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BAD BOUNDARY BEHAVIOR IN STAR INVARIANT SUBSPACES I

ANDREAS HARTMANN & WILLIAM T. ROSS

ABSTRACT. We discuss the boundary behavior of functions in backward shift invariant subspaces $(BH^2)^\perp$, where B is a Blaschke product. Extending some results of Ahern and Clark, we are particularly interested in the growth rates of functions at points of the spectrum of B where B does not admit a derivative in the sense of Carathéodory.

1. INTRODUCTION

For a Blaschke product

$$B(z) = \prod_{n \geq 1} b_{\lambda_n}(z), \quad b_\lambda(z) = \frac{|\lambda|}{\lambda} \frac{\lambda - z}{1 - \bar{\lambda}z},$$

with zeros $\Lambda = \{\lambda_n\}_{n \geq 1}$, repeated according to multiplicity, let us recall this following theorem of Ahern and Clark [AC70] about the “good” non-tangential boundary behavior of functions in the model spaces $(BH^2)^\perp$ [Nik86] of the Hardy space H^2 of the open unit disk \mathbb{D} [Dur70, Gar07].

Theorem 1.1 ([AC70]). *For a Blaschke product B with zeros $\{\lambda_n\}_{n \geq 1}$ and $\zeta \in \mathbb{T} := \partial\mathbb{D}$, the following are equivalent:*

(1) *Every $f \in (BH^2)^\perp$ has a non-tangential limit at ζ , i.e.,*

$$f(\zeta) := \angle \lim_{\lambda \rightarrow \zeta} f(\lambda) \text{ exists.}$$

(2) *B has an angular derivative in the sense of Carathéodory at ζ , i.e.,*

$$\angle \lim_{z \rightarrow \zeta} B(z) = \eta \in \mathbb{T} \quad \text{and} \quad \angle \lim_{z \rightarrow \zeta} B'(z) \text{ exists.}$$

(3) *The following condition holds*

$$(1.2) \quad \sum_{n \geq 1} \frac{1 - |\lambda_n|}{|\zeta - \lambda_n|^2} < \infty.$$

(4) *The family of reproducing kernels for $(BH^2)^\perp$*

$$k_\lambda^B(z) := \frac{1 - \overline{B(\lambda)}B(z)}{1 - \bar{\lambda}z}$$

is uniformly norm bounded in each fixed Stolz domain

$$\Gamma_{\alpha, \zeta} = \left\{ z \in \mathbb{D} : \frac{|z - \zeta|}{1 - |z|} < \alpha \right\}, \quad \alpha \in (1, \infty).$$

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This is only a partial statement of the Ahern-Clark result. They went on further to characterize the existence of non-tangential boundary limits of the derivatives (up to a given order) of functions in $(BH^2)^\perp$. They also discuss the boundary behavior of functions in $(IH^2)^\perp$ where I is a general inner function and not necessarily a Blaschke product. Of course there is the well-known result (see e.g. [Nik86, p. 78]) which says that every $f \in (BH^2)^\perp$ has an analytic continuation across $\mathbb{T} \setminus \sigma(B)$, where

$$\sigma(B) = \left\{ |z| \leq 1 : \lim_{\lambda \rightarrow z} |B(\lambda)| = 0 \right\}$$

is the spectrum of B .

The aim of this paper is to consider the growth of functions in $(BH^2)^\perp$ at the points $\zeta \in \mathbb{T}$ where (1.2) fails. Thus, as in the title of this paper, we are looking at the “bad” boundary behavior of functions from $(BH^2)^\perp$. First let us get a handle on the worst behavior we can expect. Indeed, it is well known that every $f \in H^2$ satisfies the growth condition

$$(1.3) \quad |f(\lambda)| = o\left(\frac{1}{\sqrt{1-|\lambda|}}\right), \quad \lambda \in \Gamma_{\alpha, \zeta}.$$

As seen in the Ahern-Clark theorem, functions in $(BH^2)^\perp$ can be significantly better behaved depending on the distribution of the zeros of B . To fix our ideas, let $\zeta = 1$. A trivial observation is the following: For every $f \in (BH^2)^\perp$ and $\lambda \in \mathbb{D}$ we have

$$(1.4) \quad |f(\lambda)| = |\langle f, k_\lambda^B \rangle| \leq \|f\| \left(\frac{1 - |B(\lambda)|^2}{1 - |\lambda|^2} \right)^{1/2},$$

and by duality

$$\sup_{\|f\| \leq 1} |\langle f, k_\lambda^B \rangle| = \|k_\lambda^B\| = \left(\frac{1 - |B(\lambda)|^2}{1 - |\lambda|^2} \right)^{1/2}.$$

In the above, $\|\cdot\|$ denotes the usual norm in H^2 . So, in order to give an upper estimate of the admissible growth in $(BH^2)^\perp$ in a Stolz angle, we have to control $\|k_\lambda^B\|$ in such an angle which involves controlling how fast $|B(z)|$ goes to 1 in $\Gamma_{\alpha, 1}$.

Let us first make the obvious observation that if there exists a sequence $\{z_n\}_{n \geq 1} \subset \Gamma_{\alpha, 1}$ with $z_n \rightarrow \zeta = 1$ and with $|B(z_n)| \leq \delta < 1$, then clearly $\|k_{z_n}^B\| \rightarrow \infty$ and so, by basic functional analysis, there must be an $f \in (BH^2)^\perp$ without a finite non-tangential limit at 1. We actually have

$$\|k_{z_n}^B\| \asymp \|k_{z_n}\| = \frac{1}{\sqrt{1 - |z_n|^2}},$$

where

$$k_\lambda(z) := \frac{1}{1 - \bar{\lambda}z}, \quad z \in \mathbb{D},$$

is the reproducing kernel for H^2 . So, in this situation, the maximal possible growth of reproducing kernels in Hardy spaces is attained. Thus the subtlety occurs, for example, when

$$\angle \lim_{z \rightarrow 1} B(z) = \eta \in \mathbb{T}$$

which is implied by the Frostman condition [CL66]

$$(1.5) \quad \sum_{n \geq 1} \frac{1 - |\lambda_n|}{|1 - \lambda_n|} < \infty.$$

Observe the power in the denominator in (1.5) with respect to the Ahern-Clark result (1.2).

The main results of this paper will be non-tangential growth estimates of functions in $(BH^2)^\perp$ via non-tangential growth estimates of the norms of the kernel functions. Our main result (Theorem 3.8) will be an estimate of the form

$$\|k_r^B\| \asymp h(r), \quad r \rightarrow 1^-,$$

for some $h : [0, 1) \rightarrow \mathbb{R}_+$ which depends on the position of the zeros of the Blaschke product B near 1. This will, of course via (1.4), yield the estimate

$$|f(r)| \lesssim h(r), \quad f \in (BH^2)^\perp, \quad r \rightarrow 1^-.$$

To get a handle on the sharpness of this growth estimate, we will show (Theorem 4.13) that for every $\varepsilon > 0$, there exists an $f \in (BH^2)^\perp$ satisfying

$$|f(r)| \gtrsim \frac{h(r)}{\log^{1+\varepsilon} h(r)}, \quad r \rightarrow 1^-.$$

In certain situations, it is possible to replace $\log^{1+\varepsilon} h(r)$ by $\log h(r) \log^{1+\varepsilon} \log h(r)$ without ever getting rid of a logarithmic term. We do not know whether this logarithmic gap is optimal. Still, it allows to show that a certain sequence of reproducing kernels cannot form an unconditional sequence (see Remark 2.13(1) and Section 5).

The two basic types of Blaschke sequences $\{\lambda_n\}_{n \geq 1}$ we will be considering here, for which we can get concise estimates of h , are

$$(1.6) \quad \lambda_n = (1 - x_n 2^{-2n}) e^{i2^{-n}}, \quad x_n \downarrow 0,$$

which approaches 1 very tangentially, and

$$(1.7) \quad \lambda_n = (1 - \theta_n^2) e^{i\theta_n}, \quad \sum_{n \geq 1} \theta_n < \infty,$$

which approaches 1 along an oricycle. As an example of the types of results we will obtain, we will prove that when $x_n = 1/n$ in (1.6), we have the upper estimate (see Example (4.11)(1))

$$|f(r)| \lesssim \sqrt{\log \log \frac{1}{1-r}}, \quad r \rightarrow 1^-,$$

for all $f \in (BH^2)^\perp$ while when $\theta_n = 1/n^\alpha$, $\alpha > 1$, in (1.7), we have the estimate (see Example (4.33)(1))

$$|f(r)| \lesssim \frac{1}{(1-r)^{\frac{1}{2\alpha}}}, \quad r \rightarrow 1^-.$$

Compare these two results to the growth rate in (1.3) of a generic H^2 function.

This is the first of two papers on “bad” boundary behavior of $(IH^2)^\perp$ (I inner) functions near a fixed point on the circle. This paper considers the case when I is a Blaschke product. The next paper will consider the case when I is a general inner function where different types of estimates and very different methods are used yielding however less precise results in the Blaschke product situation than in this paper.

2. WHAT CAN BE EXPECTED

Before discussing the growth of functions in $(BH^2)^\perp$ or more generally in $(IH^2)^\perp$, we should consider the following three generic situations:

- the Hardy space itself,
- $(IH^2)^\perp$ at a point $\zeta \in \mathbb{T}$ where I vanishes with some decrease conditions,
- $(BH^2)^\perp$ when B is an interpolating Blaschke product with zeros on a radius.

Let us start with the Hardy space situation. It is well known that for every $f \in H^2$ and $\lambda \in \mathbb{D}$, we have

$$(2.1) \quad |f(\lambda)| = |\langle f, k_\lambda \rangle| \leq \|f\| \|k_\lambda\| = \frac{\|f\|}{\sqrt{1-|\lambda|}}.$$

Also well-known is the slightly better estimate:

$$(2.2) \quad |f(\lambda)| = o\left(\frac{1}{\sqrt{1-|\lambda|}}\right), \quad \lambda \in \Gamma_{\alpha, \zeta}.$$

The proof of this is short: Indeed we suppose that $\zeta = 1$. Since H^∞ , the bounded analytic functions on \mathbb{D} , is dense in H^2 , given $\varepsilon > 0$ and $f \in H^2$, we can choose a $g \in H^\infty$ with $\|g - f\| \leq \varepsilon$. Then

$$\begin{aligned} \sqrt{1-r}|f(r)| &= \sqrt{1-r}|f(r) - g(r)| + \sqrt{1-r}|g(r)| \\ &\leq \sqrt{1-r}\|f - g\| \frac{1}{\sqrt{1-r}} + C_g \sqrt{1-r} \\ &\leq \varepsilon + o(1). \end{aligned}$$

The little-oh condition in (2.2) is, in a sense, sharp. Though the following result is most likely known, we include a proof here for completeness.

Proposition 2.3. *For every function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ strictly increasing to infinity and such that φ^2 is concave¹, there exists an $F \in H^2$ such that*

$$\overline{\lim}_{r \rightarrow 1^-} \sqrt{1-r} \varphi\left(\frac{1}{1-r}\right) |F(r)| > 0.$$

Proof. In order to construct the desired function F we will begin by constructing its real part f which is the Poisson extension of a certain boundary function $\omega : (0, 1] \rightarrow \mathbb{R}_+$ defined by

$$\omega = \sum_{n \geq 1} \frac{1}{\varphi(1/t_n) \sqrt{t_n}} \chi_{[t_n, t_{n-1})}.$$

In the above, $\{t_n\}_{n \geq 1}$ is a suitable sequence of positive numbers (to be determined in a moment) which decrease to zero and $t_1 = 1$. Let us show how to find this sequence for the given function φ .

We have two requirements. The first one is that $\omega \in L^2$. This translates to

$$0 \leq \int_0^1 \omega^2(t) dt = \sum_{n \geq 1} \frac{t_{n-1} - t_n}{\varphi^2(1/t_n) t_n} = \sum_{n \geq 1} \frac{1}{\varphi^2(1/t_n)} \left(\frac{t_{n-1}}{t_n} - 1 \right) < \infty.$$

¹The concavity assumption is not very restrictive since we are interested in very slow growth of φ .

With

$$(2.4) \quad \beta_n := \frac{t_{n-1}}{t_n} - 1,$$

the requirement $\omega \in L^2$ becomes

$$(2.5) \quad \sum_{n \geq 1} \frac{\beta_n}{\varphi^2(1/t_n)} < \infty.$$

In order to simplify the following computations we set $\tau_n := 1/t_n$ which we need increasing to infinity if we want our function ω to be well defined. This will be our second requirement.

It is immediate, provided that $\tau_n \rightarrow \infty$ when $n \rightarrow \infty$, that the sequence $\{\beta_n\}_{n \geq 1}$ defined by

$$(2.6) \quad \beta_n := \varphi^2(\tau_n) \left(\frac{1}{\varphi^2(\tau_n)} - \frac{1}{\varphi^2(\tau_{n+1})} \right) = 1 - \frac{\varphi^2(\tau_n)}{\varphi^2(\tau_{n+1})}$$

satisfies (2.5) (since (2.5) reduces to a telescoping series). Comparing (2.4) and (2.6) we obtain a recurrence formula

$$(2.7) \quad 1 - \frac{\varphi^2(\tau_n)}{\varphi^2(\tau_{n+1})} = \frac{\tau_n}{\tau_{n-1}} - 1,$$

which we write as

$$\frac{\varphi^2(\tau_{n+1}) - \varphi^2(\tau_n)}{\tau_{n+1} - \tau_n} \frac{\tau_{n+1} - \tau_n}{\tau_n - \tau_{n-1}} = \frac{\varphi^2(\tau_{n+1})}{\tau_{n+1}} \frac{\tau_{n+1}}{\tau_{n-1}},$$

or equivalently

$$(2.8) \quad \frac{\varphi^2(\tau_{n+1}) - \varphi^2(\tau_n)}{\tau_{n+1} - \tau_n} \frac{\tau_{n+1}}{\varphi^2(\tau_{n+1})} = \frac{\tau_{n+1}}{\tau_{n-1}} \frac{\tau_n - \tau_{n-1}}{\tau_{n+1} - \tau_n}.$$

The concavity of φ^2 implies that

$$\frac{\varphi^2(\tau_{n+1}) - \varphi^2(\tau_n)}{\tau_{n+1} - \tau_n} \leq \frac{\varphi(\tau_{n+1})}{\tau_{n+1}},$$

so that the left hand side of (2.8) is less than or equal to one. Hence

$$\frac{\tau_{n+1}}{\tau_{n-1}} \frac{\tau_n - \tau_{n-1}}{\tau_{n+1} - \tau_n} \leq 1$$

and so

$$(2.9) \quad \tau_{n+1} - \tau_n \geq \frac{\tau_{n+1}}{\tau_{n-1}} (\tau_n - \tau_{n-1}).$$

Let us now argue that $\{\tau_n\}_{n \geq 1}$ is an increasing sequence. First we set $\tau_1 > \tau_0 = 1$. Then, we assume, for the sake of induction, that $\tau_n \geq \tau_{n-1}$. The identity in (2.7) implies that $\varphi^2(\tau_{n+1}) \geq \varphi^2(\tau_n)$ and, since φ^2 is strictly increasing, we get $\tau_{n+1} \geq \tau_n$. Thus $\{\tau_n\}_{n \geq 1}$ is an increasing sequence. Using (2.9) yields

$$\tau_{n+1} - \tau_n \geq \tau_n - \tau_{n-1}.$$

and so $\tau_n \uparrow \infty$ which implies that $t_n \downarrow 0$. Thus our second requirement is met.

Before constructing the function F realizing the required growth at t_n , we have to check one more additional property: ω is decreasing on $(0, 1]$. This is equivalent to

$$\frac{\sqrt{\tau_n}}{\varphi(\tau_n)} \leq \frac{\sqrt{\tau_{n+1}}}{\varphi(\tau_{n+1})},$$

or

$$\frac{\varphi^2(\tau_{n+1})}{\tau_{n+1}} \leq \frac{\varphi^2(\tau_n)}{\tau_n},$$

and again this is a consequence of the concavity of φ^2 .

Now, setting

$$f(e^{it}) := \begin{cases} \omega(t) & t \in [0, 1], \\ 0 & \text{otherwise,} \end{cases}$$

we obtain a (positive) L^2 -function f on \mathbb{T} (which we extend to \mathbb{D} using the Poisson kernel). Its harmonic conjugate \tilde{f} is also (real-valued) in L^2 . Hence $F = f + i\tilde{f} \in H^2$. Setting $r_n = 1 - t_n$ and letting $I_r = \{e^{it} : |t| \leq 1 - r\}$ be the Privalov shadow of r on \mathbb{T} , we get

$$|F(r_n)| \geq |f(r_n)| = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - r_n^2}{|e^{it} - r_n|^2} f(e^{it}) dm(e^{it}) \geq \frac{1}{2\pi} \int_{I_{r_n}} \frac{1 - r_n^2}{|e^{it} - r_n|^2} f(e^{it}) dm(e^{it}).$$

Finally, since ω is decreasing on $(0, 1]$ we get

$$|F(r_n)| \geq \omega(1 - r_n) \frac{1}{2\pi} \int_0^{1-r_n} \frac{1 - r_n^2}{|e^{it} - r_n|^2} dt \simeq \omega(1 - r_n) = \omega(t_n) = \frac{1}{\sqrt{(1 - r_n)\varphi(1/(1 - r_n))}}. \quad \blacksquare$$

The next observation is that in certain model spaces $(IH^2)^\perp$ we have the same boundary behavior as in H^2 .

Theorem 2.10. *Let I be an inner function such that $|I(r)| = o(\sqrt{1 - r})$. For every function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ strictly increasing to infinity and such that φ^2 is concave, there is an $F_1 \in (IH^2)^\perp$ such that*

$$\lim_{r \rightarrow 1^-} \sqrt{1 - r} \varphi\left(\frac{1}{1 - r}\right) |F_1(r)| > 0.$$

Proof. With φ fixed, let F as well as $\{r_n\}_{n \geq 1}$ be as in Proposition 2.3. Then we set $F_1 = P_I F \in (IH^2)^\perp$ so that $F = F_1 + Ih$ for a suitable $h \in H^2$. Hence

$$\begin{aligned} |F_1(r_n)| &= |P_I F(r_n)| \geq |F(r_n)| - |I(r_n)h(r_n)| \\ &\gtrsim \frac{1}{\sqrt{1 - r_n}\varphi(1/(1 - r_n))} - \frac{|I(r_n)|}{\sqrt{1 - r_n}} \\ &\simeq \frac{1}{\sqrt{1 - r_n}\varphi(1/(1 - r_n))}. \quad \blacksquare \end{aligned}$$

Remark 2.11. (1) An inner function whose singular part has an associated singular measure with a point mass at 1 will easily satisfy the hypothesis of Theorem 2.10.

(2) The condition $|I(r)| = o(\sqrt{1 - r})$ can be weakened but one would need to place some restriction on the types of growth functions φ allowed. To avoid being too wordy and technical we did not state the theorem in its greatest generality.

From Theorem 2.10 we see that in certain model spaces $(IH^2)^\perp$, we can attain the same maximal growth as in H^2 . Contrast this with the following result.

Proposition 2.12. *Let B be the Blaschke product with zeros $\rho_n = 1 - 2^{-n}$. Then*

$$|f(\rho_n)| = \varepsilon_n \frac{1}{\sqrt{1 - \rho_n}}, \quad \forall n \in \mathbb{N},$$

for $f \in (BH^2)^\perp$ if and only if $\{\varepsilon_n\}_{n \geq 1} \in \ell^2$.

Remark 2.13. (1) In this situation we cannot reach arbitrarily slow growth for φ in Theorem 2.10 since we would have to require

$$\varphi\left(\frac{1}{1-\rho_n}\right) = \frac{1}{\varepsilon_n}$$

with $\{\varepsilon_n\}_{n \geq 1} \in \ell^2$.

(2) The estimate in Proposition 2.12 extends to a whole Stolz angle $\Gamma_{1,\alpha}$:

$$|f(z)| \lesssim \varepsilon_n \frac{1}{\sqrt{1-|z|}}$$

for $|b_{\rho_n}(z)| < \delta$ where δ is some fixed constant depending on the opening α of the Stolz angle.

Proof of Proposition 2.12. In order to prove this result we first need to recall that an unconditional basis (or sequence) $\{x_k\}_{k \geq 1}$ in a Hilbert space is an isomorphic image of an orthogonal basis (or sequence). We allow non-normalized sequences. We can associate its so-called Gram matrix:

$$G := \left(\left\langle \frac{x_n}{\|x_n\|}, \frac{x_m}{\|x_m\|} \right\rangle \right)_{n,m \geq 1}.$$

It is well known [Nik02, Exercice C3.3.1(d)] that if $\{x_n\}_{n \geq 1}$ is an unconditional basis (sequence) then G represents an isomorphism from ℓ^2 onto ℓ^2 .

Let us transfer this to our situation. Consider $x_n = k_{\rho_n}$, and let G be the associated Gram matrix. It is well known that $\{\rho_n\}_{n \geq 1}$ is an interpolating sequence and so $\{k_{\rho_n}\}_{n \geq 1}$ is an unconditional basis for $(BH^2)^\perp$. Hence every $f \in (BH^2)^\perp$ can be written as

$$f = f_\alpha := \sum_{n \geq 1} \alpha_n \frac{k_{\rho_n}}{\|k_{\rho_n}\|}, \quad \alpha = \{\alpha_n\}_{n \geq 1} \in \ell^2,$$

with $\|f_\alpha\|^2 \asymp \sum_{n \geq 1} |\alpha_n|^2 < \infty$. Notice that

$$f_\alpha(\rho_N) = \sum_{n \geq 1} \alpha_n \frac{k_{\rho_n}(\rho_N)}{\|k_{\rho_n}\|} = \|k_{\rho_N}\| \sum_{n \geq 1} \alpha_n \frac{\langle k_{\rho_n}, k_{\rho_N} \rangle}{\|k_{\rho_n}\| \|k_{\rho_N}\|} = \|k_{\rho_N}\| (G\alpha)_N.$$

Hence for every $\alpha \in \ell^2$, we have

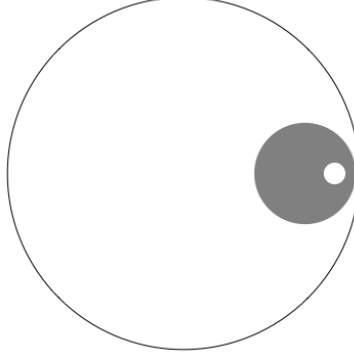
$$(2.14) \quad f_\alpha(\rho_N) = \|k_{\rho_N}\| (G\alpha)_N$$

and $G\alpha \in \ell^2$. Conversely, for every $\beta \in \ell^2$, we can set $\alpha = G^{-1}\beta$ and find a $f_\alpha \in (BH^2)^\perp$ satisfying (2.14). \blacksquare

3. A GENERAL GROWTH RESULT FOR $(BH^2)^\perp$

We will start by introducing a growth parameter associated with a Blaschke sequence $\Lambda = \{\lambda_n\}_{n \geq 1} \subset \mathbb{D}$ and a boundary point $\zeta \in \mathbb{T}$. Let us again set

$$\rho_N := 1 - \frac{1}{2^N}, \quad N \in \mathbb{N}.$$

FIGURE 1. An example of a domain $\Gamma_n^{N,1}$.

For every $N \in \mathbb{N}$ and $n \in \mathbb{Z}$, set

$$(3.1) \quad \Gamma_n^{N,\zeta} := \left\{ z \in \mathbb{D} : \frac{1 - |z|^2}{|\zeta - \rho_N z|^2} \in \left[\frac{1}{2^{n+1}}, \frac{1}{2^n} \right] \right\}.$$

This is a kind of pseudohyperbolic annulus (see Figure 1). A simple computation shows that $\frac{1 - |z|^2}{|\zeta - \rho_N z|^2} = c$ if and only if

$$\left| z - \frac{c\rho}{1 + c\rho^2} \zeta \right|^2 = \frac{1 - c(1 - \rho^2)}{(1 + c\rho^2)^2}.$$

From here it can be observed that necessarily $c \leq \frac{1}{1 - \rho^2}$ which means that $\Gamma_n^{N,\zeta}$ is empty when

$$\frac{1}{2^{n+1}} \geq \frac{1}{1 - \rho_N^2} \geq \frac{1}{2(1 - \rho_N)} = 2^{N-1}.$$

So we assume that $n \geq -N$.

For simplicity, we will assume from now on that $\zeta = 1$ and set

$$\Gamma_n^N := \Gamma_n^{N,1}.$$

Define

$$\alpha_{N,n} := \#(\Lambda \cap \Gamma_n^N)$$

to be the number of points in $\Lambda \cap \Gamma_n^N$. We now define the following growth parameter σ_N as

$$\sigma_N := \sum_{n \in \mathbb{Z}} \frac{\alpha_{N,n}}{2^n} = \sum_{n \geq -N} \frac{\alpha_{N,n}}{2^n}.$$

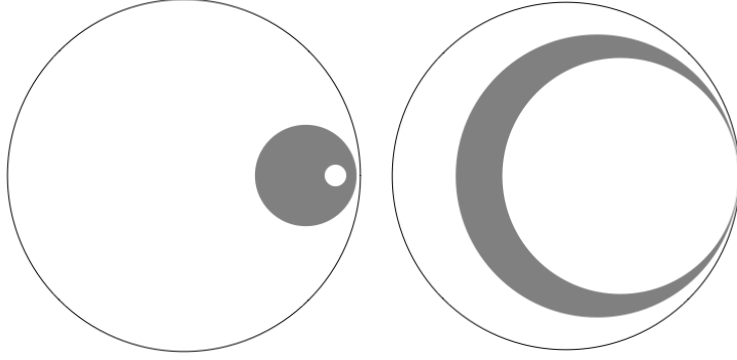
For each $\lambda \in \Lambda \cap \Gamma_n^N$ we have, by definition (see (3.1)),

$$\frac{1}{2^n} \asymp \frac{1 - |\lambda|^2}{|1 - \rho_N \lambda|^2}$$

and so, since there are $\alpha_{N,n}$ points in $\Lambda \cap \Gamma_n^N$, we have

$$\sum_{n \geq -N} \frac{1}{2^n} \#(\Lambda \cap \Gamma_n^N) \asymp \sum_{n \geq -N} \sum_{\lambda \in \Lambda \cap \Gamma_n^N} \frac{1 - |\lambda|^2}{|1 - \rho_N \lambda|^2}.$$

But since $\{\Gamma_n^N\}_{n \geq -N}$ is a partition of \mathbb{D} (see Figure 2) we get

FIGURE 2. The domains Γ_n^N , $-N \leq n$ cover \mathbb{D} .

$$\sum_{n \geq -N} \sum_{\lambda \in \Lambda \cap \Gamma_n^N} \frac{1 - |\lambda|^2}{|1 - \rho_N \lambda|^2} = \sum_{n \geq 1} \frac{1 - |\lambda_n|^2}{|1 - \rho_N \lambda_n|^2}.$$

Putting this all together we arrive at

$$(3.2) \quad \sigma_N \asymp \sum_{n \geq 1} \frac{1 - |\lambda_n|^2}{|1 - \rho_N \lambda_n|^2}.$$

In order to relate σ_N to the norms of the reproducing kernels for $(BH^2)^\perp$ we need the following result.

Lemma 3.3. *If B is a Blaschke product with zeros $\lambda_n = r_n e^{i\theta_n}$ and $\angle \lim_{z \rightarrow 1} B(z) = \eta \in \mathbb{T}$, then*

$$\|k_r^B\|^2 \asymp \sum_{n \geq 1} \frac{1 - r_n^2}{|e^{i\theta_n} - r r_n|^2}, \quad r \in (0, 1).$$

(The estimate extends naturally to a Stolz angle.)

Recall that (1.5) implies $\angle \lim_{z \rightarrow 1} B(z) = \eta \in \mathbb{T}$.

Proof. Since $\angle \lim_{z \rightarrow 1} B(z) = \eta \in \mathbb{T}$, the zeros of B (after some point) can not lie in $\Gamma_{\alpha,1}$. Thus if

$$b_\lambda(z) = \frac{z - \lambda}{1 - \bar{\lambda}z},$$

then $\inf_{n \geq 1} |b_{\lambda_n}(r)| \geq \delta > 0$ and so

$$\log \frac{1}{|b_\lambda(r)|^2} \asymp 1 - |b_{\lambda_n}(r)|^2.$$

It is a well known fact that

$$1 - |b_{\lambda_n}(r)|^2 = \frac{(1 - r^2)(1 - |\lambda_n|^2)}{|1 - r \bar{\lambda}_n|^2} = \frac{(1 - r^2)(1 - |\lambda_n|^2)}{|e^{i\theta_n} - r r_n|^2}.$$

Thus

$$\begin{aligned} \log |B(r)|^{-2} &= \sum_{n \geq 1} \log \frac{1}{|b_{\lambda_n}(z)|^2} \asymp \sum_{n \geq 1} \frac{(1 - |\lambda_n|^2)(1 - |r|^2)}{|1 - \bar{\lambda}_n r|^2} \\ &\asymp (1 - r^2) \sum_{n \geq 1} \frac{(1 - r_n^2)}{|e^{i\theta_n} - r r_n|^2}. \end{aligned}$$

Since $|B(r)| \rightarrow 1$ when $r \rightarrow 1^-$ the latter quantity goes to 0 and so

$$\begin{aligned} \|k_r^B\|^2 &= \frac{1 - |B(r)|^2}{1 - |r|^2} = \frac{1 - e^{\log |B(r)|^2}}{1 - r^2} = \frac{1 - (1 + \log |B(r)|^2 + o(\log |B(r)|^2))}{1 - r^2} \\ &\asymp -\frac{\log |B(r)|^2}{1 - r^2} \\ &\asymp \sum_n \frac{1 - r_n^2}{|e^{i\theta_n} - r r_n|^2}. \end{aligned} \quad \blacksquare$$

Combine (3.2) with the above lemma to get the two-sided estimate

$$(3.4) \quad \sigma_N \asymp \|k_{\rho_N}^B\|^2.$$

Now if the zeros $\{\lambda_n\}_{n \geq 1}$ of B satisfy the Ahern-Clark condition (1.2) then, by Theorem 1.1, the sequence $\{\|k_{\rho_N}^B\|\}_{N \geq 1}$ is uniformly bounded and, by (3.4), so is $\{\sigma_N\}_{N \geq 1}$.

In this paper we would like to discuss the situation when $\{\sigma_N\}_{N \geq 1}$ is unbounded. Let us assume that $\{\sigma_N\}_{N \geq 1}$ also satisfies the regularity condition

$$(3.5) \quad 0 < m := \inf_N \frac{\sigma_{N+1}}{\sigma_N} \leq M := \sup_N \frac{\sigma_{N+1}}{\sigma_N} < \infty.$$

Suppose $\varphi_\sigma^0 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is the continuous piecewise affine function defined by

$$\varphi_\sigma^0(N) := \sigma_N, \quad N \in \mathbb{N}.$$

Looking at (3.5) we deduce that for $x \in [N, N+1)$,

$$\sigma_N \asymp \min(\varphi_\sigma^0(N), \varphi_\sigma^0(N+1)) \leq \varphi_\sigma^0(x) \leq \max(\varphi_\sigma^0(N), \varphi_\sigma^0(N+1)) \asymp \sigma_N.$$

Now set

$$(3.6) \quad \varphi_\sigma(y) := \varphi_\sigma^0\left(\log_2 \frac{1}{1-y}\right), \quad y \in [0, 1),$$

(meaning that we identify the natural numbers \mathbb{N} with the sequence $\{1 - 2^{-N}\}_{N \geq 1}$ in \mathbb{D}). Then we also get, for $y \in [1 - 2^{-N}, 1 - 2^{-(N+1)})$,

$$(3.7) \quad \varphi_\sigma(y) \asymp \varphi_\sigma(1 - 2^{-N}).$$

Theorem 3.8. *Let $\Lambda = \{\lambda_n\}_{n \geq 1} \subset \mathbb{D}$ be a Blaschke sequence with associated growth sequence $\sigma = \{\sigma_N\}_{N \geq 1}$ at $\zeta = 1$ satisfying the regularity condition (3.5), and B the Blaschke product with zeros Λ . Then*

$$\|k_z^B\| \asymp \sqrt{\varphi_\sigma(|z|)}, \quad z \in \Gamma_{\alpha,1}.$$

Consequently, every $f \in (BH^2)^\perp$ satisfies

$$|f(z)| = |\langle f, k_z \rangle| \lesssim \sqrt{\varphi_\sigma(|z|)}, \quad z \in \Gamma_{\alpha,1}.$$

Proof. The main ingredient of this proof is (3.4), which immediately gives us the required estimate for $\rho_N = 1 - 1/2^N$:

$$\|k_{\rho_N}^B\|^2 \asymp \sigma_N = \varphi_0(N) = \varphi_\sigma(\rho_N).$$

In order to get the same estimate for $z \in \Gamma_{\alpha,1}$ we need the following well known estimate

$$(3.9) \quad |b_\lambda(\mu)| \leq \varepsilon < 1 \Rightarrow \frac{1-\varepsilon}{1+\varepsilon} \leq \frac{|1-\bar{\lambda}z|}{|1-\bar{\mu}z|} \leq \frac{1+\varepsilon}{1-\varepsilon}, \quad z \in \mathbb{D}.$$

Now let $z \in \Gamma_{\alpha,1}$ and suppose that $|z| > 1/2$. Then there exists an N such that

$$|b_z(\rho_N)| = |b_z(1 - 2^{-N})| \leq \delta < 1$$

(where δ only depends on the opening of the Stolz angle). Hence

$$\|k_z^B\|^2 \asymp \sum_{n \geq 1} \frac{1-r_n^2}{|e^{i\theta_n} - zr_n|^2} \asymp \sum_{n \geq 1} \frac{1-r_n^2}{|e^{i\theta_n} - rr_n|^2} \asymp \|k_{\rho_N}\|^2,$$

and so

$$\|k_z^B\|^2 \asymp \|k_{\rho_N}^B\|^2 \asymp \sigma_N = \varphi_\sigma(\rho_N) \asymp \varphi_\sigma(|z|). \quad \blacksquare$$

We will discuss several examples later on where we estimate the norm of the reproducing kernel for certain sequences Λ . In particular, we will see that for every increasing concave function φ , there is a sequence Λ with associated growth sequence σ such that $\varphi = \varphi_\sigma$. As in the Hardy spaces it does not seem possible to show that the upper bound is sharp. For a certain class of sequences we will give a general way of finding functions that reach the maximal growth up to some logarithmic gap (see Corollary 4.17).

We will now assume that the zeros $\{\lambda_n\}_{n \geq 1}$ of B satisfy

$$\sum_{n=1}^{\infty} \frac{1-|\lambda_n|}{|1-\lambda_n|} < \infty$$

and so

$$\angle \lim_{z \rightarrow 1} B(z) = \eta \in \mathbb{T}$$

(Recall Frostman's theorem from (1.5)). In particular, this means that there can only be a finite number of zeros of B in any Stolz angle with vertex at 1. This brings up another observation. We have $\overline{\lim}_{N \rightarrow \infty} |B(\rho_N)| = 1$. In such a situation it is notoriously difficult to decide whether or not $\{k_{\rho_N}^B\}_{N \geq 1}$ is an unconditional basis (or sequence) for $(BH^2)^\perp$. We will discuss this issue in Section 5.

4. EXAMPLES

The two basic types of Blaschke sequences we will consider here are

$$\lambda_n = (1 - x_n 2^{-2n}) e^{i2^{-n}}, \quad x_n \downarrow 0,$$

which approaches 1 very tangentially, and

$$\lambda_n = (1 - \theta_n^2) e^{i\theta_n}, \quad \sum_{n \geq 1} \theta_n < \infty,$$

which (essentially) approaches 1 along an oricycle. Needed in our analysis is this trivial Pythagorean type theorem.

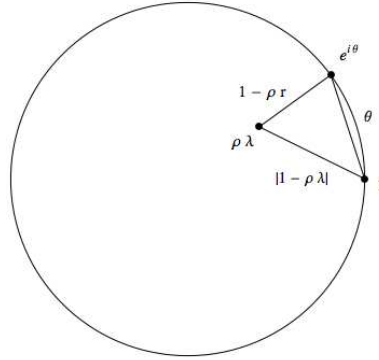


FIGURE 3. A Pythagorean type theorem.

Lemma 4.1 (Pythagorean type theorem). *If $\lambda = r e^{i\theta}$, $r \in (0, 1)$, $\rho \in (0, 1]$, then*

$$|1 - \rho\lambda|^2 \asymp (1 - \rho r)^2 + \theta^2, \quad \rho \approx 1, r \approx 1, \theta \approx 0.$$

Proof. If \vec{a} is the vector from $e^{i\theta}$ to $\rho\lambda$, \vec{b} is the vector from $e^{i\theta}$ to 1, and ψ is the angle between \vec{a} and \vec{b} , elementary vector calculus will show that $\cos \psi \asymp \theta$ as $\theta \rightarrow 0$. Thus we are justified viewing the angle between \vec{a} and \vec{b} as a right angle and the vector from $\rho\lambda$ to 1 as the hypotenuse (see Figure 4). From the Pythagorean theorem, along with the obvious estimate $|e^{i\theta} - 1| \asymp \theta$, we have the desired estimate. ■

Also needed in our analysis is the following trivial fact from functional analysis – which follows from the fact that any two norms on a finite dimensional Banach space are equivalent:

$$(4.2) \quad (|z|^2 + |w|^2)^{1/2} \asymp |z| + |w|, \quad z, w \in \mathbb{C}$$

(constants independent on z and w).

First class of examples: We will start with the first class of examples: $\Lambda = \{\lambda_k\}_{k \geq 1}$ with $\lambda_k = r_k e^{i\theta_k}$ and

$$(4.3) \quad 1 - r_k = x_k \theta_k^2, \quad \theta_k = \frac{1}{2^k}, \quad k \in \mathbb{N}.$$

We will assume that $x_k \downarrow 0$ so that in particular the sequence Λ goes tangentially to 1. The quicker x_k goes to zero, the more tangential the sequence Λ . This also implies that

$$\sum_{n \geq 1} (1 - |\lambda_n|) = \sum_{n \geq 1} (1 - r_n) = \sum_{n \geq 1} \frac{x_n}{2^{2n}} < \infty,$$

and so Λ is indeed a Blaschke sequence. Observe

$$\begin{aligned} |1 - \lambda_k| &\asymp \sqrt{(1 - r_k)^2 + \theta_k^2} \quad (\text{Lemma 4.1}) \\ &\asymp (1 - r_k) + \theta_k \quad (\text{by (4.2)}) \\ &\asymp x_k \theta_k^2 + \theta_k \\ &\asymp \theta_k. \end{aligned}$$

Hence

$$\sum_{n \geq 1} \frac{1 - |\lambda_n|}{|1 - \lambda_n|} \asymp \sum_{n \geq 1} \frac{1 - r_n}{\theta_k} = \sum_{n \geq 1} \theta_n x_n < \infty$$

and so condition (1.5) is satisfied implying $\angle \lim_{z \rightarrow 1} B(z) = \eta \in \mathbb{T}$. Similarly,

$$\sum_{n \geq 1} \frac{1 - |\lambda_n|}{|1 - \lambda_n|^2} \asymp \sum_{n \geq 1} x_n.$$

So, in view of the Ahern-Clark result (1.2), we will be interested in the “bad behavior” situation when $\sum_{n \geq 1} x_n = +\infty$.

It turns out that for these sequences we can give an explicit estimate for σ_N .

Proposition 4.4. *Let Λ be a sequence as in (4.3). Then*

$$\sigma_N \asymp \sum_{k=1}^N x_k.$$

Proof. Since $x_n \downarrow 0$ we can assume without loss of generality that $x_n < 1$ for all $n \geq 1$. Fix $N \in \mathbb{N}$. For each $n \in \mathbb{N}$, define

$$\gamma_n := \left\{ k \in \mathbb{N} : k \leq N \text{ and } \frac{1}{2^{n+1}} \leq x_k < \frac{1}{2^n} \right\}$$

and

$$\beta_{N,n} := \#\gamma_n.$$

For the above fixed $N \in \mathbb{N}$ let n_N be the smallest integer such that

$$(4.5) \quad n > n_N \Rightarrow \gamma_n = \emptyset.$$

This implies that

$$x_N \in \left[\frac{1}{2^{n_N+1}}, \frac{1}{2^{n_N}} \right).$$

Clearly for $k \in \gamma_n$, $x_k \simeq 1/2^n$, and so

$$(4.6) \quad \sum_{k=1}^N x_k = \sum_{n=1}^{n_N} \sum_{k \in \gamma_n} x_k \simeq \sum_{n=-N}^{n_N} \frac{\beta_{N,n}}{2^n}.$$

For each $n \in \mathbb{Z}$, we now count the number of points of Λ in

$$\Gamma_n^N = \left\{ z \in \mathbb{D} : \frac{1 - |z|^2}{|1 - \rho_N z|^2} \in \left[\frac{1}{2^{n+1}}, \frac{1}{2^n} \right] \right\}.$$

We have the two-sided estimate

$$\frac{1 - |\lambda_k|^2}{|1 - \rho_N \lambda_k|^2} \asymp \frac{x_k \theta_k^2}{|1 - \rho_N \lambda_k|^2}.$$

Now from Lemma 4.1 and the estimate in (4.2) we have

$$\begin{aligned} |1 - \rho_N \lambda_k|^2 &= |e^{i\theta_k} - \rho_N r_k|^2 \asymp (\theta_k + (1 - \rho_N r_k))^2 = (\theta_k + (1 - \rho_N(1 - x_k \theta_k^2)))^2 \\ &= (\theta_k + (1 - \rho_N) + \rho_N x_k \theta_k^2)^2, \end{aligned}$$

and since $x_k \theta_k^2 \ll \theta_k$ when $k \rightarrow \infty$, we get

$$(4.7) \quad |1 - \rho_N \lambda_k|^2 \asymp (\theta_k + (1 - \rho_N))^2,$$

Hence

$$(4.8) \quad \begin{aligned} \frac{1 - |\lambda_k|^2}{|1 - \rho_N \lambda_k|^2} &\asymp \frac{x_k \theta_k^2}{(\theta_k + (1 - \rho_N))^2} = \frac{x_k \theta_k^2}{(\theta_k + \theta_N)^2} \asymp \begin{cases} \frac{x_k \theta_k^2}{\theta_k^2} & \text{if } k \leq N \\ \frac{x_k \theta_k^2}{\theta_N^2} & \text{if } k > N \end{cases} \\ &\asymp \begin{cases} x_k & \text{if } k \leq N, \\ \frac{x_k \theta_k^2}{\theta_N^2} & \text{if } k > N. \end{cases} \end{aligned}$$

Since we are assuming $x_n < 1$, these expressions are bounded by 1 so that we do not need to consider Γ_n^N for negative n (we could even suppose x_n sufficiently small so that we do not need to bother with constants that might appear in the above estimates).

Let us now estimate the number of $\lambda_k \in \Gamma_n^N$. This is equivalent to evaluate the number of k for which

$$(4.9) \quad \frac{1}{2^n} \simeq \begin{cases} x_k, & \text{if } k \leq N \\ \frac{x_k \theta_k^2}{\theta_N^2}, & \text{if } k > N. \end{cases}$$

Observe that this is not an identity so that it cannot be directly used to compute explicitly $\alpha_{N,n}$ or $\beta_{N,n}$.

Set

$$\begin{aligned} \tilde{\alpha}_{N,n} &:= \# \left\{ k \leq N : \frac{1}{2^{n+1}} \leq x_k < \frac{1}{2^n} \right\} + \# \left\{ k \geq N+1 : \frac{1}{2^{n+1}} \leq x_k \theta_k^2 / \theta_N^2 < \frac{1}{2^n} \right\} \\ &= \begin{cases} \beta_{N,n} + \# \{ k \geq N+1 : \frac{1}{2^{n+1}} \leq x_k \theta_k^2 / \theta_N^2 < \frac{1}{2^n} \} & \text{if } n \leq n_N, \\ \# \{ k \geq N+1 : \frac{1}{2^{n+1}} \leq x_k \theta_k^2 / \theta_N^2 < \frac{1}{2^n} \} & \text{if } n > n_N. \end{cases} \end{aligned}$$

Since in (4.9) we have estimates with constants that do neither depend on n nor on N , there exists a fixed constant $M \in \mathbb{N}$ such that

$$\sum_{k=n-M}^{n+M} \alpha_{N,k} \asymp \sum_{k=n-M}^{n+M} \tilde{\alpha}_{N,k}.$$

In particular,

$$\sigma_N = \sum_{n \geq 1} \frac{1}{2^n} \alpha_{N,n} \asymp \sum_{n \geq 1} \frac{1}{2^n} \tilde{\alpha}_{N,n}.$$

While we do not really need the following observation, we note that for $k \geq N+1$ and since $x_n \downarrow 0$, we have

$$\frac{x_k \theta_k^2}{\theta_N^2} \leq \frac{x_k}{4} < \frac{1}{2} x_k \leq \frac{1}{2} x_N < \frac{1}{2^{n_N+1}}$$

by the construction of n_N from (4.5). When $n \leq n_N$ then $\frac{1}{2^{n+1}} \geq \frac{1}{2^{n_N+1}}$. Hence, in this case

$$\frac{x_k \theta_k^2}{\theta_N^2} < \frac{1}{2^{n+1}},$$

implying that $\{k \geq N+1 : \frac{1}{2^{n+1}} \leq x_k \theta_k^2 / \theta_N^2 < \frac{1}{2^n}\} = \emptyset$. So

$$\tilde{\alpha}_{N,n} = \begin{cases} \beta_{n,N} & \text{if } n \leq n_N \\ \# \{ k \geq N+1 : \frac{1}{2^{n+1}} \leq x_k \theta_k^2 / \theta_N^2 < \frac{1}{2^n} \} & \text{if } n > n_N. \end{cases}$$

On the other hand

$$\sum_{n>n_N} \frac{1}{2^n} \# \left\{ k \geq N+1 : \frac{1}{2^{n+1}} \leq x_k \theta_k^2 / \theta_N^2 < \frac{1}{2^n} \right\} \lesssim \sum_{k \geq N+1} \frac{x_k \theta_k^2}{\theta_N^2} \leq x_{N+1}.$$

which remains bounded as $N \rightarrow \infty$.

Hence

$$\begin{aligned} \sigma_N &= \sum_{n \geq 1} \frac{1}{2^n} \alpha_{N,n} \asymp \sum_{n \geq 1} \frac{1}{2^n} \tilde{\alpha}_{N,n} \asymp \sum_{n=1}^{n_N} \frac{1}{2^n} \beta_{N,n} + \sum_{n>n_N} \frac{1}{2^n} \# \left\{ k \geq N+1 : \frac{1}{2^{n+1}} \leq x_k \theta_k^2 / \theta_N^2 \right\} \\ &\asymp \sum_{n=1}^{n_N} \frac{1}{2^n} \beta_{N,n} \asymp \sum_{n=1}^N x_n. \quad \blacksquare \end{aligned}$$

Let us combine Theorem 3.8 and Proposition 4.4 into the following corollary.

Corollary 4.10. *Consider the Blaschke product whose zeros are*

$$\lambda_n = (1 - x_n 2^{-2n}) e^{i2^{-n}}, \quad x_n \downarrow 0.$$

If

$$\sigma_N = \sum_{n=1}^N x_n$$

and

$$0 < \inf_N \frac{\sigma_{N+1}}{\sigma_N} \leq \sup_N \frac{\sigma_{N+1}}{\sigma_N} < +\infty$$

and φ_0 is a concave function with $\varphi_0(N) = \sigma_N$ and

$$\varphi(y) = \varphi_0 \left(\log_2 \frac{1}{1-y} \right),$$

then

$$\|k_z^B\| \asymp \sqrt{\varphi(|z|)}, \quad z \in \Gamma_{\alpha,1},$$

and so every f in $(BH^2)^\perp$ satisfies

$$|f(z)| \lesssim \sqrt{\phi(|z|)}, \quad z \in \Gamma_{\alpha,1}.$$

Example 4.11. (1) If B is a Blaschke product whose zeros are

$$\lambda_n = (1 - x_n 2^{-2n}) e^{i2^{-n}}, \quad x_n = \frac{1}{n},$$

then

$$\sigma_N = \sum_{n=1}^N \frac{1}{n} \asymp \log N$$

and so every $f \in (BH^2)^\perp$ satisfies the growth condition

$$|f(r)| \lesssim \sqrt{\log \log \frac{1}{1-r}}, \quad r \rightarrow 1^-.$$

(2) If the zeros of B are

$$\lambda_n = (1 - x_n 2^{-2n}) e^{i2^{-n}}, \quad x_n = \frac{1}{n \log n},$$

then $\sigma_N \asymp \log \log N$ and so every f in $(BH^2)^\perp$ satisfies the growth condition

$$|f(r)| \lesssim \sqrt{\log \log \log \frac{1}{1-r}}, \quad r \rightarrow 1^-.$$

Remark 4.12. It is possible to prove the estimates in Example 4.11, and others like it, without using Theorem 3.8 and the regions Γ_n^N , by splitting the sum

$$\|k_r\|^2 \asymp \sum_{n \geq 1} \frac{1 - r_n^2}{|e^{i\theta_n} - r r_n|^2}$$

and using the equivalence in (4.8). In this case we can also weaken the original hypothesis $x_n \downarrow 0$ to

$$x_n \text{ is bounded}$$

which will be useful later. Here is how it works: In order to estimate $\|k_{\rho_N}\|^2$, we are interested in the sum

$$\sum_{k \geq 1} \frac{1 - |\lambda_k|^2}{|1 - \rho_N \lambda_k|^2}.$$

By (4.8),

$$\frac{1 - |\lambda_k|^2}{|1 - \rho_N \lambda_k|^2} \asymp \begin{cases} x_k & k \leq N \\ \frac{x_k \theta_k^2}{\theta_N^2} & k \geq N. \end{cases}$$

Hence, we can split the sum into two parts

$$\sum_{k \geq 0} \frac{1 - |\lambda_k|^2}{|1 - \rho_N \lambda_k|^2} \asymp \sum_{k \leq N} x_k + 2^{2N} \sum_{k \geq N+1} x_k \theta_k^2.$$

The first term is exactly σ_N while the second is bounded by a uniform constant (recall that we are assuming x_n is bounded and $\theta_k = 2^{-k}$) and hence negligible with respect to σ_N which is supposed to tend to infinity. This short and elegant argument unfortunately does not apply to the general case where it is not clear in what order we have to sum an arbitrary sequence.

We would now like to consider the sharpness of the growth in Corollary 4.10.

Theorem 4.13. *Suppose we are in the situation of Corollary 4.10 or Remark 4.12. Then for every $\varepsilon > 0$, there exists an $f \in (BH^2)^\perp$ such that*

$$|f(z)| \gtrsim \sqrt{\frac{\varphi(|z|)}{\log^{1+\varepsilon} \varphi(|z|)}}, \quad z \in \Gamma_{\alpha,1}.$$

Proof. Functions $f \in (BH^2)^\perp$ behave rather nicely if the sequence Λ is interpolating. To see this, recall first that x_n is decreasing so that

$$\overline{\lim}_{k \rightarrow \infty} \frac{x_{k+1}}{x_k} \leq 1.$$

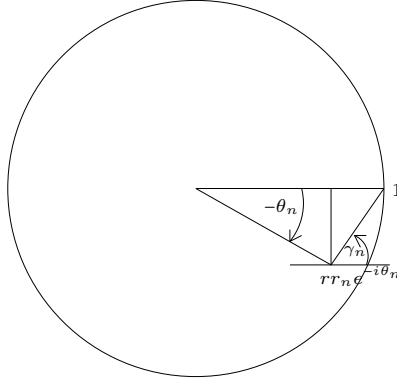


FIGURE 4. angles

Hence

$$|b_{r_k}(r_{k+1})| = \frac{x_k 2^{-2k} - x_{k+1} 2^{-2(k+1)}}{x_k 2^{-2k} + x_{k+1} 2^{-2(k+1)}} = \frac{1 - \frac{1}{4} \frac{x_{k+1}}{x_k}}{1 + \frac{1}{4} \frac{x_{k+1}}{x_k}} \geq 1 - \frac{1}{4} = \frac{3}{4} \text{ (asymptotically).}$$

Thus the sequence of moduli is pseudohyperbolically separated which implies that the sequence of moduli is interpolating as will be the one spread out by the arguments i.e., Λ .

Since we know that Λ is an interpolating sequence, we also know that the normalized reproducing kernels

$$K_n := \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|} = \frac{\sqrt{1 - |\lambda_n|^2}}{1 - \overline{\lambda_n} z}, \quad n \in \mathbb{N},$$

form an unconditional basis for $(BH^2)^\perp$. This is essentially the result by Shapiro and Shields [SS61], see also [Nik02, Section 3] and in particular [Nik02, Exercice C3.3.3(c)]. Hence for every $f \in (BH^2)^\perp$, there is a sequence $\alpha := \{\alpha_n\}_{n \geq 1} \in \ell^2$ such that

$$(4.14) \quad f_\alpha(z) := \sum_{n \geq 1} \alpha_n \frac{k_{\lambda_n}(z)}{\|k_{\lambda_n}\|} = \sum_{n \geq 1} \alpha_n \frac{\sqrt{1 - r_n^2}}{1 - r_n e^{-i\theta_n} z}.$$

We will examine this series for $z = r \in [0, 1)$ (it could be necessary at some point to require $r \geq r_0 > 0$). In what follows we will assume that $\alpha_n > 0$. Note first that the argument $1 - e^{-i\theta_n} r r_n$ is positive (this is γ_n in Figure 5).

Fix $\rho_N = 1 - 2^{-N}$. Then

$$(4.15) \quad f_\alpha(\rho_N) = \sum_{n=1}^N \alpha_n \frac{\sqrt{1 - r_n^2}}{1 - \rho_N r_n e^{-i\theta_n}} + \sum_{n > N} \alpha_n \frac{\sqrt{1 - r_n^2}}{1 - \rho_N r_n e^{-i\theta_n}}.$$

Let us first show that the second term is bounded by a constant. By definition $1 - r_n = x_n \theta_n^2 = x_n 2^{-2n}$, and from (4.7) $|e^{i\theta_n} - \rho_N r_n| \asymp \theta_n + (1 - \rho_N) \asymp 1 - \rho_N$ for $n \geq N$. In particular,

$$\left| \sum_{n > N} \alpha_n \frac{\sqrt{1 - r_n^2}}{1 - \rho_N r_n e^{-i\theta_n}} \right| \leq \sum_{n > N} \alpha_n \frac{\sqrt{1 - r_n^2}}{|e^{i\theta_n} - \rho_N r_n|} \asymp \sum_{n > N} \alpha_n \frac{\sqrt{x_n} \theta_n}{1 - \rho_N} = 2^N \sum_{n > N} \alpha_n \sqrt{x_n} \frac{1}{2^n}.$$

Now since the terms $\alpha_n \sqrt{x_n}$ are bounded, the last expression is uniformly bounded in N by a positive constant M .

Consider the first sum in (4.15). To begin with we will show that for $1 \leq n \leq N$ the argument of $1 - e^{-i\theta_n} \rho_N r_n$ is uniformly close to $\pi/2$ (or at least from a certain n_0 on), meaning that $1 - e^{-i\theta_n} \rho_N r_n$ points in a direction uniformly close to the positive imaginary axis. Set $\gamma_n = \arg(1 - \rho_N r_n e^{-i\theta_n})$, then

$$\begin{aligned} \tan \gamma_n &= \frac{r_n \rho_N \sin \theta_n}{1 - r_n \rho_N \cos \theta_n} \simeq \frac{\theta_n}{1 - (1 - x_n \theta_n^2)(1 - \theta_N)(1 - \theta_n^2/2 + o(\theta_n^2))} \\ &= \frac{\theta_n}{x_n \theta_n^2 + \theta_N + \theta_n^2/2 + o(\theta_n^2)} \asymp \frac{\theta_n}{\theta_n^2 + \theta_N} \asymp \begin{cases} \frac{1}{\theta_n} & \text{if } n \leq N/2 \\ \frac{\theta_n}{\theta_N} & \text{if } N/2 < n \leq N. \end{cases} \\ &= \begin{cases} 2^n & \text{if } n \leq N/2 \\ 2^{N-n} & \text{if } N/2 < n \leq N. \end{cases} \\ &\geq 1. \end{aligned}$$

Hence the argument of $1 - \rho_N r_n e^{-i\theta_n}$ is uniformly bounded away from zero and less than $\pi/2$ so that

$$1 \geq \sin \arg(1 - \rho_N r_n e^{-i\theta_n}) \geq \eta > 0.$$

In particular, for $1 \leq n \leq N$,

$$\left| \operatorname{Im} \frac{1}{1 - \rho_N r_n e^{-i\theta_n}} \right| \asymp \frac{1}{|1 - \rho_N r_n e^{-i\theta_n}|} \asymp \frac{1}{\theta_n + (1 - \rho_N)} \asymp \frac{1}{\theta_n}.$$

This implies that

$$\begin{aligned} |f_\alpha(\rho_N)| &= \left| \sum_{n \geq 1} \alpha_n \frac{\sqrt{1 - r_n^2}}{1 - \rho_N r_n e^{-i\theta_n}} \right| \geq \left| \sum_{n=1}^N \alpha_n \frac{\sqrt{1 - r_n^2}}{1 - \rho_N r_n e^{-i\theta_n}} \right| - \left| \sum_{n > N} \alpha_n \frac{\sqrt{1 - r_n^2}}{1 - \rho_N r_n e^{-i\theta_n}} \right| \\ &\geq \left| \sum_{n=1}^N \alpha_n \frac{\sqrt{1 - r_n^2}}{1 - \rho_N r_n e^{-i\theta_n}} \right| - M \asymp \sum_{n=1}^N \alpha_n \sqrt{1 - r_n^2} \times \left| \operatorname{Im} \frac{1}{1 - \rho_N r_n e^{-i\theta_n}} \right| - M \\ &\asymp \sum_{n=1}^N \alpha_n \frac{\sqrt{x_n} \theta_n}{\theta_n} - M \\ &= \sum_{n=1}^N \alpha_n \sqrt{x_n} - M. \end{aligned}$$

As we will see, for a specific choice of sequence $\{\alpha_n\}_{n \geq 1}$, the sum $\sum_{n=1}^N \alpha_n \sqrt{x_n}$ tends to infinity implying that in such a situation the constant M is negligible and

$$|f_\alpha(\rho_N)| \gtrsim \sum_{n=1}^N \alpha_n \sqrt{x_n}.$$

Let us discuss the following choice for α_n :

$$\alpha_n := \sqrt{\frac{x_n}{\sigma_n \log^{1+\varepsilon} \sigma_n}}.$$

We need to show two things (i) we get the desired lower estimate in the statement of the theorem; and (ii) $\{\alpha_n\}_{n \geq 1} \in \ell^2$. Let us begin with the lower estimate. Observe that σ_N is increasing and so

$$\begin{aligned} \sum_{n=1}^N \alpha_n \sqrt{x_n} &= \sum_{n=1}^N \frac{\sqrt{x_n}}{\sqrt{\sigma_n \log^{1+\varepsilon} \sigma_n}} \sqrt{x_n} = \sum_{n=1}^N \frac{x_n}{\sqrt{\sigma_n \log^{1+\varepsilon} \sigma_n}} \geq \frac{1}{\sqrt{\sigma_N \log^{1+\varepsilon} \sigma_N}} \sum_{n=1}^N x_n \\ &= \frac{\sigma_N}{\sqrt{\sigma_N \log^{1+\varepsilon} \sigma_N}} \\ &= \sqrt{\frac{\sigma_N}{\log^{1+\varepsilon} \sigma_N}}. \end{aligned}$$

To show that $\{\alpha_n\}_{n \geq 1} \in \ell^2$, observe that

$$\sum_{n=1}^N |\alpha_n|^2 = \sum_{n=1}^N \frac{x_n}{\sigma_n \log^{1+\varepsilon} \sigma_n} = \sum_{n=1}^N \frac{\sigma_n - \sigma_{n-1}}{\sigma_n \log^{1+\varepsilon} \sigma_n},$$

where we have set $\sigma_0 = 0$.

Claim: we have the estimate

$$(4.16) \quad \frac{\sigma_n - \sigma_{n-1}}{\sigma_n \log^{1+\varepsilon} \sigma_n} \leq \frac{1}{\varepsilon} \left(\frac{1}{\log^\varepsilon \sigma_{n-1}} - \frac{1}{\log^\varepsilon \sigma_n} \right).$$

Indeed, since σ_n is increasing, we can suppose $1 \leq x := \sigma_{n-1} \leq y := \sigma_n$ for n sufficiently big. We also introduce the auxiliary function

$$g(t) = \frac{1}{\log^\varepsilon(t)}, \quad t \in [1, \infty).$$

Then

$$\frac{1}{\log^\varepsilon \sigma_{n-1}} - \frac{1}{\log^\varepsilon \sigma_n} = g(x) - g(y) = g'(\eta)(x - y)$$

for some $\eta \in [x, y]$. Since

$$g'(t) = -\frac{\varepsilon}{t \log^{1+\varepsilon} t},$$

we know that $-g'$ is a decreasing function and so

$$g(x) - g(y) = g'(\eta)(x - y) = -\frac{\varepsilon}{\eta \log^{1+\varepsilon} \eta} (x - y) = \varepsilon \frac{y - x}{\eta \log^{1+\varepsilon} \eta} \geq \varepsilon \frac{y - x}{y \log^{1+\varepsilon} y},$$

which, after a little rearranging, is exactly (4.16).

To finish our proof that $\{\alpha_n\}_{n \geq 1} \in \ell^2$ we note that

$$\begin{aligned} \sum_{n=2}^N \alpha_n^2 &= \sum_{n=2}^N \frac{\sigma_n - \sigma_{n-1}}{\sigma_n \log^{1+\varepsilon} \sigma_n} \leq \frac{1}{\varepsilon} \sum_{n=2}^N \left(\frac{1}{\log^\varepsilon \sigma_{n-1}} - \frac{1}{\log^\varepsilon \sigma_n} \right) = \frac{1}{\varepsilon} \left(\frac{1}{\log^\varepsilon \sigma_1} - \frac{1}{\log^\varepsilon \sigma_N} \right) \\ &\leq \frac{1}{\varepsilon \log^\varepsilon \sigma_1}. \quad \blacksquare \end{aligned}$$

If one looks closely at the proof of Theorem 4.13 one sees the proof of the following.

Corollary 4.17. *For every concave function $\varphi_0 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ tending to infinity there exists a sequence $\Lambda \subset \mathbb{D}$ tending to 1 such that if B is the Blaschke product associated with Λ , then*

(i) for every $z \in \Gamma_{\alpha,1}$ we have

$$\|k_z^I\| \simeq \sqrt{\varphi(|z|)}$$

where we define φ as in (3.6) by

$$\varphi(y) := \varphi_0\left(\log_2 \frac{1}{1-y}\right), \quad y \in (0, 1).$$

In particular for each Stolz domain $\Gamma_{\alpha,1}$ we have

$$|f(z)| \lesssim \sqrt{\varphi(|z|)}, \quad f \in (BH^2)^\perp.$$

(ii) For every $\varepsilon > 0$, there exists an $f \in (BH^2)^\perp$ such that for $z \in \Gamma_{\alpha,1}$

$$|f(z)| \gtrsim \sqrt{\frac{\varphi(|z|)}{\log^{1+\varepsilon} \varphi(|z|)}},$$

the constants being independent of z .

Remark 4.18. Pushing the computations a little bit further, it is possible to narrow down this gap to an estimate such as

$$\sqrt{\frac{\sigma_N}{\log \sigma_N (\log \log \sigma_N)^{1+\varepsilon}}}.$$

In order to discuss the optimality of Theorem 4.13 we will be interested in the following question: for which sequences $\varepsilon_n \downarrow 0$ does there exist a sequence $\{\alpha_n\}_{n \geq 1} \in \ell^2$ such that

$$(4.19) \quad \sum_{n=1}^N \alpha_n \sqrt{x_n} = \varepsilon_N \sigma_N ?$$

We will do this here in the special case $x_n = 1$ for every $n \in \mathbb{N}$ so that in particular $\sigma_N = N$ (this situation is allowed by Remark 4.12). Hence the question becomes: for which sequences $\varepsilon_n \downarrow 0$ does there exist a sequence $\{\alpha_n\}_{n \geq 1} \in \ell^2$ such that

$$(4.20) \quad \sum_{n=1}^N \alpha_n = \varepsilon_N \sqrt{N} ?$$

The following result shows that we cannot do much better than Corollary 4.17.

Proposition 4.21. *Suppose $1 - r_n = \theta_n^2 = 1/2^{2n}$ and let B be the Blaschke product associated with $\Lambda = \{\lambda_n\}_{n \geq 1} = \{r_n e^{i\theta_n}\}_{n \geq 1}$. Let $\{\varepsilon_n\}_{n \geq 1}$ be a convex sequence decreasing to zero. If there exists an*

$$f_\alpha := \sum_{n \geq 1} \alpha_n \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|} \in (BH^2)^\perp$$

with

$$(4.22) \quad \sum_{n=1}^N \alpha_n = \varepsilon_N \sqrt{N}, \quad N = 1, 2, \dots,$$

then

$$\sum_{n \geq 1} \frac{\varepsilon_n^2}{n} < \infty.$$

Remark 4.23. (1) This means that under the conditions of the proposition we cannot choose $\{\varepsilon_n\}_{n \geq 1}$ to be a sequence decreasing arbitrarily slowly to zero. For instance, if we were to choose $\varepsilon_n = 1/\log^\alpha n$, then we would need $\alpha > 1/2$ which is, in a sense, optimal in view of the preceding corollary.

(2) Note that in the preceding proposition the growth condition is required for $\sum_{n=1}^N \alpha_n$ and not for $f_\alpha(\rho_N)$ itself which can differ from the sum by uniformly bounded constants. There is *a priori* no reason that we can get the required growth on $f_\alpha(\rho_N)$. If $\{k_{\rho_N}^B\}_{N \geq 1}$ were an unconditional sequence, then this would be possible. We include a discussion on unconditional bases in Section 5 showing that $\{k_{\rho_N}^B\}_{N \geq 1}$ is in general not an unconditional basis for $(BH^2)^\perp$.

Proof of Proposition 4.21. From (4.22) we define α_n to be

$$\alpha_n = \varepsilon_n \sqrt{n} - \varepsilon_{n-1} \sqrt{n-1},$$

and since $f_\alpha \in K_B$, we need $\{\alpha_n\}_{n \geq 1} \in \ell^2$. We have

$$\begin{aligned} \varepsilon_n \sqrt{n} - \varepsilon_{n-1} \sqrt{n-1} &= \varepsilon_n \sqrt{n} \left(1 - \frac{\varepsilon_{n-1}}{\varepsilon_n} \sqrt{1 - \frac{1}{n}} \right) \\ &= \varepsilon_n \sqrt{n} \left(1 - \frac{\varepsilon_{n-1}}{\varepsilon_n} \left(1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right) \right) \right) \\ &= \varepsilon_n \sqrt{n} \left(1 - \frac{\varepsilon_{n-1}}{\varepsilon_n} \left(1 - \frac{1}{2n} \right) \right) - \varepsilon_{n-1} O\left(\frac{1}{n^{3/2}}\right). \end{aligned}$$

Since $\{\varepsilon_n\}_{n \geq 1}$ is bounded, the last term above is always in ℓ^2 . Let us discuss the first term:

$$\varepsilon_n \sqrt{n} \left(1 - \frac{\varepsilon_{n-1}}{\varepsilon_n} \left(1 - \frac{1}{2n} \right) \right) = \sqrt{n} \left(\varepsilon_n - \varepsilon_{n-1} + \frac{\varepsilon_{n-1}}{2n} \right) = \sqrt{n}(\varepsilon_n - \varepsilon_{n-1}) + \frac{\varepsilon_{n-1}}{2\sqrt{n}}.$$

Observe that since $\{\varepsilon_n\}_{n \geq 1}$ is a positive decreasing sequence, the sequence $\{\gamma_n\}_{n \geq 1}$ defined by $\gamma_n = \varepsilon_n - \varepsilon_{n-1}$ is summable. We also note that $\{\gamma_n\}_{n \geq 1}$ is decreasing since $\{\varepsilon_n\}_{n \geq 1}$ is assumed to be convex. From general results it follows thus that $0 \leq \gamma_n \lesssim 1/n$, or equivalently $M := \sup_n (n\gamma_n) < \infty$. Hence

$$\sum_n \left(\sqrt{n}(\varepsilon_n - \varepsilon_{n-1}) \right)^2 = \sum_n n\gamma_n^2 \leq M \sum_n \gamma_n < \infty.$$

So we necessarily have

$$\sum_{n \geq 1} \frac{\varepsilon_n^2}{n} \asymp \sum_{n \geq 2} \frac{\varepsilon_{n-1}^2}{4n} < \infty. \quad \blacksquare$$

Second class of examples: In the preceding class of examples from (4.3), we slowed down the growth of functions in $(BH^2)^\perp$ by controlling the ‘‘tangentiality’’ of the sequence (given by the speed of convergence to zero of x_n). Our second class of examples are of the following type:

$$(4.24) \quad \lambda_n = r_n e^{i\theta_n}, \quad 1 - r_n = \theta_n^2, \quad \sum_{n \geq 1} \theta_n < \infty,$$

where θ_n will now be adjusted to trigger the growth speed of $(BH^2)^\perp$ -functions. Asymptotically this sequence is in the oricycle $\{z \in \mathbb{D} : |z - 1/2| = 1/2\}$. We also note that

$$\sum_{n \geq 1} (1 - |\lambda_n|) = \sum_{n \geq 1} \theta_n^2 < \infty$$

so indeed $\{\lambda_n\}_{n \geq 1}$ is a Blaschke sequence. Moreover,

$$\sum_{n \geq 1} \frac{1 - |\lambda_n|}{|1 - \lambda_n|} \asymp \sum_{n \geq 1} \frac{\theta_n^2}{\theta_n} = \sum_{n \geq 1} \theta_n < \infty$$

and so, by (1.5), $\lim_{r \rightarrow 1^-} B(r) = \eta \in \mathbb{T}$. Still further, we have

$$\sum_{n \geq 1} \frac{1 - |\lambda_n|}{|1 - \lambda_n|^2} \asymp \frac{\theta_n^2}{\theta_n^2} = \infty$$

so $\{\lambda_n\}_{n \geq 1}$ does not satisfy the hypothesis (1.2) of the Ahern-Clark theorem. Thus we can expect bad behavior of functions from $(BH^2)^\perp$.

As in (4.8), we have

$$\frac{1 - |\lambda_k|^2}{|1 - r\lambda_k|^2} \asymp \frac{1 - r_k}{(1 - r)^2 + \theta_k^2} = \frac{\theta_k^2}{(1 - r)^2 + \theta_k^2} \asymp \begin{cases} 1 & \text{if } (1 - r) \leq \theta_k \\ \frac{\theta_k^2}{(1 - r)^2} & \text{if } (1 - r) > \theta_k. \end{cases}$$

Instead of computing the number of points in each Γ_n^N , we will again use the simpler argument given in Remark 4.12 which is better suited to our sequences defined by (4.24). Now, using Lemma 3.3, the splitting gives:

$$(4.25) \quad \|k_r^B\|^2 \asymp \sum_{k \geq 1} \frac{1 - |\lambda_k|^2}{|1 - r\lambda_k|^2} \asymp \sum_{\{k: (1-r) \leq \theta_k\}} 1 + \frac{1}{(1-r)^2} \sum_{\{k: (1-r) > \theta_k\}} \theta_k^2.$$

Theorem 4.26. *Let $\{\sigma_N\}_{N \geq 1}$ be a sequence of positive numbers strictly increasing to infinity either concave or convex for $N \geq N_0$, where N_0 is a suitable integer, and*

$$(4.27) \quad \sigma_{N+1} \leq 2^\beta \sigma_N, \quad N \in \mathbb{N},$$

for some $\beta \in (0, 1)$. Then there exists a sequence $\{\theta_k\}_{k \geq 1} \in \ell^1$ such that

$$\|k_{\rho_N}^B\| \asymp \sqrt{\sigma_N},$$

where B is the Blaschke product associated with $\Lambda = \{\lambda_k\}_{k \geq 1}$ and $\lambda_k = r_k e^{i\theta_k}$, $1 - r_k = \theta_k^2$.

Proof. Let $\{\sigma_N\}_{N \geq 1}$ be as in the theorem, and let

$$\psi : [0, +\infty) \longrightarrow [0, +\infty)$$

be a continuous increasing function such that

$$(4.28) \quad \psi(N) = \sigma_N, \quad N \in \mathbb{N}.$$

We could, for example, choose ψ to be the continuous piecewise affine function defined in the nodes by (4.28). Since ψ is continuous, strictly increasing to infinity on $[0, +\infty)$, it admits an inverse function ψ^{-1} . By assumption the sequence $\{\psi^{-1}(k)\}_{k \geq k_0}$ is concave or convex (k_0 has to satisfy $\psi(N_0) \leq k_0$). Set

$$\theta_k = 2^{-\psi^{-1}(k)}, \quad k \in \mathbb{N}.$$

We do need to show that $\{\theta_n\}_{n \geq 1} \in \ell^1$ but this will come out of our analysis below. Let us consider the first sum in (4.25) (with $r = \rho_N$):

$$\sum_{\{k: (1-\rho_N) \leq \theta_k\}} 1 = \sum_{\{k: 1/2^N \leq 1/2^{\psi^{-1}(k)}\}} 1 = \sum_{\{k: \psi^{-1}(k) \leq N\}} 1 = \sum_{\{k: k \leq \psi(N)\}} 1 = \psi(N) = \sigma_N.$$

We have to consider the second sum in (4.25):

$$\frac{1}{(1 - \rho_N)^2} \sum_{\{k: (1 - \rho_N) > \theta_k\}} \theta_k^2 = 2^{2N} \sum_{\{k: \psi^{-1}(k) \geq N+1\}} 2^{-2\psi^{-1}(k)} = 2^{2N} \sum_{\{k \geq \psi(N+1)\}} 2^{-2\psi^{-1}(k)}.$$

This separates into two cases:

Case 1: $\{\sigma_N\}_{N \geq N_0}$ is concave.

Letting K be a fixed integer such that $\psi(N_0) \leq K$, the sequence $\{\psi^{-1}(k)\}_{k \geq K}$ is convex and so, for every $k \geq K$,

$$\psi^{-1}(k+1) - \psi^{-1}(k) \geq \psi^{-1}(K) - \psi^{-1}(K) =: c > 0.$$

Now, for every $N \geq N_0$ there is $k_N \geq K$ such that $k_N - 1 < \psi(N+1) \leq k_N$. Clearly for $k \geq k_N$, $\psi^{-1}(k) \geq c(k - k_N) + \psi^{-1}(k_N)$, and hence

$$\sum_{\{k \geq \psi(N+1)\}} 2^{-2\psi^{-1}(k)} \leq \frac{1}{2^{2\psi^{-1}(k_N)}} \sum_{\{k \geq k_N\}} 2^{-2c(k - k_N)} \asymp \frac{1}{2^{2\psi^{-1}(k_N)}} \leq \frac{1}{2^{2\psi^{-1}(\psi(N+1))}} = \frac{4}{2^{2N}}$$

which yields

$$2^{2N} \sum_{\{k \geq \psi(N+1)\}} 2^{-2\psi^{-1}(k)} \leq 4.$$

So in this case, the second sum in (4.25) is at most a constant (which is independent of N). Since a constant is negligible with respect to σ_N when $N \rightarrow \infty$, the theorem is proved in this case. A slight modification of the above estimates will also show that $\{\theta_n\}_{n \geq 1} \in \ell^1$ which, as mentioned earlier, is needed for $\{\lambda_n\}_{n \geq 1}$ to be a Blaschke sequence with $\lim_{r \rightarrow 1^-} B(r) = \eta \in \mathbb{T}$.

Case 2: $\{\sigma_N\}_{N \geq N_0}$ is convex

In this case ψ^{-1} is a concave (increasing) function. So when k is big enough we have

$$\psi^{-1}(k+1) - \psi^{-1}(k) \leq M,$$

for a suitable constant M . In particular

$$2^{-2\psi^{-1}(k)} \geq \int_k^{k+1} 2^{-2\psi^{-1}(t)} dt \geq 2^{-2\psi^{-1}(k+1)} \geq 2^{-2M} 2^{-2\psi^{-1}(k)},$$

so that we can switch to integrals. We have

$$(4.29) \quad \sum_{k \geq \psi(N+1)} 2^{-2\psi^{-1}(k)} \asymp \int_{\psi(N+1)}^{\infty} 2^{-2\psi^{-1}(t)} dt.$$

Let us do the change of variables $u = \psi^{-1}(t)$ so that for almost all $u > 0$,

$$\frac{1}{(\psi^{-1})'(t)} = \psi'(u),$$

so that when $y \geq 0$,

$$\int_{\psi(y)}^{+\infty} \frac{1}{2^{2\psi^{-1}(t)}} dt = \int_y^{+\infty} \frac{1}{2^{2u}} \frac{du}{(\psi^{-1})'(t)} = \int_y^{\infty} \psi'(u) 2^{-2u} du.$$

Integration by parts gives

$$\begin{aligned} \int_y^A \psi'(u)2^{-2u} du &= [\psi(u)2^{-2u}]_y^A + 2 \ln 2 \int_y^A \psi(u)2^{-2u} du \\ &= \frac{\psi(A)}{2^{2A}} - \frac{\psi(y)}{2^{2y}} + 2 \ln 2 \int_y^A \psi(u)2^{-2u} du. \end{aligned}$$

Note that from condition (4.27) it follows that $\sigma_N \lesssim 2^N$ and thus $\psi(A) \lesssim 2^A$. Hence, letting A go to infinity we get

$$\int_y^\infty \psi'(u)2^{-2u} du = -\frac{\psi(y)}{2^{2y}} + 2 \log 2 \int_y^\infty \psi(u)2^{-2u} du.$$

Claim: For every $t \geq 0$,

$$\frac{\psi(t+1)}{\psi(t)} \leq 2^{2\beta}.$$

This is immediate from the growth of ψ and Condition (4.27): Suppose that $t \in [N-1, N)$, then

$$\frac{\psi(t+1)}{\psi(t)} \leq \frac{\psi(N+1)}{\psi(N-1)} = \frac{\sigma_{N+1}}{\sigma_{N-1}} \leq 2^{2\beta}.$$

As a consequence of this observation we have for every $0 < y \leq u$

$$\frac{\psi(u)}{\psi(y)} \lesssim 2^{2\beta(u-y)}.$$

Indeed, if $u = y + n + \delta$ with $\delta \in [0, 1)$ and $n \in \mathbb{N}$, then

$$\frac{\psi(u)}{\psi(y)} = \frac{\psi(y+n+\delta)}{\psi(y)} \leq \frac{\psi(y+n+1)}{\psi(y)} \leq 2^{2\beta(n+1)} = 2^{2\beta(n+\delta)} 2^{2\beta(1-\delta)} \simeq 2^{2\beta(u-y)}.$$

Hence

$$\int_y^\infty \psi'(u)2^{-2u} du = \frac{\psi(y)}{2^{2y}} \int_y^\infty \frac{\psi(u)/\psi(y)}{2^{2(u-y)}} du,$$

where the last integral is uniformly bounded in y by a constant:

$$\int_y^\infty \frac{\psi(u)/\psi(y)}{2^{2(u-y)}} du \lesssim \int_y^\infty \frac{2^{2\beta(u-y)}}{2^{2(u-y)}} du = \int_y^\infty \frac{1}{2^{2(1-\beta)(u-y)}} du = \int_0^\infty \frac{1}{2^{2(1-\beta)y}} du = \frac{1}{2(1-\beta) \log 2}.$$

Hence

$$\begin{aligned} \int_y^\infty \psi'(u)2^{-2u} du &\leq -\frac{\psi(y)}{2^{2y}} + 2 \log 2 \int_y^\infty \psi(u)2^{-2u} du \leq -\frac{\psi(y)}{2^{2y}} + \frac{1}{1-\beta} \frac{\psi(y)}{2^{2y}} \\ (4.30) \qquad \qquad \qquad &= \frac{\beta}{1-\beta} \frac{\psi(y)}{2^{2y}} \end{aligned}$$

Now using (4.29), we get

$$\begin{aligned} \frac{1}{(1-\rho_N)^2} \sum_{\{k:1-\rho_N < \theta_k\}} \theta_k^2 &= 2^{2N} \sum_{\{k:\psi^{-1}(k) \geq N+1\}} 2^{-2\psi^{-1}(k)} = 2^{2N} \sum_{k \geq \psi(N+1)} 2^{-2\psi^{-1}(k)} \\ &\leq 2^{2N} \int_{\psi(N+1)}^\infty 2^{-2\psi^{-1}(t)} dt = 2^{2N} \int_{N+1}^\infty \psi'(u)2^{-2u} du \\ (4.31) \qquad \qquad \qquad &\lesssim 2^{2N} \frac{\psi(N+1)}{2^{2(N+1)}} = \frac{\sigma_{N+1}}{4} \leq \frac{2^\beta}{4} \sigma_N. \end{aligned}$$

Since the first sum in (4.25) is equal to σ_N and the second one is controlled by σ_N , the theorem is proved. \blacksquare

Remark 4.32. Note that the Blaschke condition for Λ is given by

$$\sum_k (1 - |\lambda_k|^2) \simeq \sum_k (1 - r_k) = \sum_k \theta_k^2 = \sum_k 2^{-2\psi^{-1}(k)} < \infty.$$

It can be seen for instance from (4.30) and (4.31) that the condition $0 < \beta < 1$ (condition (4.27)) guarantees that Λ is a Blaschke sequence.

Example 4.33. Here is a list of examples. The sequences $\{\sigma_N\}_{N \geq 1}$ are all either concave or convex.

- (1) Let $\sigma_N = 2^{N/\alpha}$, $N = 1, 2, \dots$, where $\alpha > 1$ (this is needed for (4.27)). Then, we can choose $\psi(t) = 2^{t/\alpha}$. Hence

$$\theta_k = 2^{-\psi^{-1}(k)} = 2^{-\alpha \log k} = \frac{1}{k^\alpha}$$

(logarithms are taken in the basis 2). Hence, with this choice of arguments, we get

$$\|k_{\rho_N}^B\| \simeq 2^{N/2\alpha} = \frac{1}{(1 - \rho_N)^{1/2\alpha}},$$

which by similar arguments as given earlier can be extended to every $r \in (0, 1)$, i.e.,

$$|f(r)| \lesssim \frac{1}{(1 - r)^{1/2\alpha}}, \quad f \in (BH^2)^\perp.$$

We thus obtain all power growths beyond the limiting case 1/2.

- (2) Let $\sigma_N = N^\alpha$, $N = 1, 2, \dots$, where $\alpha > 0$. Then, we can choose $\psi(t) = t^\alpha$. Hence

$$\theta_k = 2^{-\psi^{-1}(k)} = 2^{-k^{1/\alpha}}.$$

Hence, with this choice of arguments, we get

$$\|k_{\rho_N}^B\| \simeq N^{\alpha/2} = \left(\log \frac{1}{1 - \rho_N} \right)^{\alpha/2},$$

which by similar arguments as given earlier can be extended to every $r \in (0, 1)$, i.e.,

$$|f(r)| \lesssim \left(\log \frac{1}{1 - r} \right)^{\alpha/2}, \quad f \in (BH^2)^\perp.$$

In the special case $\alpha = 1$ we obtain logarithmic growth.

- (3) Let $\sigma_N = \log^2 N$, $N \geq 2$. Then, we can choose $\psi(t) = \log^2 t$. Hence

$$\theta_k = 2^{-\psi^{-1}(k)} = 2^{-2\sqrt{k}}.$$

Hence, with this choice of arguments, we get for N big enough

$$\|k_{\rho_N}^B\| \simeq \log N = \log \log \frac{1}{1 - \rho_N},$$

which, by similar arguments as given earlier, can be extended to every $r \in (0, 1)$. Hence we get double logarithmic growth.

5. A FINAL REMARK ON UNCONDITIONAL BASES

Since a central piece of our discussions was the behavior of the reproducing kernel at ρ_N , one could ask whether or not $\{k_{\rho_N}^B\}_{N \geq 1}$ was an unconditional bases (or sequence) of reproducing kernels for $(BH^2)^\perp$. Most of the material needed here has already been introduced in Section 2 and in particular in Proposition 2.12.

We consider now the following situation which is different from that in Section 2: let $x_n = k_{\rho_n}^B$ and G be the associated Gram matrix. Suppose that $\{k_{\rho_n}^B\}_{n \geq 1}$ were an unconditional basis (or sequence) for $(BH^2)^\perp$. Then every $f \in (BH^2)^\perp$ could be written as

$$f = f_\alpha := \sum_{n \geq 1} \alpha_n \frac{k_{\rho_n}^B}{\|k_{\rho_n}^B\|}, \quad \alpha = \{\alpha_n\}_{n \geq 1} \in \ell^2,$$

with $\|f_\alpha\|^2 \asymp \sum_{n \geq 1} |\alpha_n|^2 < \infty$. As before we want to estimate $f = f_\alpha$ at ρ_N :

$$f_\alpha(\rho_N) = \sum_{n \geq 1} \alpha_n \frac{k_{\rho_n}^B(\rho_N)}{\|k_{\rho_n}^B\|} = \|k_{\rho_N}^B\| \sum_{n \geq 1} \alpha_n \frac{\langle k_{\rho_n}^B, k_{\rho_N}^B \rangle}{\|k_{\rho_n}^B\| \|k_{\rho_N}^B\|} = \|k_{\rho_N}^B\| (G\alpha)_N.$$

Again we observe that for every $\alpha \in \ell^2$, we have

$$f_\alpha(\rho_N) = \|k_{\rho_N}^B\| (G\alpha)_N$$

where $G\alpha \in \ell^2$, and for every ℓ^2 -sequence β we could find an $f \in (BH^2)^\perp$ such that

$$\frac{f(\rho_N)}{\|k_{\rho_N}^B\|} = \beta_N.$$

However, recall from Corollary 4.17 that that for $\varepsilon > 0$ there is a function f_α with

$$|f_\alpha(\rho_N)| \gtrsim \sqrt{\frac{\sigma_N}{\log^{1+\varepsilon} \sigma_N}}$$

(we refer to that corollary for notation). Hence we would have

$$\beta_N := \frac{|f_\alpha(\rho_N)|}{\|k_{\rho_N}^B\|} \asymp \frac{|f_\alpha(\rho_N)|}{\sqrt{\sigma_N}} \gtrsim \frac{1}{\log^{(1+\varepsilon)/2} \sigma_N}.$$

Now choosing, for instance, $x_n = 1$ yields $\sigma_N = N$, or $x_n = 1/n$ yields $\sigma_N \simeq \log N$ etc. In all these cases $\{1/\log^{(1+\varepsilon)/2} \sigma_N\}_{N \geq 1}$ is obviously not in ℓ^2 . As a result, we can conclude that in the above examples $\{k_{\rho_N}^B\}_{N \geq 1}$ cannot be an unconditional basis for $(BH^2)^\perp$ (nor an unconditional sequence since the functions in Corollary 4.17 were constructed using the reproducing kernels).

It should be noted that the problem of deciding whether or not a sequence of reproducing kernels is an unconditional basis (or sequence) in a model space is a difficult problem related to the Carleson condition and invertibility of Toeplitz operators. We do not want to go into details here, but the situation is even more difficult in our context where $\overline{\lim}_N |B(\rho_N)| = 1$. See [Nik02, Chapter D4] for more about this.

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