resulting from the spatial discretization of the 2D operator. The recognition that there is one pair velocity-pressure solution of the homogeneous Stokes or Unsteady Stokes equations opens the way to a superposition or influence matrix technique. This technique is implemented and is used to provide solutions to classical configurations such as the channel flow for instance.

GOVERNING EQUATIONS

We assume that the governing equations are those of an incompressible flow of a newtonian fluid under the Boussinesq approximation. These equations are made dimensionless using H as unit length, Vref= $(\kappa/H)\,Ra^{1/2}$ as reference velocity where κ is the thermal diffusivity, $(H^2/\kappa)\,Ra^{-1/2}$ as reference time and ΔT as unit temperature difference. In dimensionless form, these equations read:

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{\partial P}{\partial x}+\frac{\mathrm{Pr}}{Ra^{1/2}}\Delta u$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{\partial P}{\partial z}+\frac{\mathrm{Pr}}{Ra^{1/2}}\Delta u+\mathrm{Pr}\,T$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+w\frac{\partial T}{\partial z}=\frac{1}{Ra^{1/2}}\Delta T$$

where u and w stand for, respectively, the dimensionless horizontal and vertical component velocities, P the pressure and T the dimensionless temperature. Ra is the Rayleigh number $(=g\beta\Delta TH^3/\nu\kappa)$, and Pr is the Prandtl number $(=\nu/\kappa)$

Let us assume that these equations are discretized in time with a time step Δt using an implicit discretization of the linear diffusive terms and an explicit treatment of the nonlinear terms. Thus at each time step, the velocity and pressure satisfy the following linear system of equations known as the unsteady Stokes problem:

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0$$

$$H_u u-\frac{\partial P}{\partial x}=Su$$

$$H_w w-\frac{\partial P}{\partial z}=Sw$$

where H_u and H_w are the elliptic operators $(\Delta-\lambda I)$ with λ a constant that depends on Δt and Ra. Typically λ is equal to $C\,Ra^{1/2}/\Delta t$ with C a constant that depends on the specific time stepping scheme. At steady state, the velocity and pressure satisfy the same system with $\lambda = 0$. This elliptic system has to be supplemented with boundary conditions. Let us

consider for instance the channel flow configuration shown in fig. 1 with

$\frac{\partial}{\partial z} w = 0$ and $u = 0$ at entrance and outlet.

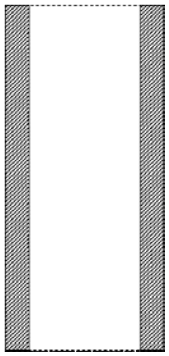

Figure1
Sketch of the channel geometry

ANALYSIS

Continuous equations: It is well known than the corresponding Stokes problem is singular. This is due to the fact that the 3-uplet (u,w,P)=(0,0,1) satisfies the homogeneous equations whatever the imposed boundary conditions. This corresponds to the fact that in an incompressible flow, the pressure is defined up an arbitrary constant. Now the basic question is the following: do the imposed boundary conditions allow for other linearly independent modes solution of the homogeneous equations? Or, in other words, do the imposed boundary conditions increase the size of the kernel of the Stokes operator? For the channel flow with the above specified boundary conditions, the answer is obvious as is known from any undergraduate textbook : a Poiseuille flow, with parabolic w-velocity component in x and linear pressure in z, satisfies the homogeneous steady Stokes problem, making the kernel of the steady Stokes problem of dimension at least 2. This means that if a steady state solution of the inhomogeneous equations is obtained in some way, it is not unique since another solution equal to the previous one plus any Poiseuille is solution of the inhomogeneous equations. This indeterminacy probably explains some of the discrepancies that have been observed in the literature. Now the following questions arise. Are there other "boundary modes" in the kernel? What happens for the type of geometry considered by Kettleborough [3] for instance? What happens for flows around heated bodies? As soon as the geometry becomes less trivial, the answer has to be obtained directly from the discretized equations. Looking at this question numerically may raise new difficulties owing to the classical question of the compatibility of the velocity and pressure spaces, which may add additional spurious modes in addition to those of the continuous problem.

Discretized equations: We will restrict ourselves here to the classical staggered grid discretization, which is known to provide compatible velocity-pressure approximation spaces, giving rise to the only constant pressure mode in the case of Dirichlet boundary conditions. The restriction to the staggered grid discretization may seem arbitrary, but it is not our goal to study this question in general but rather to provide a general framework which can be used in other types of spatial discretization, arising from finite element or spectral type approximations. Consider the unit square $\Omega = [0,1]x[0,1]$. Let Ω be covered with NxM cells. With the staggered grid arrangement, the component u is defined by Nu = (N+1)xM degrees of freedom, the component w is defined by Nw = Nx(M+1) degrees of freedom, and the pressure by Np = NxM. The Stokes operator is thus of order (Nu+Nw+Np). The boundary conditions are imposed in the appropriate way. In order to explicitly build the operator, all fields are considered as 1D vectors resulting from the natural ordering. The explicit construction of the Stokes operator allows one to perform its Singular Value Decomposition. The SVD allows one to fully characterize the matrix, as it provides the dimension of the kernel of the operator, and orthonormal basis of its kernel and of its image as well as those of its transpose. More precisely any matrix S, here square, can be written as $S = U \Lambda V^t$ where Λ is the diagonal matrix of the singular values. The number of null singular values gives the dimension of the kernel of S, and the columns of V corresponding to the null singular values form an orthonormal basis of Ker(S) while the columns of U corresponding to the non zero singular values form an orthonormal basis of Im(S). Since $S^t = V \Lambda U^t$, $Ker(S^t)$ and $Im(S^t)$ are characterized as well, allowing a complete determination of the domain of S and of its range.

The SVD of the steady Stokes operator of the channel with its specified boundary condition was performed with Scilab for values of N and M small enough for the problem to remain tractable but large enough to get a generic answer. For the channel flow is suffices that N and M be larger than 3. The number of null singular values and hence the number of modes of the kernel is 2. Although the two modes that come out of the SVD are mixed, the

boundary condition mode can be obtained if one assumes (and checks) that the intrinsic mode is the mode (0, 0, 1) and performing the orthonormalisation. It is verified that this mode is the discrete counterpart of the continuous Poiseuille flow, although even in this simple case, it has to be determined numerically due to the location of the velocity component at half grid cells.

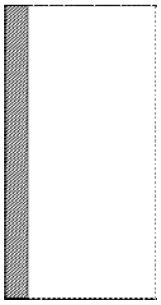

Figure 2
Sketch of a heated vertical plate. The artificial boundaries are the dotted lines.

In the case of a heated vertical plate (Fig. 2), the dimension of the kernel is also equal to 2. The mode of the kernel is a 1D flow with slip at the outer boundary, corresponding to a linear pressure gradient.

Additional computations were performed for other types of geometries. Let us consider the geometry sketched in fig.3 which corresponds to the case studied by Kettleborough [3] in order to set the entrance conditions at artificial boundaries away from the channel entrance. We impose the following boundary conditions

$$\frac{\partial}{\partial n} V.n = 0$$

$$\frac{\partial}{\partial n} V.\tau = 0$$

where n and τ are the unit vectors normal and tangential to the open boundaries. The use of a cell phase function allowed us to systematize the investigation of the various geometrical shapes shown below.

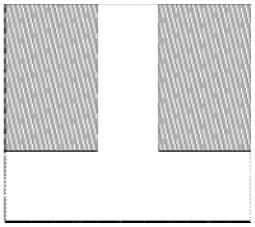

Figure 3
Sketch of the channel geometry considered in [3]. The artificial boundaries are the dotted lines.

Performing the SVD of the Stokes operator in this case reveals that its kernel is of dimension 3, which leaves 2 undetermined Stokes modes in addition to the constant pressure mode. Let us further consider the case for the flow around a heated body such as sketched in Fig.4. If we impose Neuman conditions on the normal component on all four external boundaries, the number of null singular values is equal to 4 yielding 3 undetermined Stokes modes in addition to the constant pressure mode.

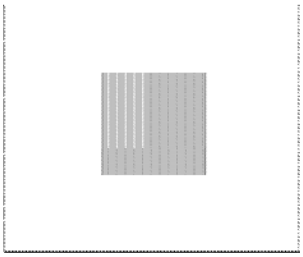

Figure 4
Configuration of a heated body

A SUPERPOSITION METHOD

General: Recognizing that with the specified boundary conditions the steady state solution of the Stokes problem is undefined up to a linear combination of the non-trivial Stokes modes solution of the homogeneous Stokes problem opens the way to the determination of a correct solution with the help of a superposition method. Assuming a particular solution has been obtained, one can superpose to this solution a linear combination of the non-trivial Stokes modes so as to satisfy additional constraints on the solution. The number of these constraints has to be equal to the dimension of the kernel. The additional constraints themselves depend on the modeling assumptions, as will be shown below. The present superposition method will thus allow one to compute the unique solution that satisfies the additional constraints which depend on the modeling assumption.

Channel flow: Assume a particular solution has been obtained. On can superpose to this solution a linear combination of the Poiseuille flow requiring that the pressure drop between inlet and outlet is equal to a specified value, which is the global condition that the channel flow has to satisfy. This value can be either zero, which says that the motion pressure outside the channel remains constant or to -½ G^2 if one takes into account the pressure drop

needed to accelerate the fluid from infinity to the channel entrance [4]. One could then proceed as follows. A particular solution is obtained through some procedure, either through time integration or solving iteratively for the steady state equations. When steady state is obtained the mean pressure drop between inlet and outlet is computed and the multiplicative constant of the Poiseuille flow is determined so as to produce the required pressure drop. The Poiseuille flow times this constant is then added to the particular solution, yielding the "true" solution.

In fact it turns out that it is not so easy to compute directly the particular steady solution precisely because this solution is not completely defined. In particular it may continuously keep shifting in time and very often diverge, in particular when the chimney effect is triggered. It is better then to control the solution in a time dependent way by using the superposition principle at each time step. In that case the boundary condition mode is no longer the steady Poiseuille flow but its unsteady counterpart that has to be determined numerically. That mode corresponds to (0, w(x), z) where w is solution of

$$\frac{\partial^2 w}{\partial x^2} - \lambda w = 1$$

with w(0)=w(1)=0. In this version, the superposition method is done at each time step. This results in a stable algorithm. It should be noted that the unsteady Stokes mode has to be computed only once (assuming Δt is constant), and the additional cost of the superposition principle is completely negligible. Sample computations were performed to test the effectiveness of this unsteady superposition principle.

We consider the specific case of an asymmetric channel (fig. 5) investigated experimentally in [5] and was considered as a benchmark exercise in the french convection community [6]. The numerical algorithm integrates the equations in unsteady form using a prediction-projection splitting procedure. The prediction step uses an implicit treatment of the viscous terms coupled with an explicit treatment of the convective terms. The projection step requires solving a Poisson type equation for the pressure correction which is done using a multigrid algorithm.

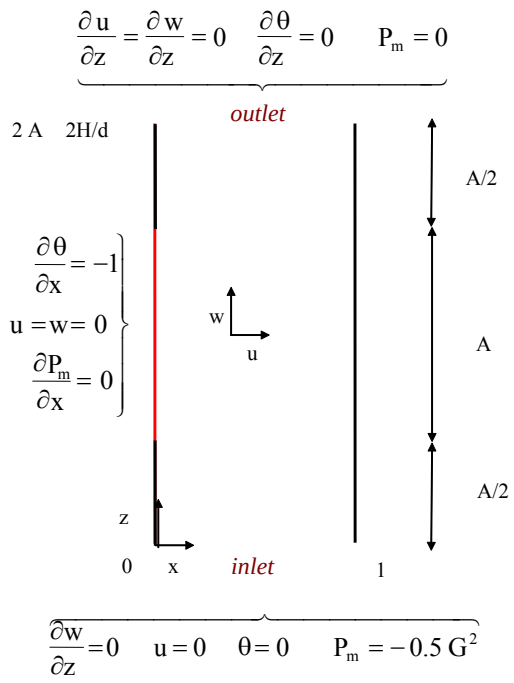

Figure 5
Asymmetric channel hated at uniform heat flux on half its vertical left wall.

We have computed the solution of the governing equations with the specified boundary conditions for one set of values of parameters corresponding to the benchmark exercise: A=5, Ra_w=5 10^5. Fig. 6 presents the w-velocity profile at various heights, showing that the flow enters with a parabolic profile and exits with a boundary layer profile. The plot also shows that above a certain height downward velocities are found which correspond to the existence of a recirculation zone with fluid entering from the top. The existence of such recirculation zones with fluid entering from top has been the subject of long debate in the heat transfer literature (see [7]). Needless to say that the confident and accurate numerical simulation of these subtle configurations requires an appropriate treatment of the inlet and outlet conditions. Fig. 7 shows the vertical pressure distribution at mid-width. It starts at -0.5 G^2, first shows a linear drop corresponding to fully developed flow, followed by a monotonous increase up to the end value.

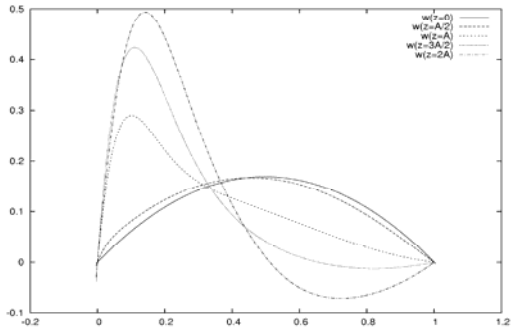

Figure 6
Vertical velocity profiles at various heights

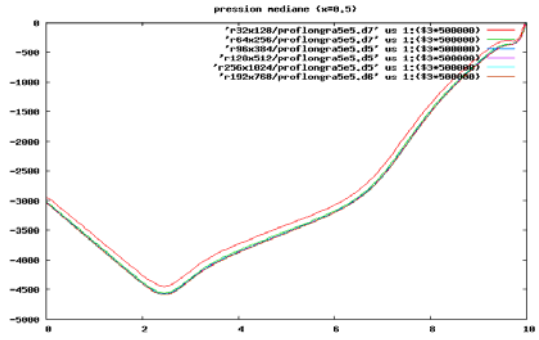

Figure 7
Pressure distribution at mid-width. Convergence with mesh refinement is shown

Let us mention that for a Ra_w number of 10^7, a global unsteadiness of the solution was observed as shown in Fig. 8 that presents the time variation of the Nusselt number. The figure shows that the asymptotic evolution of the Nusselt number displays repetitive large amplitude fluctuations of period approximately equal to 400 convective time units. The dynamics is made of a long decrease followed by a sudden jump to the maximum value. We have been able to relate this behavior to a slow increase in time of the recirculation zone with corresponding slow decrease of the through flow rate until the heat does no longer evacuate through the upper boundary. Since the channel is heated at uniform heat flux the heat then continues to accumulate in the upper part of the channel until the buoyancy forces suddenly take over, evacuate all the accumulated heat and restore the initial situation. This dynamics was found with several resolutions up to 130x512.

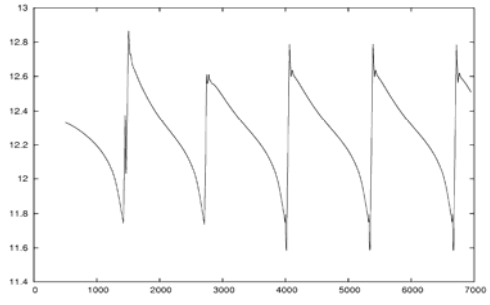

Figure 8
Time evolution of Nusselt number (time in the figure is in units of $(W^2/\kappa)Ra_w^{-1/2}$)

Boundary layer flow: To further assess the validity of the superposition principle we have addressed the configuration of a vertical natural convection boundary layer along a heated vertical plate. This configuration turns out to be much more demanding since the flow has to be allowed to flow across the artificial boundary facing the heated plate in order to provide the flow rate needed for the boundary layer to grow as it should do. Let us recall that the boundary layer thickness should grow as $z^{1/4}$ and the vertical velocity like $z^{1/2}$. The global flow rate thus grows like $z^{3/4}$ and the horizontal velocity at infinity should thus decrease like $z^{-1/4}$ like the Nusselt number (see [2] for instance). It is also known that the boundary layer equations admit a similarity solution first produced in [9].
This computation has considered a rectangular domain of vertical aspect ratio A with artificial boundaries such a sketched in fig. 9, on which normal derivatives of both velocity components and temperature were set to zero except on the bottom boundary OA where the temperature was set to zero. The left boundary was considered as a wall, adiabatic for $0<z<1/4$, heated at uniform temperature for $z>1/4$.
The superposition principle consisted of using the quasi-Poiseuille flow corresponding to

$$\frac{\partial^2 w}{\partial x^2} - \lambda w = 1$$

with w(0) and w'(1/A)=0 to control the w-velocity component in one point of the boundary AB. Other attempts to control the pressure difference did not result in stable computations. We have considered a domain of vertical aspect ratio 10 and a nominal Rayleigh number of 10^{10} based on the total height of the domain, corresponding to a local Rayleigh number at the top of the plate equal to 4.2×10^9. The spatial resolution is 64x256 with a stretched half-cosine grid in x and a uniform grid in z.

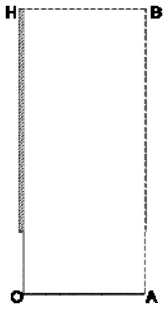

Figure 9
Computational domain for heated plate calculation

Fig. 10 presents the vertical velocity profiles at various heights along the plate, showing the development of the boundary layer and that all the flow comes from the lateral boundary since the vertical velocity at z=1/4 is zero. Figure 10 shows that the computed solution verifies the similarity conditions.

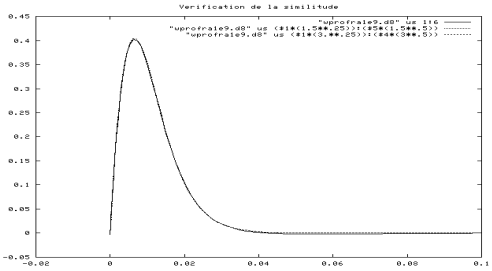

Figure 10
Vertical velocity profiles at various heights scaled with similarity variables

OTHER CONFIGURATIONS

Vented cavity: Consider a vented cavity like that sketched on figure 11, where the flow will be due to the fact that the inner body is heated. In this case the dimension of the kernel is equal to 2 and the non homogeneous Stokes mode corresponding to Neuman conditions on the velocity at both apertures is shown in figure 12. In consists of a flow entering and exiting through the apertures driven by the corresponding pressure field.

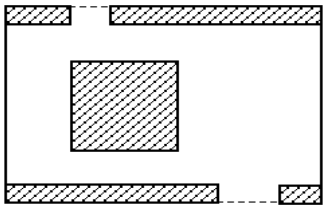

Figure 11
Sketch of vented cavity

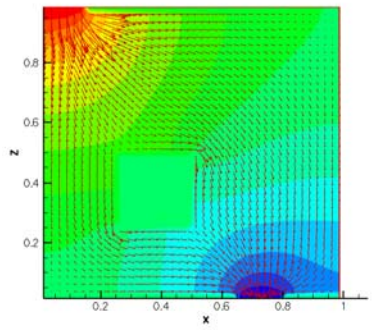

Figure 12
Non homogeneous Stokes mode

A full nonlinear solution obtained with the internal heated body corresponding to equal mean pressures at inlet and oulet is shown in figure 13.

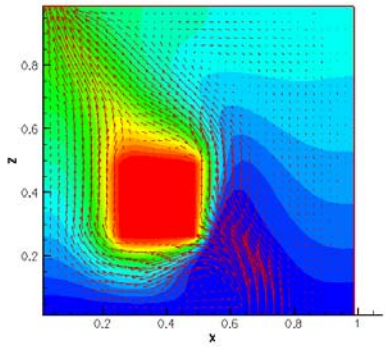

Figure 13
Fully non linear solution for Ra=10^5

Pipe flow with multiple outlets: A classical flow configuration is that of pipe flow with multiple outlets which raises the issue of the way the inlet flow rate splits between the various branches. If one does not want to specify a priori the flow rates in the various branches, one is led to impose Neuman type conditions on the normal component at the various outlets. Consider for instance the flow geometry sketched in figure 14, consisting of one inlet and two successive branches with 4 outlets.

Figure 14
Sketch of pipe flow with 4 outlets

With neuman type conditions imposed on the normal components of the velocity in the outlet planes, the dimension of the kernel is 4, as many as free outlets, yielding 3 non homogeneous Stokes modes. Such one mode is shown in Figure 15

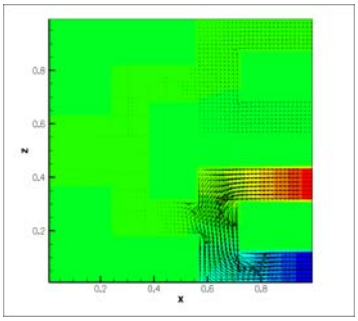

Figure 15
One typical non homogeneous Stokes mode

A superposition algorithm was implemented consisting of controlling three scalars, the differences in mean pressure between the 4 outlets (3 scalars). A full nonlinear solution for a Reynolds number of 200 is shown in figure 16.

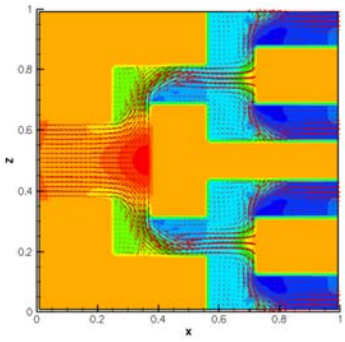

Figure 16
Nonlinear solution for a Reynolds number of 200

CONCLUSIONS

We have revisited the issue of the computation of incompressible flows in open domains or partially enclosed domains when the forces responsible of the motion are located within the computation domain. We have shown that the imposition of Neumann type boundary conditions may lead to one or several non-trivial combinations of velocity-pressure fields which satisfy the homogeneous Stokes operator, in addition to the unavoidable constant pressure mode. This recognition leads to proposing an algorithm in which the solution is sought as a combination of particular solution of the inhomogeneous Stokes or unsteady Stokes problem plus a linear combination of the modes of the kernel so as to satisfy global conditions such as a mean pressure difference between outlet and inlet. The algorithm was tested on two classical configurations, a channel flow and a heated vertical plate, demonstrating its effectiveness and its efficiency. The algorithm can also more generally apply for pipe networks with one or several outlets, which is the unique way not to specify a priori the partition of the flow rate between the several exits.

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