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# PARTIAL NORMALIZATIONS OF COXETER ARRANGEMENTS AND DISCRIMINANTS

MICHEL GRANGER, DAVID MOND, AND MATHIAS SCHULZE

*To the memory of V.I. Arnol'd*

ABSTRACT. We study natural partial normalization spaces of Coxeter arrangements and discriminants and relate their geometry to representation theory. The underlying ring structures arise from Dubrovin's Frobenius manifold structure which is lifted (without unit) to the space of the arrangement. We also describe an independent approach to these structures via duality of maximal Cohen–Macaulay fractional ideals. In the process, we find 3rd order differential relations for the basic invariants of the Coxeter group. Finally, we show that our partial normalizations give rise to new free divisors.

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## INTRODUCTION

V.I. Arnol'd was the first to identify the singularities of type  $A$ ,  $D$  and  $E$  as the simple singularities – those that are adjacent to only finitely many other types. He also uncovered the links between the Coxeter groups of type  $B_n$ ,  $C_n$  and  $F_4$  and boundary singularities, see [Arn79]. In the latter paper his formulæ for generators of  $\text{Der}(-\log D)$  parallels K. Saito's definition of free divisors. Along with Brieskorn, Dynkin, Gelfan'd, and Gabriel, Arnol'd revealed the ADE list as one of the central squares in mathematical heaven, where representation theory, algebra, geometry and topology converge. As with so many of Arnol'd's contributions, his work on this topic has given rise to a huge range of further work by others.

If  $f: X = (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0) = S$  is a complex function singularity of type  $A_n$ ,  $D_n$ ,  $E_6$ ,  $E_7$  or  $E_8$ , then the discriminant  $D$  in the base  $B = (\mathbb{C}^\mu, 0)$  of a miniversal deformation  $F: X \times B \rightarrow S$  of  $f$  is isomorphic to the discriminant of the Coxeter group of the same name. This is only the most superficial feature of the profound link between singularity theory and the geometry of Coxeter groups which Arnol'd helped to make clear.

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The starting point of this paper is the fact, common to Coxeter groups and singularities, that a Saito matrix  $K$  of the discriminant  $D$  is also the presentation of a ring. As  $D$  is a free divisor, the module of logarithmic vector fields  $\text{Der}(-\log D)$  has a basis consisting of  $\mu$  vector fields, and  $K$  is the  $\mu \times \mu$ -matrix of its coefficients with respect to a basis for the module of all vector fields  $\Theta_B$ . On the singularity theory side these two roles are well known. Let  $h$  be a defining equation for  $D$ . Then  $K$  appears in the exact sequence

$$0 \longrightarrow \mathcal{O}_B^\mu \xrightarrow{K} \Theta_B \xrightarrow{dh} J_D \longrightarrow 0$$

which defines  $\text{Der}(-\log D)$  as the vector fields which preserve the ideal of  $D$ . If  $\Sigma \subset X \times B$  denotes the relative critical locus defined by the relative Jacobian ideal  $J_F^{\text{rel}}$  of  $F$ , and  $\Sigma^0 = \Sigma \cap V(F)$ , then  $K$  also appears in the exact sequence

$$0 \longrightarrow \text{Der}(-\log D) \xrightarrow{K} \Theta_B \xrightarrow{dF} \mathcal{O}_{\Sigma^0} \longrightarrow 0$$

in which  $dF$  maps a vector field  $\eta \in \Theta_B$  to the function  $dF(\tilde{\eta})$  on  $\Sigma^0$ , where  $\tilde{\eta}$  is a lift of  $\eta$  to  $X \times B$ . As  $\mathcal{O}_\Sigma$  is free over  $\mathcal{O}_B$  of rank  $\mu$ , we can make the identifications

$$\mathcal{O}_\Sigma \cong \mathcal{O}_B^\mu \cong \Theta_B,$$

and reinterpret  $K$  as the matrix of the  $\mathcal{O}_B$ -linear operator induced on  $\mathcal{O}_\Sigma$  by multiplication by  $F$ , whose cokernel is also, evidently,  $\mathcal{O}_{\Sigma^0}$ .

Coxeter groups of type ADE correspond to miniversal deformations of singularities of function germs  $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ , and the groups of type  $B_k, C_k$  and  $F_4$  correspond in a similar way to boundary singularities, for which a similar argument shows that the cokernel of  $K$  is naturally a ring.

Nevertheless, also for the remaining Coxeter groups  $G_2, I_2(p), H_3$  and  $H_4$ , the cokernel of the Saito matrix carries a natural ring structure. This can be seen in two ways. The first involves the Frobenius structure constructed on the orbit space by Dubrovin in [Dub98], following K. Saito. Here a principal ingredient is a fibre-wise multiplication on the tangent bundle, which coincides with the multiplication coming from  $\mathcal{O}_\Sigma$  in the singularity case. We recall the necessary details of Dubrovin's construction, following C. Hertling's account in [Her02], in Section 4.

The second route is purely algebraic, and does not require the machinery of the Frobenius structure. Instead, it relies on embedding  $\text{coker } K$  into the total ring of fractions  $Q(\mathcal{O}_D)$  and on the duality functor  $-\vee = \text{Hom}_{\mathcal{O}_D}(-, \mathcal{O}_D)$  on maximal Cohen–Macaulay fractional ideals, and harnesses the multiplication of  $Q(\mathcal{O}_D)$ . In order for this construction to work it is necessary and sufficient that the  $(\mu - 1) \times (\mu - 1)$  minors of  $K$  satisfy the so-called *rank condition* (rc), described in [Cat84] and later in [MP89], which we recall in Section 1. Its significance lies in guaranteeing that

$$\text{End}_{\mathcal{O}_D}(J_D) \hookrightarrow \text{Hom}_{\mathcal{O}_D}(J_D, \mathcal{O}_D)$$

is surjective; the 2-periodicity of an  $\mathcal{O}_D$ -free resolution then allows the identification of  $\text{coker}(K)$  with  $\text{Hom}_{\mathcal{O}_D}(J, \mathcal{O}_D)$  and hence with the ring  $\text{End}_{\mathcal{O}_D}(J_D)$ . This ring is naturally contained in  $Q(\mathcal{O}_D)$  with composition of endomorphisms corresponding to multiplication.

It is in principle possible to prove that the rank condition holds for  $K$  by explicit calculation in  $\mathcal{O}_B$ , but calculations are easier if we begin in the space on which the Coxeter group acts, and then deduce the result for  $K$ . We prove

**Theorem.** *Let  $\mathcal{A}$  be the reflection arrangement of a Coxeter group  $W$  in a vector space  $V$ , and let  $A$  be a Saito matrix for  $\mathcal{A}$ . Then  $A^t$  satisfies the rank condition, and, in consequence,  $\text{coker}(A^t)$  has a natural ring structure.*

Note that here it is the transpose of the Saito matrix whose cokernel is a ring. Saito matrices for the discriminants of Coxeter groups are symmetric, so this issue does not arise there. In Section 3 we use invariant theory due originally to Solomon to reduce (rc) for  $A^t$  to a condition relating the Gorenstein algebra structure of the fibre of  $p: V \rightarrow V/W$  over 0, and its  $W$ -module structure as the regular representation. This is a 3rd order partial differential condition on the basic invariants which we call the *Hessian rank condition* (hrc). In Theorem 3.12, we prove it in all cases except for  $E_6, E_7$  and  $E_8$ , where we conjecture it to be true.

In Section 4 we take the opposite route: beginning with the multiplicative structure on  $\Theta_B$  and  $\text{coker}(K)$  coming from Dubrovin's Frobenius structure, we endow both  $\Theta_V$  and  $\text{coker}(A)$  with a multiplication, and  $\Theta_V$  with a  $\Theta_B$ -module structure, whose crucial feature is that the derivative  $tp: \Theta_V \rightarrow \Theta_B \otimes_{\mathcal{O}_B} \mathcal{O}_V$  of  $p$  is  $\Theta_B$ -linear. From this, our (rc) follows in all cases. On  $\Theta_V$ , but not on  $\text{coker}(A^t)$ , this multiplication lacks a neutral element.

The space  $\text{Spec}(\text{coker}(K))$  is normal exactly in the ADE case; on the other hand  $\text{Spec}(\text{coker}(A^t))$  is normal only in the case of  $A_1$ . We discuss the geometry of these two spaces, and their link with the representation theory in Section 2.3.

In our final section we show that by adding to  $D$  a divisor which pulls back to the conductor of the ring extension  $\mathcal{O}_D \rightarrow \text{coker}(K)$ , we obtain a new free divisor (Theorem 5.5). This was already shown on the singularity side in [MS10]. The preimage in  $V$  of this free divisor is a free divisor containing the reflection arrangement (Corollary 5.6).

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## 1. RING STRUCTURES ON COKERNELS OF SQUARE MATRICES

**1.1. Fractional ideals.** Let  $R$  be a regular local or graded  $\mathbb{C}$ -algebra with maximal ideal  $\mathfrak{m}$ ,  $D = V(\Delta)$  a divisor on  $X = \text{Spec } R$ , and set  $R_D := R/R\Delta$ .

**Definition 1.1.** A *fractional ideal* (over  $R_D$ ) is a finitely generated  $R_D$ -submodule of the total ring of fractions  $Q(R_D)$  which contains a non-zero-divisor.

**Proposition 1.2** ([dJvS90, Prop. (1.7)]). *The duality functor  $-^\vee := \text{Hom}_{R_D}(-, R_D)$  preserves the class of fractional ideals, reverses inclusions, and is an involution on the class of maximal Cohen–Macaulay fractional ideals.*

**Notation 1.3.** Let  $\Lambda = (\Lambda_j^i)$  be an  $\ell \times \ell$  matrix with entries in  $R$  and  $\Delta = \det \Lambda \neq 0$ , and let  $(m_j^i) = \text{ad } \Lambda^t$  be its transpose adjoint, that is,  $m_j^i$  equals  $(-1)^{i+j}$  times the minor determinant of  $\Lambda$  obtained by deleting row  $i$  and column  $j$ . We denote by  $\Lambda'$  the matrix obtained from  $\Lambda$  by deleting its last row.

Let  $F^k(M)$  be the  $k$ th Fitting ideal of  $M = \text{coker } \Lambda$ , that is, the ideal in  $R$  generated by the  $(\ell - k) \times (\ell - k)$  minors of  $\Lambda$ . By  $g_1, \dots, g_\ell$  we denote the images in  $M$  of the standard basis elements of  $R^\ell$ . Write  $F_D^k(M) = F_{R_D}^k(M) = F^k(M)R_D$ .

By  $M$  being a ring, we will mean that it has a ring structure with respect to which  $R_D$ , embedded via  $r \mapsto r \cdot 1_M$ , is a subring.

Note that by Cramer’s rule we have

$$(1.1) \quad m_j^i g_k = m_j^k g_i.$$

In the following we recall the proof that when  $D$  is reduced then  $M$  is a fractional ideal. This requires some preparation on zero divisors in  $R_D$ .

**Lemma 1.4.** *An element  $r \in R$  is zero in  $R_D$  if  $r \in \Delta R_{\mathfrak{p}}$  for all minimal primes  $\mathfrak{p}$  over  $\Delta$ . In particular,  $r$  is not a zero-divisor in  $R_D$  if  $r \notin \mathfrak{p}R_{\mathfrak{p}}$  for all such primes.*

*Proof.* The ring  $R_D$  is Cohen–Macaulay and hence unmixed, that is, all associated primes of  $R_D$  are minimal. □

**Lemma 1.5.** *If  $D$  is reduced then  $F_D^1(M)$  contains a non-zero-divisor. In case  $M$  is a ring and  $g_\ell = 1_M$ , some linear combination  $U_\ell = \sum_j u_j m_j^\ell$  is not a zero-divisor in  $R_D$ .*

*Proof.* Let  $\mathfrak{p}$  define a prime divisor of  $D$ . By assumption  $R_{\mathfrak{p}}$  is regular and  $D$  reduced. So by Cohen’s structure theorem, the completion of  $R_{\mathfrak{p}}$  is a formal power series ring  $\widehat{R}_{\mathfrak{p}} = \mathbb{C}[[t]]$  with  $t = \Delta$ . Considering  $\Lambda$  as a map from  $X$  to matrix space  $\text{Spec } \mathbb{C}[X_j^i]$ , we can write  $\Delta = \det \circ \Lambda$ , from which we obtain by the chain rule

$$1 = \frac{\partial \Delta}{\partial t} = \sum_{i,j} \frac{\partial \det}{\partial X_j^i} \circ \Lambda \cdot \frac{\partial \Lambda_j^i}{\partial t} = \sum_{i,j} m_j^i \frac{\partial \Lambda_j^i}{\partial t}$$

So there is an  $m_j^i$  which is not in  $t\mathbb{C}[[t]] = \widehat{\Delta R}_{\mathfrak{p}}$  and hence not in  $\Delta R_{\mathfrak{p}}$ . Then the claim follows from Lemma 1.4 by taking a suitable linear combination of the  $m_j^i$ . If  $M$  is a ring and  $g_\ell = 1_M$ , the latter can be written using (1.1) as

$$\sum_{i,j} u_{i,j} m_j^i = \sum_{i,j} u_{i,j} g_i m_j^\ell = \sum_j u_j m_j^\ell = U_\ell$$

where  $u_j = \sum_i u_{i,j} g_i$ . □

**Lemma 1.6.** *Assume that  $D$  is reduced. Given  $u_1, \dots, u_\ell \in R_D$ , let  $U_k = \sum_j u_j m_j^k$  for  $k = 1, \dots, \ell$ . Suppose that for some  $k$ ,  $U_k$  is not a zero divisor in  $R_D$ . Then  $g_1, \dots, g_\ell$  have the same  $R_D$ -relations as  $U_1, \dots, U_\ell$  and as the columns  $(m_1^p, \dots, m_\ell^p)$  of  $\text{ad } \Lambda$ . If, moreover,  $M$  is a ring and  $g_\ell$  is its multiplicative neutral element, then  $U_\ell$  is not a zero divisor in  $R_D$ .*

*Proof.* Notice first the exactness of the 2-periodic sequence

$$(1.2) \quad \cdots \longrightarrow R_D^\ell \xrightarrow{\Lambda} R_D^\ell \xrightarrow{\text{ad } \Lambda} R_D^\ell \xrightarrow{\Lambda} \cdots$$

Generically on  $D$ ,  $\Lambda$  has rank  $\ell - 1$  by Lemmas 1.4 and 1.5, and hence  $\text{ad } \Lambda$  has rank 1. It follows using Lemma 1.4 that  $\sum_p \alpha_p U_p = 0$  in  $R_D$  if and only if  $\sum_p \alpha_p m_j^p = 0$  for all  $j$ . By the exactness of (1.2), this is equivalent to  $(\alpha_1, \dots, \alpha_\ell)^t$  being in the column space of  $\Lambda$ , and thus to  $\sum_k \alpha_k g_k = 0$ . This proves the first claim.

Now if  $M$  is a ring and  $g_\ell = 1_M$ , then equation (1.1) yields  $U_k = g_k U_\ell$ . The second claim follows.  $\square$

**Proposition 1.7.** *If  $D$  is reduced, then  $M$  is isomorphic to a fractional ideal between  $R_D$  and  $Q(R_D)$ .*

*Proof.* Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$  be the minimal primes over  $\Delta$ . By left and right-multiplication of  $\Lambda$  by invertible matrices (i.e. choosing a new set of generators of  $M$ , and a new set of generators for the relations among these generators), one can arrange that  $m_\ell^i \notin \mathfrak{p}_k$  for any  $k$ . By Lemma 1.4, this element is then a non-zero divisor in  $R_D$ . To see this, fix  $k$  and  $j$  and consider the set

$$\{(\alpha_1, \dots, \alpha_\ell) \in \mathbb{C}^\ell \mid \sum_i \alpha_i m_j^i \in \mathfrak{p}_k\}.$$

By Lemma 1.5, this set is algebraic and not equal to  $\mathbb{C}^\ell$  for some  $j(k)$ . As  $\mathbb{C}$  is infinite, there exist  $(\alpha_1, \dots, \alpha_\ell) \in \mathbb{C}^\ell$  such that for each  $k = 1, \dots, r$ , there is a  $j(k)$  such that

$$\sum_i \alpha_i m_{j(k)}^i \notin \mathfrak{p}_k.$$

Left-multiply  $\text{ad } \Lambda^t$  by some  $P \in \text{GL}_\ell(\mathbb{C})$  with last row  $(\alpha_1, \dots, \alpha_\ell)$  (this corresponds to left-multiplying  $\Lambda$  by a unit times  $(P^{-1})^t$ ). Now  $m_{j(k)}^\ell \notin \mathfrak{p}_k R_{\mathfrak{p}_k}$  for any  $k$ . The sets

$$\{(\beta^1, \dots, \beta^\ell) \in \mathbb{C}^\ell \mid \sum_j \beta^j m_j^\ell \in \mathfrak{p}_k\}$$

depending on  $k$  are then once again algebraic and not equal to  $\mathbb{C}^\ell$ . Thus there exists  $(\beta^1, \dots, \beta^\ell)$  such that  $\sum_j \beta^j m_j^\ell \notin \mathfrak{p}_k$  for all  $k = 1, \dots, r$ . As before, right-multiply  $\text{ad } \Lambda^t$  by  $Q \in \text{GL}_\ell(\mathbb{C})$  with last column  $(\beta^1, \dots, \beta^\ell)$ . Now  $m_\ell^\ell$  is not a zero divisor in  $R_D$ .

By Lemma 1.6,  $M$  embeds into  $Q(R_D)$  by sending  $g_k$  to  $m_\ell^k / m_\ell^\ell$  for  $k = 1, \dots, \ell$ . Since  $g_\ell$  is sent to  $1_{R_D}$ , the image of  $M$  contains a non zero divisor, and thus  $M$  is a fractional ideal as claimed.  $\square$

## 1.2. Rank condition.

**Definition 1.8.** We say that the *rank condition* (rc) holds for  $\Lambda$  if, possibly after left multiplication by some invertible matrix over  $R$ , the ideal  $F^1(M)$  is generated by the maximal minors of the matrix obtained from  $\Lambda$  by deleting one of its rows, and  $\text{grade}(F^1(M)) \geq 2$ .

Note (rc) implies that  $F_D^1(M)$  is a maximal Cohen–Macaulay  $\mathcal{O}_D$ -module, by the Hilbert–Burch theorem.

**Proposition 1.9.** *(rc) is a property of  $M$ .*

*Proof.* Let  $P$  be an invertible  $\ell \times \ell$  matrix with entries in  $R$ . Then

$$(\text{ad}(\Lambda P))^t = (\text{ad } \Lambda)^t (\text{ad } P)^t,$$

so if all entries in  $(\text{ad } \Lambda)^t$  are linear combinations of the entries in the last row then the same is true for  $(\text{ad}(\Lambda P))^t$ . Thus if (rc) holds for  $\Lambda$  then it holds for  $\Lambda P$ .

By linear algebra in  $\mathbb{C} = R/\mathfrak{m}$ , left and right multiplication by invertible matrices brings  $\Lambda$  to the form  $\begin{pmatrix} \Lambda_0 & 0 \\ 0 & I_r \end{pmatrix}$  where  $I_r$  is the  $r \times r$  identity matrix and all entries of  $\Lambda_0$  lie in  $\mathfrak{m}$ . Evidently (rc) holds for  $\Lambda$  if and only if it holds for  $\Lambda_0$ . So given a second square presentation matrix  $\Lambda'$  of  $M$ , we may assume that both  $\Lambda$  and  $\Lambda'$  have entries in  $\mathfrak{m}$  and are hence minimal presentations. Then by Nakayama's lemma  $\Lambda$  and  $\Lambda'$  have the same size, and there are invertible  $\ell \times \ell$  matrices  $P$  and  $Q$  such that  $\Lambda' = P\Lambda Q$ . It follows that (rc) holds for  $\Lambda$  if and only if it holds for  $\Lambda'$ .  $\square$

As the proof of Proposition 1.7 makes clear, many different embeddings of  $M \hookrightarrow Q(R_D)$  are possible. In case  $M$  has a multiplicative structure making it into a ring, there is no reason why any of these embeddings should be a multiplicative homomorphism. Nevertheless if  $\Lambda$  satisfies (rc), there is an embedding which achieves just this. We shall need the following

**Lemma 1.10** ([dJvS90, Prop. 1.10]). *Assume that the ideal  $I \subset R$  generated by the maximal minors of  $\Lambda'$  has codimension 2. Then the ideal  $I_D = IR_D$  has free resolution of the form*

$$(1.3) \quad 0 \longrightarrow R^\ell \xrightarrow{\Lambda^t} R^\ell \xrightarrow{(m_1^\ell, \dots, m_\ell^\ell)} I_D \longrightarrow 0.$$

*Proof.* Consider the following diagram:

$$(1.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & R^{\ell-1} & \xrightarrow{\Lambda^t} & R^\ell & \xrightarrow{(m_1^\ell, \dots, m_\ell^\ell)} & I & \longrightarrow & 0 \\ & & \downarrow & & \parallel & & \downarrow & & \\ 0 & \longrightarrow & R^\ell & \xrightarrow{\Lambda^t} & R^\ell & \xrightarrow{(m_1^\ell, \dots, m_\ell^\ell)} & I_D & \longrightarrow & 0 \end{array}$$

The upper row is exact by the Hilbert-Burch theorem; the lower row is evidently a complex, and exact except perhaps in the center. But expanding  $\Delta = \det \Lambda$  by its last row, shows that  $\Delta \in I$ , and thus generates the kernel of the projection  $I \twoheadrightarrow I_D$ . So the lower sequence in (1.4) is exact in the middle too.  $\square$

The following theorem show that, provided  $D$  is reduced, (rc) is equivalent to  $M$  being a ring.

**Theorem 1.11** ([MP89, Thm. 3.4, Prop. 3.14], [dJvS90, Lem. 1.6, Cor. 1.11.(iii), Thm. 1.12]). *Assume that  $D$  is reduced. Then (rc) for  $M$  implies that  $F_D^1(M)$  is a maximal Cohen–Macaulay fractional ideal with dual*

$$F_D^1(M)^\vee \cong \text{End}_{R_D}(F_D^1(M)) \cong M.$$

*In particular, composition of endomorphisms defines a multiplication in  $M$  making  $M$  into a subring of  $Q(R_D)$ . Conversely, (rc) holds if  $M$  is a ring.*

*Proof.* Suppose that  $F^1(M)$  is generated by the maximal minors of  $\Lambda'$ . Then Lemma 1.10 yields a presentation

$$(1.5) \quad 0 \longrightarrow R^\ell \xrightarrow{\Lambda^t} R^\ell \xrightarrow{(m_1^\ell, \dots, m_\ell^\ell)} F_D^1(M) \longrightarrow 0.$$

In particular,  $F_D^1(M)$  is a maximal Cohen–Macaulay  $R_D$ -module. It is a fractional ideal contained in  $R_D$  by Lemma 1.5, and hence  $F_D^1(M)^\vee$  is a fractional ideal containing  $R_D$  by Proposition 1.2.

Dualizing (1.5) with respect to  $R_D$  gives the exact sequence

$$0 \longrightarrow F_D^1(M)^\vee \longrightarrow R_D^\ell \xrightarrow{\Lambda} R_D^\ell$$

so  $F_D^1(M)^\vee \cong \ker_{R_D} \Lambda \cong \text{coker}_{R_D} \Lambda = M$  by exactness of (1.2). However  $\ker_{R_D} \Lambda = \text{im}_{R_D} \text{ad } \Lambda$ , again by exactness of (1.2). Thus  $F_D^1(M)^\vee$  is generated by the homomorphisms  $\varphi_1, \dots, \varphi_\ell$ , where

$$(1.6) \quad \varphi_i(m_j^\ell) = m_j^i, \quad j = 1, \dots, \ell.$$

All of these homomorphisms in fact map into  $F_D^1(M)$ , so  $F_D^1(M)^\vee \cong \text{End}_{R_D}(F_D^1(M))$ . So  $M$  is a ring as claimed. For the converse statement we refer to [MP89, Thm. 3.4].  $\square$

**1.3. Rings associated to free divisors.** We shall now assume that  $D = V(\Delta)$  is an Euler homogeneous free divisor in  $X = (\mathbb{C}^\ell, 0)$  with Saito matrix  $A$ . Then we have an exact sequence

$$(1.7) \quad 0 \longrightarrow R^\ell \xrightarrow{A} R^\ell \xrightarrow{(\Delta_1, \dots, \Delta_\ell)} R_D \longrightarrow R_D/J_D \longrightarrow 0,$$

where  $\Delta_j := \partial \Delta / \partial x_j$  for  $j = 1, \dots, \ell$ , and  $J_D := R_D J_\Delta$  is the Jacobian ideal of  $D$ . Note that  $\Delta \in J_\Delta$  by the Euler relation. By adding multiples of the Euler vector field  $\chi = \delta_1$  to the remaining members  $\delta_2, \dots, \delta_\ell$  of a Saito basis of  $D$ , we may assume that these annihilate  $\Delta$ . We shall assume that  $A$  is obtained from such a basis, and denote by  $A'$  the submatrix of coefficients of  $\delta_2, \dots, \delta_\ell$ . We say that  $D$  satisfies (rc) if (rc) holds for  $\Lambda = A^t$ . In this case, we write

$$\tilde{R}_D := M = \text{coker } \Lambda \subset Q(R_D)$$

for the ring of Theorem 1.11.

*Remark 1.12.* Free divisors arising as the discriminant of reflection arrangements (see Section 2.1 below), or as the discriminant in the base of a versal deformation of an isolated singularity of a holomorphic function, admit a symmetric Saito matrix. In such a case,  $M \cong J_D$  by (1.7).

**Lemma 1.13.** *We have  $m_j^1 = \frac{\Delta_j}{\deg \Delta}$  for  $j = 1, \dots, \ell$ .*

*Proof.* By Cramer's rule the logarithmic 1-form  $\omega = \omega_1 := \frac{1}{\Delta} \sum_j m_j^1 dx_j$  satisfies

$$\langle \omega, \delta_j \rangle = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{if } j = 2, \dots, \ell, \end{cases}$$

as does  $\omega = \frac{1}{\deg \Delta} \frac{d\Delta}{\Delta}$ . □

It is well known that for any algebraic or analytic space  $D$  satisfying Serre's condition S2, the fractional ideal  $\text{End}_{R_D}(J_D^\vee)$  is naturally contained in the integral closure of  $R_D$  in  $Q(R_D)$ , and the inclusion  $R_D \hookrightarrow \text{End}_{R_D}(J_D^\vee)$  gives a partial normalization (see for example [Vas98, Ch. 2, §2; Ch. 6, §2]. Grauert and Remmert showed in [GR71] (see also [GR84, Ch. 6, §5]) that for analytic spaces,  $R_D = \text{End}_{R_D}(J_D^\vee)$  precisely at the normal points of  $D$ , and the analogous result for algebraic spaces was shown by Vasconcelos in [Vas91].

**Proposition 1.14.** *If the free divisor  $D$  satisfies (rc) then  $\tilde{R}_D \cong \text{End}_{R_D}(J_D) \cong \text{End}_{R_D}(J_D^\vee)$ .*

*Proof.* The presentation (1.5) of  $F_D^1(M)$  coincides with that of  $J_D$  in (1.7); it follows that as  $R_D$ -modules the two ideals are isomorphic. Hence, by Theorem 1.11,

$$\tilde{R}_D = \text{End}_{R_D}(F_D^1(M)) \cong \text{End}_{R_D}(J_D).$$

Since  $D$  is free,  $J_D$  is maximal Cohen–Macaulay, and then reflexive by [dJvS90, Prop. (1.7) iii]. So dualizing induces an isomorphism  $\text{End}_{R_D}(J_D) \cong \text{End}_{R_D}(J_D^\vee)$ . □

*Remark 1.15.* The map  $\varphi_1 \in \text{End}_{R_D}(F_D^1(M))$  described in the proof of Theorem 1.11 gives an explicit isomorphism  $F_D^1(M) \cong J_D$ . Indeed,  $\varphi_1(m_j^1) = \frac{\Delta_j}{\deg \Delta}$  by Lemma 1.13.

However the example of the discriminant of the reflection group  $B_3$  shows that, even under the hypotheses of Proposition 1.14, it is not necessarily the case that the other generators  $\varphi_i$  of  $\text{End}_{R_D}(F_D^1(M))$ ,  $i = 2, \dots, \ell$  defined in (1.6) are isomorphisms onto their image.

A Saito matrix for the discriminant  $D$  of  $B_3$  is given by

$$A := \begin{pmatrix} x & -4x^2 + 18y & -xy + 27z \\ 2y & xy + 27z & -2y^2 + 18xz \\ 3z & 6xz & 6yz \end{pmatrix} = \Lambda^t.$$

One checks that (rc) holds for  $D$ . Indeed, the ideal of maximal minors of  $\Lambda$  with its third row deleted,

$$\langle x^2y - 4y^2 + 3xz, x^2z - 3yz, xyz - 9z^2 \rangle,$$

is equal to  $F^1(M)$ . On the other hand deleting the second row from  $\Lambda$  gives the ideal

$$\langle x^2z - 3yz, xyz - 9z^2 \rangle.$$

Evidently the two ideals are not isomorphic as  $R_D$ -modules.

In contrast, for irreducible free divisors we have

**Proposition 1.16.** *Assume that in addition to the hypotheses of Proposition 1.14,  $D$  is irreducible and is not isomorphic to the Cartesian product of a smooth space with a variety of dimension  $< \ell - 1$ . Then each of the maps  $\varphi_i$  in (1.6) is an isomorphism onto its image. Let  $I_i$  denote the ideal of maximal minors of  $\Lambda$  with its  $i$ th row deleted. Then, for each  $i = 1, \dots, \ell$ ,  $R/I_i = R_D/I_i R_D$  is a Cohen–Macaulay ring with support  $D_{\text{Sing}}$ .*

*Proof.* Because  $\Delta \in I_i$ , the  $(\ell - 1)$ -dimensional components of  $V(I_i)$  are among the components of  $D$ . Since  $\Delta$  is irreducible, the only component possible is  $D$  itself. But then because  $D$  is reduced, we would have  $I_i \subset \langle \Delta \rangle$ . This is absurd, for by hypothesis all entries of  $\Lambda$  lie in the maximal ideal, and  $\Delta = \sum_j \Lambda_j^i m_j^i$ . Thus  $V(I_i)$  is purely  $\ell - 2$ -dimensional. From this the result now follows by Lemma 1.10. □

Our Propositions 1.14 and 1.16 are closely related to [Vas98, Prop. 6.15]:

**Proposition 1.17.** *Suppose that that  $R_D/J_D$  is Cohen–Macaulay of codimension 2 (this is equivalent to  $D$  being a free divisor). Then*

$$(1.8) \quad J_D \cdot \text{Hom}_{R_D}(J_D, R_D) = F_D^1(M).$$

Here both ideals  $J_D$  and  $\text{Hom}_{R_D}(J_D, R_D)$  are viewed as fractional ideals in  $Q(R_D)$ .

The left hand side of (1.8) is the so-called *trace ideal* of  $J_D$ ; it is the set

$$\{\varphi(g) \mid \varphi \in \text{Hom}_{R_D}(J_D, R_D), g \in J_D\}.$$

Buchweitz, Ebeling and Graf von Bothmer give a criterion under which, for a free divisor  $D$  appearing as the discriminant in the base-space of a versal deformation of a singularity, the ring  $\text{End}_{R_D}(J_D)$  coincides with the normalization  $\bar{R}_D$  of  $R_D$ :

**Proposition 1.18** ([BEGvB09, Thm. 2.5, Rmk. 2.6]). *If  $D \subset S$  is the discriminant in the smooth base-space of a versal deformation  $f: X \rightarrow S$  and the module of  $f$ -liftable vector fields in  $\text{Der}_S$  is free, then provided  $\text{codim}_S f(X_{\text{Sing}}) \geq 2$ , this module coincides with  $\text{Der}(-\log D)$ . If in fact  $\text{codim}_S f(X_{\text{Sing}}) \geq 3$ , then  $\text{End}_{R_D}(J_D) = \bar{R}_D$ .*

## 2. RING STRUCTURES ASSOCIATED WITH COXETER GROUPS

**2.1. Review of Coxeter groups.** For more details on the material reviewed in this section, we refer to the book of Humphreys [Hum90]. Let  $V_{\mathbb{R}}$  be an  $\ell$ -dimensional  $\mathbb{R}$ -vector space and let  $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ . Consider a finite group  $W \subset \text{GL}(V)$  generated by reflections defined over  $\mathbb{R}$ , that is,  $W$  arises from  $W \subset \text{GL}(V_{\mathbb{R}})$  by extension of scalars. Any such representation  $W$  decomposes into a direct sum of irreducible representations, and  $W$  is irreducible if and only if the corresponding root system is so. The irreducible isomorphism types are  $A_k, B_k, D_k, E_6, E_7, E_8, F_4, G_2, H_3, H_4$ , and  $I_2(k)$ .

The group  $W$  acts naturally on the symmetric algebra  $S := \mathbb{C}[V]$  by the contragredient action, and we denote by  $R := S^W$  the corresponding graded ring of invariants. By a choice of linear basis, we identify  $S = \mathbb{C}[x_1, \dots, x_{\ell}]$ . The natural inclusion  $R \subset S$  turns  $S$  into a finite  $R$ -module of rank  $\#W$ . The averaging operator

$$(2.1) \quad \#: S \rightarrow R, \quad g \mapsto g^{\#} := \frac{1}{\#W} \sum_{w \in W} g^w$$

defines a section of this inclusion.

By Chevalley’s theorem ([Hum90, Thm. 3.5]),  $R$  is a polynomial algebra  $R = \mathbb{C}[p_1, \dots, p_{\ell}]$  where

$$(2.2) \quad p_1, \dots, p_{\ell} \in R$$

are homogeneous  $W$ -invariant polynomials. We set

$$(2.3) \quad \deg p_i = m_i + 1 = w_i$$

and assume that  $m_1 \leq \dots \leq m_{\ell}$ . Then the *degrees*  $w_i$ , or the *exponents*  $m_i$ , are uniquely determined and

$$(2.4) \quad \sum_{i=1}^{\ell} m_i = \#\mathcal{A}$$

where  $\mathcal{A}$  is the arrangement of reflection hyperplanes of  $W$  ([Hum90, Thm. 3.9]).

We make this more precise in the case  $W$  is irreducible. Then the eigenvalues of any Coxeter element are  $\exp(2\pi i \frac{m_i}{h})$  where  $h$  is the Coxeter number ([Hum90, Thm. 3.19]). Moreover,

$$(2.5) \quad 1 = m_1 < m_2 \leq \dots \leq m_{\ell-1} < m_{\ell} = h - 1,$$

$$(2.6) \quad m_i + m_{\ell-i+1} = h.$$

In particular, this implies that  $\sum_{i=1}^{\ell} m_i = \frac{\ell h}{2}$ . For  $m_1 = 1$ , the  $W$ -invariant 2-form  $p_1$  is unique up to a constant factor. By a choice of sign, it determines a unique  $W$ -invariant Euclidean inner product  $(\cdot, \cdot)$  on  $V$ , which turns  $W$  into a subgroup of  $\text{O}(V)$  and serves to identify  $V$  and  $V^*$ . With respect to dual bases of  $V$  and  $V^*$  we notice that the two corresponding inner products have mutually inverse matrices. At the level of  $V^*$ , we denote by

$$\Gamma := ((x_i, x_j)) = ((dx_i, dx_j))$$



the (symmetric) matrix of  $(\cdot, \cdot)$  with respect to coordinates  $x_1, \dots, x_\ell$ . In suitable coordinates

$$(2.7) \quad p_1 = \sum_{i=1}^{\ell} x_i^2, \quad (x, y) = \sum_{i=1}^{\ell} x_i y_i, \quad \Gamma = (\delta_{i,j}).$$

We refer to such coordinates as standard coordinates. In case  $W$  is reducible, we have the above situation on each of the irreducible summands separately.

**2.2. Reflection arrangement and discriminant.** Geometrically the finiteness of  $S$  over  $R$  means that the map

$$(2.8) \quad V = \text{Spec } S \xrightarrow{p} \text{Spec } R = V/W$$

is finite of degree  $\#W$ . We identify the reflection arrangement  $\mathcal{A}$  of  $W$  with its underlying variety  $\bigcup \mathcal{A} = \bigcup_{H \in \mathcal{A}} H$ . Let  $\Delta$  be a reduced defining equation for  $\mathcal{A}$ , and denote by  $D = p(\mathcal{A})$  the discriminant. An anti-invariant of  $W$  is a relative invariant  $f \in S$  with associated character  $\det^{-1}$ , that is,  $wf = \det^{-1}(w)f$  for all  $w \in W$ . The following crucial fact due to Solomon [Sol63, §3, Lem.] (see also ([Hum90, Prop. 3.13(b)])) implies that  $\Delta^2$  is a reduced defining equation for  $D$ .

**Theorem 2.1** (Solomon).  *$R\Delta$  is the set of all anti-invariants.* □

A second fundamental fact, due to K. Saito [Sai93, §3], is the following

**Theorem 2.2** (Saito). *For irreducible  $W$ ,  $\Delta^2$  is a monic polynomial in  $p_\ell$  of degree  $\ell$ , that is,*

$$\Delta^2 = \sum_{k=0}^{\ell} a_{\ell-k}(p_1, \dots, p_{\ell-1}) p_\ell^k, \quad \text{with } a_0 = 1. \quad \square$$

We denote by  $\text{Der}_S$  and  $\text{Der}_R$  the modules of vector fields on  $V = \text{Spec } S$  and  $V/W = \text{Spec } R$  respectively. The group  $W$  acts naturally on  $\text{Der}_S$ . Terao [Ter83] showed that each  $\theta \in \text{Der}(-\log D)$  has a unique lifting  $p^{-1}(\theta)$  to  $V$  and that the set of lifted vector fields is

$$p^{-1} \text{Der}(-\log D) = (\text{Der}_S)^W, \quad p^* \text{Der}(-\log D) = (\text{Der}_S)^W \otimes_R S = \text{Der}(-\log \mathcal{A}),$$

and both  $\mathcal{A}$  and  $D$  are free divisors. This can be seen as follows: We denote by

$$(2.9) \quad J := (\partial_{x_j}(p_i))$$

the Jacobian matrix of  $p$  in (2.8) with respect to the coordinates  $x_1, \dots, x_\ell$  and  $p_1, \dots, p_\ell$ . Via the identification of the 1-form  $dp_i$  with a vector field  $\eta_i$  such that  $(dp_i, v) = \langle \eta_i, v \rangle$ ,

$$(2.10) \quad \begin{aligned} dp_i &= \sum_{j=1}^{\ell} \partial_{x_j}(p_i) dx_j \leftrightarrow \eta_i = \sum_{j=1}^{\ell} \langle \eta_i, dx_j \rangle \partial_{x_j} = \sum_{j=1}^{\ell} (dp_i, dx_j) \partial_{x_j} \\ &= \sum_{k,j=1}^{\ell} \partial_{x_k}(p_i) (dx_k, dx_j) \partial_{x_j} = \sum_{k,j=1}^{\ell} \partial_{x_k}(p_i) (x_k, x_j) \partial_{x_j}, \end{aligned}$$

the basic invariants define invariant vector fields  $\eta_1, \dots, \eta_\ell \in (\text{Der}_S)^W$ , which must then be in  $\text{Der}(-\log \mathcal{A})$ . By (2.10), their Saito matrix reads

$$(2.11) \quad (\eta_j(x_i)) = \Gamma J^t$$

Now  $\det J$  is an anti-invariant because  $J$  is the differential of the invariant map  $p = (p_1, \dots, p_\ell)$ . Hence,  $\det J \in \mathbb{C}^* \Delta$  by Theorem 2.1, (2.4), and the algebraic independence of the  $p_i$ . By modifying  $\Delta$ , we can therefore assume that

$$(2.12) \quad \det J = \Delta.$$

Saito's criterion ([Sai80, ]) then shows that  $\mathcal{A}$  is free with basis  $\eta_1, \dots, \eta_\ell$ . Applying the tangent map  $tp$  gives vector fields  $\delta_1, \dots, \delta_\ell \in \text{Der}_R$  such that  $\delta_j \circ p = tp(\eta_j)$  with (symmetric) Saito matrix

$$(2.13) \quad K = (K_j^i) := (\delta_j(p_i)) = J\Gamma J^t$$

with  $\det(J\Gamma J^t) \in \mathbb{C}^* \Delta^2$ . At generic points of  $\mathcal{A}$ ,  $p$  is a fold map and hence

$$(2.14) \quad \delta_1, \dots, \delta_\ell \in \text{Der}(-\log D).$$

Again Saito's criterion shows that  $D$  is a free divisor with basis  $\delta_1, \dots, \delta_\ell$ . In standard coordinates as in (2.7), this proves

**Lemma 2.3.**  $D$  admits a symmetric Saito matrix  $K = JJ^t$ .

If  $W$  is irreducible then, in standard coordinates as in (2.7),

$$(2.15) \quad \chi_w := \frac{1}{2}\delta_1 = \sum_{i=1}^{\ell} w_i p_i \partial_{p_i}.$$

We shall refer to the grading and degrees defined by this semisimple operator as  $w$ -grading and  $w$ -degrees. In particular,  $\delta_k$  is  $w$ -homogeneous of degree  $w_k - w_1$ . If  $W$  is reducible, we have a homogeneity such as (2.15) for each irreducible summand.

**2.3. Rank conditions and associated rings.** From now on we work in standard coordinates as in (2.7). Let us abbreviate

$$S_{\mathcal{A}} := S/S\Delta, \quad R_D := R/R\Delta^2.$$

Denote by  $J_{\Delta} \subset S$  and  $J_{\Delta^2} \subset R$  the Jacobian ideals of  $\Delta$  and  $\Delta^2$  respectively, and define the Jacobian ideals

$$J_{\mathcal{A}} := J_{\Delta} S_{\mathcal{A}}, \quad J_D := J_{\Delta^2} R_D$$

of  $\mathcal{A}$  and of  $D$  respectively. Consider the corresponding 1st Fitting ideals

$$(2.16) \quad I_{\mathcal{A}} := F_S^1(J_{\mathcal{A}}), \quad \tilde{I}_{\mathcal{A}} := F_{S_{\mathcal{A}}}^1(J_{\mathcal{A}}) = I_{\mathcal{A}} \cdot S_{\mathcal{A}}, \quad I_D := F_R^1(J_D), \quad \tilde{I}_D := F_{R_D}^1(J_D) = I_D \cdot R_D.$$

By (2.7), (2.11) and (2.13), we have exact sequences

$$(2.17) \quad \begin{aligned} 0 &\longrightarrow S^{\ell} \xrightarrow{J^t} S^{\ell} \longrightarrow J_{\mathcal{A}} \longrightarrow 0, \\ 0 &\longrightarrow R^{\ell} \xrightarrow{K=JJ^t} R^{\ell} \longrightarrow J_D \longrightarrow 0. \end{aligned}$$

The above Fitting ideals  $I_{\mathcal{A}}$  and  $I_D$  are generated by the sub-maximal minors of  $J$  and  $K$  respectively. Being Saito matrices,  $J^t$  and  $K$  have rank  $\ell - 1$  at smooth points of  $\mathcal{A}$  and  $D$  respectively. Therefore  $I_{\mathcal{A}}$  and  $I_D$  are ideals of grade 2 and  $\tilde{I}_{\mathcal{A}}$  and  $\tilde{I}_D$  are ideals of grade 1.

Stronger versions of the rank condition (rc) from Definition 1.8 turn out to hold for  $\mathcal{A}$  and  $D$  (see Theorem 2.12).

**Lemma 2.4.** For irreducible  $W$ , the rank condition (rc) holds for  $\mathcal{A}$  if and only if  $I_{\mathcal{A}}$  is generated by its homogeneous part of minimal degree  $\sum_{i<\ell} m_i = \frac{h\ell}{2} - h + 1$ .

*Proof.* By (2.5), the part of  $I_{\mathcal{A}}$  of minimal degree is generated by the  $\ell$ th row of  $\text{ad}(J^t)$ , corresponding to the vector field  $\eta_{\ell}$  of highest degree. From this “if” follows immediately. The converse is less obvious, and we use a theorem of Solomon ([Sol64, Thm. 2, Cor. (2a)]) that the minors of  $J$  are linearly independent over  $\mathbb{C}$ . From this it follows that if  $I_{\mathcal{A}}$  is generated by  $\ell$  minors, then these must be the minors of lowest degree.  $\square$

**Definition 2.5.** For irreducible  $W$ , we refer to the condition defined in Lemma 2.4 as the *graded rank condition* (grc) for  $\mathcal{A}$ . Analogously, we say that the (grc) holds for  $D$  if  $I_D$  is generated by the  $\ell$ th row of  $\text{ad}(K)$ , corresponding to the vector field  $\delta_{\ell}$  of highest  $w$ -degree. For reducible  $W$ , we define (grc) for both  $\mathcal{A}$  and  $D$  by requiring it, as just defined, for each irreducible summand.

Evidently (grc) implies (rc) for  $D$ . In dimension  $\ell = 2$ , (grc) holds trivially for  $\mathcal{A}$  and  $D$ :  $I_{\mathcal{A}}$  and  $I_D$  are the graded maximal ideals of  $S_{\mathcal{A}}$  and  $R_D$ , due to the presence in each case of an Euler vector field. We shall look at this case in more detail in Section 2.6.

**Proposition 2.6.** For  $\ell = 2$ , (grc) holds for  $\mathcal{A}$  and  $D$ . In particular, this covers the case where  $W$  is a direct sum of types  $A_2$ ,  $B_2 = C_2$ ,  $G_2 = I_2(6)$ , or  $I_2(k)$ .  $\square$

The remaining types will be studied explicitly in Section 3. By [MP89, Prop. 3.14], (rc) for  $\mathcal{A}$  or  $D$  yields exact sequences

$$(2.18) \quad \begin{aligned} 0 &\longrightarrow S^{\ell} \xrightarrow{J^t} S^{\ell} \longrightarrow \tilde{I}_{\mathcal{A}} \longrightarrow 0, \\ 0 &\longrightarrow R^{\ell} \xrightarrow{K} R^{\ell} \longrightarrow \tilde{I}_D \longrightarrow 0, \end{aligned}$$

and the cokernels of the dual maps  $J \in \text{End}_S(S^{\ell})$ ,  $K^t = K \in \text{End}_R(R^{\ell})$  identify naturally with

$$(2.19) \quad \tilde{S}_{\mathcal{A}} = \text{End}_{S_{\mathcal{A}}}(\tilde{I}_{\mathcal{A}}), \quad \tilde{R}_D = \text{End}_{R_D}(\tilde{I}_D)$$

respectively. In particular, these cokernels carry natural ring structures, and we set

$$(2.20) \quad \tilde{\mathcal{A}} := \text{Spec } \tilde{S}_{\mathcal{A}}, \quad \tilde{D} := \text{Spec } \tilde{R}_D.$$

*Example 2.7.* Let  $\mathcal{A}$  be the reflection arrangement for  $W$  of type  $A_1 \times \cdots \times A_1$ , that is a normal crossing divisor defined by  $\Delta = x_1 \cdots x_\ell$ . Then

$$J = J^t = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_\ell \end{pmatrix}.$$

Obviously (grc) holds true for  $\mathcal{A}$ , and

$$\tilde{I}_{\mathcal{A}} = \langle x_2 \cdots x_\ell, \dots, x_1 \cdots \hat{x}_i \cdots x_\ell, \dots, x_1 \cdots x_{\ell-1} \rangle \subset S_{\mathcal{A}} = S/Sx_1 \cdots x_\ell.$$

One easily verifies that

$$\tilde{S}_{\mathcal{A}} = \text{End}_{S_{\mathcal{A}}}(\tilde{I}_{\mathcal{A}}) \cong \mathbb{C}[x_2, \dots, x_\ell] \oplus \cdots \oplus \mathbb{C}[x_1, \dots, \hat{x}_i, \dots, x_\ell] \oplus \cdots \oplus \mathbb{C}[x_1, \dots, x_{\ell-1}].$$

So  $\tilde{\mathcal{A}} = \text{Spec } \tilde{S}_{\mathcal{A}}$  is the normalization of  $\mathcal{A}$  in this example.

Generalizing this example we have

**Lemma 2.8.** *The assignments  $W \mapsto \tilde{S}_{\mathcal{A}}$  and  $W \mapsto \tilde{R}_D$  commute with direct sums (of representations/rings).*

*Proof.* Assume that  $W = W' \oplus W''$ , and use the analogous notation to refer to the above defined objects with  $W$  replaced by  $W'$  and  $W''$  respectively. Then  $S = S' \otimes_{\mathbb{C}} S''$ ,  $J$  is a block matrix with blocks  $J'$  and  $J''$ ,  $\Delta = \Delta' \Delta''$ , hence  $I_{\mathcal{A}} = I_{\mathcal{A}'} \Delta'' + I_{\mathcal{A}''} \Delta'$  and therefore

$$\tilde{I}_{\mathcal{A}} \cong \tilde{I}_{\mathcal{A}'} \otimes_{\mathbb{C}} S'' \oplus S' \otimes_{\mathbb{C}} \tilde{I}_{\mathcal{A}''}$$

by the following Lemma 2.9. Applying  $\text{End}_{S_{\mathcal{A}}}$  yields

$$\tilde{S}_{\mathcal{A}} = \tilde{S}_{\mathcal{A}'} \otimes_{\mathbb{C}} S'' \oplus S' \otimes_{\mathbb{C}} \tilde{S}_{\mathcal{A}''}.$$

This proves the claim for  $\mathcal{A}$ ; an analogous proof works for  $D$ .  $\square$

**Lemma 2.9.** *Let  $f \in K[x] = K[x_1, \dots, x_r] \supset I$ ,  $g \in K[y] = K[y_1, \dots, y_s] \supset J$ , and  $K[x, y] = K[x_1, \dots, x_r, y_1, \dots, y_s]$ . Then*

$$\begin{aligned} (Ig + Jf)(K[x, y]/\langle fg \rangle) &\cong I(K[x]/\langle f \rangle) \otimes_K K[y] \oplus K[x] \otimes_K J(K[y]/\langle g \rangle), \\ [Pg + Qf] &\leftrightarrow [P] \oplus [Q]. \end{aligned}$$

*Proof.* One easily verifies that the given correspondence is well-defined in both directions.  $\square$

**2.4. Relation between rings for  $\mathcal{A}$  and  $D$ .** Let us assume now that  $W$  is irreducible, and that (rc) holds for  $\mathcal{A}$ . Then, by [MP89, Prop. 3.14], the rings in (2.19) can be described more explicitly as follows. We denote by

$$(2.21) \quad (m_j^i) := \text{ad}(J^t), \quad (M_j^i) := \text{ad}(K) = \text{ad}(J^t) \text{ad}(J)$$

the adjoint matrices of  $J^t$  and  $K$  respectively, and set

$$(2.22) \quad D_k = \text{deg}(m_j^k) = \sum_i m_i - m_k,$$

which is independent of  $j$ . Then

$$(2.23) \quad \tilde{S}_{\mathcal{A}} = \langle h_1, \dots, h_\ell \rangle_{S_{\mathcal{A}}} = S_{\mathcal{A}}[h_1, \dots, h_{\ell-1}]$$

where  $h_i \in \tilde{S}_{\mathcal{A}}$  is (well-)defined by

$$(2.24) \quad h_i(m_j^\ell) = m_j^i \quad \text{for all } j = 1, \dots, \ell.$$

By (2.23) there is a multiplication table  $\lambda_{i,j}^k \in S$  such that

$$(2.25) \quad h_i h_j = \sum_{k=1}^{\ell} \lambda_{i,j}^k h_k.$$

Since the grade of  $I_{\mathcal{A}}$  is 1 and (rc) holds, there is a suitable linear combination of the  $m_j^\ell$  which is a non-zero divisor in  $S_{\mathcal{A}}$ . So we can consider  $h_i \in \mathbb{Q}(S_{\mathcal{A}})$ , where  $\mathbb{Q}$  stands for the total ring of fractions, and then write in  $\mathbb{Q}(S_{\mathcal{A}})$ :

$$(2.26) \quad h_i m_j^\ell = m_j^i.$$

**Proposition 2.10.** *If  $W$  is irreducible then*

$$h_i = \frac{\partial_{p_i}(\Delta^2)}{\partial_{p_\ell}(\Delta^2)} \in \mathbb{Q}(S_{\mathcal{A}})^W.$$

*Proof.* First, differentiate  $\Delta^2 \in R$ ,

$$2\Delta d\Delta = d(\Delta^2) = \sum_{k=1}^{\ell} \partial_{p_k}(\Delta^2) dp_k$$

considered as an equality in  $\Omega_S^1$ . Then wedging with  $dp_1 \wedge \cdots \wedge \widehat{dp_i} \wedge \cdots \wedge dp_{\ell-1}$  gives

$$(-1)^i \partial_{p_i}(\Delta^2) dp_1 \wedge \cdots \wedge dp_{\ell-1} \equiv (-1)^\ell \partial_{p_\ell}(\Delta^2) dp_1 \wedge \cdots \wedge \widehat{dp_i} \wedge \cdots \wedge dp_\ell \pmod{S\Delta}.$$

Taking coefficients with respect to  $dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_\ell$  yields

$$\partial_{p_i}(\Delta^2) m_j^\ell \equiv \partial_{p_\ell}(\Delta^2) m_j^i \pmod{S\Delta}, \quad j = 1, \dots, \ell.$$

By Theorem 2.2,  $\partial_{p_\ell}(\Delta^2)$  is a non-zero divisor in  $S_{\mathcal{A}}$ , and it follows that (2.26) holds with  $\partial_{p_i}(\Delta^2)/\partial_{p_\ell}(\Delta^2)$  in place of  $h_i$ . The proposition follows.  $\square$

**Proposition 2.11.** *The (graded) rank condition for  $\mathcal{A}$  implies the graded rank condition for  $D$ .*

*Proof.* We may assume that  $W$  is irreducible; by Lemma 2.4, we may further assume that (grc) holds for  $\mathcal{A}$ . Computing in  $S_{\mathcal{A}}$  using (2.21) and (2.25) and (grc) for  $\mathcal{A}$ ,

$$(2.27) \quad M_j^i = \sum_r m_r^i m_r^j = h_i h_j \sum_r (m_r^\ell)^2 = \sum_{k,r} \lambda_{i,j}^k h_k (m_r^\ell)^2 = \sum_{k,r} \lambda_{i,j}^k m_r^k m_r^\ell = \sum_k \lambda_{i,j}^k M_k^\ell,$$

and hence

$$M_j^i = \sum_k \lambda_{i,j}^k M_k^\ell + q\Delta$$

for some  $q \in S$ . Applying the averaging operator then gives

$$(2.28) \quad M_j^i = (M_j^i)^\# = \sum_k (\lambda_{i,j}^k)^\# M_k^\ell + (q\Delta)^\#.$$

By (2.1), we have

$$(2.29) \quad (q\Delta)^\# = \frac{1}{|W|} \sum_{w \in W} q^w \Delta^w = \frac{\Delta}{|W|} \sum_{w \in W} \det(w)^{-1} q^w,$$

using the anti-invariance of  $\Delta$ . Because this is invariant,  $\frac{1}{|W|} \sum_{w \in W} \det(w)^{-1} q^w$  must be anti-invariant and therefore an  $R$ -multiple of  $\Delta$  by Theorem 2.1. Then  $(q\Delta)^\# \in R\Delta^2$  by (2.29) and it follows from (2.28) that

$$M_j^i \equiv \sum_k (\lambda_{i,j}^k)^\# M_k^\ell \pmod{R\Delta^2}$$

and hence  $I_D = I'_D + R\Delta^2$  where  $I'_D = \langle M_1^\ell, \dots, M_\ell^\ell \rangle$ . But by (2.12),  $\Delta^2 \in I'_D$  and hence  $I_D = I'_D$  as claimed.  $\square$

**Theorem 2.12.** *The graded rank condition holds true for all Coxeter arrangements and their discriminants.*

*Proof.* For  $W$  irreducible of non- $E$ -type, this follows from Propositions 2.6 and 2.11, and the discussion in Section 3. In case  $W$  is of  $E$ -type, we invoke Theorem 1.11 and Theorem 4.5 from Section 4 combined with [Bri71, Slo80] and [MP89].  $\square$

By [MP89, Cor. 3.15], we therefore have

**Theorem 2.13.**  *$\mathcal{A}$  is Cohen–Macaulay and  $\tilde{D}$  is Gorenstein.*  $\square$

Using Theorem 2.1, one verifies that the averaging operator (2.1) induces a commutative diagram of  $R$ -modules

$$\begin{array}{ccccccc}
Q(R_D) & \hookrightarrow & Q(S_{\mathcal{A}}) & \xrightarrow{\#} & Q(S_{\mathcal{A}})^W & \xleftarrow{\cong} & Q(R_D) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\tilde{R}_D & \hookrightarrow & \tilde{S}_{\mathcal{A}} & \xrightarrow{\#} & \tilde{S}_{\mathcal{A}}^W & \xleftarrow{\cong} & \tilde{R}_D \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
R_D & \hookrightarrow & S_{\mathcal{A}} & \xrightarrow{\#} & (S_{\mathcal{A}})^W & \xleftarrow{\cong} & R_D
\end{array}$$

where the dashed maps results from the following

**Proposition 2.14.** For  $g_i \in \tilde{R}_D$  defined like  $h_i \in \tilde{S}_{\mathcal{A}}$  in (2.24), we have

$$(2.30) \quad h_i = g_i = \frac{M_{\ell}^i}{M_{\ell}^{\ell}} \in Q(R_D),$$

and hence

$$(2.31) \quad \tilde{R}_D = \langle g_1, \dots, g_{\ell} \rangle_{R_D} = R_D[g_1, \dots, g_{\ell-1}] = (\tilde{S}_{\mathcal{A}})^W.$$

*Proof.* As in (2.27), we compute in  $S_{\mathcal{A}}$  that

$$M_j^i = \sum_r m_r^i m_r^j = h_i \sum_r m_r^{\ell} m_r^j = h_i M_j^{\ell}.$$

By [MP89, Thm. 3.4],  $M_{\ell}^{\ell}$  defines the conductor of  $R_D \rightarrow \tilde{R}_D$  and is therefore a non-zero divisor. Then [MP89, Thm. 3.14] yields (2.30) and (2.31) follows.  $\square$

**2.5. Local trivialization.** The integral varieties of  $\text{Der}(-\log \mathcal{A})$  and  $\text{Der}(-\log D)$  form Saito's logarithmic stratification defined in [Sai80, §3], which we denote by  $L(\mathcal{A})$  and  $L(D)$  respectively. We shall locally trivialize  $\mathcal{A}$  and  $\tilde{D}$  along logarithmic strata with slices of the same type, with  $W$  replaced by the subgroup fixing the strata. In case of  $\mathcal{A}$  the trivialization is algebraic, in case of  $\tilde{D}$ , we need to work in the analytic category.

We begin with the discussion of  $\tilde{\mathcal{A}}$ . The logarithmic stratification  $L(\mathcal{A})$  coincides, up to taking the closure of strata, with the intersection lattice of  $\mathcal{A}$ . It is a geometric lattice (ordered by reverse inclusion) whose rank function is given by the codimension in  $V$ . By  $L_k(\mathcal{A}) \subset L(\mathcal{A})$ , we denote the collection of all rank  $k$  elements.

**Definition 2.15.** For  $X \in L(\mathcal{A})$ , denote by  $W_X$  the subgroup of  $W$  generated by reflections with reflecting hyperplanes in the localization  $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\}$  of  $\mathcal{A}$  along  $X \in L(\mathcal{A})$ , and by  $\Delta_X$  the reduced defining equation of  $\mathcal{A}_X$ . We denote also by  $I_X$  the defining ideal of  $X$  in  $S_{\mathcal{A}}$ .

By [Hum90, Thm. 1.12 (d)],  $W_X$  is the group fixing  $X$  point-wise, that is

$$W_X = \bigcap_{x \in X} W_x.$$

For  $x \in V$ , let  $X(x)$  be the stratum  $X \in L(\mathcal{A})$  with  $x \in X$ . It follows that

$$W_{X(x)} = W_x$$

is the isotropy group of  $x$ .

**Proposition 2.16.** Let  $X \in L(\mathcal{A})$  and  $Y = p(X) \in L(D)$ . Then  $(\tilde{S}_{\mathcal{A}})_{I_X} = (\tilde{S}_{\mathcal{A}_X})_{I_X} = \tilde{S}_{\mathcal{A}_X/X} \otimes_{\mathbb{C}} \mathbb{C}(X)$ . In particular,  $(\tilde{S}_{\mathcal{A}})_{I_X}^X = \tilde{S}_{\mathcal{A}_X/X}$  where  $X$  is considered as a translation group.

*Proof.* Fix  $X \in L(\mathcal{A})$  and let  $Y$  be an orthogonal complement. By  $\Delta_X \in \mathbb{C}[Y]$  we denote the defining equation of  $\mathcal{A}_X$ . Then, by the product rule,

$$(J_{\mathcal{A}})_{I_X} = J_{\Delta}(S_{I_X}/S_{I_X} \Delta) = J_{\Delta_X}(S_{I_X}/S_{I_X} \Delta_X) = (J_{\mathcal{A}_X})_{I_X}.$$

Localizing a presentation, such as (2.17), at  $I_X$ , therefore shows that

$$\begin{aligned}
(I_{\mathcal{A}})_{I_X} &= (F_S^1(J_{\mathcal{A}}))_{I_X} = F_{S_{I_X}}^1((J_{\mathcal{A}})_{I_X}) \\
&= F_{S_{I_X}}^1((J_{\mathcal{A}_X})_{I_X}) = (F_S^1(J_{\mathcal{A}_X}))_{I_X} = (I_{\mathcal{A}_X})_{I_X}.
\end{aligned}$$

Then we have also  $(\tilde{I}_{\mathcal{A}})_{I_X} = (\tilde{I}_{\mathcal{A}_X})_{I_X}$  and finally,

$$\begin{aligned} (\tilde{S}_{\mathcal{A}})_{I_X} &= (\text{End}_{S_{\mathcal{A}}}(I_{\mathcal{A}}))_{I_X} = \text{End}_{S_{I_X}}((I_{\mathcal{A}})_{I_X}) \\ &= \text{End}_{S_{I_X}}((I_{\mathcal{A}_X})_{I_X}) = (\text{End}_{S_{\mathcal{A}_X}}(I_{\mathcal{A}_X}))_{I_X} = (\tilde{S}_{\mathcal{A}_X})_{I_X}. \end{aligned}$$

This proves the first equality; the second follows since  $S_{I_X} = \mathbb{C}[Y] \otimes_{\mathbb{C}} \mathbb{C}(X)$ .  $\square$

**Corollary 2.17.** *The assignment  $\mathcal{A} \mapsto \tilde{S}_{\mathcal{A}}$  is a local functor.*  $\square$

We now turn our attention to  $\tilde{D}$ . The following result holds for any free divisor, and our proof is not specific to our situation.

**Proposition 2.18.** *The ideals  $I_{\mathcal{A}}$  and  $I_D$  are stable under  $\text{Der}(-\log \mathcal{A})$  and  $\text{Der}(-\log D)$  respectively. In particular, the latter act naturally on  $\tilde{S}_{\mathcal{A}}$  and  $\tilde{R}_D$  respectively.*

*Proof.* Let  $\omega_1, \dots, \omega_\ell \in \Omega^1(\log D)$  be the dual basis of (2.14). From

$$R \ni d\omega_j(\delta_k, \delta_r) = d\omega_j \left( \delta_k, \sum_i K_r^i \partial_{p_i} \right) = \sum_i K_r^i d\omega_j(\delta_k, \partial_{p_i}),$$

(2.12) and Cramer's rule, we conclude that

$$\begin{aligned} I_D \ni d\omega_j(\delta_k, \Delta^2 \partial_{p_i}) &= \delta_k \langle \Delta^2 \omega_j, \partial_{p_i} \rangle - \Delta^2 \partial_{p_i} \langle \omega_j, \delta_k \rangle - \langle \omega_j, [\delta_k, \Delta^2 \partial_{p_i}] \rangle \\ &= \delta_k (M_j^i) + \left\langle \Delta^2 \omega_j, [\partial_{p_i}, \delta_k] - \frac{\delta_k(\Delta^2)}{\Delta^2} \partial_{p_i} \right\rangle \\ &\equiv \delta_k (M_j^i) \pmod{I_D}. \end{aligned}$$

This proves the claim for  $D$ ; the same argument works for  $\mathcal{A}$  and any free divisor.  $\square$

*Remark 2.19.* There is a transcendental argument which shows that for any divisor  $D$ , free or not,  $\text{Der}(-\log D)$  preserves the ideal  $I_k(D)$  of  $k \times k$  minors of the matrix of coefficients of a set of generators of  $\text{Der}(-\log D)$ . It is simply that each of these ideals is invariant under biholomorphic automorphisms of  $D$ , since they are Fitting ideals of the Jacobian ideal  $J_D$ . The integral flow of any vector field  $\zeta \in \text{Der}(-\log D)$  preserves  $D$ , and hence  $I_k(D)$ , from which it follows that  $\zeta \cdot I_k(D) \subset I_k(D)$ .

We can make stronger statement in the analytic category. Let  $x \in X \in L(\mathcal{A})$  and  $y = p(x) \in p(X) = Y$ . By [Ori89, §2],  $Y \in L(D)$  and  $p: X \rightarrow Y$  is a covering. By finiteness of  $W$ , there is a (Euclidean)  $W_X$ -stable neighborhood of  $x$ , in which the  $W$ -orbits are exactly the  $W_X$ -orbits. Note that  $W_X$  commutes with the translation group  $X$ . This gives

$$p_x = p_{W_X, x} \times p|_X: V_x = (V/X)_x \times X_x \rightarrow ((V/X)/W_X)_y \times Y_y.$$

Since our definition of  $\tilde{R}_D$  in (2.16) and (2.19) is compatible with passing to the analytic category, we obtain the following analytic localization statement.

**Proposition 2.20.** *Let  $x \in X \in L(\mathcal{A})$  and  $y = p(x) \in p(X) = Y \in L(D)$ , and denote by  $D_Y$  the discriminant of  $W_X$  on  $V/X$ . Then there is an equality of analytic germs  $\tilde{D}_y = \tilde{D}_{Y, y} \times Y_y$ .*  $\square$

*Remark 2.21.* Saito [Sai80, (3.6)] showed that one can always analytically trivialize the logarithmic stratification along logarithmic strata as we do in Proposition 2.20.

**Corollary 2.22.**  *$\tilde{\mathcal{A}}$  is (algebraically) and  $\tilde{D}$  (analytically) constant along logarithmic strata.*  $\square$

By [Hum90, §1.8],  $W$  acts simply transitively on the (simple) root systems and on the Weyl chambers. Choosing a simple root system defining a Weyl chamber for which  $X = X(x)$  is a face, shows that the Dynkin diagram of any isotropy group  $W_x = W_X$  is obtained by dropping from the Dynkin diagram of  $W$  the roots which are not orthogonal to  $X$ . By [Hum90, Prop. 2.2], the resulting Dynkin diagram being disconnected means that  $W$  is locally reducible at  $x$ . This discussion combined with Propositions 2.16 and 2.20 proves

**Theorem 2.23.** *Up to smooth factors, the algebraic localizations of  $\tilde{\mathcal{A}}$ , or analytic localizations of  $\tilde{D}$ , are disjoint unions of spaces of the same type associated to reflection groups whose Dynkin diagrams are the sub-diagrams of the Dynkin diagram of  $W$ .*  $\square$

For two irreducible types  $T$  and  $T'$  of  $W$ , we call  $T'$  adjacent to  $T$  and write  $T \rightarrow T'$  if the Dynkin diagram of  $T'$  is contained in that of  $T$ .

**2.6. Relation with the normalization.** We shall denote the normalizations of  $\mathcal{A}$  and  $D$  by  $\tilde{\mathcal{A}}$  and  $\tilde{D}$  respectively.

**Proposition 2.24.** *We have  $S_{\mathcal{A}} \subseteq \tilde{S}_{\mathcal{A}} \subseteq \bar{S}_{\mathcal{A}}$  and  $R_D \subseteq \tilde{R}_D \subseteq \bar{R}_D$ .*

*Proof.* This follows from the finiteness of  $\tilde{S}_{\mathcal{A}}$  and  $\tilde{R}_D$  over  $S_{\mathcal{A}}$  and  $R_D$ , see (2.23), (2.26), (2.31), (2.30).  $\square$

In the following, we describe the cases of equality in Proposition 2.24. We begin with the case  $\ell = 2$  of plane curves for irreducible  $W$ . By (2.15) and for degree reasons, this case reduces to

$$(2.32) \quad K = \begin{pmatrix} 2p_1 & hp_2 \\ hp_2 & Q \end{pmatrix}, \quad Q = ap_1^r + bp_1^s p_2, \quad r = h - 1, \quad \frac{h}{2} - 1 = s,$$

$$(2.33) \quad \Delta^2 = |K| = 2p_1 Q - h^2 p_2^2 = 2ap_1^h + 2bp_1^{h/2} p_2 - h^2 p_2^2.$$

In particular,  $b = 0$  if  $h$  is odd. Note that there are no further restrictions imposed on  $a$  and  $b$  by the requirement

$$(2.34) \quad \delta_2(\Delta^2) \in R\Delta^2$$

for  $\delta_2$  from (2.13). Indeed,  $\langle \delta_1, \delta_2 \rangle_R$  is a Lie algebra, since  $[\delta_1, \delta_2] = (h - 2)\delta_2$  by homogeneity. For generic  $(a, b)$ ,  $\Delta^2$  in (2.33) is reduced, and hence (2.34) holds true by [Sai80, Lem. 1.9]. By continuity, it holds then also for special values of  $(a, b)$ .

In Section 2.6, we shall need the following

**Proposition 2.25.** *For  $\ell = 2$ , irreducible  $W$ , and odd  $h \geq 5$ ,  $\tilde{D} \neq \bar{D}$ .*

*Proof.* In this case,

$$(2.35) \quad K = \begin{pmatrix} 2p_1 & hp_2 \\ hp_2 & ap_1^r \end{pmatrix}$$

and (2.33) specializes to

$$\Delta^2 = |K| = 2ap_1^{r+1} - h^2 p_2^2 \equiv p_1^h - p_2^2.$$

The normalization of  $D$  is given by  $p_1 = t^2$  and  $p_2 = t^h$ , and hence  $g_1 = \frac{p_2}{p_1} = t^{h-2}$  by (2.30) and (2.35). Then (2.31) becomes

$$\tilde{R}_D = R_D[g_1] = \mathbb{C}[t^2, t^{h-2}] \subsetneq \mathbb{C}[t] = \bar{R}_D.$$

$\square$

Using Theorem 2.23 and Lemma 2.8 we find

**Corollary 2.26.** *If  $W$  contains any irreducible summand of type  $H_3$ ,  $H_4$ , or  $I_2(k)$  for odd  $k$ , then  $\tilde{D} \neq \bar{D}$ .*

*Proof.* For  $W$  of type  $I_2(k)$ , we have  $h = k$  and the claim follows from Proposition 2.25. For the  $H_k$ -types, the statement follows from Theorem 2.23 and the adjacency chain  $H_4 \rightarrow H_3 \rightarrow I_2(5)$ .  $\square$

We denote  $\mathbb{C} = S/\mathfrak{m}$  where  $\mathfrak{m}$  is the graded maximal ideal in  $S$ . Then  $\tilde{\mathcal{A}}_0 = \text{Spec}(\tilde{S}_{\mathcal{A}} \otimes_S \mathbb{C})$  is the fiber of  $\tilde{\mathcal{A}}$  over  $0 \in V$ .

**Lemma 2.27.** *The group  $W$  acts trivially on the fiber  $\tilde{\mathcal{A}}_0$  of  $\tilde{\mathcal{A}}$  over  $0 \in V$ , which contains exactly as many geometric points as the number of irreducible summands of  $W$ .*

*Proof.* By (2.23),  $\tilde{S}_{\mathcal{A}} \otimes_S \mathbb{C} \cong \mathbb{C}[h_1, \dots, h_{\ell-1}]$  and by (2.30) the  $h_i$  are  $W$ -invariants. This implies the first claim. For the second statement, we may assume that  $W$  is irreducible by Lemma 2.8. Then (2.5), (2.21), and (2.26) imply that  $h_i$  has  $w$ -degree  $w_{\ell} - w_i$ . So  $\mathbb{C}[h_1, \dots, h_{\ell-1}]$  is positively graded and hence  $\tilde{\mathcal{A}}$  is a cone. As it is also finite over  $0 \in V$  due to (2.23), it must be a single geometric point as claimed.  $\square$

We denote  $\mathbb{C}_x = S/\mathfrak{m}_x$  and  $\mathbb{C}_y = R/\mathfrak{m}_y$  where  $\mathfrak{m}_x$  and  $\mathfrak{m}_y$  are the maximal ideals of  $x$  and  $y$  respectively. Then  $\tilde{\mathcal{A}}_x = \text{Spec}(\tilde{S}_{\mathcal{A}} \otimes_S \mathbb{C}_x)$  and  $\tilde{D}_y = \text{Spec}(\tilde{R}_D \otimes_R \mathbb{C}_y)$  are the fibers of  $\tilde{\mathcal{A}}$  over  $x$  and of  $\tilde{D}$  over  $y$  respectively. Combining Propositions 2.16 and 2.20, (2.23), Proposition 2.14, and Lemma 2.27, we find

**Proposition 2.28.** *The fibers  $\tilde{\mathcal{A}}_x$  and  $\tilde{D}_y$ ,  $y = p(x)$ , coincide, that is,*

$$\tilde{S}_{\mathcal{A}} \otimes_S \mathbb{C}_x = \tilde{R}_D \otimes_R \mathbb{C}_y.$$

*They are trivial  $W_x$ -modules containing exactly as many geometric points as the number of irreducible summands of  $W_x$ .*

We can now refine Proposition 2.24 for  $\mathcal{A}$ .

**Corollary 2.29.**

- (1)  $\mathcal{A} = \tilde{\mathcal{A}}$  exactly if  $\mathcal{A}$  contains only one plane (or  $W$  has type  $A_1$ ).
- (2)  $\tilde{\mathcal{A}} = \bar{\mathcal{A}}$  exactly if  $\mathcal{A}$  is Boolean (or  $W$  has type  $A_1 \times \cdots \times A_1$ ).

*Proof.*

(1) If  $\#\mathcal{A} > 1$ , pick  $x$  with  $X(x) = X \in L_2(\mathcal{A}) \neq \emptyset$ . Then  $W_X$  is of type  $A_1 \times A_1$ . So by Proposition 2.28,  $\tilde{\mathcal{A}}$  has two points over  $x$ . The converse is Example 2.7 for  $\ell = 1$ .

(2) Again one implication is Example 2.7. If  $\mathcal{A}$  is not Boolean, then  $W$  has a non- $A_1$  type irreducible summand. By Lemma 2.27, its reflection hyperplanes do not separate in  $\tilde{\mathcal{A}}$ . □

The analogue of Corollary 2.29 for  $D$  is less trivial. From [Bri71, Slo80] and [MP89] we conclude the following using Lemma 2.8.

**Theorem 2.30.** *If all irreducible summands of  $W$  are of ADE-type, then  $\tilde{D} = \bar{D}$  is smooth.* □

The following criterion shows that the conclusion of Theorem 2.30 fails for other types.

**Proposition 2.31.** *If the Dynkin diagram of  $W$  contains that of an irreducible type  $T$  whose root system  $\Phi$  is not simply laced, then  $\tilde{D} \neq \bar{D}$ .*

*Proof.* By Theorem 2.23, there is a point  $x \in V$  such that  $W_x$  is of type  $T$ , and Proposition 2.20 reduces the claim to the case where  $W$  is (irreducible) of type  $T$ . The assumption on  $T$  implies that there are at least two  $W$ -orbits in  $\Phi$ , and hence in  $\mathcal{A}$ . Thus,  $D$  is reducible and Proposition 2.28 yields the claim. □

Using Lemma 2.8 we obtain

**Corollary 2.32.** *If  $W$  contains any irreducible summands of type  $B_k, C_k, F_4, G_2 = I_2(6)$ , or  $I_2(k)$  with  $k$  even, then  $\tilde{R}_D \subsetneq \bar{R}_D$ .*

Combining Corollary 2.26, Theorem 2.30, and Corollary 2.32, proves

**Theorem 2.33.**  *$\tilde{D} = \bar{D}$  exactly if all irreducible summands of  $W$  are of ADE-type. In this case,  $\tilde{D}$  is smooth.* □

### 3. DUAL AND HESSIAN RANK CONDITIONS

Let  $F = S \cdot \mathfrak{m}_R$  be the ideal of all positive-degree  $W$ -invariants. We can identify  $S/F$  as a direct summand  $T$  of the  $W$ -module  $S$ , and setting  $S^\alpha = Tp^\alpha$ , we have

$$(3.1) \quad S = \bigoplus_{\alpha \in \mathbb{N}^\ell} S^\alpha \supset \bigoplus_{0 \neq \alpha \in \mathbb{N}^\ell} S^\alpha = F$$

as a direct sum of  $W$ -modules, where  $p = p_1, \dots, p_\ell$ . Chevalley [Che55] showed that  $T$  is the regular  $W$ -representation (see also [Sol64, p. 278]). Consider the  $W$ -modules of exterior powers

$$E_p = \bigwedge^p V^*.$$

Solomon [Sol64, Thm. 2 and footnote (2)] showed that the  $W$ -modules

$$(3.2) \quad \begin{aligned} J^j &= \langle \partial_{x_k}(p_j) \mid k = 1, \dots, \ell \rangle_{\mathbb{C}}, \\ M^j &= \langle m_k^j \mid k = 1, \dots, \ell \rangle_{\mathbb{C}}, \quad j = 1, \dots, \ell, \end{aligned}$$

project onto the irreducible summands of the isotypic component of  $S/F$  of type  $E_1 \cong V^*$  and  $E_{\ell-1} \cong V \otimes \det V$  respectively. We may and will assume that  $J^j \subset T$  and  $M^j \subset T$ . By (2.22),  $D_j$  is the homogeneous degree of  $M^j$ , while  $m_j$  is the homogeneous degree of  $J^j$ .



Let us recall the construction from the proof of [Sol64, Thm. 2]: We denote by  $I(-)$  the  $W$ -invariant part. By [Sol63], the space of  $W$ -invariant differential forms on  $V$  is

$$I(S \otimes E_p) = \sum_{i_1 < \dots < i_p} R \cdot dp_{i_1} \wedge \dots \wedge dp_{i_p}.$$

Solomon [Sol64, p. 282] considers the case where  $W$  is the Weyl group of a Lie group acting on  $V$ ; then the Killing form induces a self-duality  $E_p \cong E_p^*$ . We are only interested in the cases  $p = 1$  and  $p = \ell - 1$ , where both irreducibility and self-duality of  $E_p$  are trivial<sup>1</sup>. The self-duality of  $E_p$  induces a  $W$ -isomorphism  $S/F \otimes E_p \cong \text{Hom}_{\mathbb{C}}(E_p, S/F)$  and hence an isomorphism

$$(3.3) \quad I(S/F \otimes E_p) \cong \text{Hom}_W(E_p, S/F).$$

The image of  $dp_i$  in  $\text{Hom}_W(E_p, S/F)$  has image  $J^i$ , and the image of  $dp_1 \wedge \dots \wedge \widehat{dp_i} \wedge \dots \wedge dp_\ell$  has image  $M^i$ .

Using (3.1),

$$(3.4) \quad \bigoplus_{j=1}^{\ell} \bigoplus_{\alpha \in \mathbb{N}^{\ell}} M^j p^{\alpha} \quad \text{and} \quad \bigoplus_{j=1}^{\ell} \bigoplus_{0 \neq \alpha \in \mathbb{N}^{\ell}} M^j p^{\alpha}$$

are the isotypic components of type  $E_{\ell-1}$  of  $S$  and  $F$  respectively. In particular, we have the following

**Lemma 3.1.** *The isotypic component of  $F$  of type  $E_{\ell-1}$  lies in  $F \cdot I_{\mathcal{A}}$ .*  $\square$

It follows that (grc) can be checked modulo  $F$ :

**Lemma 3.2.** *If  $M^j \subset S \cdot M^{\ell} + F$  for all  $j = 1, \dots, \ell - 1$  then the graded rank condition holds for  $\mathcal{A}$ .*

*Proof.* Consider the maps of  $W$ -modules

$$(3.5) \quad \phi_*: \text{Hom}_{\mathbb{C}}(M^j, M^{\ell} \otimes_{\mathbb{C}} S_{D_j - D_{\ell}}) \xrightarrow{\mu_*} \text{Hom}_{\mathbb{C}}(M^j, S_{D_j}) \xrightarrow{\pi_*} \text{Hom}_{\mathbb{C}}(M^j, T_{D_j})$$

induced by the composition of  $W$ -linear maps  $\phi = \pi \circ \mu$ , where

$$\mu: S \otimes_{\mathbb{C}} S \rightarrow S \quad \text{and} \quad \pi: S \rightarrow S/F = T$$

are the product in  $S$  and the canonical projection to  $T$ . By hypothesis, there is a  $\mathbb{C}$ -linear map  $\alpha \in \text{Hom}_{\mathbb{C}}(M^j, M^{\ell} \otimes_{\mathbb{C}} S_{D_j - D_{\ell}})$  such that  $\phi_*(\alpha) \in \text{Hom}_{\mathbb{C}}(M^j, M^j)$  is the identity map. Now averaging yields

$$\gamma = \alpha^{\#} \in \text{Hom}_W(M^j, M^{\ell} \otimes_{\mathbb{C}} S_{D_j - D_{\ell}}), \quad \phi_*(\gamma) = \text{id}_{M^j}.$$

Using Lemma 3.1, we find that

$$\mu_*(\gamma) - \text{id}_{M^j} \in \text{Hom}_W(M^j, F) = \text{Hom}_W(M^j, F \cdot I_{\mathcal{A}}).$$

This proves that

$$I_{\mathcal{A}} \subset S \cdot M^{\ell} + F \cdot I_{\mathcal{A}},$$

and hence  $I_{\mathcal{A}} = S \cdot M^{\ell}$  by Nakayama's lemma.  $\square$

By Solomon's result mentioned above, the  $W$ -equivariant Gorenstein pairing on  $S/F$  induces a non-degenerate pairing of the isotypic components of type  $E_1$  and  $E_{\ell-1}$  into the unique irreducible summand of type  $E_{\ell}$ ,

$$\bigoplus_{i=1}^{\ell} J^i \otimes \bigoplus_{j=1}^{\ell} M^j \rightarrow \mathbb{C}\Delta.$$

Since the element

$$\sum_{i=1}^{\ell} \partial_{x_i}(p_j) \otimes m_i^j \in J^j \otimes M^j$$

maps to  $\Delta = \det J$  by Laplace expansion of the determinant along the  $j$ th row, we obtain induced non-degenerate pairings

$$(3.6) \quad J^j \otimes M^j \rightarrow \mathbb{C}\Delta, \quad j = 1, \dots, \ell.$$

<sup>1</sup> $E_1 \cong V^*$  is self-dual due to the  $W$ -invariant form  $p_2$  on  $V$ , and hence irreducible, since  $V$  is irreducible. Because  $\det(V)^{\otimes 2} \cong \mathbb{C}$  is the trivial representation,  $E_{\ell-1} \cong E_1^* \otimes E_{\ell} \cong V \otimes \det(V)$  is self-dual. For the same reason and irreducibility of  $V$ ,  $I(V \otimes \det(V) \otimes (V \otimes \det(V))^*) = I(V \otimes V^*) = 1$ , and hence  $E_{\ell-1}$  is irreducible.

For  $j < k$ , we have

$$(3.7) \quad \begin{aligned} \mathrm{Hom}_W(J^j, J^k) &\cong \mathrm{End}_W(E_1) \cong \mathrm{End}_W(E_{\ell-1}^* \otimes E_\ell) \\ &\cong \mathrm{End}_W(E_{\ell-1}^*) \cong \mathrm{Hom}_W(M^k, M^j), \end{aligned}$$

where  $\mu_*(\alpha) \in \mathrm{Hom}_W(J^j, J^k)$  induced by  $\alpha \in \mathrm{Hom}_W(J^j, J^j \otimes S_{m_k - m_j})$  corresponds to  $\mu_*(\beta) \in \mathrm{Hom}_W(M^k, M^j)$  induced by  $\beta = \alpha^t \in \mathrm{Hom}_W(M^k, M^k \otimes S_{D_j - D_k})$ . Because of the non-degenerate  $W$ -pairing (3.6),  $\mu_*(\alpha)$  is an isomorphism exactly if  $\mu_*(\beta)$  is one.

**Definition 3.3.** We say that the *dual (graded) rank condition* (drc) holds for  $\mathcal{A}$  if  $J^\ell \subset S \cdot J^j + F$  for all  $j = 1, \dots, \ell - 1$ .

*Remark 3.4.* The definition of (drc) is given as an equality in  $S/F$  because in general  $J^\ell \not\subset S \cdot J^j$ , though the inclusion holds trivially for  $j = 1$ . Nakayama's Lemma does not imply this stronger inclusion because unlike (grc), (drc) does not assert an inclusion of  $R$ -modules.

Now combining the preceding arguments with Lemma 3.2 proves the following

**Lemma 3.5.** *The dual and graded rank conditions are equivalent for  $\mathcal{A}$ .*

*Proof.* By the symmetry of (3.7), both implications can be proved in the same way; we show that (drc) implies (grc) for  $\mathcal{A}$ : Fix  $j \in \{1, \dots, \ell - 1\}$ . By (drc) for  $\mathcal{A}$ , we have an  $\alpha \in \mathrm{Hom}_\mathbb{C}(J^\ell, J^j \otimes S_{m_\ell - m_j})$  inducing the identity map  $\mathrm{id}_{J^\ell} = \pi_* \mu_*(\alpha) \in \mathrm{Hom}_\mathbb{C}(J^\ell, J^\ell)$ . Applying the construction from the proof of Lemma 3.2, we can turn  $\alpha$  into a  $W$ -homomorphism. The homomorphism  $\mu_*(\alpha)$  is non zero modulo  $F$  and (3.7) yields a corresponding dual map  $\mu_*(\alpha^t) \in \mathrm{Hom}_W(M^j, S_{D_j})$  induced by  $\alpha^t \in \mathrm{Hom}_W(M^j, M^\ell \otimes S_{D_j - D_\ell})$ . Thus, (grc) holds for  $\mathcal{A}$  by Lemma 3.2.  $\square$

The following property refines (grc) by a statement about the  $S$ -coefficients of  $J^j$  in the condition in Definition 3.3. By [OS88, (2.14) Lem.], the Hessian

$$\mathrm{Hess}(p): \mathrm{Der}_S \rightarrow \Omega_S^1, \quad \mathrm{Hess}(p)(\delta) = \sum_{i=1}^{\ell} \delta(\partial_{x_i}(p)) dx_i,$$

is  $W$ -equivariant for  $p \in R$ . Note that  $\mathrm{Hess}(p_1)$  is a  $W$ -isomorphism which induces our identification of  $dp_i$  with a vector field  $\eta_i$  in (2.10). By abuse of notation, we identify

$$\mathrm{Hess}(p) = \mathrm{Hess}(p) \circ \mathrm{Hess}(p_1)^{-1} \in \mathrm{End}_W(\Omega_S^1)$$

for  $p \in R$ . Using  $\Omega_S^1 = S \otimes E_1$  and passing to the quotient by  $F$ ,  $\mathrm{Hess}(p)$  then induces an element of  $\mathrm{End}_W(S/F \otimes E_1)$  and hence of  $\mathrm{End}_W(I(S/F \otimes E_1))$ . By (3.3),  $\mathrm{Hess}(p)$  thus induces a map

$$\hbar(p) \in \mathrm{End}_W(\mathrm{Hom}_W(E_1, S/F))$$

which operates on  $W$ -submodules of type  $V^*$  by passing to the image in  $\mathrm{Hom}_W(E_1, S/F)$ .

**Definition 3.6.** We say that the *Hessian (dual graded) ring condition* (Hrc) holds for  $\mathcal{A}$  if, for any  $j$ , there is an  $i$ , such that  $m_i + m_j = w_\ell$  and  $\mathrm{Hess}(p_i)(\eta_j) \notin F\Omega_S^1$ . In case  $m_1, \dots, m_\ell$  are pairwise different, this means that  $\mathrm{Hess}(p_i)(\eta_{\ell-i+1}) \notin F\Omega_S^1$ .

It is clear that  $\mathrm{Hess}(p_i)(\eta_1) = dp_i$ , so (Hrc) holds trivially in dimension  $\ell = 2$ .

**Lemma 3.7.** *The Hessian rank condition implies the dual ring condition for  $\mathcal{A}$ .*

*Proof.* (Hrc) means that  $\hbar(p_i)(J^j) \subset (S/F)_{m_\ell}$  is non-zero. By  $W$ -equivariance of  $\hbar(p_i)$ , the latter is then a non-trivial  $W$ -submodule of  $(S/F)_{m_\ell}$  of type  $E_1$ . Then it must coincide with  $J_\ell$ , which is the only such  $W$ -module in this degree by (2.5).  $\square$

Consider the symmetric group  $W$  acting on  $V' = \mathbb{C}^{\ell+1}$  by permuting coordinates  $x_0, \dots, x_\ell$ . Then the subspace  $V$  of  $V'$  defined by  $p_1$  can be identified with  $V = V'/L$ , where  $L$  is the line centralized by  $W$ , and  $W$  is of type  $A_\ell$  on  $V$ ; it is generated by the orthogonal reflections along the hyperplanes of the reflection arrangement  $\mathcal{A}$  defined by

$$\Delta = \prod_{0 \leq i < j \leq \ell} (x_i - x_j).$$

**Proposition 3.8.** *The Hessian rank condition holds for  $\mathcal{A}$  if  $W$  is of type  $A_\ell$ .*

*Proof.* By [Hum90, §3.12], the basic invariants for  $W$  on  $V'$  can be chosen as power sums

$$p_k = \frac{1}{k}(x_0^k + \cdots + x_\ell^k), \quad k = 1 \dots \ell.$$

So (Hrc) holds on  $V'$  “on the nose”, in the sense that  $\text{Hess}(p_i)(p_{\ell-i+1}) = J_\ell$  in  $S$  and not only in  $S/F$ . Indeed, we have the matrix equality

$$D(p_{\ell-i+1}) \circ \text{Hess}(p_i) = (i-1) \cdot D(p_\ell).$$

This equality is obviously invariant under orthogonal coordinate changes. Therefore (Hrc) holds also on  $V$ .  $\square$

In the case  $W$  is of type  $B_\ell$ ,  $W$  is generated by the orthogonal reflections along the hyperplanes of the reflection arrangement  $\mathcal{A}$  defined by

$$\Delta = x_1 \cdots x_\ell \prod_{1 \leq i < j \leq \ell} (x_i^2 - x_j^2).$$

**Proposition 3.9.** *The Hessian rank condition holds for  $\mathcal{A}$  if  $W$  is of type  $B_\ell$ .*

*Proof.* By [Hum90, §3.12], the basic invariants can be chosen as the power sums

$$p_k = \frac{1}{2k}(x_1^{2k} + \cdots + x_\ell^{2k}), \quad k = 1, \dots, \ell.$$

So (Hrc) holds again by the argument of Proposition 3.8.  $\square$

In the case  $W$  is of type  $D_\ell$ ,  $W$  is generated by the orthogonal reflections in the hyperplanes of the reflection arrangement  $\mathcal{A}$  defined by

$$\Delta = x_1 \cdots x_\ell \prod_{1 \leq i < j \leq \ell} (x_i \pm x_j)$$

It is a subgroup of the reflection group of type  $B_\ell$ .

**Proposition 3.10.** *The Hessian rank condition holds for  $\mathcal{A}$  if  $W$  is of type  $D_\ell$ .*

*Proof.* By [Hum90, §3.12], the basic invariants can be chosen as the power sums

$$p_k = \frac{1}{2k}(x_1^{2k} + \cdots + x_\ell^{2k}), \quad k = 1, \dots, \ell - 1,$$

together with  $p_\ell = x_1 \cdots x_\ell$ . Note that here our notation has changed slightly, and the highest weight invariant is now  $p_{\ell-1}$  rather than  $p_\ell$ . It is easy to check that for  $i = 1, \dots, \ell - 2$ ,  $D(p_i) \circ \text{Hess}(p_{\ell-i}) \equiv D(p_{\ell-1}) \pmod{\mathbb{C}^*}$ . Now consider the (evidently invariant) polynomial

$$\hat{p}_{\ell-1}(x_1, \dots, x_\ell) = D(p_\ell) \cdot D(p_\ell) = \sum_{j=1}^{\ell} x_1^2 \cdots \widehat{x_j^2} \cdots x_\ell^2 \in R,$$

whose degree is equal to that of  $p_{\ell-1}$ . We claim that  $p_{\ell-1} \equiv \hat{p}_{\ell-1} \pmod{F^2 + \mathbb{C}^*}$ . The evident equality

$$D(p_\ell) \circ \text{Hess}(p_\ell) = \frac{1}{2}D(\hat{p}_{\ell-1})$$

then implies that a similar equality holds, modulo  $F$ , with  $p_{\ell-1}$  in place of  $\hat{p}_{\ell-1}$ , completing the proof of (Hrc).

It remains only to verify the claim. Let  $\rho$  be a primitive  $2(\ell - 1)$ th root of unity and set  $a = (\rho, \rho^2, \dots, \rho^{\ell-1}, 0)$ . Then all of our basic invariants except for  $p_{\ell-1}$  vanish at  $a$ , while  $\hat{p}_{\ell-1}(a) \neq 0 \neq p_{\ell-1}(a)$ . Since  $\hat{p}_{\ell-1}$  is an invariant of the same degree as  $p_{\ell-1}$  (and this degree is greater than the degrees of the other basic invariants), the claim follows.  $\square$

**Proposition 3.11.** *The Hessian rank condition holds for  $\mathcal{A}$  if  $W$  is of type  $F_4$ ,  $H_3$  or  $H_4$ .*

*Proof.* In these cases, Macaulay2 [GS] calculations shows that (Hrc) holds for  $\mathcal{A}$ . Our calculations are based on the formulæ for the basic invariants from [Meh88].  $\square$

**Theorem 3.12.** *The Hessian rank condition holds for  $\mathcal{A}$  if  $W$  is not of type  $E_6$ ,  $E_7$ , or  $E_8$ .*

**Conjecture 3.13.** *The Hessian rank condition holds for  $\mathcal{A}$  if  $W$  is of type  $E_6$ ,  $E_7$ , or  $E_8$ .*

#### 4. F-MANIFOLDS

In this section we prove that (rc) holds for  $D$  and  $\mathcal{A}$  for all Coxeter groups. We will make use of the Frobenius manifold structure on  $V/W$ , constructed by Dubrovin in [Dub98]. However our main reference for background on Frobenius manifolds (including this result) is the book of Hertling [Her02]. In fact the only aspects of the Frobenius structure we use are the existence of an integrable structure of commutative associative  $\mathbb{C}$ -algebras on the fibers of the tangent bundle; a manifold with this structure is called by Hertling and Manin an *F-manifold*. The usage of such analytic methods will be justified in Remark 4.6, and we pass to the analytic category without changing our notation.

For any  $n$ -dimensional F-manifold  $M$ , the multiplication on  $TM$  is encoded by an  $n$ -dimensional subvariety of  $T^*M$ , the *analytic spectrum*  $L$ , as follows: for each point  $p \in M$ , points in  $T_p^*M$  determine  $\mathbb{C}$ -linear maps  $T_p M \rightarrow \mathbb{C}$ ; among these, a finite number are  $\mathbb{C}$ -algebra homomorphisms. These finitely many points in each fiber of  $T^*M$  piece together to form  $L$ . Thus the composite

$$(4.1) \quad \text{Der}_M \rightarrow \pi_* \mathcal{O}_{T^*M} \rightarrow \pi_* \mathcal{O}_L$$

is an isomorphism of rings. The multiplication  $\circ$  in  $TM$  satisfies the integrability property

$$\text{Lie}_{X \circ Y}(\circ) = X \circ \text{Lie}_Y(\circ) + Y \circ \text{Lie}_X(\circ).$$

Provided the multiplication is generically semi-simple, as is the case for the structure constructed by Dubrovin and Hertling, this implies that  $L$  is Lagrangian. This in turn means that the restriction to  $L$  of the canonical action form  $\alpha$  on  $T^*M$  is closed and therefore exact. A *generating function* for  $L$  is any function  $F \in \mathcal{O}_L$  such that  $dF = \alpha|_L$ . A generating function determines an *Euler field*  $E$  on  $M$ , namely a vector field mapped to  $F$  by the isomorphism (4.1). The discriminant of  $M$  is defined by any of the following equivalent characterizations:

- (1)  $D = \pi(F^{-1}(0))$ ,
- (2)  $D$  is the set of points  $x \in M$  where the endomorphism  $E \circ : T_x M \rightarrow T_x M$  is not invertible.

Similarly, the module  $\text{Der}(-\log D)$  may be viewed as either

- (1) the set of vector fields whose image under the isomorphism (4.1) vanishes on  $F^{-1}(0)$ , or equivalently as
- (2) the image in  $\text{Der}_M$  of multiplication by  $E$ .

This yields

**Lemma 4.1.** *The cokernel  $\tilde{R}_D = \text{coker } K$  of the Saito matrix  $K$  of  $D$  acquires a ring structure as quotient of the Frobenius manifold multiplication in  $\text{Der}_R$ .*

*Proof.* The matrix of multiplication by  $E$  with respect to the basis  $\partial_{x_1}, \dots, \partial_{x_\ell}$  of  $\text{Der}_R$  is  $K$ . Thus

$$(4.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & R^\ell & \xrightarrow{K} & R^\ell & \longrightarrow & \tilde{R}_D \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \text{Der}_R & \xrightarrow{E \circ} & \text{Der}_R & \longrightarrow & \text{Der}_R / \text{Der}_R(-\log D) \longrightarrow 0 \end{array}$$

is a presentation of  $\text{Der}_R / E \circ \text{Der}_R = \text{Der}_R / \text{Der}_R(-\log D)$ , which is itself isomorphic to  $\pi_* \mathcal{O}_{F^{-1}(0)}$ .  $\square$

Recall from (2.9) that  $J: S^\ell \rightarrow S^\ell$  is the matrix of the morphism

$$(4.3) \quad tp: \text{Der}_S \rightarrow p^* \text{Der}_R = \text{Der}_R \otimes S, \quad tp\left(\sum_{j=1}^n \eta_j \partial_{x_j}\right) = \sum_{i=1}^n \sum_{j=1}^n \eta_j \partial_{x_j}(p_i) \partial_{p_i},$$

defined by left composition (of vector fields as sections of  $TV$ ) with  $dp$ . Both  $tp$  and  $\omega p: \text{Der}_R \rightarrow p^* \text{Der}_R$ , defined by right composition with  $p$ , are familiar in singularity theory. By definition, we have

$$(4.4) \quad \chi \in \text{Der}_R \text{ has lift } \eta \in \text{Der}_S \iff tp(\eta) = \omega p(\chi).$$

Using Lemma 2.3, (4.2), and the obvious identifications, there is a commutative diagram of  $S$ -modules

$$(4.5) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & J_\Delta & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \text{Der}_S & \xrightarrow{tp} & \text{Der}_R \otimes_R S & \longrightarrow & \tilde{S}_\mathcal{A} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & J^t & & = & & \\ 0 & \longrightarrow & R^\ell \otimes_R S & \xrightarrow{K \otimes 1} & \text{Der}_R \otimes_R S & \longrightarrow & \tilde{R}_D \otimes_R S \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & L \\ & & & & & & \uparrow \\ & & & & & & 0 \end{array}$$

in which  $L$  is simply defined as a kernel. Both rows here are exact: the upper row defines  $\tilde{S}_\mathcal{A}$ , and the lower row is the tensor product with the flat  $R$ -module  $S$  of the short exact sequence defining  $\tilde{R}_D$ . Now  $\tilde{R}_D \otimes_R S$ , as a tensor product of rings, has a natural ring structure; to show that  $\tilde{S}_\mathcal{A}$  is a ring, it will be enough to show that  $L$  is an ideal of  $\tilde{R}_D \otimes_R S$ . Although, by the Snake Lemma,  $L \cong J_\Delta$  and is thus an  $S$ -submodule of  $\tilde{R}_D \otimes_R S$ , this is not enough for our purposes here. Nevertheless

**Lemma 4.2.**  *$L$  is an ideal of  $\tilde{R}_D \otimes_R S$ .*

The proof of Lemma 4.2 requires some preparations. Before beginning these, we clarify the relationship between  $\tilde{S}_\mathcal{A}$  and  $\tilde{R}_D \otimes_R S_\mathcal{A}$ . In general they are not isomorphic, and the space  $\text{Spec } \tilde{S}_\mathcal{A}$  is not the fiber product  $\text{Spec}(\tilde{R}_D \times_D \mathcal{A})$ . For  $\tilde{R}_D \otimes_R S_\mathcal{A}$  is the cokernel of  $1 \otimes \Delta: \tilde{R}_D \otimes_R S \rightarrow \tilde{R}_D \otimes_R S$ , and using the epimorphism  $\text{Der}_R \otimes_R S \rightarrow \tilde{R}_D \otimes_R S$  we find that there is an epimorphism  $\text{Der}_R \otimes_R S \rightarrow \tilde{R}_D \otimes_R S_\mathcal{A}$ , whose kernel is equal to  $\text{Der}_R \otimes_R S \Delta + \text{Der}(-\log D) \otimes_R S$ . Both summands here are contained in the image of  $tp: \text{Der}_S \rightarrow \text{Der}_R \otimes_R S$ , the first by Cramer's rule and the second because every vector field  $\eta \in \text{Der}(-\log D)$  is liftable via  $p$ . Thus  $\tilde{S}_\mathcal{A}$  is a quotient of  $\tilde{R}_D \otimes_R S_\mathcal{A}$ . The kernel of the projection  $\tilde{R}_D \otimes_R S_\mathcal{A} \rightarrow \tilde{S}_\mathcal{A}$  is the quotient

$$\ker := tp(\text{Der}_S) / (\text{Der}(-\log D) \otimes_R S + \text{Der}_R \otimes_R S \Delta).$$

At a generic point  $x \in \mathcal{A}$  this vanishes: here  $p$  is a fold map, right-left-equivalent to

$$(x_1, \dots, x_\ell) \mapsto (x_1, \dots, x_{\ell-1}, x_\ell^2)$$

and an easy local calculation shows that in this case  $\ker_x = 0$ . However, if  $p$  has multiplicity  $> 2$  at  $x$  then  $\ker_x \neq 0$ . For example at an  $A_2$  point,  $p$  is right-left equivalent to

$$(x_1, \dots, x_\ell) \mapsto (x_1^2 + x_1 x_2 + x_2^2, x_1 x_2 (x_1 + x_2), x_3, \dots, x_\ell);$$

$tp(\text{Der}_S)$  is generated by  $\partial_{p_3}, \dots, \partial_{p_\ell}$  together with

$$(2x_1 + x_2)\partial_{p_1} + (2x_1 x_2 + x_2^2)\partial_{p_2}, (x_1 + 2x_2)\partial_{p_1} + (x_1^2 + 2x_1 x_2)\partial_{p_2},$$

while the coefficients of  $\partial_{p_1}$  in the generators of  $\text{Der}(-\log D) \otimes_R S + \text{Der}_R \otimes_R S \Delta$  are at least quadratic in  $x_1, \dots, x_\ell$ . In fact, assuming Lemma 4.2, we have

**Theorem 4.3.**  $\tilde{\mathcal{A}} = (\tilde{D} \times_D \mathcal{A})_{\text{red}}$

*Proof.*  $\tilde{S}_\mathcal{A} = \text{coker } tp$ , with  $tp$  as in (4.5), is a maximal Cohen–Macaulay  $S_\mathcal{A}$ -module of rank 1. This means that at a smooth point of  $\mathcal{A}$ ,  $\tilde{S}_\mathcal{A}$  is isomorphic to  $S_\mathcal{A}$ , and is thus reduced. As  $\tilde{S}_\mathcal{A}$  is finite over  $S_\mathcal{A}$ , its depth over itself (assuming it is a ring) is equal to its depth over  $S_\mathcal{A}$ . Since it is therefore a Cohen–Macaulay ring, generic reducedness implies reducedness.  $\square$

Now we prepare for the proof of Lemma 4.2.

**Proposition 4.4.**

(1) *The Frobenius multiplication in  $\text{Der}_R$  can be lifted to  $\text{Der}_S$ , though without multiplicative unit.*

- (2) The same procedure makes  $\text{Der}_S$  into a  $\text{Der}_R$ -module.  
(3) The map  $tp$  in (4.3) is  $\text{Der}_R$ -linear, with respect to the structure in (2) and Frobenius multiplication induced on  $\text{Der}_R \otimes_R S$ .

*Proof.* By (4.4), for a multiplication in  $\text{Der}_S$ , (1) means that

$$(4.6) \quad tp(\eta_1 \circ \eta_2) = \omega p(\chi_1 \circ \chi_2)$$

where  $\eta_i \in \text{Der}_S$  is a lift of  $\chi_i \in \text{Der}_R$  for  $i = 1, 2$ . Similarly, the scalar multiplication for (2) must satisfy

$$(4.7) \quad tp(\chi \cdot \eta) = \omega p(\chi \circ \xi)$$

where  $\chi \in \text{Der}_R$  and  $\eta \in \text{Der}_S$  is a lift of  $\xi \in \text{Der}_R$ .

Locally, at a point  $v \in V \setminus \mathcal{A}$ ,  $p$ ,  $tp$  and  $\omega p$  are isomorphisms, so there is nothing to prove. Now suppose  $v \in H$  is a generic point on a reflecting hyperplane  $H \in \mathcal{A}$ , with  $p(v)$  outside the bifurcation set  $B$ . Then in a neighborhood of  $p(v)$  in  $V/W$ , we may take canonical coordinates  $u_1, \dots, u_\ell$  (cf. [Her02, 2.12.(ii)]). These are characterized by the property that the vector fields  $e_i := \partial_{u_i}$ ,  $i = 1, \dots, \ell$  satisfy  $e_i \circ e_j = \delta_{i,j} \cdot e_i$ . With respect to these coordinates  $D$  is a normal crossing divisor (cf. [Her02, 4.1]), though  $D$  is smooth at  $p(v)$  by choice of  $v$ . By [Her02, Cor. 4.6], the tangent space  $T_{p(v)}D$  is spanned by  $\ell - 1$  of these idempotent vector fields, and the remaining idempotent, which we label  $e_1$ , is normal to it. The map  $p_v: (V, v) \rightarrow (V/W, p(v))$  has multiplicity 2, critical set  $H$  and set of critical values  $D$ , from which it follows that  $d_v p: T_v H \rightarrow T_{p(v)}D$  is an isomorphism. Since we have fixed our coordinate system on  $(V/W, p)$ , we are free to choose only the coordinates on  $(V, v)$ . Define  $x_i = u_i \circ p$  for  $i = 2, \dots, \ell$ . To extend these to a coordinate system on  $(V, v)$ , we may take as  $x_1$  any function whose derivative at  $v$  is linearly independent of  $d_v x_2, \dots, d_v x_\ell$ . This means we may take as  $x_1$  any defining equation of the critical set (the hyperplane  $H$ ) of  $p$  at  $v$ . With respect to these coordinates on  $(V, v)$  and  $(V/W, p(v))$ ,  $p$  takes the form

$$(4.8) \quad p_v(x_1, \dots, x_\ell) = (f(x_1, \dots, x_\ell), x_2, \dots, x_\ell).$$

As  $p_v$  has critical set  $\{x_1 = 0\}$  and discriminant  $\{u_1 = 0\}$ , both  $f$  and  $\partial_{x_1}(f)$  vanish along  $\{x_1 = 0\}$ . Thus  $f(x) = x_1^2 g(x)$  for some  $g \in \mathcal{O}_{V,v}$ . Since  $p$  has multiplicity 2 at  $v$ ,  $g(0) \neq 0$ . Now replace the coordinate  $x_1$  by  $x_1 g(x)^{1/2}$ . With respect to these new coordinates, which we still call  $x_1, \dots, x_\ell$ ,  $p_v$  becomes a standard fold:

$$p_v(x_1, \dots, x_\ell) = (x_1^2, x_2, \dots, x_\ell).$$

We can now explicitly calculate the multiplication in  $\text{Der}_S$ , locally at  $v$ :

$$\begin{cases} tp_v(x_1 \partial_{x_1}) = \omega p_v(2u_1 \partial_{u_1}), \\ tp_v(\partial_{x_i}) = \omega p_v(\partial_{u_i}), \end{cases} \quad \text{for } i = 2, \dots, \ell.$$

So (4.6) implies

$$\begin{aligned} tp_v((x_1 \partial_{x_1}) \circ (x_1 \partial_{x_1})) &= \omega p_v((2u_1 \partial_{u_1}) \circ (2u_1 \partial_{u_1})) \\ &= \omega p_v(4u_1^2 \partial_{u_1}) = \omega p_v(2u_1(2u_1 \partial_{u_1})) = tp_v((2x_1^2)x_1 \partial_{x_1}), \end{aligned}$$

and hence  $x_1 \partial_{x_1} \circ x_1 \partial_{x_1} = 2x_1^3 \partial_{x_1}$ . So in order that (4.6) should hold, we are forced to define

$$\partial_{x_i} \circ \partial_{x_j} = \begin{cases} 2x_1 \partial_{x_1}, & \text{for } i = j = 1, \\ \delta_{i,j} \cdot \partial_{x_i}, & \text{otherwise.} \end{cases}$$

Since the multiplication in  $\text{Der}_V$  is locally uniquely defined by (4.6) outside codimension 2, it extends to  $V$  by Hartog's Extension Theorem. This proves (1), and (2) is obtained by an analogous argument using (4.7).

Finally, (3) follows from (4.4) and (4.7) on  $V \setminus \mathcal{A}$ , and therefore holds everywhere.  $\square$

*Proof of Lemma 4.2.* Suppose that  $m := \sum_i \tilde{r}_i \otimes s_i \in L$ . Lift it to  $\sum_i \xi_i \otimes s_i \in \text{Der}_R \otimes_R S$ . As  $m \in L$ , there exists some  $\eta \in \text{Der}_S$  such that

$$(4.9) \quad \sum_i \xi_i \otimes s_i = tp(\eta).$$

Clearly, for any  $s \in S$ , we have

$$(1 \otimes s) \cdot \left( \sum_i \xi_i \otimes s_i \right) = \sum_i r_i \otimes (ss_i) = tp(s\eta),$$

and thus  $(1 \otimes s) \cdot (\sum_i \xi_i \otimes s_i) \in L$ . To complete the proof that  $L$  is an ideal of  $\tilde{R}_D \otimes_R S$ , we must show that, for any  $\tilde{r} \in \tilde{R}_D$ , we also have

$$(4.10) \quad (\tilde{r} \otimes 1) \cdot (\sum_i \tilde{r}_i \otimes s_i) = \sum_i (\tilde{r}\tilde{r}_i) \otimes s_i \in L.$$

Lift  $\tilde{r} \in \tilde{R}_D$  to  $\xi \in \text{Der}_R$ . By Lemma 4.1,  $\sum_i (\tilde{r}\tilde{r}_i) \otimes s_i$  is the image of  $\sum_i (\xi \circ \xi_i) \otimes s_i$  in  $\tilde{R}_D \otimes_R S$ , where  $\xi \circ \xi_i$  is the Frobenius product in  $\text{Der}_R$ . But Proposition 4.4 applied to (4.9) yields

$$\sum_i (\xi \circ \xi_i) \otimes s_i = tp(\xi \cdot \eta)$$

which implies (4.10) using (4.5).  $\square$

We have proved

**Theorem 4.5.** *The cokernel  $\tilde{S}_{\mathcal{A}} = \text{coker } J$  of the transposed Saito matrix  $J$  of  $\mathcal{A}$  is a ring compatibly with its structure as an  $S$ -module.*  $\square$

*Remark 4.6.* Even though our proof uses complex analytic methods – for example, using canonical coordinates in the proof of Proposition 4.4 – the conclusion is valid over any field over which the basic invariants are defined. For the conclusion that  $\text{coker } J$  is a ring is equivalent to the condition (grc), which is a condition on ideal membership. We have shown that for each  $i, j \in \{1, \dots, \ell\}$ , the equation

$$(4.11) \quad m_j^i = C_1 m_1^\ell + \dots + C_\ell m_\ell^\ell$$

in unknown functions  $C_1, \dots, C_\ell$  has a solution in which the  $C_i$  are germs of complex analytic functions at 0.

Let  $\mathbb{K}$  be a subfield of  $\mathbb{C}$  containing the coefficients of the basic invariants  $p_j$ , so that the coefficients of the polynomials  $m_j^i$  all lie in  $\mathbb{K}$ . We claim that (4.11) has solutions  $C_i \in \mathbb{K}[V]$ . To see this, first note that since the  $m_j^i$  are all homogeneous, each  $C_i$  can be replaced by its graded part of degree  $D_i - D_\ell$  (see (2.22)). Let  $\mathbb{K}[V]_d \subset \mathbb{K}[V]$  be the vector space of all polynomials of degree  $d$ . The map

$$A : (\mathbb{K}[V]_{D_i - D_\ell})^\ell \rightarrow \mathbb{K}[V]_{D_i}, \quad A(C_1, \dots, C_\ell) = \sum_{j=1}^{\ell} C_j m_j^\ell,$$

is  $\mathbb{K}$ -linear. Therefore the solvability of (4.11) in  $\mathbb{K}[V]$  reduces to a simple theorem of linear algebra, which can be rephrased more abstractly as follows: Let  $A : \mathbb{K}^m \rightarrow \mathbb{K}^n$  be a  $\mathbb{K}$ -linear map, and suppose  $\mathbb{K} \subset \mathbb{L}$  is a field extension. Then

$$\text{im}(A \otimes_{\mathbb{K}} 1_{\mathbb{L}}) \cap \mathbb{K}^n = \text{im}(A).$$

This is easy to see: suppose  $\mathbf{b} \in \mathbb{K}^n$  and we can solve

$$(4.12) \quad A\mathbf{x} = \mathbf{b}$$

for  $\mathbf{x} \in \mathbb{L}^m$ . Let  $(A|\mathbf{b})$  be the augmented matrix of the system of equations. Row operations over  $\mathbb{K}$  on the matrix  $(A|\mathbf{b})$ , and column operations over  $\mathbb{K}$  involving the columns of  $A$  alone, bring  $(A|\mathbf{b})$  to the form

$$(4.13) \quad \begin{pmatrix} I_s & 0 & \mathbf{b}' \\ 0 & 0 & \mathbf{b}'' \end{pmatrix}$$

where  $s = \text{rk } A$ ,  $\mathbf{b}' = (b'_1, \dots, b'_s)^t$  and  $\mathbf{b}'' = (b''_1, \dots, b''_{n-s})^t$ . There are now two cases. If  $s < n$  and  $\mathbf{b}'' \neq 0$  then (4.12) has no solution. If  $s = n$  or  $\mathbf{b}'' = 0$  then a solution over  $\mathbb{K}$  can be read off from (4.13). So if (4.12) has a solution in  $\mathbb{L}^m$ , we are in the second case and there is a solution in  $\mathbb{K}^m$ .

## 5. FREE AND ADJOINT DIVISORS

In [MS10] a new class of free divisors was constructed using the recipe “discriminant + adjoint”. If  $D$  is the discriminant in the base of a miniversal deformation of a weighted homogeneous hypersurface singularity (subject to some numerical conditions on the weights) and  $D'$  is an adjoint divisor, in the sense that the pull-back of  $D'$  to the normalization  $\Sigma^0$  of  $D$  is the conductor of the ring extension  $\mathcal{O}_D \rightarrow \mathcal{O}_{\Sigma^0}$ , then  $D + D'$  is a free divisor ([MS10, Thm. 1.3]). The singularities to which this applies include those of type ADE. In this section we point out that essentially the same construction works for the other Coxeter groups. We have to replace the normalization by the space  $\tilde{D}$  from (2.20), and take, as  $D'$ , a divisor pulling back to the conductor of the ring extension  $\mathcal{O}_D \hookrightarrow \mathcal{O}_{\tilde{D}}$ . The construction lifts to the representation space  $V$ , giving a new free divisor strictly containing the reflection arrangement.

We keep the notations from Section 2.1 and work in standard coordinates as in (2.7).

**Lemma 5.1.** *With a suitable choice of basic invariants in (2.2), the linear part of the Saito matrix  $K = JJ^t$  of  $D$  from (2.17) is symmetric of the form*

$$(5.1) \quad \bar{K} = \begin{pmatrix} w_1 p_1 & w_2 p_2 & \cdots & \cdots & w_{\ell-1} p_{\ell-1} & w_\ell p_\ell \\ w_2 p_2 & \star & \cdots & \star & \alpha_{\ell-1} p_\ell & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \star & \ddots & \ddots & \ddots & \vdots \\ w_{\ell-1} p_{\ell-1} & \alpha_2 p_\ell & \ddots & \ddots & \ddots & \vdots \\ w_\ell p_\ell & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

where  $\alpha_2, \dots, \alpha_{\ell-1} \in \mathbb{C}^*$  with  $\alpha_i = \alpha_{\ell+1-i}$ . Moreover the only entries in this matrix equal to non-zero constant multiples of  $p_\ell$  lie along the anti-diagonal.

*Remark 5.2.* This matrix shows the linearized convolution of the basic invariants  $p_1, \dots, p_\ell$  as described in [Arn79]. In the light of the translator's notes on pages 22 and 21 of [Arn79] it seems likely our statement here was already known to Arnol'd and Givental.

*Proof.* The first row and column of (5.1) can be read from (2.15). It remains to show the triangular form of  $\bar{K}$  and that the anti-diagonal entries, and only these, are non-zero constant multiples of  $p_\ell$ . By inspection, the degree of  $K_j^i$  is  $w_i + w_j - w_1$ . By (2.3), (2.5) and (2.6), the degree of  $K_j^i$  with  $i + j = \ell + 1$  equals  $h = w_\ell$ , and hence  $\bar{K}_j^i = \alpha_j p_\ell$  for some  $\alpha_j \in \mathbb{C}$ . Provided  $W$  is not of type  $D_{2k}$ , the degrees  $w_1, \dots, w_\ell$  of the basic invariants are pairwise distinct. It follows that:

- All  $K_j^i$  with  $i + j > \ell + 1$  have degree strictly between  $w_\ell$  and  $2w_\ell$  and hence have a linear part equal to zero. In particular,  $\bar{K}$  has the claimed triangular shape.
- All  $K_j^i$  with  $i + j < \ell + 1$  have degree less than  $w_\ell$ , and hence do not involve  $p_\ell$ .

But by (2.12), (2.13), and Theorem 2.2,  $\det K = \Delta^2$  is a monic polynomial of degree  $\ell$  in  $p_\ell$ . It follows that  $\alpha_2 \cdots \alpha_{\ell-2} \neq 0$ . Finally the symmetry property  $\alpha_i = \alpha_{\ell+1-i}$  comes from the symmetry of  $K$ .

In the case of  $D_{2k}$ , the same argument shows that the  $p_\ell$ -coefficient matrix of  $\bar{K}$  is a constant symmetric anti-diagonal block matrix, where  $i$  and  $j$  are in the same block exactly if  $w_i = w_j$ . By the procedure in the proof of [MS10, Lem. 3.6] it can be turned into a symmetric anti-diagonal matrix by linear algebra on the basic invariants.  $\square$

*Remark 5.3.* By (2.5), the minor  $M_\ell^\ell$  is not changed by the change of basic invariants in Proposition 5.1.

As in (2.21) and (2.16) using Theorem 2.12, we set, for  $\bar{K}$  as in (5.1),

$$(\bar{M}_j^i) := \text{ad}(\bar{K}), \quad \bar{I}_D := \langle \bar{M}_1^\ell, \dots, \bar{M}_\ell^\ell \rangle.$$

**Lemma 5.4.**  $dM_\ell^\ell(\text{Der}(-\log D)) = I_D$ .

*Proof.* The strategy is the same as in the proof of the analogous result in [MS10]. We replace  $\delta_i$  by its linear part  $\bar{\delta}_i$  whose coefficients are in the  $i$ th row/column of  $\bar{K}$  in (5.1). Then it suffices to prove that the inclusion

$$(5.2) \quad d\bar{M}_\ell^\ell(\langle \bar{\delta}_1, \dots, \bar{\delta}_\ell \rangle) \subseteq \bar{I}_D.$$

obtained from Proposition 2.18 is an equality. The polynomial expansion of the minor  $\bar{M}_{\ell-i+1}^\ell$  contains the distinguished monomial  $p_i p_\ell^{\ell-2}$  with non-zero coefficient. This monomial does not appear in the expansion of  $\bar{M}_j^\ell$  for  $j \neq i$ . In particular the expansion of  $\bar{M}_\ell^\ell$  contains the monomial  $p_1 p_\ell^{\ell-2}$ , with coefficient  $(-1)^{\ell-2} \iota w_1 \alpha$ , where  $\iota$  is the sign of the order-reversing permutation of  $1, \dots, \ell - 1$ , and we abbreviate

$$\alpha := \alpha_2 \cdots \alpha_{\ell-1}.$$

We claim that  $d\bar{M}_\ell^\ell(\bar{\delta}_i)$  contains the monomial  $p_i p_\ell^{\ell-2}$  with non-zero coefficient, and no other of the distinguished monomials. This shows that (5.2) is an equality and proves the lemma.

Contributions to the coefficient of  $p_j p_\ell^{\ell-2}$  in the expansion of  $d\bar{M}_\ell^\ell(\bar{\delta}_i)$  arise as follows:

- (1) By applying the derivation  $p_j \partial_{p_1}$  to the monomial  $p_1 p_\ell^{\ell-2}$ . This happens only when  $i = j$ , and in this case the resulting contribution to the coefficient of  $p_j p_\ell^{\ell-2}$  is

$$\delta_{i,j} (-1)^{\ell-2} \iota w_i w_1 \alpha.$$



- (2) By applying the derivation  $p_\ell \partial_{p_k}$  to the monomial  $p_j p_k p_\ell^{\ell-3}$ . This derivation appears in  $\bar{\delta}_i$  only if  $k = \ell - i + 1$ , and then with coefficient  $\alpha_i$ ; also this monomial appears in  $\bar{M}_\ell^\ell$  only if  $k = \ell - j + 1$ , and hence  $i = j$ . If  $2j = \ell + 1$ , the monomial  $p_j p_{\ell-i+1} p_\ell^{\ell-3}$  appears in the expansion of  $\bar{M}_\ell^\ell$  with coefficient

$$\delta_{i,j}(-1)^{\ell-1} \iota w_j w_{\ell-j+1} \alpha / \alpha_j,$$

otherwise, it appears twice with that coefficient. The resulting contribution to the coefficient of  $p_j p_\ell^{\ell-2}$  in  $d\bar{M}_\ell^\ell(\bar{\delta}_i)$  is

$$\delta_{i,j}(-1)^{\ell-1} \iota \alpha w_j w_{\ell-j+1}$$

if  $2j = \ell + 1$ , or twice this if  $2j \neq \ell + 1$ .

Therefore  $p_j p_\ell^{\ell-2}$  can appear in  $d\bar{M}_\ell^\ell(\bar{\delta}_i)$  with non-zero coefficient only if  $i = j$ , and in this case the coefficient is non-zero provided

$$\begin{cases} w_1 \neq w_j, & \text{if } 2j = \ell + 1, \\ w_1 \neq 2w_{\ell-j+1}, & \text{if } 2j \neq \ell + 1. \end{cases}$$

These conditions hold by (2.5).  $\square$

**Theorem 5.5.** *Let  $D' = \{M_\ell^\ell = 0\}$ . Then  $D + D'$  is a free divisor.*

*Proof.* Here the proof is identical to the proof of the comparable result of [MS10, Prop. 3.10]. By Lemma 5.4, there are vector fields  $\tilde{\delta}_1, \dots, \tilde{\delta}_\ell \in \text{Der}(\log D)$  such that

$$(5.3) \quad dM_\ell^\ell(\tilde{\delta}_i) = M_i^\ell.$$

We may take  $\tilde{\delta}_\ell$  equal to a constant multiple of the Euler vector field  $\delta_1$ . Since  $\delta_1, \dots, \delta_\ell$  is a basis of  $\text{Der}(-\log D)$ , there exist  $B_j^i \in R$  such that  $\tilde{\delta}_i = \sum_j B_j^i \delta_j$ . By the proof of Lemma 5.4, the matrix  $B = (B_j^i)$  is invertible. Note that the Saito matrix of the basis  $\tilde{\delta}_1, \dots, \tilde{\delta}_\ell$  is then  $KB$ . Let  $K'$  be obtained from the matrix  $K$  by deleting its last column. The columns of  $K'$  give relations among the generators  $M_1^\ell, \dots, M_\ell^\ell$  of  $I_D$ , by Cramer's rule.

For each relation  $\sum_i \lambda_i M_i^\ell = 0$ , (5.3) gives

$$\sum_i \lambda_i \tilde{\delta}_i(M_\ell^\ell) = dM_\ell^\ell(\sum_i \lambda_i \tilde{\delta}_i) = \sum_i \lambda_i M_i^\ell = 0,$$

so

$$\sum_i \lambda_i \tilde{\delta}_i \in \text{Der}(-\log D) \cap \text{Der}(-\log D') = \text{Der}(-\log(D + D')).$$

Because  $\tilde{\delta}_\ell$  is a scalar multiple of  $\delta_1$ , we also have  $\tilde{\delta}_\ell \in \text{Der}(-\log(D + D'))$ . Let  $K''$  denote the matrix formed by adjoining to  $K'$  the extra column  $(0, \dots, 0, 1)^t$ . Thus the columns of the  $\ell \times \ell$  matrix  $KBK''$  are the coefficients of vector fields in  $\text{Der}(-\log(D + D'))$ , and  $\det(KBK'') \equiv \Delta^2 M_\ell^\ell \pmod{\mathbb{C}^*}$  where  $\Delta^2 = \det K$  is a reduced equation for  $D$ . Now provided

- (1)  $M_\ell^\ell$  is reduced, and
- (2)  $M_\ell^\ell$  and  $\Delta^2$  have no common factor,

it follows from Saito's criterion that  $D + D'$  is a free divisor, and the vector fields represented by the columns of  $KBK''$  form a free basis for  $\text{Der}(-\log(D + D'))$ .

By [MP89, Cor. 3.15],  $M_\ell^\ell$  generates (over  $\tilde{R}_D$ ) the conductor ideal of the map  $\tilde{D} \rightarrow D$ . It follows that  $D \cap D' = V(I_D) = \text{Sing}(D)$  has codimension 2, and hence (2) holds. It suffices to check (1) at generic points of  $\text{Sing}(D)$ . Using Proposition 2.20, this reduces to checking (1) in the case  $\ell = 2$  discussed in Section 2.6. But in this case  $M_2^2 = 2p_1$  is reduced by (2.32).  $\square$

**Corollary 5.6.**  *$\mathcal{A} + p^{-1}(D')$  is a free divisor.*

*Proof.* We continue with the notation of the proof of Theorem 5.5. Consider the vector fields represented by the columns of  $J^t(BK'') \circ p$ . Since  $JJ^tBK'' = KBK''$ , these vector fields are lifts to  $V$  of the vector fields represented by the columns of  $KBK''$ ; they are therefore logarithmic with respect to  $p^{-1}(D')$ . Since they are linear combinations of the columns of  $J^t$  they are logarithmic with respect to  $\mathcal{A}$ , and thus with respect to  $\mathcal{A} + p^{-1}(D')$ .

By (2.12),  $\det J = \Delta$  is a reduced equation of  $\mathcal{A}$ . Since  $\det K'' = \pm M_\ell^\ell$  is reduced and, along  $V(M_\ell^\ell)$ ,  $p$  is generically a submersion (for the critical set of  $p$  is  $\mathcal{A}$ , which meets  $V(M_\ell^\ell \circ p)$  only in codimension 2),  $\det(K'' \circ p)$  is a reduced equation for  $V(M_\ell^\ell \circ p)$ . As  $\det B \in \mathbb{C}^*$ ,  $\det(J^t(BK'') \circ p)$  is therefore a reduced equation for  $\mathcal{A} + p^{-1}(D')$ , and the corollary follows by Saito's criterion.  $\square$

*Example 5.7.* The reflection arrangement for  $A_n$  consists of the intersection of  $V := \{\sum_{i=1}^{n+1} x_i = 0\} \subset \mathbb{C}^{n+1}$  with the union of the hyperplanes  $\{x_i = x_j\}$ . For  $A_2$ , the composite equation  $M_\ell^\ell \circ p$  defining  $p^{-1}(D')$  in Corollary 5.6 is equal, on  $V$ , to the second elementary symmetric function,  $\sigma_2$ . For  $A_3$ , this becomes  $8\sigma_2\sigma_4 - 9\sigma_3^2 - 2\sigma_2^3$ .

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