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Michel Granger, David Mond, Mathias Schulze

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1 **PARTIAL NORMALIZATIONS OF**
2 **COXETER ARRANGEMENTS AND DISCRIMINANTS**

3 MICHEL GRANGER, DAVID MOND, AND MATHIAS SCHULZE

To the memory of V.I. Arnol'd

ABSTRACT. We study natural partial normalization spaces of Coxeter arrangements and discriminants and relate their geometry to representation theory. The underlying ring structures arise from Dubrovin's Frobenius manifold structure which is lifted (without unit) to the space of the arrangement. We also describe an independent approach to these structures via duality of maximal Cohen–Macaulay fractional ideals. In the process, we find 3rd order differential relations for the basic invariants of the Coxeter group. Finally, we show that our partial normalizations give rise to new free divisors.

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21 INTRODUCTION

22 V.I. Arnol'd was the first to identify the singularities of type ADE , that is A_ℓ , D_ℓ , E_6 , E_7 or E_8 , as
23 the simple singularities – those that are adjacent to only finitely many other types. He also uncovered
24 the links between the Coxeter groups of type B_ℓ , C_ℓ and F_4 and boundary singularities, see [Arn79].
25 In the latter paper his formulæ for generators of the module of logarithmic vector fields $\text{Der}(-\log D)$
26 along the discriminant D parallels K. Saito's definition of free divisors. Along with Brieskorn, Dynkin,
27 Gelfan'd, and Gabriel, Arnol'd revealed the ADE list as one of the central piazzas in mathematical
28 heaven, where representation theory, algebra, geometry and topology converge. As with so many of
29 Arnol'd's contributions, his work on this topic has given rise to a huge range of further work by others.

30 Let $f: X = (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0) = S$ be a complex function singularity of type ADE and let $F: X \times B \rightarrow S$
31 be a miniversal deformation of f with base $B = (\mathbb{C}^\mu, 0)$. Writing $f_u := F(-, u)$, the discriminant $D \subset B$
32 is the set of parameter values $u \in B$ such that $f_u^{-1}(0)$ is singular. It is isomorphic to the discriminant
33 of the Coxeter group W of the same name. Here the discriminant is the set of exceptional orbits in
34 the orbit space V/W . This is only the most superficial feature of the profound link between singularity
35 theory and the geometry of Coxeter groups which Arnol'd helped to make clear.

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36 The starting point of this paper is the fact, common to Coxeter groups and singularities, that D is
 37 a free divisor (see e.g. [Her02, §4.3]) with a symmetric Saito matrix K whose cokernel is a ring in the
 38 singularity case. By definition the Saito matrix K is the $\mu \times \mu$ -matrix whose columns are the coefficient
 39 vectors of a basis of $\text{Der}(-\log D)$ with respect to a basis of the module $\text{Der}_B := \text{Der}_{\mathbb{C}}(\mathcal{O}_B)$ of vector
 40 fields on B .

On the singularity theory side these two roles are well known. Let h be a defining equation for D .
 Then K appears in the exact sequence

$$0 \longrightarrow \mathcal{O}_B^\mu \xrightarrow{K} \text{Der}_B \xrightarrow{dh} J_D \longrightarrow 0$$

which defines $\text{Der}(-\log D)$ as the vector fields which preserve the ideal of D . Let $\pi : \Sigma \rightarrow B$ denote
 the restriction of the projection $X \times B \rightarrow B$. If $\Sigma \subset X \times B$ is the relative critical locus defined by the
 Jacobian ideal J_F^{rel} of F relative to B , and $\Sigma^0 := \Sigma \cap V(F)$ so that $D = \pi(\Sigma^0)$, then K also appears in
 the exact sequence

$$(0.1) \quad 0 \longrightarrow \mathcal{O}_B^\mu \xrightarrow{K} \text{Der}_B \xrightarrow{dF} \pi_* \mathcal{O}_{\Sigma^0} \longrightarrow 0$$

in which dF maps a vector field $\eta \in \text{Der}_B$ to the function $dF(\tilde{\eta})$ on Σ^0 , where $\tilde{\eta}$ is a lift of η to $X \times B$,
 and $\pi : \Sigma \rightarrow B$ is the restriction of the projection $X \times B \rightarrow B$. As $\pi_* \mathcal{O}_{\Sigma}$ is free over \mathcal{O}_B of rank μ , we
 can make the identifications

$$\pi_* \mathcal{O}_{\Sigma} \cong \mathcal{O}_B^\mu \cong \text{Der}_B,$$

41 and reinterpret K as the matrix of the \mathcal{O}_B -linear operator induced on $\pi_* \mathcal{O}_{\Sigma}$ by multiplication by F ,
 42 whose cokernel is also, evidently, $\pi_* \mathcal{O}_{\Sigma^0}$.

43 Similar to the case of ADE singularities and corresponding Coxeter groups, Coxeter groups of type
 44 B_k and F_4 are linked with boundary singularities, for which a similar argument shows that the cokernel
 45 of K is naturally a ring. Also for these and the remaining Coxeter groups $I_2(k)$, H_3 and H_4 , the cokernel
 46 of K carries a natural ring structure. The simplest way to see this involves the Frobenius structure
 47 constructed on the orbit space by Dubrovin in [Dub98], following K. Saito. Here the key ingredient is
 48 a fiber-wise multiplication on the tangent bundle, which coincides with the multiplication coming from
 49 \mathcal{O}_{Σ} in the ADE singularity case. We recall the necessary details of Dubrovin's construction, following
 50 C. Hertling's account in [Her02], in Section 2, in preparation for the proof of our main result. This states
 51 that also the cokernel of a transposed Saito matrix for the reflection arrangement of a Coxeter group
 52 carries a natural ring structure.

53 **Theorem 0.1.**

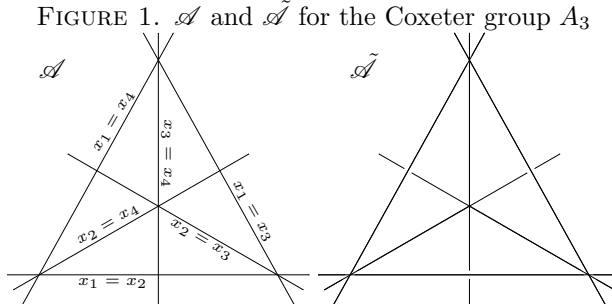
- 54 (1) Let \mathcal{A} be the reflection arrangement of a Coxeter group W acting on the vector space $V \cong \mathbb{C}^\ell$, let
 55 p_1, \dots, p_ℓ be generators of the ring of W -invariant polynomials, homogeneous in each irreducible
 56 component of V , and let J be the Jacobian matrix of the map (p_1, \dots, p_ℓ) , which is a transposed
 57 Saito matrix for \mathcal{A} . Then $\text{coker } J$ has a natural structure of $\mathbb{C}[V]$ -algebra.
 58 (2) Denoting $\text{Spec coker } J$ by $\tilde{\mathcal{A}}$, we have
 59 (i) $\tilde{\mathcal{A}}$ is finite and birational over \mathcal{A} (and thus lies between \mathcal{A} and its normalization).
 60 (ii) For $x \in \mathcal{A}$, let W_x be the stabilizer of x in W and let X be the flat of \mathcal{A} containing
 61 x . There is a natural bijection between the geometric fiber of $\tilde{\mathcal{A}}$ over x and the set of
 62 irreducible summands in the representation of W_x on V/X .
 63 (iii) Under the bijection of (2ii), smooth points of $\tilde{\mathcal{A}}$ correspond to representations of type A_1 .

64 *Example 0.2.*

65 (1) In the case of A_2 , the arrangement \mathcal{A} consists of three concurrent coplanar lines. In this case $\tilde{\mathcal{A}}$
 66 is isomorphic to the union of the three coordinate axes in 3-space, for this is the only connected curve
 67 singularity regular and birational over \mathcal{A} but not isomorphic to it. More generally, in the case of A_ℓ ,
 68 with $\binom{\ell+1}{2}$ reflecting hyperplanes, $\tilde{\mathcal{A}}$ is isomorphic to the codimension-2 subspace arrangement in
 69 $(\ell+1)$ -space consisting of the $(\ell-1)$ -planes $L_{i,j} := \{x_i = x_j = 0\}$ for $1 \leq i < j \leq \ell+1$. The projection
 70 $x \mapsto x - x^\sharp$, where x^\sharp is x averaged by the action of the symmetric group S_ℓ permuting coordinates,
 71 gives an S_ℓ -equivariant map of $\tilde{\mathcal{A}}$ to the standard arrangement $\mathcal{A} \subset \{\sum_{i=1}^{\ell+1} x_i = 0\}$, sending $L_{i,j}$
 72 isomorphically to $\{x_i = x_j\}$. We return to this example, and prove these assertions, in Subsection 4.5.

73 (2) Figure 1 shows a 2-dimensional section of the hyperplane arrangement \mathcal{A} for A_3 , on the left, and,
 74 on the right, a topologically accurate view of the preimage of this section in $\tilde{\mathcal{A}}$.

FIGURE 1. \mathcal{A} and $\tilde{\mathcal{A}}$ for the Coxeter group A_3



75 The planes $\{x_{i_1} = x_{i_2}\}$ and $\{x_{i_3} = x_{i_4}\}$ meet orthogonally if i_1, i_2, i_3 and i_4 are all different, and the
76 reflections in these planes commute; it follows that at a point x in the stratum $\{x_{i_1} = x_{i_2} \neq x_{i_3} = x_{i_4}\}$,
77 the representation is of type $A_1 \oplus A_1$ and by (2ii) of Theorem 0.1 above, the fiber of $\tilde{\mathcal{A}}$ over x consists
78 of two points. In each of these pictures there are four nodes of valency three. In the left hand picture,
79 each lies in a 1-dimensional stratum in $\tilde{\mathcal{A}}$ where the local representation is of type A_2 , so that locally \mathcal{A}
80 consists of three planes in 3-space, meeting along a common line. The preimage of this stratum in $\tilde{\mathcal{A}}$
81 is a line, along which $\tilde{\mathcal{A}}$ is locally isomorphic to the union of the three planes $\langle e_1, e_4 \rangle, \langle e_2, e_4 \rangle$ and $\langle e_3, e_4 \rangle$
82 in 4-space.

83 It would be interesting to find explicit embeddings of the space $\tilde{\mathcal{A}}$ in the remaining cases.

84 To prove the theorem, beginning with the multiplicative structure on Der_B and $\text{coker}(K)$ coming from
85 Dubrovin's Frobenius structure, we endow both Der_V and $\text{coker } J$ with a multiplication, and Der_V with
86 a Der_B -module structure, whose crucial feature is that the derivative $tp: \text{Der}_V \rightarrow \text{Der}_B \otimes_{\mathcal{O}_B} \mathcal{O}_V$ of p is
87 Der_B -linear. On Der_V , but not on $\text{coker } J$, this multiplication lacks a neutral element.

88 Nevertheless, the first evidence for the theorem was found by an entirely different route not involv-
89 ing Dubrovin's Frobenius structure. This was based on the fact that the cokernel of the linear map
90 $S^\ell \xrightarrow{\Lambda} S^\ell$ defined by a square matrix Λ has a natural S -algebra structure if and only if the so-called
91 *rank condition* (rc) holds. This is a purely algebraic condition on the adjugate matrix of Λ , which can
92 be checked by explicit calculation. We explain this in general in Section 3.

93 In Section 4, we then specialize to the case where Λ is the Jacobian matrix J of the basic invariants
94 of a Coxeter group \mathcal{A} , or the Saito matrix of the discriminant D of a Coxeter group. The space $\tilde{D} =$
95 $\text{Spec coker } K$ is normal (indeed smooth) exactly in the *ADE*-case; on the other hand $\tilde{\mathcal{A}} = \text{Spec coker } J$
96 is normal only in the case of A_1 . We discuss the geometry of these two spaces, and their link with
97 the representation theory. In particular we compare them with the normalizations of D and \mathcal{A} in
98 Subsection 4.4.

99 In Section 5, our earlier approach to the main theorem lead to an interesting problem on Coxeter
100 groups. The algebra of the fiber over 0 of the projection $p: V \rightarrow V/W$ carries two structures: that of a
101 zero-dimensional Gorenstein algebra and that of the regular W -representation. It is not clear how these
102 two structures are related: which irreducible components of the same W -isomorphism type admit an
103 isomorphism induced by the algebra structure? The following consequence of Theorem 0.1, whose proof
104 is completed by Proposition 5.7, answers this question in a special case.

Corollary 0.3. *Let W be an irreducible Coxeter group in $\text{GL}(V)$ with homogeneous basic invariants p_1, \dots, p_ℓ , ordered by increasing degree, and let F be the ideal in $\mathbb{C}[V]$ generated by p_1, \dots, p_ℓ . Then for each $j = 1, \dots, \ell$, there exists an $\ell \times \ell$ -matrix A_j with entries in $\mathbb{C}[V]$ such that*

$$\left(\frac{\partial p_\ell}{\partial x_1}, \dots, \frac{\partial p_\ell}{\partial x_\ell} \right) = \left(\frac{\partial p_j}{\partial x_1}, \dots, \frac{\partial p_j}{\partial x_\ell} \right) A_j \pmod{F \cdot (\mathbb{C}[V])^\ell}.$$

105 In all cases except for E_6, E_7 and E_8 , we give an explicit formula for the matrices A_j in Corollary 0.3:
106 they are Hessians of basic invariants. This statement is a 3rd order partial differential condition on
107 the basic invariants which we call the *Hessian rank condition* (Hrc). Besides the missing proof for the
108 E -types, which would lead to a self contained algebraic proof of Theorem 0.1, it would be interesting to
109 know whether (Hrc) is a new condition or can be explained in the framework of Frobenius manifolds.

110 In our final Section 6, we show that by adding to D a divisor which pulls back to the conductor of
111 the ring extension $\mathcal{O}_D \rightarrow \text{coker } K$, we obtain a new free divisor (Theorem 6.5). This was already shown

112 on the singularity side in [MS10]. The preimage in V of this free divisor is a free divisor containing the
 113 reflection arrangement (Corollary 6.6).

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 115 “Research in Pairs” stays in 2010 and 2011. The authors are grateful to the referee for forceful, detailed
 116 and helpful comments on an earlier version.

1. REVIEW OF COXETER GROUPS

118 For more details on the material reviewed in this section, we refer to the book of Humphreys [Hum90].
 119 Let $V_{\mathbb{R}}$ be an ℓ -dimensional \mathbb{R} -vector space and let $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Consider a finite group $W \subset \mathrm{GL}(V)$
 120 generated by reflections defined over \mathbb{R} . Any such representation W decomposes into a direct sum of
 121 irreducible representations, and W is irreducible if and only if the corresponding root system is. The
 122 irreducible isomorphism types are $A_{\ell}, B_{\ell}, D_{\ell}, E_6, E_7, E_8, F_4, G_2 = I_2(6), H_3, H_4$, and $I_2(k)$.

The group W acts naturally on the symmetric algebra $S := \mathbb{C}[V]$ by the contragredient action, and we
 denote by $R := S^W$ the corresponding graded ring of invariants. By a choice of linear basis, we identify
 S with $\mathbb{C}[x_1, \dots, x_{\ell}]$. The natural inclusion $R \subset S$ turns S into a finite R -module of rank $\#W$. The
 averaging operator

$$(1.1) \quad \#: S \rightarrow R, \quad g \mapsto g^{\#} := \frac{1}{\#W} \sum_{w \in W} g^w$$

123 defines a section of this inclusion.

By Chevalley’s theorem ([Hum90, Thm. 3.5]), R is a polynomial algebra $R = \mathbb{C}[p_1, \dots, p_{\ell}]$ where
 p_1, \dots, p_{ℓ} are homogeneous W -invariant polynomials in S . We set

$$(1.2) \quad \deg p_i = m_i + 1 = w_i$$

and assume that $m_1 \leq \dots \leq m_{\ell}$. Then the *degrees* w_i , or the *exponents* m_i , are uniquely determined
 and

$$(1.3) \quad \sum_{i=1}^{\ell} m_i = \#\mathcal{A}$$

124 where \mathcal{A} is the arrangement of reflection hyperplanes of W ([Hum90, Thm. 3.9]).

We make this more precise in the case W is irreducible. Then the eigenvalues of any Coxeter element
 are $\exp(2\pi i \frac{m_i}{h})$ where h is the Coxeter number ([Hum90, Thm. 3.19]). Moreover,

$$(1.4) \quad 1 = m_1 < m_2 \leq \dots \leq m_{\ell-1} < m_{\ell} = h - 1,$$

$$(1.5) \quad m_i + m_{\ell-i+1} = h.$$

In particular, this implies that $\sum_{i=1}^{\ell} m_i = \frac{\ell h}{2}$. For $m_1 = 1$, the W -invariant 2-form p_1 is unique up to a
 constant factor. By a choice of a positive multiple of p_1 , it determines a unique W -invariant Euclidean
 inner product (\cdot, \cdot) on $V_{\mathbb{R}}$, which turns W into a subgroup of $O(V_{\mathbb{R}})$ and serves to identify $V_{\mathbb{R}}$ and $V_{\mathbb{R}}^*$.
 With respect to dual bases of $V_{\mathbb{R}}$ and $V_{\mathbb{R}}^*$ we notice that the two corresponding inner products have
 mutually inverse matrices. At the level of V^* , we denote by

$$\Gamma := ((x_i, x_j)) = ((dx_i, dx_j))$$

the (symmetric) matrix of (\cdot, \cdot) with respect to coordinates x_1, \dots, x_{ℓ} . In suitable coordinates

$$(1.6) \quad p_1 = \sum_{i=1}^{\ell} x_i^2, \quad (x, y) = \sum_{i=1}^{\ell} x_i y_i, \quad \Gamma = (\delta_{i,j}).$$

125 We refer to such coordinates as standard coordinates. In case W is reducible, we have the above situation
 126 on each of the irreducible summands separately.

Geometrically the finiteness of S over R means that the map

$$(1.7) \quad V = \mathrm{Spec} S \xrightarrow{p} \mathrm{Spec} R = V/W$$

127 is finite of degree $\#W$. We identify the reflection arrangement \mathcal{A} of W with its underlying variety
 128 $\bigcup_{H \in \mathcal{A}} H$. Let Δ be a reduced defining equation for \mathcal{A} , and denote by $D = p(\mathcal{A})$ the discriminant. An
 129 *anti-invariant* of W is a relative invariant $f \in S$ with associated character \det^{-1} , that is, $wf = \det^{-1}(w)f$
 130 for all $w \in W$. The following crucial fact due to Solomon [Sol63, §3, Lem.] (see also ([Hum90, Prop.
 131 3.13(b)])) implies that Δ^2 is a reduced defining equation for D .

132 **Theorem 1.1** (Solomon). $R\Delta$ is the set of all anti-invariants. □

133 A second fundamental fact, due to K. Saito [Sai93, §3], is the following

Theorem 1.2 (Saito). For irreducible W , Δ^2 is a monic polynomial in p_ℓ of degree ℓ , that is,

$$\Delta^2 = \sum_{k=0}^{\ell} a_{\ell-k}(p_1, \dots, p_{\ell-1})p_\ell^k, \quad \text{with } a_0 = 1. \quad \square$$

We denote by Der_S and Der_R the modules of vector fields on $V = \text{Spec } S$ and $V/W = \text{Spec } R$ respectively. The group W acts naturally on Der_S . Terao [Ter83] showed that each $\theta \in \text{Der}(-\log D)$ has a unique lifting $p^{-1}(\theta)$ to V and that the set of lifted vector fields is

$$p^{-1} \text{Der}(-\log D) = (\text{Der}_S)^W, \quad p^* \text{Der}(-\log D) = (\text{Der}_S)^W \otimes_R S = \text{Der}(-\log \mathcal{A}),$$

and both \mathcal{A} and D are free divisors. This can be seen as follows: We denote by

$$(1.8) \quad J := (\partial_{x_j}(p_i))$$

the Jacobian matrix of p in (1.7) with respect to the coordinates x_1, \dots, x_ℓ and p_1, \dots, p_ℓ . Via the identification of the 1-form dp_i with a vector field η_i such that $(dp_i, v) = \langle \eta_i, v \rangle$,

$$(1.9) \quad \begin{aligned} dp_i &= \sum_{j=1}^{\ell} \partial_{x_j}(p_i) dx_j \leftrightarrow \eta_i = \sum_{j=1}^{\ell} \langle \eta_i, dx_j \rangle \partial_{x_j} = \sum_{j=1}^{\ell} (dp_i, dx_j) \partial_{x_j} \\ &= \sum_{k,j=1}^{\ell} \partial_{x_k}(p_i) (dx_k, dx_j) \partial_{x_j} = \sum_{k,j=1}^{\ell} \partial_{x_k}(p_i) (x_k, x_j) \partial_{x_j}, \end{aligned}$$

the basic invariants define invariant vector fields $\eta_1, \dots, \eta_\ell \in (\text{Der}_S)^W$, which must then be in $\text{Der}(-\log \mathcal{A})$. By (1.9), their Saito matrix reads

$$(1.10) \quad (\eta_j(x_i)) = \Gamma J^t$$

Now $\det J$ is an anti-invariant because J is the differential of the invariant map $p = (p_1, \dots, p_\ell)$. Hence, $\det J \in \mathbb{C}^* \Delta$ by Theorem 1.1, (1.3), and the algebraic independence of the p_i . By scaling p , we can therefore assume that

$$(1.11) \quad \det J = \Delta.$$

Saito's criterion ([Sai80,]) then shows that \mathcal{A} is free with basis η_1, \dots, η_ℓ . Applying the tangent map tp (see (2.3)) gives vector fields $\delta_1, \dots, \delta_\ell \in \text{Der}_R$ such that $\delta_j \circ p = tp(\eta_j)$ with (symmetric) Saito matrix

$$(1.12) \quad K = (K_j^i) := (\delta_j(p_i)) = J \Gamma J^t$$

with $\det(J \Gamma J^t) \in \mathbb{C}^* \Delta^2$. At generic points of \mathcal{A} , p is a fold map and hence

$$(1.13) \quad \delta_1, \dots, \delta_\ell \in \text{Der}(-\log D).$$

134 Again Saito's criterion shows that D is a free divisor with basis $\delta_1, \dots, \delta_\ell$. In standard coordinates as in
135 (1.6), this proves

136 **Lemma 1.3.** D admits a symmetric Saito matrix $K = J J^t$.

If W is irreducible then, in standard coordinates as in (1.6),

$$(1.14) \quad \chi_w := \frac{1}{2} \delta_1 = \sum_{i=1}^{\ell} w_i p_i \partial_{p_i}.$$

137 We shall refer to the grading defined by this semisimple operator as the w -grading. In particular, δ_k is
138 w -homogeneous of degree $w_k - w_1$. If W is reducible, we have a homogeneity such as (1.14) for each
139 irreducible summand.

Throughout the paper we will abbreviate

$$S_{\mathcal{A}} := S/S\Delta, \quad R_D := R/R\Delta^2.$$

141 In this section we prove Theorem 0.1. We will make use of the Frobenius manifold structure on
 142 V/W , constructed by Dubrovin in [Dub98]. However our main reference for background on Frobenius
 143 manifolds (including this result) is the book of Hertling [Her02]. In fact the only aspects of the Frobenius
 144 structure we use are the existence of an integrable structure of commutative associative \mathbb{C} -algebras on
 145 the fibers of the tangent bundle; a manifold with this structure is called by Hertling and Manin an
 146 *F-manifold*. This notion is much simpler than that of Frobenius manifold, omitting as it does all of the
 147 metric properties, and the connections, which make the definition of Frobenius manifold so complicated.
 148 Following Hertling, we use local analytic methods, and in particular local analytic coordinate changes,
 149 in order to make use of normal forms. Such analytic methods will be justified in Remark 2.5, and we
 150 pass to the analytic category without changing our notation.

The following account summarizes parts of [Her02, Ch. 2]. For any n -dimensional F-manifold M , the multiplication on TM is encoded by an n -dimensional subvariety of T^*M , the *analytic spectrum* L , as follows: for each point $p \in M$, points in T_p^*M determine \mathbb{C} -linear maps $T_pM \rightarrow \mathbb{C}$; among these, a finite number are \mathbb{C} -algebra homomorphisms. These finitely many points in each fiber of T^*M piece together to form L . Thus the composite

$$(2.1) \quad \text{Der}_M \rightarrow \pi_* \mathcal{O}_{T^*M} \rightarrow \pi_* \mathcal{O}_L$$

151 is an isomorphism of \mathbb{C} -algebras ([Her02, Thm. 2.3]).

The multiplication \circ in TM satisfies the integrability property

$$\text{Lie}_{X \circ Y}(\circ) = X \circ \text{Lie}_Y(\circ) + Y \circ \text{Lie}_X(\circ).$$

152 Provided the multiplication is generically semi-simple, as is the case for the structure constructed by
 153 Dubrovin and Hertling, this implies that L is Lagrangian ([Her02, Theorem 3.2]). This in turn means
 154 that the restriction to L of the canonical action form α on T^*M is closed and therefore exact. A
 155 *generating function* for L is any function $F \in \mathcal{O}_L$ such that $dF = \alpha|_L$. A generating function determines
 156 an *Euler field* E on M , namely a vector field mapped to F by the isomorphism (2.1). The discriminant
 157 of M is defined by any of the following equivalent characterizations:

$$(1) \quad D = \pi(F^{-1}(0)),$$

$$(2) \quad D \text{ is the set of points } x \in M \text{ where the endomorphism } E \circ : T_x M \rightarrow T_x M \text{ is not invertible.}$$

160 Similarly, the module $\text{Der}(-\log D)$ may be viewed as either

$$(1) \quad \text{the set of vector fields whose image under the isomorphism (2.1) vanishes on } F^{-1}(0), \text{ or equivalently as}$$

$$(2) \quad \text{the image in } \text{Der}_M \text{ of multiplication by } E.$$

164 This yields the well-known

165 **Lemma 2.1.** *The cokernel $\tilde{R}_D = \text{coker } K$ of the Saito matrix K of D acquires an R -algebra structure as quotient of the Frobenius manifold multiplication in Der_R .*

Proof. The matrix of multiplication by E with respect to the basis $\partial_{x_1}, \dots, \partial_{x_\ell}$ of Der_R is K . Thus

$$(2.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & R^\ell & \xrightarrow{K} & R^\ell & \longrightarrow & \tilde{R}_D \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \text{Der}_R & \xrightarrow{E \circ} & \text{Der}_R & \longrightarrow & \text{Der}_R / \text{Der}_R(-\log D) \longrightarrow 0 \end{array}$$

167 is a presentation of $\text{Der}_R / E \circ \text{Der}_R = \text{Der}_R / \text{Der}_R(-\log D)$, which is itself isomorphic to $\pi_* \mathcal{O}_{F^{-1}(0)}$. \square

168 We will denote $\text{Spec } \tilde{R}_D$ by \tilde{D} .

Recall from (1.8) that $J: S^\ell \rightarrow S^\ell$ is the matrix of the morphism

$$(2.3) \quad tp: \text{Der}_S \rightarrow p^* \text{Der}_R = \text{Der}_R \otimes_R S, \quad tp\left(\sum_{j=1}^n \eta_j \partial_{x_j}\right) = \sum_{i=1}^n \sum_{j=1}^n \eta_j \partial_{x_j}(p_i) \partial_{p_i},$$

defined by left composition (of vector fields as sections of TV) with dp . The following diagram, in which the vertical arrows are bundle projections, helps to keep track of these morphisms. Sections of $p^* \text{Der}_R$

are maps from bottom left to top right making the lower triangle in the diagram commute.

$$(2.4) \quad \begin{array}{ccc} TV & \xrightarrow{dp} & T(V/W) \\ \downarrow & & \downarrow \\ V & \xrightarrow{p} & V/W \end{array}$$

Both $tp : \text{Der}_S \rightarrow p^* \text{Der}_R$ and $\omega p : \text{Der}_R \rightarrow p^* \text{Der}_R$, defined by right composition with p , are familiar in singularity theory. By definition,

$$(2.5) \quad \chi \in \text{Der}_R \text{ lifts to } \eta \in \text{Der}_S \iff tp(\eta) = \omega p(\chi).$$

Using Lemma 1.3, (2.2), and the obvious identifications, there is a commutative diagram of S -modules

$$(2.6) \quad \begin{array}{ccccccc} & & J_\Delta & & & & \\ & & \uparrow & & & & \\ 0 & \longrightarrow & \text{Der}_S & \xrightarrow{tp} & \text{Der}_R \otimes_R S & \longrightarrow & \tilde{S}_{\mathcal{A}} \longrightarrow 0 \\ & & \uparrow J^t & & \uparrow = & & \uparrow \\ 0 & \longrightarrow & R^\ell \otimes_R S & \xrightarrow{K \otimes 1} & \text{Der}_R \otimes_R S & \longrightarrow & \tilde{R}_D \otimes_R S \longrightarrow 0 \end{array}$$

169 Both rows here are exact: the upper row defines $\tilde{S}_{\mathcal{A}}$, and the lower row is the tensor product with the
 170 flat R -module S of the short exact sequence defining \tilde{R}_D . Now $\tilde{R}_D \otimes_R S$, as a tensor product of rings,
 171 has a natural ring structure; to show that $\tilde{S}_{\mathcal{A}}$ is a ring, it will be enough to show

172 **Lemma 2.2.** *The image of tp is an ideal of $\text{Der}_R \otimes_R S$.*

173 We prove Lemma 2.2 by showing that the Frobenius multiplication in Der_R lifts to a $p^* \text{Der}_R$ -module
 174 structure on Der_S , and that $tp : \text{Der}_S \rightarrow \text{Der}_R \otimes_R S$ is Der_R -linear.

175 **Proposition 2.3.**

- 176 (1) *The Frobenius multiplication in Der_R can be lifted to Der_S , though without multiplicative unit.*
 177 (2) *The same procedure makes Der_S into a Der_R -module.*
 178 (3) *The map tp in (2.3) is Der_R -linear, with respect to the structure in (2) and Frobenius multipli-
 179 cation induced on $\text{Der}_R \otimes_R S$.*

Proof. By (2.5), for a multiplication in Der_S , (1) means that

$$(2.7) \quad tp(\eta_1 \circ \eta_2) = \omega p(\chi_1 \circ \chi_2)$$

where $\eta_i \in \text{Der}_S$ is a lift of $\chi_i \in \text{Der}_R$ for $i = 1, 2$. Similarly, the scalar multiplication for (2) must satisfy

$$(2.8) \quad tp(\chi \cdot \eta) = \omega p(\chi \circ \xi)$$

180 where $\chi \in \text{Der}_R$ and $\eta \in \text{Der}_S$ is a lift of $\xi \in \text{Der}_R$.

Locally, at a point $v \in V \setminus \mathcal{A}$, p , tp and ωp are isomorphisms, so there is nothing to prove. Now suppose $v \in H$ is a generic point on a reflecting hyperplane $H \in \mathcal{A}$, with $p(v)$ outside the bifurcation set B . In a neighborhood of $p(v)$ in V/W , we may take canonical coordinates u_1, \dots, u_ℓ (cf. [Her02, 2.12.(ii)]). These are characterized by the property that the vector fields $e_i := \partial_{u_i}$, $i = 1, \dots, \ell$ satisfy $e_i \circ e_j = \delta_{i,j} \cdot e_i$. By [Her02, Cor. 4.6], the tangent space $T_{p(v)}D$ is spanned by $\ell - 1$ of these idempotent vector fields, and the remaining idempotent, which we label e_1 , is normal to it. The map $p_v : (V, v) \rightarrow (V/W, p(v))$ has multiplicity 2, critical set H and set of critical values D , from which it follows that $d_v p : T_v H \rightarrow T_{p(v)}D$ is an isomorphism. Since we have fixed our coordinate system on $(V/W, p)$, we are free to choose only the coordinates on (V, v) . Define $x_i = u_i \circ p$ for $i = 2, \dots, \ell$. To extend these to a coordinate system on (V, v) , we may take as x_1 any function whose derivative at v is linearly independent of $d_v x_2, \dots, d_v x_\ell$. This means we may take as x_1 any defining equation of the critical set (the hyperplane H) of p at v . With respect to these coordinates, p takes the form

$$(2.9) \quad p_v(x_1, \dots, x_\ell) = (f(x_1, \dots, x_\ell), x_2, \dots, x_\ell).$$

As p_v has critical set $\{x_1 = 0\}$ and discriminant $\{u_1 = 0\}$, both f and $\partial_{x_1}(f)$ vanish along $\{x_1 = 0\}$. Thus $f(x) = x_1^2 g(x)$ for some $g \in \mathcal{O}_{V,v}$. Since p has multiplicity 2 at v , $g(0) \neq 0$. Now replace the

coordinate x_1 by $x_1 g(x)^{1/2}$. With respect to these new coordinates, which we still call x_1, \dots, x_ℓ , p_v becomes a standard fold:

$$p_v(x_1, \dots, x_\ell) = (x_1^2, x_2, \dots, x_\ell).$$

We can now explicitly calculate the multiplication in Der_S , locally at v :

$$\begin{cases} tp_v(x_1 \partial_{x_1}) = \omega p_v(2u_1 \partial_{u_1}), \\ tp_v(\partial_{x_i}) = \omega p_v(\partial_{u_i}), \end{cases} \quad \text{for } i = 2, \dots, \ell.$$

So (2.7) implies

$$\begin{aligned} tp_v((x_1 \partial_{x_1}) \circ (x_1 \partial_{x_1})) &= \omega p_v((2u_1 \partial_{u_1}) \circ (2u_1 \partial_{u_1})) \\ &= \omega p_v(4u_1^2 \partial_{u_1}) = \omega p_v(2u_1(2u_1 \partial_{u_1})) = tp_v((2x_1^2)x_1 \partial_{x_1}), \end{aligned}$$

and hence $x_1 \partial_{x_1} \circ x_1 \partial_{x_1} = 2x_1^3 \partial_{x_1}$. So in order that (2.7) should hold, we are forced to define

$$\partial_{x_i} \circ \partial_{x_j} = \begin{cases} 2x_1 \partial_{x_1}, & \text{for } i = j = 1, \\ \delta_{i,j} \cdot \partial_{x_i}, & \text{otherwise.} \end{cases}$$

181 Since the multiplication in Der_V is uniquely defined by (2.7) outside codimension 2, it extends to V by
182 Hartog's Extension Theorem. This proves (1); (2) is obtained by an analogous argument using (2.8).

183 Finally, (3) follows from (2.5) and (2.8) on $V \setminus \mathcal{A}$, and therefore holds everywhere. \square

Proof of Lemma 2.2. Let $\xi \in \text{Der}_S$, $g \in S$ and $\eta \in \text{Der}_R$. By Proposition 2.3.(3) and the evident S -linearity of the lifted Frobenius multiplication,

$$(\eta \otimes_R g) \cdot tp(\xi) = tp(\eta \circ g\xi).$$

184 \square

185 We have proved the following result, which implies (1) of Theorem 0.1.

186 **Theorem 2.4.** *The cokernel $\tilde{S}_{\mathcal{A}} = \text{coker } J$ of the transposed Saito matrix of \mathcal{A} is an $S_{\mathcal{A}}$ -algebra.* \square

187 *Remark 2.5.* Even though our proof uses complex analytic methods, such as canonical coordinates in the
188 proof of Proposition 2.3, the conclusion is valid over any field over which the basic invariants are defined.
189 We show this in Section 3 below by proving that the fact that $\text{coker } J$ is an S -algebra is equivalent to a
190 condition on ideal membership, the so-called *rank condition* (rc).

We end this section by clarifying the relationship between $\tilde{S}_{\mathcal{A}}$ and $\tilde{R}_D \otimes_R S_{\mathcal{A}}$. In general they are not isomorphic, and the space $\text{Spec } \tilde{S}_{\mathcal{A}}$ is not the fiber product $\text{Spec}(\tilde{R}_D \times_D \mathcal{A})$. For $\tilde{R}_D \otimes_R S_{\mathcal{A}}$ is the cokernel of $1 \otimes \Delta: \tilde{R}_D \otimes_R S \rightarrow \tilde{R}_D \otimes_R S$, and using the epimorphism $\text{Der}_R \otimes_R S \rightarrow \tilde{R}_D \otimes_R S$ we find that there is an epimorphism $\text{Der}_R \otimes_R S \rightarrow \tilde{R}_D \otimes_R S_{\mathcal{A}}$, whose kernel is equal to $\text{Der}_R \otimes_R S\Delta + \text{Der}(-\log D) \otimes_R S$. Both summands here are contained in the image of $tp: \text{Der}_S \rightarrow \text{Der}_R \otimes_R S$, the first by Cramer's rule and the second because every vector field $\eta \in \text{Der}(-\log D)$ is liftable via p . Thus $\tilde{S}_{\mathcal{A}}$ is a quotient of $\tilde{R}_D \otimes_R S_{\mathcal{A}}$. The kernel N of the projection $\tilde{R}_D \otimes_R S_{\mathcal{A}} \rightarrow \tilde{S}_{\mathcal{A}}$ is the quotient

$$N := tp(\text{Der}_S) / (\text{Der}(-\log D) \otimes_R S + \text{Der}_R \otimes_R S\Delta).$$

At a generic point $x \in \mathcal{A}$ this vanishes: here p is a fold map, right-left-equivalent to

$$(x_1, \dots, x_\ell) \mapsto (x_1, \dots, x_{\ell-1}, x_\ell^2)$$

and an easy local calculation shows that in this case $N_x = 0$. However, if p has multiplicity > 2 at x then $N_x \neq 0$. For example at an A_2 point, p is right-left equivalent to

$$(x_1, \dots, x_\ell) \mapsto (x_1^2 + x_1 x_2 + x_2^2, x_1 x_2 (x_1 + x_2), x_3, \dots, x_\ell);$$

$tp(\text{Der}_S)$ is generated by $\partial_{p_3}, \dots, \partial_{p_\ell}$ together with

$$(2x_1 + x_2)\partial_{p_1} + (2x_1 x_2 + x_2^2)\partial_{p_2}, (x_1 + 2x_2)\partial_{p_1} + (x_1^2 + 2x_1 x_2)\partial_{p_2},$$

191 while the coefficients of ∂_{p_1} in the generators of $\text{Der}(-\log D) \otimes_R S + \text{Der}_R \otimes_R S\Delta$ are at least quadratic
192 in x_1, \dots, x_ℓ . In fact, assuming Lemma 2.2, we have

193 **Theorem 2.6.** $\tilde{\mathcal{A}} = (\tilde{D} \times_D \mathcal{A})_{\text{red}}$

194 *Proof.* $\tilde{S}_{\mathcal{A}} = \text{coker } tp$, with tp as in (2.6), is a maximal Cohen–Macaulay $S_{\mathcal{A}}$ -module of rank 1. This
 195 means that at a smooth point of \mathcal{A} , $\tilde{S}_{\mathcal{A}}$ is isomorphic to $S_{\mathcal{A}}$, and is thus reduced. As $\tilde{S}_{\mathcal{A}}$ is finite over
 196 $S_{\mathcal{A}}$, its depth over itself (assuming it is a ring) is equal to its depth over $S_{\mathcal{A}}$. Since it is therefore a
 197 Cohen–Macaulay ring, generic reducedness implies reducedness. \square

198 For later use we note that by [MP89, Cor. 3.15], we have

199 **Theorem 2.7.** $\tilde{\mathcal{A}}$ is Cohen–Macaulay and \tilde{D} is Gorenstein. \square

200

3. ALGEBRA STRUCTURES ON COKERNELS OF SQUARE MATRICES

201 **3.1. Rank condition.** In this subsection we recall a condition on the rows of the adjugate of a square
 202 matrix over a ring R , which is equivalent to that matrix presenting an R -algebra, at least in the local
 203 and local graded cases. It is the key to proving Corollary 0.3 in the Introduction.

204 Let R be an ℓ -dimensional (graded) local Cohen–Macaulay ring with maximal (graded) ideal \mathfrak{m} . In
 205 the graded local case, we assume that all R -modules are graded and all R -linear maps are homogeneous.

206 Let A be an $\ell \times \ell$ -matrix over R with transpose $\Lambda := A^t$. We consider both A and Λ as R -linear maps
 207 $R^\ell \rightarrow R^\ell$. Assume that $\Delta := \det A$ is a reduced non-zero-divisor and set $D = V(\Delta)$. By Cramer’s rule
 208 Δ annihilates $M := \text{coker } A$ which is hence a module over $R_D := R/R\Delta$. For any ideal $I \subseteq R$, we denote
 209 by $I_D := R_D I$ its image in R_D . By $Q_D := Q(R_D)$, we denote the total ring of fractions of R_D .

210 The k -th Fitting ideal of M over R , $F^k(M)$, is the ideal of R generated by the $(\ell - k) \times (\ell - k)$ -minors
 211 of A . It is an invariant of M , and independent of the presentation A . We denote by m_j^i the generator of
 212 $F^1(M)$ obtained from A by deleting row i and column j . Note that $F_D^k(M)$ is the k ’th Fitting ideal of
 213 M over R_D . For properties of Fitting ideals, see e.g. [Eis95, Ch. 20].

214 **Definition 3.1.** We say that the *rank condition* (rc) holds for A if $\text{grade } F^1(M) \geq 2$ and $F^1(M)$ is equal
 215 to the ideal of maximal minors of the matrix obtained from A by deleting one of its rows, possibly after
 216 left multiplication of A by some invertible matrix over R .

217 Note that (rc) implies that $F_D^1(M)$ is a maximal Cohen–Macaulay R_D -module, by the Hilbert–Burch
 218 theorem.

219 It turns out that (rc) depends only on the module $M = \text{coker } A$, and not on the choice of presentation
 220 A . This is a consequence of the following two theorems, which also make clear the reason for our interest
 221 in the condition (rc).

222 **Theorem 3.2** ([MP89, Thm. 3.4]). *If M is an R_D -algebra then (rc) holds for A .* \square

223 The proof in [MP89] shows that if M is an R_D -algebra by e, m_2, \dots, m_ℓ , where e is the multiplicative
 224 identity of M , and A is a presentation of M with respect to these generators, then $F^1(M)$ is equal to
 225 the ideal of maximal minors of A with its first row deleted.

226 The converse theorem also holds. A proof, due to de Jong and van Straten, can be found in [MP89,
 227 Prop. 3.14]. We will use some of the notions introduced there, however, and so we give a sketch, based
 228 on the accounts there and in [dJvS90].

Recall that a *fractional ideal* U (over R_D) is a finitely generated R_D -submodule of Q_D which contains
 a non-zero-divisor and that

$$(3.1) \quad \text{Hom}_{R_D}(U, V) = [V :_{Q_D} U]$$

is a fractional ideal, for any two fractional ideals U and V . We shall use this identification implicitly. In
 particular, the duality functor

$$(-)^\vee := \text{Hom}_{R_D}(-, R_D)$$

229 preserves fractional ideals. It is inclusion reversing and a duality on maximal Cohen–Macaulay fractional
 230 ideals (see [dJvS90, Prop. 1.7]).

Theorem 3.3. *If (rc) holds for A then M is a fractional ideal generated by $\varphi_1, \dots, \varphi_\ell \in Q_D$ where*

$$(3.2) \quad \varphi_i m_j^\ell = m_j^i, \quad i, j = 1, \dots, \ell,$$

231 *and an R_D -subalgebra of Q_D isomorphic to $\text{End}_{R_D}(F_D^1(M))$.*

Proof. Using (rc) for A , Lemma 3.4 (below) yields a presentation

$$(3.3) \quad 0 \longrightarrow R^\ell \xrightarrow{\Lambda} R^\ell \xrightarrow{(m_1^\ell, \dots, m_\ell^\ell)} F_D^1(M) \longrightarrow 0.$$

In particular, $F_D^1(M)$ is a maximal Cohen–Macaulay R_D -module of rank 1, and therefore can be viewed as a fractional ideal. As $F_D^1(M)$ is contained in R_D , $F_D^1(M)^\vee$ is a fractional ideal containing R_D . Dualizing (3.3) with respect to R_D gives the exact sequence

$$0 \longrightarrow F_D^1(M)^\vee \longrightarrow R_D^\ell \xrightarrow{A} R_D^\ell .$$

There is also a 2-periodic exact sequence

$$\cdots \longrightarrow R_D^\ell \xrightarrow{A} R_D^\ell \xrightarrow{\text{ad } A} R_D^\ell \xrightarrow{A} \cdots .$$

Therefore,

$$F_D^1(M)^\vee \cong \ker_{R_D} A \cong \begin{cases} \text{coker}_{R_D} A = M, \\ \text{im}_{R_D} \text{ad } A = F_D^1(M). \end{cases}$$

232 and hence $F_D^1(M)^\vee \cong \text{End}_{R_D}(F_D^1(M))$. From this all the statements follow. \square

233 In Subsection 4.2 we identify the generators in Theorem 3.3 in the case that D is the reflection
234 arrangement or discriminant of an irreducible Coxeter group.

Lemma 3.4 ([dJvS90, Prop. 1.10]). *Suppose that the ideal I (generated by the maximal minors of the matrix A with one row deleted) has grade 2. Then there is a free resolution*

$$(3.4) \quad 0 \longrightarrow R^\ell \xrightarrow{\Lambda} R^\ell \xrightarrow{(m_1^\ell, \dots, m_\ell^\ell)} I_D \longrightarrow 0 . \quad \square$$

We can now make good the promise we made in Remark 2.5: that Theorem 2.4 is valid over any field over which the basic invariants p_1, \dots, p_ℓ are defined. From Theorems 2.4 and 3.2 it follows that (rc) holds for $\text{coker } A$: for each $i, j \in \{1, \dots, \ell\}$, the equation

$$(3.5) \quad m_j^i = C_1 m_1^\ell + \cdots + C_\ell m_\ell^\ell$$

235 in unknown functions C_1, \dots, C_ℓ has a solution in which the C_i are germs of complex analytic functions
236 at 0.

237 Let \mathbb{K} be a subfield of \mathbb{C} containing the coefficients of the basic invariants p_j , so that the coefficients
238 of the polynomials m_j^i all lie in \mathbb{K} . We claim that (3.5) has solutions $C_i \in \mathbb{K}[V]$. From this claim, the
239 existence of the S -algebra structure on $\text{coker } A$ follows by Theorem 3.3.

To prove the claim, first note that since the m_j^i are all homogeneous, each C_i can be replaced by its graded part of degree $D_i - D_\ell$ (see (4.6)). Let $\mathbb{K}[V]_d \subset \mathbb{K}[V]$ be the vector space of all polynomials of degree d . The map

$$A : (\mathbb{K}[V]_{D_i - D_\ell})^\ell \rightarrow \mathbb{K}[V]_{D_i}, \quad A(C_1, \dots, C_\ell) = \sum_{j=1}^{\ell} C_j m_j^\ell,$$

is \mathbb{K} -linear. Therefore the solvability of (3.5) in $\mathbb{K}[V]$ reduces to a simple theorem of linear algebra, which can be rephrased more abstractly as follows: Let $A : \mathbb{K}^m \rightarrow \mathbb{K}^n$ be a \mathbb{K} -linear map, and suppose $\mathbb{K} \subset \mathbb{L}$ is a field extension. Then

$$\text{im}(A \otimes_{\mathbb{K}} 1_{\mathbb{L}}) \cap \mathbb{K}^n = \text{im}(A).$$

240 We leave the proof of this to the reader.

3.2. Rings associated to free divisors. In this subsection we make some general observations about the algebra presented by the transpose of a Saito matrix of a free divisor. Let $D = V(\Delta)$ be a free divisor in $(\mathbb{C}^\ell, 0)$ with Saito matrix A . Then we have an exact sequence

$$(3.6) \quad 0 \longrightarrow R^\ell \xrightarrow{A} R^\ell \xrightarrow{(\Delta_1, \dots, \Delta_\ell)} R_D \longrightarrow R_D/J_D \longrightarrow 0 ,$$

where $\Delta_j := \partial\Delta/\partial x_j$ for $j = 1, \dots, \ell$, and $J_D := R_D J_\Delta$ is the Jacobian ideal of D . Now assume also that D is Euler homogeneous. By adding multiples of the Euler vector field $\chi = \delta_1$ to the remaining members $\delta_2, \dots, \delta_\ell$ of a Saito basis of D , we may assume that these annihilate Δ . We shall assume that A is obtained from such a basis. We say that D satisfies (rc) if (rc) holds for $\Lambda = A^t$. In this case, we write

$$\tilde{R}_D := M = \text{coker } \Lambda \subset Q_D$$

241 for the ring of Theorem 3.3.

242 It is well known that for any algebraic or analytic space D satisfying Serre's condition S2, the frac-
 243 tional ideal $\text{End}_{R_D}(J_D^\vee)$ is naturally contained in the integral closure of R_D in Q_D , and the inclusion
 244 $R_D \hookrightarrow \text{End}_{R_D}(J_D^\vee)$ gives a partial normalization (see for example [Vas98, Ch. 2, §2; Ch. 6, §2]. Grauert
 245 and Remmert showed in [GR71] (see also [GR84, Ch. 6, §5]) that for analytic spaces, $R_D = \text{End}_{R_D}(J_D^\vee)$
 246 precisely at the normal points of D , and the analogous result for algebraic spaces was shown by Vascon-
 247 celos in [Vas91].

248 **Proposition 3.5.** *If the free divisor D satisfies (rc) then $\tilde{R}_D \cong \text{End}_{R_D}(J_D) \cong \text{End}_{R_D}(J_D^\vee)$.*

Proof. First, recall the well known fact that for $j = 1, \dots, \ell$, equality

$$(3.7) \quad m_j^1 = \frac{\Delta_j}{\deg \Delta}.$$

which follows from the fact that by Cramer's rule the logarithmic 1-form $\omega_1 := \frac{1}{\Delta} \sum_{j=1}^{\ell} m_j^1 dx_j$ satisfies

$$\langle \omega_1, \delta_j \rangle = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{if } j = 2, \dots, \ell, \end{cases}$$

249 as does $\frac{1}{\deg \Delta} \frac{d\Delta}{\Delta}$.

Next, Lemma 3.4 yields a presentation

$$0 \longrightarrow R^\ell \xrightarrow{A} R^\ell \xrightarrow{(m_1^\ell, \dots, m_\ell^\ell)} F_D^1(M) \longrightarrow 0.$$

This coincides with that of J_D in (3.6); it follows that as R_D -modules, $F_D^1(M)$ and J_D are isomorphic. Hence, by Theorem 3.3,

$$\tilde{R}_D = \text{End}_{R_D}(F_D^1(M)) \cong \text{End}_{R_D}(J_D).$$

250 Since D is free, J_D is maximal Cohen–Macaulay, and then reflexive by [dJvS90, Prop. (1.7) iii)]. So
 251 dualizing induces an isomorphism $\text{End}_{R_D}(J_D) \cong \text{End}_{R_D}(J_D^\vee)$. \square

252 *Remark 3.6.* The map $\varphi_1 \in \text{End}_{R_D}(F_D^1(M))$ described in the proof of Theorem 3.3 gives an explicit
 253 isomorphism $F_D^1(M) \cong J_D$. Indeed, $\varphi_1(m_j^\ell) = \frac{\Delta_j}{\deg \Delta}$ by Lemma 3.7.

254 However the following example, of the discriminant of the reflection group B_3 , shows that, even
 255 under the hypotheses of Proposition 3.5, it is not necessarily the case that the other generators φ_i of
 256 $\text{End}_{R_D}(F_D^1(M))$, $i = 2, \dots, \ell$, defined in (3.2) are isomorphisms onto their image.

A Saito matrix for the discriminant D of B_3 is given by

$$A := \begin{pmatrix} x & -4x^2 + 18y & -xy + 27z \\ 2y & xy + 27z & -2y^2 + 18xz \\ 3z & 6xz & 6yz \end{pmatrix} = \Lambda^t.$$

Because this satisfies (rc),

$$\tilde{I}_D = \langle x^2y - 4y^2 + 3xz, x^2z - 3yz, xyz - 9z^2 \rangle,$$

is equal to the ideal of maximal minors of Λ with its third column deleted. On the other hand the ideal of maximal minors of A with its second column deleted is

$$\langle x^2z - 3yz, xyz - 9z^2 \rangle.$$

257 Evidently the two ideals are not isomorphic as R_D -modules.

258 In contrast, for irreducible free divisors we have

259 **Proposition 3.7.** *Assume that in addition to the hypotheses of Proposition 3.5, D is irreducible and is
 260 not isomorphic to the Cartesian product of a smooth space with a variety of dimension $< \ell - 1$. Then
 261 each of the maps φ_i in (3.2) is an isomorphism onto its image. Let I_i denote the ideal of maximal minors
 262 of A with its i 'th row deleted. Then, for each $i = 1, \dots, \ell$, $R/I_i = R_D/I_i R_D$ is a Cohen–Macaulay ring
 263 with support D_{Sing} .*

264 *Proof.* Because $\Delta \in I_i$, the $(\ell - 1)$ -dimensional components of $V(I_i)$ are among the components of
 265 D . Since Δ is irreducible, the only component possible is D itself. But then because D is reduced,
 266 we would have $I_i \subset \langle \Delta \rangle$. This is absurd, for by hypothesis all entries of A lie in the maximal ideal,
 267 and $\Delta = \sum_{j=1}^{\ell} A_j^i m_j^i$. Thus $V(I_i)$ is purely $\ell - 2$ -dimensional. From this the result now follows by
 268 Lemma 3.4. \square

269 Our Propositions 3.5 and 3.7 are closely related to [Vas98, Prop. 6.15]:

Proposition 3.8. *If D is a free divisor, then*

$$(3.8) \quad J_D \cdot \text{Hom}_{R_D}(J_D, R_D) = F_D^1(M).$$

270 Here both ideals J_D and $\text{Hom}_{R_D}(J_D, R_D)$ are viewed as fractional ideals in Q_D . □

The left hand side of (3.8) is the so-called *trace ideal* of J_D ; it is the set

$$\{\varphi(g) \mid \varphi \in \text{Hom}_{R_D}(J_D, R_D), g \in J_D\}.$$

271 Buchweitz, Ebeling and Graf von Bothmer give a criterion under which, for a free divisor D appearing
272 as the discriminant in the base-space of a versal deformation of a singularity, the ring $\text{End}_{R_D}(J_D)$
273 coincides with the normalization \bar{R}_D of R_D :

274 **Proposition 3.9** ([BEGvB09, Thm. 2.5, Rmk. 2.6]). *If $D \subset S$ is the discriminant in the smooth base-
275 space of a versal deformation $f: X \rightarrow S$ and the module of f -liftable vector fields in Der_S is free, then
276 provided $\text{codim}_S f(X_{\text{Sing}}) \geq 2$, this module coincides with $\text{Der}(-\log D)$. If in fact $\text{codim}_S f(X_{\text{Sing}}) \geq 3$,
277 then $\text{End}_{R_D}(J_D) = \bar{R}_D$.*

4. RING STRUCTURES ASSOCIATED WITH COXETER GROUPS

4.1. **Rank conditions and associated rings.** We return to the situation of Section 1. From now on we work in standard coordinates as in (1.6). Denote by $J_\Delta \subset S$ and $J_{\Delta^2} \subset R$ the Jacobian ideals of Δ and Δ^2 respectively, and define the Jacobian ideals

$$J_{\mathcal{A}} := J_\Delta S_{\mathcal{A}}, \quad J_D := J_{\Delta^2} R_D$$

of \mathcal{A} and of D respectively. Consider the corresponding 1st Fitting ideals

$$(4.1) \quad I_{\mathcal{A}} := F_S^1(J_{\mathcal{A}}), \quad \tilde{I}_{\mathcal{A}} := F_{S_{\mathcal{A}}}^1(J_{\mathcal{A}}) = I_{\mathcal{A}} \cdot S_{\mathcal{A}}, \quad I_D := F_R^1(J_D), \quad \tilde{I}_D := F_{R_D}^1(J_D) = I_D \cdot R_D.$$

By (1.6), (1.10) and (1.12), we have exact sequences

$$(4.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & S^\ell & \xrightarrow{J^t} & S^\ell & \longrightarrow & J_{\mathcal{A}} \longrightarrow 0, \\ & & & & & & \\ 0 & \longrightarrow & R^\ell & \xrightarrow{K=JJ^t} & R^\ell & \longrightarrow & J_D \longrightarrow 0. \end{array}$$

279 The above Fitting ideals $I_{\mathcal{A}}$ and I_D are generated by the sub-maximal minors of J and K respectively.
280 Being Saito matrices, J^t and K have rank $\ell - 1$ at smooth points of \mathcal{A} and D respectively. Therefore
281 $I_{\mathcal{A}}$ and I_D are ideals of grade 2 and $\tilde{I}_{\mathcal{A}}$ and \tilde{I}_D are ideals of grade 1.

282 A more precise version of the rank condition (rc) from Definition 3.1 holds for \mathcal{A} and D :

283 **Lemma 4.1.** *For irreducible W , $I_{\mathcal{A}}$ is generated by the maximal minors of the matrix obtained from J
284 by omitting its ℓ 'th row. This is its homogeneous part of minimal degree $\sum_{i < \ell} m_i = \frac{h\ell}{2} - h + 1$.*

285 *Proof.* By a theorem of Solomon [Sol64, Thm. 2, Cor. (2a)] the minors of J are linearly independent over
286 \mathbb{C} . As $I_{\mathcal{A}}$ is generated by ℓ minors, these must then be the minors of lowest degree. □

287 **Definition 4.2.** For irreducible W , we refer to the condition defined in Lemma 4.1 as the *graded rank
288 condition* (grc) for \mathcal{A} . Analogously, we say that the (grc) holds for D if I_D is generated by the entries
289 in the ℓ 'th row of $\text{ad}(K)$, once again the maximal minors of the matrix obtained by omitting from K
290 the highest weight vector field δ_ℓ . For reducible W , we define (grc) for both \mathcal{A} and D by requiring it,
291 as just defined, for each irreducible summand.

292 In dimension $\ell = 2$, (grc) holds trivially for \mathcal{A} and D : $I_{\mathcal{A}}$ and I_D are the graded maximal ideals of
293 $S_{\mathcal{A}}$ and R_D , due to the presence in each case of an Euler vector field. We shall look at this case in more
294 detail in Subsection 4.4.

By Lemma 3.4, (rc) for \mathcal{A} and D yields exact sequences

$$(4.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & S^\ell & \xrightarrow{J^t} & S^\ell & \longrightarrow & \tilde{I}_{\mathcal{A}} \longrightarrow 0, \\ & & & & & & \\ 0 & \longrightarrow & R^\ell & \xrightarrow{K} & R^\ell & \longrightarrow & \tilde{I}_D \longrightarrow 0. \end{array}$$

The cokernels of the dual maps $J \in \text{End}_S(S^\ell)$, $K^t = K \in \text{End}_R(R^\ell)$ are the algebras

$$(4.4) \quad \tilde{S}_{\mathcal{A}} = \text{End}_{S_{\mathcal{A}}}(\tilde{I}_{\mathcal{A}}), \quad \tilde{R}_D = \text{End}_{R_D}(\tilde{I}_D),$$

295 of Theorem 2.4 and of Lemma 2.1, respectively. Recall that we write $\tilde{\mathcal{A}} = \text{Spec } \tilde{S}_{\mathcal{A}}$ and $\tilde{D} = \text{Spec } \tilde{R}_D$.

Example 4.3. Let \mathcal{A} be the reflection arrangement for W of type $A_1 \times \cdots \times A_1$. In suitable coordinates this is a normal crossing divisor defined by $\Delta = x_1 \cdots x_\ell$. Then $J = J^t = \text{diag}(x_1, \dots, x_\ell)$ and

$$\tilde{S}_{\mathcal{A}} = \text{coker } J = \mathbb{C}[x_2, \dots, x_\ell] \oplus \mathbb{C}[x_1, x_3, \dots, x_\ell] \oplus \cdots \oplus \mathbb{C}[x_1, \dots, x_{\ell-1}].$$

296 Generalizing this example we have

297 **Lemma 4.4.** *The assignments $W \mapsto \tilde{S}_{\mathcal{A}}$ and $W \mapsto \tilde{R}_D$ commute with direct sums (of representa-*
298 *tions/rings).*

Proof. Assume that $W = W' \oplus W''$, and use the analogous notation to refer to the above defined objects with W replaced by W' and W'' respectively. Then $S = S' \otimes_{\mathbb{C}} S''$, J is a block matrix with blocks J' and J'' , $\Delta = \Delta' \Delta''$, hence $I_{\mathcal{A}} = I_{\mathcal{A}'} \Delta'' + I_{\mathcal{A}''} \Delta'$ and therefore

$$\tilde{I}_{\mathcal{A}} \cong \tilde{I}_{\mathcal{A}'} \otimes_{\mathbb{C}} S'' \oplus S' \otimes_{\mathbb{C}} \tilde{I}_{\mathcal{A}''}$$

by the following Lemma 4.5. Applying $\text{End}_{S_{\mathcal{A}}}$ yields

$$\tilde{S}_{\mathcal{A}} = \tilde{S}_{\mathcal{A}'} \otimes_{\mathbb{C}} S'' \oplus S' \otimes_{\mathbb{C}} \tilde{S}_{\mathcal{A}''}.$$

299 This proves the claim for \mathcal{A} ; an analogous proof works for D . □

Lemma 4.5. *Let $f \in K[x] = K[x_1, \dots, x_r] \supset I$, $g \in K[y] = K[y_1, \dots, y_s] \supset J$, and $K[x, y] = K[x_1, \dots, x_r, y_1, \dots, y_s]$. Then*

$$(Ig + Jf)(K[x, y]/\langle fg \rangle) \cong I(K[x]/\langle f \rangle) \otimes_K K[y] \oplus K[x] \otimes_K J(K[y]/\langle g \rangle),$$

$$[Pg + Qf] \leftrightarrow [P] \oplus [Q].$$

300 *Proof.* One easily verifies that the given correspondence is well-defined in both directions. □

4.2. Relation of rings for \mathcal{A} and D . Let us assume now that W is irreducible. Then the algebras $\tilde{S}_{\mathcal{A}}$ and \tilde{R}_D can be described more explicitly as follows. We denote by

$$(4.5) \quad (m_j^i) := \text{ad}(J^t), \quad (M_j^i) := \text{ad}(K) = \text{ad}(J^t) \text{ad}(J)$$

the adjoint matrices of J^t and K respectively, and set

$$(4.6) \quad D_k = \text{deg}(m_j^k) = \sum_{i=1}^{\ell} m_i - m_k.$$

Abbreviating $h_i := \varphi_i^{\mathcal{A}} \in Q_{\mathcal{A}}$ and $g_i := \varphi_i^D \in Q_D$ for $i = 1, \dots, \ell$, Theorem 3.3 reads

$$(4.7) \quad h_i m_j^{\ell} = m_j^i, \quad g_i M_j^{\ell} = M_j^i, \quad i, j = 1, \dots, \ell,$$

$$(4.8) \quad \tilde{S}_{\mathcal{A}} = \langle h_1, \dots, h_{\ell} \rangle_{S_{\mathcal{A}}} = S_{\mathcal{A}}[h_1, \dots, h_{\ell-1}], \quad \tilde{R}_D = \langle g_1, \dots, g_{\ell} \rangle_{R_D} = R_D[g_1, \dots, g_{\ell-1}].$$

Proposition 4.6. *If W is irreducible then*

$$h_i = \frac{\partial_{p_i}(\Delta^2)}{\partial_{p_{\ell}}(\Delta^2)} \in Q_{\mathcal{A}}^W.$$

Proof. First, differentiate $\Delta^2 \in R$,

$$2\Delta d\Delta = d(\Delta^2) = \sum_{k=1}^{\ell} \partial_{p_k}(\Delta^2) dp_k$$

considered as an equality in Ω_S^1 . Then wedging with $dp_1 \wedge \cdots \wedge \widehat{dp_i} \wedge \cdots \wedge dp_{\ell-1}$ gives

$$(-1)^{i-1} \partial_{p_i}(\Delta^2) dp_1 \wedge \cdots \wedge dp_{\ell-1} + (-1)^{\ell-1} \partial_{p_{\ell}}(\Delta^2) dp_1 \wedge \cdots \wedge \widehat{dp_i} \wedge \cdots \wedge dp_{\ell} \equiv 0 \pmod{S\Delta}.$$

Taking coefficients with respect to $dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{\ell}$ yields

$$\partial_{p_i}(\Delta^2) m_j^{\ell} \equiv \partial_{p_{\ell}}(\Delta^2) m_j^i \pmod{S\Delta}, \quad j = 1, \dots, \ell.$$

301 By Theorem 1.2, $\partial_{p_{\ell}}(\Delta^2)$ is a non-zero-divisor in $S_{\mathcal{A}}$, and the claim follows from (4.7). □

Using Theorem 1.1, one verifies that the averaging operator (1.1) induces a commutative diagram of R -modules

$$\begin{array}{ccccccc}
Q_D & \hookrightarrow & Q_{\mathcal{A}} & \xrightarrow{\#} & Q_{\mathcal{A}}^W & \xleftarrow{\cong} & Q_D \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\tilde{R}_D & \hookrightarrow & \tilde{S}_{\mathcal{A}} & \dashrightarrow & \tilde{S}_{\mathcal{A}}^W & \xleftarrow{\cong} & \tilde{R}_D \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
R_D & \hookrightarrow & S_{\mathcal{A}} & \xrightarrow{\#} & (S_{\mathcal{A}})^W & \xleftarrow{\cong} & R_D
\end{array}$$

where the dashed maps result from the following proposition.

Proposition 4.7. *We have*

$$(4.9) \quad h_i = g_i = \frac{M_{\ell}^i}{M_{\ell}^{\ell}} \in Q_D,$$

and hence

$$(4.10) \quad \tilde{R}_D = (\tilde{S}_{\mathcal{A}})^W.$$

Proof. Using (4.5) we have $M_{\ell}^i = \sum_r m_r^i m_r^{\ell}$. By (4.7), this is equal to $h_i \sum_r m_r^{\ell} m_r^{\ell}$ and therefore to $h_i M_{\ell}^{\ell}$. By [MP89, Thm. 3.4], M_{ℓ}^{ℓ} generates the conductor of $R_D \hookrightarrow \tilde{R}_D$ and is therefore not a zero-divisor on R_D or $S_{\mathcal{A}}$. Therefore, $h_i = M_{\ell}^i / M_{\ell}^{\ell} = g_i$ by (4.7) and (4.10) follows using (4.8). \square

4.3. Local trivialization. The integral varieties of $\text{Der}(-\log \mathcal{A})$ and $\text{Der}(-\log D)$ form Saito's logarithmic stratification defined in [Sai80, §3], which we denote by $L(\mathcal{A})$ and $L(D)$ respectively. We shall locally trivialize $\tilde{\mathcal{A}}$ and \tilde{D} along logarithmic strata with slices of the same type, with W replaced by the subgroup fixing the strata. In the case of $\tilde{\mathcal{A}}$ the trivialization is algebraic, while in the case of \tilde{D} we need to work in the analytic category.

We begin with the discussion of $\tilde{\mathcal{A}}$. The logarithmic stratification $L(\mathcal{A})$ coincides, up to taking the closure of strata, with the intersection lattice of \mathcal{A} . It is a geometric lattice (ordered by reverse inclusion) whose rank function is given by the codimension in V . By $L_k(\mathcal{A}) \subset L(\mathcal{A})$, we denote the collection of all rank k elements.

Definition 4.8. For $X \in L(\mathcal{A})$, denote by W_X the subgroup of W generated by reflections with reflecting hyperplanes in the localization $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\}$ of \mathcal{A} along $X \in L(\mathcal{A})$, and by Δ_X the reduced defining equation of \mathcal{A}_X . We denote also by I_X the defining ideal of X in $S_{\mathcal{A}}$.

By [Hum90, Thm. 1.12 (d)], W_X is the group fixing X point-wise, that is

$$W_X = \bigcap_{x \in X} W_x.$$

For $x \in V$, let $X(x)$ be the stratum $X \in L(\mathcal{A})$ with $x \in X$. It follows that

$$W_{X(x)} = W_x$$

is the isotropy group of x .

Proposition 4.9. *If $X \in L(\mathcal{A})$ then $(\tilde{S}_{\mathcal{A}})_{I_X} = (\tilde{S}_{\mathcal{A}_X})_{I_X} = \tilde{S}_{\mathcal{A}_X/X} \otimes_{\mathbb{C}} \mathbb{C}(X)$. In particular, $\tilde{S}_{\mathcal{A}_X/X} = (\tilde{S}_{\mathcal{A}})_{I_X}^X$ where the upper index “ X ” means X considered as a translation group.*

Proof. Fix $X \in L(\mathcal{A})$ and let Y be its orthogonal complement. By $\Delta_X \in \mathbb{C}[Y]$ we denote the defining equation of \mathcal{A}_X . Then, by the product rule,

$$(J_{\mathcal{A}})_{I_X} = J_{\Delta}(S_{I_X}/S_{I_X} \Delta) = J_{\Delta_X}(S_{I_X}/S_{I_X} \Delta_X) = (J_{\mathcal{A}_X})_{I_X}.$$

Localizing a presentation, such as (4.2), at I_X , therefore shows that

$$\begin{aligned}
(I_{\mathcal{A}})_{I_X} &= (F_S^1(J_{\mathcal{A}}))_{I_X} = F_{S_{I_X}}^1((J_{\mathcal{A}})_{I_X}) \\
&= F_{S_{I_X}}^1((J_{\mathcal{A}_X})_{I_X}) = (F_S^1(J_{\mathcal{A}_X}))_{I_X} = (I_{\mathcal{A}_X})_{I_X}.
\end{aligned}$$

Then we have also $(\tilde{I}_{\mathcal{A}})_{I_X} = (\tilde{I}_{\mathcal{A}_X})_{I_X}$ and finally,

$$\begin{aligned} (\tilde{S}_{\mathcal{A}})_{I_X} &= (\text{End}_{S_{\mathcal{A}}}(I_{\mathcal{A}}))_{I_X} = \text{End}_{S_{I_X}}((I_{\mathcal{A}})_{I_X}) \\ &= \text{End}_{S_{I_X}}((I_{\mathcal{A}_X})_{I_X}) = (\text{End}_{S_{\mathcal{A}_X}}(I_{\mathcal{A}_X}))_{I_X} = (\tilde{S}_{\mathcal{A}_X})_{I_X}. \end{aligned}$$

321 This proves the first equality; the second follows since $S_{I_X} = \mathbb{C}[Y] \otimes_{\mathbb{C}} \mathbb{C}(X)$. \square

322 **Corollary 4.10.** *The assignment $\mathcal{A} \mapsto \tilde{S}_{\mathcal{A}}$ is a local functor.* \square

323 We now turn our attention to \tilde{D} . The following result holds for any free divisor, and our proof is not
324 specific to our situation.

325 **Proposition 4.11.** *The ideals $I_{\mathcal{A}}$ and I_D are stable under $\text{Der}(-\log \mathcal{A})$ and $\text{Der}(-\log D)$ respectively.
326 In particular, the latter act naturally on $\tilde{S}_{\mathcal{A}}$ and \tilde{R}_D respectively.*

Proof. Let $\omega_1, \dots, \omega_\ell \in \Omega^1(\log D)$ be the dual basis of (1.13). From

$$R \ni d\omega_j(\delta_k, \delta_r) = d\omega_j \left(\delta_k, \sum_{i=1}^{\ell} K_r^i \partial_{p_i} \right) = \sum_{i=1}^{\ell} K_r^i d\omega_j(\delta_k, \partial_{p_i}),$$

(1.11) and Cramer's rule, we conclude that

$$\begin{aligned} I_D \ni d\omega_j(\delta_k, \Delta^2 \partial_{p_i}) &= \delta_k \langle \Delta^2 \omega_j, \partial_{p_i} \rangle - \Delta^2 \partial_{p_i} \langle \omega_j, \delta_k \rangle - \langle \omega_j, [\delta_k, \Delta^2 \partial_{p_i}] \rangle \\ &= \delta_k (M_j^i) + \left\langle \Delta^2 \omega_j, [\partial_{p_i}, \delta_k] - \frac{\delta_k(\Delta^2)}{\Delta^2} \partial_{p_i} \right\rangle \\ &\equiv \delta_k (M_j^i) \pmod{I_D}. \end{aligned}$$

327 This proves the claim for D ; the same argument works for \mathcal{A} and any free divisor. \square

328 *Remark 4.12.* There is a transcendental argument which shows that for any divisor D , free or not,
329 $\text{Der}(-\log D)$ preserves the ideal $I_k(D)$ of $k \times k$ minors of the matrix of coefficients of a set of generators
330 of $\text{Der}(-\log D)$. It is simply that each of these ideals is invariant under biholomorphic automorphisms
331 of D , since they are Fitting ideals of the Jacobian ideal J_D . The integral flow of any vector field
332 $\zeta \in \text{Der}(-\log D)$ preserves D , and hence $I_k(D)$, from which it follows that $\zeta \cdot I_k(D) \subset I_k(D)$.

We can improve on Proposition 4.9 in the analytic category. Let $x \in X \in L(\mathcal{A})$ and $y = p(x) \in p(X) = Y$. By [Orl89, §2], $Y \in L(D)$ and $p: X \rightarrow Y$ is a covering. By finiteness of W , there is a (Euclidean) W_X -stable neighborhood of x , in which the W -orbits are exactly the W_X -orbits. Note that W_X commutes with the translation group X . This gives

$$p_x = p_{W_X, x} \times p|_X: V_x = (V/X)_x \times X_x \rightarrow ((V/X)/W_X)_y \times Y_y.$$

333 Since our definition of \tilde{R}_D in (4.1) and (4.4) is compatible with passing to the analytic category, we
334 obtain the following analytic localization statement.

335 **Proposition 4.13.** *Let $x \in X \in L(\mathcal{A})$ and $y = p(x) \in p(X) = Y \in L(D)$, and denote by D_Y the
336 discriminant of W_X on V/X . Then there is an isomorphism of analytic germs $\tilde{D}_y \cong \tilde{D}_{Y, y} \times Y_y$.* \square

337 *Remark 4.14.* Saito [Sai80, (3.6)] showed that one can always analytically trivialize the logarithmic
338 stratification along logarithmic strata as we do in Proposition 4.13.

339 **Corollary 4.15.** *$\tilde{\mathcal{A}}$ is (algebraically) and \tilde{D} (analytically) constant over logarithmic strata.* \square

340 By [Hum90, §1.8], W acts simply transitively on the (simple) root systems and on the Weyl chambers.
341 Choosing a simple root system defining a Weyl chamber of which $X = X(x)$ is a face, shows that the
342 Dynkin diagram of any isotropy group $W_x = W_X$ is obtained by dropping from the Dynkin diagram of
343 W the roots which are not orthogonal to X . By [Hum90, Prop. 2.2], the connected components of the
344 resulting Dynkin diagram are in bijection with the irreducible factors of W_x . This discussion combined
345 with Propositions 4.9 and 4.13 proves

Theorem 4.16. *Let $X \in L(\mathcal{A})$ and let $Y = p(X)$. Let W_1, \dots, W_r be the irreducible Coxeter groups
whose Dynkin diagrams are the connected components of the sub-diagram of the Dynkin diagram of W
formed by the vertices corresponding to simple roots orthogonal to X . Let $\mathcal{A}_1, \dots, \mathcal{A}_r$ and D_1, \dots, D_r be*

their reflection arrangements and discriminants, and let ℓ_i be the dimension of the standard representation of W_i . Then the algebraic localization of \mathcal{A} along X , and the analytic localization of \bar{D} along Y , are isomorphic, respectively, to the disjoint unions

$$\bigsqcup_{i=1}^r \tilde{\mathcal{A}}_i \times \mathbb{C}^{\ell-\ell_i} \quad \text{and} \quad \bigsqcup_{i=1}^r \tilde{D}_i \times \mathbb{C}^{\ell-\ell_i}. \quad \square$$

346 **4.4. Relation with the normalization.** We denote the normalizations of \mathcal{A} and D by $\tilde{\mathcal{A}}$ and \bar{D}
347 respectively.

348 **Proposition 4.17.** *We have $S_{\mathcal{A}} \subseteq \tilde{S}_{\mathcal{A}} \subseteq \bar{S}_{\mathcal{A}}$ and $R_D \subseteq \tilde{R}_D \subseteq \bar{R}_D$.*

349 *Proof.* This follows from the finiteness and birationality of $\tilde{S}_{\mathcal{A}}$ and \tilde{R}_D over $S_{\mathcal{A}}$ and R_D , see (4.8), (4.7),
350 (4.10), (4.9). \square

351 In the following, we describe the cases of equality in Proposition 4.17.

We begin with the case $\ell = 2$ of plane curves for irreducible W . By (1.14) and for degree reasons, this case reduces to

$$(4.11) \quad K = \begin{pmatrix} 2p_1 & hp_2 \\ hp_2 & Q \end{pmatrix}, \quad Q = ap_1^r + bp_1^s p_2, \quad r = h - 1, \quad \frac{h}{2} - 1 = s,$$

$$(4.12) \quad \Delta^2 = |K| = 2p_1 Q - h^2 p_2^2 = 2ap_1^h + 2bp_1^{h/2} p_2 - h^2 p_2^2.$$

In particular, $b = 0$ if h is odd. Note that there are no further restrictions imposed on a and b by the requirement

$$(4.13) \quad \delta_2(\Delta^2) \in R\Delta^2$$

352 for δ_2 from (1.12). Indeed, $\langle \delta_1, \delta_2 \rangle_R$ is a Lie algebra, since $[\delta_1, \delta_2] = (h - 2)\delta_2$ by homogeneity. For
353 generic (a, b) , Δ^2 in (4.12) is reduced, and hence (4.13) holds true by [Sai80, Lem. 1.9]. By continuity,
354 it holds then also for special values of (a, b) .

355 **Proposition 4.18.** *For $\ell = 2$, irreducible W , and odd $h \geq 5$, $\tilde{D} \neq \bar{D}$.*

Proof. In this case,

$$(4.14) \quad K = \begin{pmatrix} 2p_1 & hp_2 \\ hp_2 & ap_1^r \end{pmatrix}$$

and (4.12) specializes to

$$\Delta^2 = |K| = 2ap_1^{r+1} - h^2 p_2^2 \equiv p_1^h - p_2^2.$$

The normalization of D is given by $p_1 = t^2$ and $p_2 = t^h$, and hence $g_1 = \frac{p_2}{p_1} = t^{h-2}$ by (4.9) and (4.14). Then (4.10) becomes

$$\tilde{R}_D = R_D[g_1] = \mathbb{C}[t^2, t^{h-2}] \subsetneq \mathbb{C}[t] = \bar{R}_D. \quad \square$$

356 Using Theorem 4.16 and Lemma 4.4 we find

357 **Corollary 4.19.** *If W contains any irreducible summand of type H_3 , H_4 , or $I_2(k)$ for odd k , then*
358 $\tilde{D} \neq \bar{D}$.

359 *Proof.* For W of type $I_2(k)$, we have $h = k$ and the claim follows from Proposition 4.18. For the H_k -types,
360 the statement follows from Theorem 4.16 and the adjacency chain $H_4 \rightarrow H_3 \rightarrow I_2(5)$. \square

361 We write $\mathbb{C}_0 = S/\mathfrak{m}$ where \mathfrak{m} is the graded maximal ideal in S . Then $\tilde{\mathcal{A}}_0 = \text{Spec}(\tilde{S}_{\mathcal{A}} \otimes_S \mathbb{C}_0)$ is the
362 fiber of $\tilde{\mathcal{A}}$ over $0 \in V$.

363 **Lemma 4.20.** *The group W acts trivially on the fiber $\tilde{\mathcal{A}}_0$ of $\tilde{\mathcal{A}}$ over $0 \in V$, which contains exactly as*
364 *many geometric points as the number of irreducible summands of W .*

365 *Proof.* By (4.8), $\tilde{S}_{\mathcal{A}} \otimes_S \mathbb{C}_0 \cong \mathbb{C}[h_1, \dots, h_{\ell-1}]$ and by Proposition 4.7 the h_i are W -invariants. This
366 implies the first claim. For the second statement, we may assume that W is irreducible by Lemma 4.4.
367 Then (1.4), (4.5), and (4.7) imply that h_i has w -degree $w_{\ell} - w_i$. So $\mathbb{C}[h_1, \dots, h_{\ell-1}]$ is positively graded
368 and hence $\tilde{\mathcal{A}}$ is a cone. As it is also finite over $0 \in V$ due to (4.8), it must be a single geometric point
369 as claimed. \square

370 We write $\mathbb{C}_x = S/\mathfrak{m}_x$ and $\mathbb{C}_y = R/\mathfrak{m}_y$ where \mathfrak{m}_x and \mathfrak{m}_y are the maximal ideals of S at x and of R at
371 y . Then $\tilde{\mathcal{A}}_x = \text{Spec}(\tilde{S}_{\mathcal{A}} \otimes_S \mathbb{C}_x)$ and $\tilde{D}_y = \text{Spec}(\tilde{R}_D \otimes_R \mathbb{C}_y)$ are the fibers of $\tilde{\mathcal{A}}$ over x and of \tilde{D} over y
372 respectively. Combining Propositions 4.9 and 4.13, (4.8), Proposition 4.7, and Lemma 4.20, we find

Proposition 4.21. *The fibers $\tilde{\mathcal{A}}_x$ and \tilde{D}_y , $y = p(x)$, coincide, that is,*

$$\tilde{S}_{\mathcal{A}} \otimes_S \mathbb{C}_x = \tilde{R}_D \otimes_R \mathbb{C}_y.$$

373 *They are trivial W_x -modules containing exactly as many geometric points as the number of irreducible*
374 *summands of W_x .*

375 We can now refine Proposition 4.17 for \mathcal{A} .

376 **Corollary 4.22.**

- 377 (1) $\mathcal{A} = \tilde{\mathcal{A}}$ exactly if \mathcal{A} contains only one plane (or W has type A_1).
378 (2) $\tilde{\mathcal{A}} = \mathcal{A}$ exactly if \mathcal{A} is Boolean (or W has type $A_1 \times \cdots \times A_1$).

379 *Proof.*

380 (1) If $\#\mathcal{A} > 1$, pick x with $X(x) = X \in L_2(\mathcal{A}) \neq \emptyset$. Then W_X is of type $A_1 \times A_1$. So by
381 Proposition 4.21, $\tilde{\mathcal{A}}$ has two points over x . The converse is Example 4.3 for $\ell = 1$.

382 (2) Again one implication is Example 4.3. If \mathcal{A} is not Boolean, then W has an non- A_1 type irreducible
383 summand. By Lemma 4.20, its reflection hyperplanes do not separate in $\tilde{\mathcal{A}}$.
384 □

385 The analogue of Corollary 4.22 for D is less trivial.

386 **Theorem 4.23.** *$\tilde{D} = \bar{D}$ exactly if all irreducible summands of W are of ADE-type. In this case, \tilde{D} is*
387 *smooth.*

388 *Proof.* If W is of type ADE, then by [Bri71, Slo80] V/W can be identified with the base space of a
389 versal deformation of a singularity of the same type. Then by (0.1) $\tilde{D} = \Sigma^0$ is a smooth space and hence
390 $\tilde{D} = \bar{D}$. If W is reducible, with all irreducible summands of type ADE, then by Proposition 4.4 \tilde{D} is
391 the disjoint union of the spaces corresponding to the summands.

392 Conversely, consider an irreducible W not of type ADE and not covered by Corollary 4.19, that is,
393 of type B_ℓ , C_ℓ , F_4 , or $I_2(k)$ with k even. Then there are at least two W -orbits in \mathcal{A} , D is reducible,
394 and \bar{D} has at least two connected components. On the other hand \tilde{D} is connected, by Lemma 4.20 and
395 Proposition 4.21. Thus $\tilde{D} \neq \bar{D}$. By Proposition 4.4 this conclusion applies to reducible W also. □

396 **4.5. Example 0.2 revisited.** In Example 0.2 we asserted that in the case of A_ℓ , the space $\tilde{\mathcal{A}}$ is
397 isomorphic to the union of the coordinate $(\ell - 1)$ -planes in $\mathbb{C}^{\ell+1}$. We now prove this. Let us denote this
398 union by L_ℓ , and denote by s and t the natural projections $L_\ell \rightarrow \mathcal{A}$ and $\tilde{\mathcal{A}} \rightarrow \mathcal{A}$. Recall that a space
399 X is *weakly normal* if every continuous function which is holomorphic on the smooth part of X is in fact
400 holomorphic on all of X .

401 **Lemma 4.24.** *The space L_ℓ is Cohen–Macaulay and weakly normal.*

402 *Proof.* Cohen–Macaulayness is well known, and follows from the Hilbert–Burch theorem: the ideal I_ℓ of
403 functions vanishing on L_ℓ is $\langle x_2 \cdots x_{\ell+1}, x_1 x_3 \cdots x_{\ell+1}, \dots, x_1 \cdots x_\ell \rangle$, and it is easy to obtain this as the
404 ideal of maximal minors of an $\ell \times (\ell + 1)$ matrix. For weak normality, we use induction on ℓ : the space
405 L_2 is the union of the coordinate axes in 3-space, and weak normality can easily be checked here. Now
406 suppose $\ell \geq 3$ and that the statement is true for L_j with $j < \ell$, and let f be a continuous function on
407 L_ℓ , holomorphic on the regular part. In a neighborhood of each point $x \in L_\ell \setminus \{0\}$, L_ℓ is isomorphic to
408 a product $L_j \times \mathbb{C}^{\ell-j}$ for some $j < \ell$. It follows from the induction hypothesis that L_ℓ is weakly normal
409 at these points, and therefore f is holomorphic at x . Since L_ℓ is Cohen–Macaulay, Hartogs’s Theorem
410 holds and therefore f is holomorphic also at 0. □

411 **Proposition 4.25.** *In the case of the reflection arrangement for A_ℓ , the space $\tilde{\mathcal{A}}$ is isomorphic to L_ℓ .*

412 *Proof.* The key step is to show that the combinatorial structure of L_ℓ and $\tilde{\mathcal{A}}$ is the same. For then by
413 the universal property of weak normality, there exists an analytic map $\pi: (L_\ell, 0) \rightarrow (\tilde{\mathcal{A}}, 0)$ such that
414 $s = t \circ \pi$. This map is a homeomorphism, and therefore has a (topological) inverse. To see that the
415 inverse is analytic, it is enough, once again by Hartogs’s Theorem, to prove it so outside codimension
416 2. It is clearly so over regular points of \mathcal{A} , since here the projections $\tilde{\mathcal{A}} \rightarrow \mathcal{A}$ and $L_\ell \rightarrow \mathcal{A}$ are both
417 bianalytic. The codimension-1 singularities of \mathcal{A} are of type $A_1 + A_1$ (a normal crossing of 2 branches,

418 with reducible representation) and A_2 . Over points of the first kind, both $\tilde{\mathcal{A}}$ and L_ℓ are regular, and so
 419 π^{-1} is analytic. Over points of the second kind, the computation carried out in Example 0.2 shows that
 420 π is an isomorphism.

To see that the combinatorial structure is the same, recall that $s(x) = x - x^\sharp$. Let $a = (a_1, \dots, a_{\ell+1}) \in \mathcal{A}$. Then

$$s^{-1}(a) = \{x \in L_\ell \mid x = a + \lambda(1, \dots, 1) \text{ for some } \lambda \in \mathbb{C}\}.$$

421 Now $a + \lambda(1, \dots, 1) \in L_{i,j}$ if and only if $\lambda = -a_i = -a_j$. Thus a has a preimage for each value λ
 422 such that two or more of the coordinates a_i take the value $-\lambda$. Thus, preimages are in bijection with
 423 equivalence classes $I \subset \{1, \dots, \ell + 1\}$ under the equivalence relation $i \sim j$ if $x_i = x_j$. This is the same
 424 equivalence relation which determines the decomposition of the isotropy group of x in W into a direct
 425 product of indecomposable factors. By part (2ii) of Theorem 0.1, the set of equivalence classes is in
 426 natural bijection with the geometric fiber of $\tilde{\mathcal{A}}$ over x . \square

427 It would be interesting to know if the space $\tilde{\mathcal{A}}$ is weakly normal for other Coxeter arrangements.

428

5. DUAL AND HESSIAN RANK CONDITIONS

Let $F = S \cdot \mathfrak{m}_R$ be the ideal of all positive-degree W -invariants. We can identify S/F with a direct
 summand T of the W -module S , and setting $S^\alpha = T \cdot p^\alpha$, we have

$$(5.1) \quad S = \bigoplus_{\alpha \in \mathbb{N}^\ell} S^\alpha \supset \bigoplus_{0 \neq \alpha \in \mathbb{N}^\ell} S^\alpha = F$$

as a direct sum of W -modules, where $p = p_1, \dots, p_\ell$. Chevalley [Che55] showed that T is the regular
 W -representation (see also [Sol64, p. 278]). Consider the W -modules of exterior powers

$$E_p = \bigwedge^p V^*.$$

Solomon [Sol64, Thm. 2 and footnote (2)] showed that the isotypic components of S/F of type $E_1 \cong V^*$
 and $E_{\ell-1} \cong V \otimes \det V$ are the direct sums of the projections to S/F of the W -modules

$$(5.2) \quad \begin{aligned} J^j &= \langle \partial_{x_k}(p_j) \mid k = 1, \dots, \ell \rangle_{\mathbb{C}}, \\ M^j &= \langle m_k^j \mid k = 1, \dots, \ell \rangle_{\mathbb{C}}, \quad j = 1, \dots, \ell, \end{aligned}$$

429 respectively. We may and will assume that $J^j \subset T$ and $M^j \subset T$. By (1.2) and (4.6), D_j is the
 430 homogeneous degree of M^j , while m_j is the homogeneous degree of J^j .

Let us recall the construction from the proof of [Sol64, Thm. 2]: We denote by $I(-)$ the W -invariant
 part. By [Sol63], the space of W -invariant differential forms on V is

$$I(S \otimes E_p) = \sum_{i_1 < \dots < i_p} R \cdot dp_{i_1} \wedge \dots \wedge dp_{i_p}.$$

Solomon [Sol64, p. 282] considers the case where W is the Weyl group of a Lie group acting on V ;
 then the Killing form induces a self-duality $E_p \cong E_p^*$. We are only interested in the cases $p = 1$ and
 $p = \ell - 1$, where both irreducibility and self-duality of E_p are trivial¹. The self-duality of E_p induces a
 W -isomorphism $S/F \otimes E_p \cong \text{Hom}_{\mathbb{C}}(E_p, S/F)$ and hence an isomorphism

$$(5.3) \quad I(S/F \otimes E_p) \cong \text{Hom}_W(E_p, S/F).$$

431 The image of dp_i in $\text{Hom}_W(E_p, S/F)$ has image J^i , and the image of $dp_1 \wedge \dots \wedge \widehat{dp_i} \wedge \dots \wedge dp_\ell$ has image
 432 M^i .

Using (5.1),

$$(5.4) \quad \bigoplus_{j=1}^{\ell} \bigoplus_{\alpha \in \mathbb{N}^\ell} M^j p^\alpha \quad \text{and} \quad \bigoplus_{j=1}^{\ell} \bigoplus_{0 \neq \alpha \in \mathbb{N}^\ell} M^j p^\alpha$$

433 are the isotypic components of type $E_{\ell-1}$ of S and F respectively. In particular, we have the following

434 **Lemma 5.1.** *The isotypic component of F of type $E_{\ell-1}$ lies in $F \cdot I_{\mathcal{A}}$. \square*

¹ $E_1 \cong V^*$ is self-dual due to the W -invariant form p_2 on V , and hence irreducible, since V is irreducible. Because
 $\det(V)^{\otimes 2} \cong \mathbb{C}$ is the trivial representation, $E_{\ell-1} \cong E_1^* \otimes E_\ell \cong V \otimes \det(V)$ is self-dual. For the same reason and irreducibility
 of V , $I(V \otimes \det(V) \otimes (V \otimes \det(V))^*) = I(V \otimes V^*) = 1$, and hence $E_{\ell-1}$ is irreducible.

435 It follows that (grc) can be checked modulo F .

436 **Definition 5.2.** We say that the *graded rank condition mod F* holds for \mathcal{A} if $M^j \subset M^\ell + F$ for all
437 $j = 1, \dots, \ell - 1$.

438 **Lemma 5.3.** *The graded rank condition mod F is equivalent to the graded rank condition for \mathcal{A} .*

Proof. Consider the maps of W -modules

$$(5.5) \quad \phi_*: \text{Hom}_{\mathbb{C}}(M^j, M^\ell \otimes_{\mathbb{C}} S_{D_j - D_\ell}) \xrightarrow{\mu_*} \text{Hom}_{\mathbb{C}}(M^j, S_{D_j}) \xrightarrow{\pi_*} \text{Hom}_{\mathbb{C}}(M^j, T_{D_j})$$

induced by the composition of W -linear maps $\phi = \pi \circ \mu$, where

$$\mu: S \otimes_{\mathbb{C}} S \rightarrow S \quad \text{and} \quad \pi: S \twoheadrightarrow S/F = T$$

are the product in S and the canonical projection to T . By hypothesis, there is a \mathbb{C} -linear map $\alpha \in \text{Hom}_{\mathbb{C}}(M^j, M^\ell \otimes_{\mathbb{C}} S_{D_j - D_\ell})$ such that $\phi_*(\alpha) \in \text{Hom}_{\mathbb{C}}(M^j, M^j)$ is the identity map. Now averaging yields

$$\gamma = \alpha^\# \in \text{Hom}_W(M^j, M^\ell \otimes_{\mathbb{C}} S_{D_j - D_\ell}), \quad \phi_*(\gamma) = \text{id}_{M^j}.$$

Using Lemma 5.1, we find that

$$\mu_*(\gamma) - \text{id}_{M^j} \in \text{Hom}_W(M^j, F) = \text{Hom}_W(M^j, F \cdot I_{\mathcal{A}}).$$

This proves that

$$I_{\mathcal{A}} \subset S \cdot M^\ell + F \cdot I_{\mathcal{A}},$$

439 and hence $I_{\mathcal{A}} = S \cdot M^\ell$ by Nakayama's lemma. \square

By Solomon's result mentioned above, the W -equivariant Gorenstein pairing on S/F induces a non-degenerate pairing of the isotypic components of type E_1 and $E_{\ell-1}$ into the unique irreducible summand of type $E_\ell \cong \det(V)$,

$$\bigoplus_{i=1}^{\ell} J^i \otimes \bigoplus_{j=1}^{\ell} M^j \rightarrow \mathbb{C} \cdot \Delta.$$

Since the element

$$\sum_{i=1}^{\ell} \partial_{x_i}(p_j) \otimes m_i^j \in J^j \otimes M^j$$

maps to $\Delta = \det J$ by Laplace expansion of the determinant along the j 'th row, we obtain induced non-degenerate pairings

$$(5.6) \quad J^j \otimes M^j \rightarrow \mathbb{C} \cdot \Delta, \quad j = 1, \dots, \ell.$$

For $j < k$, we have

$$(5.7) \quad \begin{aligned} \text{Hom}_W(J^j, J^k) &\cong \text{End}_W(E_1) \cong \text{End}_W(E_{\ell-1}^* \otimes E_\ell) \\ &\cong \text{End}_W(E_{\ell-1}^*) \cong \text{Hom}_W(M^k, M^j), \end{aligned}$$

440 where $\mu_*(\alpha) \in \text{Hom}_W(J^j, J^k)$ induced by $\alpha \in \text{Hom}_W(J^j, J^j \otimes S_{m_k - m_j})$ corresponds to $\mu_*(\beta) \in$
441 $\text{Hom}_W(M^k, M^j)$ induced by $\beta = \alpha^t \in \text{Hom}_W(M^k, M^k \otimes S_{D_j - D_k})$. Note here that $m_k - m_j = D_j - D_k$
442 by (4.6). Because of the non-degenerate W -pairing (5.6), $\mu_*(\alpha)$ is an isomorphism exactly if $\mu_*(\beta)$ is an
443 isomorphism.

444 **Definition 5.4.** We say that the *dual (graded) rank condition* (drc) holds for \mathcal{A} if $J^\ell \subset S \cdot J^j + F$ for
445 all $j = 1, \dots, \ell - 1$.

446 *Remark 5.5.* The definition of (drc) is given as an equality in S/F because in general $J^\ell \not\subset S \cdot J^j$, though
447 the inclusion holds trivially for $j = 1$.

448 **Lemma 5.6.** *The graded rank condition mod F is equivalent to the dual rank condition for \mathcal{A} .*

449 *Proof.* We show that (grc) mod F implies (drc). The opposite implication is proved in just the same
450 way. Fix $j \in \{1, \dots, \ell - 1\}$. By (grc) mod F , there is a $\beta \in \text{Hom}_{\mathbb{C}}(M^j, M^\ell \otimes S_{D_j - D_\ell})$ inducing the
451 identity map $\text{id}_{M^j} = \pi_* \mu_*(\beta) \in \text{Hom}_{\mathbb{C}}(M^j, M^j)$. By averaging, we can turn β into a W -homomorphism.
452 The homomorphism $\mu_*(\beta)$ is non-zero modulo F and (5.7) yields a corresponding dual map $\mu_*(\alpha) \in$
453 $\text{Hom}_W(J^j, S_{D_\ell})$ induced by $\alpha := \beta^t \in \text{Hom}_W(J^\ell, J^j \otimes S_{m_\ell - m_j})$. This shows that (drc) holds. \square

454 By Lemma 5.3, we deduce the following equivalence that combined with Theorems 2.4 and 3.3 and
455 Lemma 4.1 proves Corollary 0.3.

456 **Proposition 5.7.** *The dual graded rank condition is equivalent to the (graded) rank condition for \mathcal{A} . \square*

The following property refines (grc) by a statement about the S -coefficients of J^j in the condition in Definition 5.4. By [OS88, (2.14) Lem.], the Hessian

$$\text{Hess}(p): \text{Der}_S \rightarrow \Omega_S^1, \quad \text{Hess}(p)(\delta) := \sum_{i=1}^{\ell} \delta(\partial_{x_i}(p)) dx_i,$$

is W -equivariant for $p \in R$. Note that $\text{Hess}(p_1)$ is a W -isomorphism which induces our identification of dp_i with a vector field η_i in (1.9). By abuse of notation, we identify

$$\text{Hess}(p) = \text{Hess}(p) \circ \text{Hess}(p_1)^{-1} \in \text{End}_W(\Omega_S^1)$$

for $p \in R$. Using $\Omega_S^1 = S \otimes E_1$ and passing to the quotient by F , $\text{Hess}(p)$ then induces an element of $\text{End}_W(S/F \otimes E_1)$ and hence of $\text{End}_W(I(S/F \otimes E_1))$. By (5.3), $\text{Hess}(p)$ thus induces a map

$$\hat{h}(p) \in \text{End}_W(\text{Hom}_W(E_1, S/F))$$

457 which operates on W -submodules of type V^* by passing to the image in $\text{Hom}_W(E_1, S/F)$.

458 **Definition 5.8.** We say that the *Hessian (dual graded) ring condition* (Hrc) holds for \mathcal{A} if, for any j ,
459 there is an i , such that $m_i + m_j = w_\ell$ and $\text{Hess}(p_i)(\eta_j) \notin F\Omega_S^1$. In case m_1, \dots, m_ℓ are pairwise different,
460 this means that $\text{Hess}(p_i)(\eta_{\ell-i+1}) \notin F\Omega_S^1$.

461 **Lemma 5.9.** *The Hessian rank condition implies the dual ring condition for \mathcal{A} .*

462 *Proof.* (Hrc) means that $\hat{h}(p_i)(J^j) \subset (S/F)_{m_\ell}$ is non-zero. By W -equivariance of $\hat{h}(p_i)$, the latter is then
463 a non-trivial W -submodule of $(S/F)_{m_\ell}$ of type E_1 . Then it must coincide with J_ℓ , which is the only
464 such W -module in this degree by (1.4). \square

465 **Theorem 5.10.** *The Hessian rank condition holds for \mathcal{A} if W is not of type E_6, E_7 , or E_8 .*

466 *Proof.* It is clear that $\text{Hess}(p_i)(\eta_1) = dp_i$, so (Hrc) holds trivially in dimension $\ell = 2$. For the A -
467 and B -types, it is an easy exercise to verify (Hrc) using [Hum90, §3.12]. In case of F_4, H_3 and H_4 ,
468 Macaulay2 [GS] calculations, based on the formulæ for basic invariants given by Mehta [Meh88], show
469 that (Hrc) holds for \mathcal{A} .

Let us now prove (Hrc) for W of type D_ℓ . By [Hum90, §3.12], the basic invariants can be chosen as the power sums

$$p_k = \frac{1}{2k}(x_1^{2k} + \dots + x_\ell^{2k}), \quad k = 1, \dots, \ell - 1,$$

together with $p_\ell = x_1 \cdots x_\ell$. Note the change of notation turning $p_{\ell-1}$ into the highest degree invariant. It is easy to check that $D(p_i) \circ \text{Hess}(p_{\ell-i}) \equiv D(p_{\ell-1}) \pmod{\mathbb{C}^*}$ for $i = 1, \dots, \ell - 2$. We now replace $p_{\ell-1}$ by the invariant polynomial

$$\hat{p}_{\ell-1}(x_1, \dots, x_\ell) = D(p_\ell) \cdot D(p_\ell) = \sum_{j=1}^{\ell} x_1^2 \cdots \widehat{x_j^2} \cdots x_\ell^2 \in R$$

of the same degree. We claim that $p_{\ell-1} \equiv \hat{p}_{\ell-1} \pmod{F^2 + \mathbb{C}^*}$. In the evident equality

$$2 \cdot D(p_\ell) \circ \text{Hess}(p_\ell) = D(\hat{p}_{\ell-1})$$

470 we can then replace $\hat{p}_{\ell-1}$ by $p_{\ell-1}$ modulo F , completing the proof of (Hrc).

471 In order to verify the claim, let ρ be a primitive $2(\ell-1)$ 'th root of unity and set $a = (\rho, \rho^2, \dots, \rho^{\ell-1}, 0)$.
472 Then all of our basic invariants except for $p_{\ell-1}$ vanish at a , while $\hat{p}_{\ell-1}(a) \neq 0 \neq p_{\ell-1}(a)$. Since
473 $\deg \hat{p}_{\ell-1} = \deg p_{\ell-1} > \deg p_i$ for all $i \neq \ell - 1$ by (1.4), the claim follows. \square

474 Computing limitations oblige us to leave open the following conjecture.

475 **Conjecture 5.11.** *The Hessian rank condition holds for \mathcal{A} if W is of type E_6, E_7 , or E_8 .*

In [MS10] a new class of free divisors was constructed using the recipe “discriminant + adjoint”. If D is the discriminant in the base of a miniversal deformation of a weighted homogeneous hypersurface singularity (subject to some numerical conditions on the weights) and D' is an adjoint divisor, in the sense that the pull-back of D' to the normalization Σ^0 of D is the conductor of the ring extension $\mathcal{O}_D \rightarrow \mathcal{O}_{\Sigma^0}$, then $D + D'$ is a free divisor ([MS10, Thm. 1.3]). The singularities to which this applies include those of type ADE . In this section we point out that essentially the same construction works for the other Coxeter groups. We have to replace the normalization \bar{D} by the space \tilde{D} of Lemma 2.1 (though recall that $\bar{D} = \tilde{D}$ for Coxeter groups of type ADE), and take, as D' , a divisor pulling back to the conductor of the ring extension $\mathcal{O}_D \hookrightarrow \mathcal{O}_{\tilde{D}}$. The construction lifts to the representation space V , giving a new free divisor strictly containing the reflection arrangement.

We keep the notations from Section 1 and work in standard coordinates as in (1.6).

Lemma 6.1. *With a suitable choice of basic invariants p_1, \dots, p_ℓ , the linear part \bar{K} of the Saito matrix $K = JJ^t$ of D from (4.2) is symmetric of the form*

$$(6.1) \quad \bar{K} = \begin{pmatrix} w_1 p_1 & w_2 p_2 & \cdots & \cdots & w_{\ell-1} p_{\ell-1} & w_\ell p_\ell \\ w_2 p_2 & \star & \cdots & \star & \alpha_{\ell-1} p_\ell & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \star & \ddots & & & \vdots \\ w_{\ell-1} p_{\ell-1} & \alpha_2 p_\ell & \ddots & & & \vdots \\ w_\ell p_\ell & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

where $\alpha_2, \dots, \alpha_{\ell-1} \in \mathbb{C}^*$ with $\alpha_i = \alpha_{\ell+1-i}$. Moreover the only entries in this matrix equal to non-zero constant multiples of p_ℓ lie along the anti-diagonal.

Remark 6.2. This matrix shows the linearized convolution of the basic invariants p_1, \dots, p_ℓ as described in [Arn79].

Proof. The first row and column of (6.1) can be read from (1.14). It remains to show the triangular form of \bar{K} and that the anti-diagonal entries, and only these, are non-zero constant multiples of p_ℓ . By inspection, the degree of K_j^i is $w_i + w_j - w_1$. By (1.2), (1.4) and (1.5), the degree of K_j^i with $i + j = \ell + 1$ equals $h = w_\ell$, and hence $\bar{K}_j^i = \alpha_j p_\ell$ for some $\alpha_j \in \mathbb{C}$. Provided W is not of type D_{2k} , the degrees w_1, \dots, w_ℓ of the basic invariants are pairwise distinct. It follows that:

- All K_j^i with $i + j > \ell + 1$ have degree strictly between w_ℓ and $2w_\ell$ and hence have a linear part equal to zero. In particular, \bar{K} has the claimed triangular shape.
- All K_j^i with $i + j < \ell + 1$ have degree less than w_ℓ , and hence do not involve p_ℓ .

But by (1.11), (1.12), and Theorem 1.2, $\det K = \Delta^2$ is a monic polynomial of degree ℓ in p_ℓ . It follows that $\alpha_2 \cdots \alpha_{\ell-2} \neq 0$. Finally the symmetry property $\alpha_i = \alpha_{\ell+1-i}$ comes from the symmetry of K .

In the case of D_{2k} , the same argument shows that the p_ℓ -coefficient matrix of \bar{K} is a constant symmetric anti-diagonal block matrix, where i and j are in the same block exactly if $w_i = w_j$. By the procedure in the proof of [MS10, Lem. 3.6] it can be turned into a symmetric anti-diagonal matrix by linear algebra on the basic invariants. \square

Remark 6.3. By (1.4), the minor M_ℓ^ℓ is not changed by the change of basic invariants in Proposition 6.1.

For \bar{K} as in (6.1), we set

$$(\bar{M}_j^i) := \text{ad}(\bar{K}), \quad \bar{I}_D := \langle \bar{M}_1^\ell, \dots, \bar{M}_\ell^\ell \rangle.$$

Note that because (rc) holds, $\bar{I}_D = \langle \bar{M}_j^i \mid 1 \leq i, j \leq \ell \rangle$.

Lemma 6.4. $dM_\ell^\ell(\text{Der}(-\log D)) = \bar{I}_D$.

Proof. The strategy is the same as in the proof of the analogous result in [MS10]. We replace δ_i by its linear part $\bar{\delta}_i$ whose coefficients are in the i 'th row/column of \bar{K} in (6.1). Then it suffices to prove that the inclusion

$$(6.2) \quad d\bar{M}_\ell^\ell(\langle \bar{\delta}_1, \dots, \bar{\delta}_\ell \rangle) \subseteq \bar{I}_D.$$

obtained from Proposition 4.11 is an equality. The polynomial expansion of the minor $\bar{M}_{\ell-i+1}^\ell$ contains the distinguished monomial $p_i p_\ell^{\ell-2}$ with non-zero coefficient. This monomial does not appear in the

511 expansion of \bar{M}_j^ℓ for $j \neq i$. In particular the expansion of \bar{M}_ℓ^ℓ contains the monomial $p_1 p_\ell^{\ell-2}$, with
 512 coefficient $(-1)^{\ell-2} \iota w_1 \alpha$, where ι is the sign of the order-reversing permutation of $1, \dots, \ell - 1$, and
 513 $\alpha := \alpha_2 \cdots \alpha_{\ell-1}$.

514 We claim that $d\bar{M}_\ell^\ell(\bar{\delta}_i)$ contains the monomial $p_i p_\ell^{\ell-2}$ with non-zero coefficient, and no other of the
 515 distinguished monomials. This shows that (6.2) is an equality and proves the lemma.

516 Contributions to the coefficient of $p_j p_\ell^{\ell-2}$ in the expansion of $d\bar{M}_\ell^\ell(\bar{\delta}_i)$ arise as follows:

- (1) By applying the derivation $p_j \partial_{p_1}$ to the monomial $p_1 p_\ell^{\ell-2}$. This happens only when $i = j$, and
 in this case the resulting contribution to the coefficient of $p_j p_\ell^{\ell-2}$ is

$$\delta_{i,j} (-1)^{\ell-2} \iota w_i w_1 \alpha.$$

- (2) By applying the derivation $p_\ell \partial_{p_k}$ to the monomial $p_j p_k p_\ell^{\ell-3}$. This derivation appears in $\bar{\delta}_i$ only if
 $k = \ell - i + 1$, and then with coefficient α_i ; also this monomial appears in \bar{M}_ℓ^ℓ only if $k = \ell - j + 1$,
 and hence $i = j$. If $2j = \ell + 1$, the monomial $p_j p_{\ell-i+1} p_\ell^{\ell-3}$ appears in the expansion of \bar{M}_ℓ^ℓ with
 coefficient

$$\delta_{i,j} (-1)^{\ell-1} \iota w_j w_{\ell-j+1} \alpha / \alpha_j,$$

otherwise, it appears twice with that coefficient. The resulting contribution to the coefficient of
 $p_j p_\ell^{\ell-2}$ in $d\bar{M}_\ell^\ell(\bar{\delta}_i)$ is

$$\delta_{i,j} (-1)^{\ell-1} \iota \alpha w_j w_{\ell-j+1}$$

517 if $2j = \ell + 1$, or twice this if $2j \neq \ell + 1$.

Therefore $p_j p_\ell^{\ell-2}$ can appear in $d\bar{M}_\ell^\ell(\bar{\delta}_i)$ with non-zero coefficient only if $i = j$, and in this case the
 coefficient is non-zero provided

$$\begin{cases} w_1 \neq w_j, & \text{if } 2j = \ell + 1, \\ w_1 \neq 2w_{\ell-j+1}, & \text{if } 2j \neq \ell + 1. \end{cases}$$

518 These conditions hold by (1.4). □

519 **Theorem 6.5.** *Let $D' = \{M_\ell^\ell = 0\}$. Then $D + D'$ is a free divisor.*

Proof. Here the proof is identical to the proof of the comparable result of [MS10, Prop. 3.10]. By
 Lemma 6.4, there are vector fields $\tilde{\delta}_1, \dots, \tilde{\delta}_\ell \in \text{Der}(\log D)$ such that

$$(6.3) \quad dM_\ell^\ell(\tilde{\delta}_i) = M_i^\ell.$$

520 We may take $\tilde{\delta}_\ell$ equal to a constant multiple of the Euler vector field δ_1 . Since $\delta_1, \dots, \delta_\ell$ is a basis of
 521 $\text{Der}(-\log D)$, there exist $B_j^i \in R$ such that $\tilde{\delta}_i = \sum_{j=1}^\ell B_j^i \delta_j$. By the proof of Lemma 6.4, the matrix
 522 $B = (B_j^i)$ is invertible. Note that the Saito matrix of the basis $\tilde{\delta}_1, \dots, \tilde{\delta}_\ell$ is then KB . Let K' be obtained
 523 from the matrix K by deleting its last column. The columns of K' give relations among the generators
 524 $M_1^\ell, \dots, M_\ell^\ell$ of I_D , by Cramer's rule.

For each relation $\sum_{i=1}^\ell \lambda_i M_i^\ell = 0$, (6.3) gives

$$\sum_{i=1}^\ell \lambda_i \tilde{\delta}_i(M_\ell^\ell) = dM_\ell^\ell\left(\sum_{i=1}^\ell \lambda_i \tilde{\delta}_i\right) = \sum_{i=1}^\ell \lambda_i M_i^\ell = 0,$$

so

$$\sum_{i=1}^\ell \lambda_i \tilde{\delta}_i \in \text{Der}(-\log D) \cap \text{Der}(-\log D') = \text{Der}(-\log(D + D')).$$

525 Because $\tilde{\delta}_\ell$ is a scalar multiple of δ_1 , we also have $\tilde{\delta}_\ell \in \text{Der}(-\log(D + D'))$. Let K'' denote the matrix
 526 formed by adjoining to K' the extra column $(0, \dots, 0, 1)^t$. Thus the columns of the $\ell \times \ell$ matrix KBK''
 527 are the coefficients of vector fields in $\text{Der}(-\log(D + D'))$, and $\det(KBK'') \equiv \Delta^2 M_\ell^\ell \pmod{\mathbb{C}^*}$ where
 528 $\Delta^2 = \det K$ is a reduced equation for D . Now provided

- (1) M_ℓ^ℓ is reduced, and
 530 (2) M_ℓ^ℓ and Δ^2 have no common factor,

531 it follows from Saito's criterion that $D + D'$ is a free divisor, and the vector fields represented by the
 532 columns of KBK'' form a free basis for $\text{Der}(-\log(D + D'))$.

533 By [MP89, Cor. 3.15], M_ℓ^ℓ generates (over \tilde{R}_D) the conductor ideal of the map $\tilde{D} \rightarrow D$. It follows that
 534 $D \cap D' = V(I_D) = \text{Sing}(D)$ has codimension 2, and hence (2) holds. It suffices to check (1) at generic
 535 points of $\text{Sing}(D)$. Using Proposition 4.13, this reduces to checking (1) in the case $\ell = 2$ discussed in
 536 Section 4.4. But in this case $M_2^2 = 2p_1$ is reduced by (4.11). □

537 **Corollary 6.6.** $\mathcal{A} + p^{-1}(D')$ is a free divisor.

538 *Proof.* We continue with the notation of the proof of Theorem 6.5. Consider the vector fields represented
539 by the columns of $J^t(BK'') \circ p$. Since $JJ^tBK'' = KBK''$, these vector fields are lifts to V of the vector
540 fields represented by the columns of KBK'' ; they are therefore logarithmic with respect to $p^{-1}(D')$.
541 Since they are linear combinations of the columns of J^t they are logarithmic with respect to \mathcal{A} , and
542 thus with respect to $\mathcal{A} + p^{-1}(D')$.

543 By (1.11), $\det J = \Delta$ is a reduced equation of \mathcal{A} . Since $\det K'' = \pm M_\ell^\ell$ is reduced and, along $V(M_\ell^\ell)$,
544 p is generically a submersion (for the critical set of p is \mathcal{A} , which meets $V(M_\ell^\ell \circ p)$ only in codimension
545 2), $\det(K'' \circ p)$ is a reduced equation for $V(M_\ell^\ell \circ p)$. As $\det B \in \mathbb{C}^*$, $\det(J^t(BK'') \circ p)$ is therefore a
546 reduced equation for $\mathcal{A} + p^{-1}(D')$, and the corollary follows by Saito's criterion. \square

547 *Example 6.7.* The reflection arrangement for A_n consists of the intersection of $V := \{\sum_{i=1}^{n+1} x_i = 0\} \subset$
548 \mathbb{C}^{n+1} with the union of the hyperplanes $\{x_i = x_j\}$. For A_2 , the composite equation $M_\ell^\ell \circ p$ defining
549 $p^{-1}(D')$ in Corollary 6.6 is equal, on V , to the second elementary symmetric function, σ_2 . For A_3 , this
550 becomes $8\sigma_2\sigma_4 - 9\sigma_3^2 - 2\sigma_2^3$.

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- 610 MICHEL GRANGER, UNIVERSITÉ D'ANGERS, DÉPARTEMENT DE MATHÉMATIQUES, LAREMA, CNRS UMR n°6093, 2 Bd
611 LAVOISIER, 49045 ANGERS, FRANCE
612 *E-mail address:* granger@univ-angers.fr
- 613 D. MOND, MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY CV47AL, ENGLAND
614 *E-mail address:* D.M.Q.Mond@warwick.ac.uk
- 615 M. SCHULZE, DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OK 74078, UNITED STATES
616 *E-mail address:* mschulze@math.okstate.edu