Half-turn symmetric FPLs with rare couplings and tilings of hexagons

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JEAN-CHRISTOPHE AVAL, PHILIPPE DUCHON

Abstract. In this work, we put to light a formula that relies the number of fully packed loop configurations (FPLs) associated to a given coupling π to the number of half-turn symmetric FPLs (HTFPLs) of even size whose coupling is a punctured version of the coupling π. When the coupling π is the coupling with all arches parallel π_0 (the "rarest" one), this formula states the equality of the number of corresponding HTFPLs to the number of cyclically-symmetric plane partition of the same size. We provide a bijective proof of this fact. In the case of HTFPLs odd size, and although there is no similar expression, we study the number of HTFPLs whose coupling is a slit version of π_0, and put to light new puzzling enumerative coincidence involving countings of tilings of hexagons and various symmetry classes of FPLs.

Introduction

Fully packed loop configurations (FPLs) are ubiquitous objects which are fascinating both in the world of theoretical physics (they appear in the so-called six-vertex ice model) and in the world of combinatorics (they are in bijection with alternating sign matrices, which are the center of an intense research for years). In 2004 Razumov and Stroganov [16] stated a remarkable conjecture that relies the stationary distribution of the $O(1)$-dense loop model to the enumeration of FPLs according to their coupling. After several years of efforts, their formula was only recently proved by Cantini and Sportiello [1] by means of a purely combinatorial method using the operation of gyration discovered by Wieland [17]. Following Razumov and Stroganov's investigations, de Gier [9] gave in 2005 an analogous conjectural formula for the same model with half-turn symmetry constraints.

When we compare Razuomv-Stroganov's and de Gier's formula (for the even size), we are led to the following interesting expression: the number of FPLs of size n and coupling π is equal to the sum of the

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numbers of half-turn symmetric FPLs (HTFPLs) of size $2n$ and coupling a punctured version of π. A special case is when the coupling is the rarest one π_0 (with the arches all parallel), where this expression reduces to an equality between the number of half-turn symmetric FPLs of size $2n$ with their coupling being a punctured version of π_0 and the number of cyclically symmetric plane partition of size $2n$. We are able to prove this assertion *bijectively*.

In the case of the odd size, there is no natural expression between couplings of HTFPLs and asymmetric FPLs. Nevertheless, we may study the number of HTFPLs of size $2n+1$ whose couplings are slit versions of π_0. Using a factorization principle due to Ciucu [3], we are lead to evaluate the number of tilings with losenges of portions of some hexagonal regions. These numbers of tilings may be expressed through determinants [12]. Surprisingly, we put to light that several determinant expressions are proved or conjectured to be equal to the number of symmtry classes of FPLs!

This paper is organized as follows: Section 1 presents all definitions relative to FPLs and their couplings, Section 2 deals with the case of even-sized HTFPLs, Section 3 presents the problem studied in the case of the odd size, together with its reduction to the evaluation of determinants and presents new intriguing results and conjectures of equinumeration between certain tilings and symmetry classes of FPLs.

1. Definitions

1.1. **FPLs and their couplings.** A *fully-packed loop configuration* (FPL for short) of size N is a subgraph of the $N \times N$ square lattice, where each internal vertex has degree exactly 2. The set of edges forms a set of closed loops and paths ending at the boundary vertices. The boundary conditions are the alternating conditions: boundary vertices also have degree 2 when boundary edges (edges that connect the finite square lattice to the rest of the $\mathbb{Z}^2$ lattice) are taken into account, and these boundary edges, when going around the grid, are alternatingly "in" and "out" of the FPL. For definiteness, we use the convention that the top edge along the left border is always "in". Thus, exactly $2N$ boundary edges act as endpoints for paths, and the FPL consists of N noncrossing paths and an indeterminate number of closed loops.

Any FPL f of size N has a *coupling* $\pi(f)$, which is a partition of the set of integers $\{1 \dots 2N\}$ into pairs, defined as follows: first label the endpoints of the open loops 1 to $2N$ in clockwise or counterclockwise order (for definiteness, we use counterclockwise order, starting with the top left endpoint); then the link pattern $\pi(f)$ will include pair (i, j) if

and only if f contains a loop whose two endpoints are labeled i and j. Because the loops are noncrossing, the coupling satisfies the noncrossing condition: if a link pattern π contains two pairs (i, j) and k, l, then one cannot have $i < k < j < l$. The possible link patterns for FPLs of size N are thus counted by the Catalan numbers $C_N = \frac{1}{N+1}\binom{2N}{N}$. Figure 1 gives an example of an FPL together with its coupling. We shall denote by $A(N; \pi)$ the number of FPLs of size$N$ which afford coupling π, and by $A(N)$ the total number of FPLs, which is equal, because of the bijection between FPL and alternating-sign matrices [18] to:

$$A(N) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}\,. \tag{1}$$

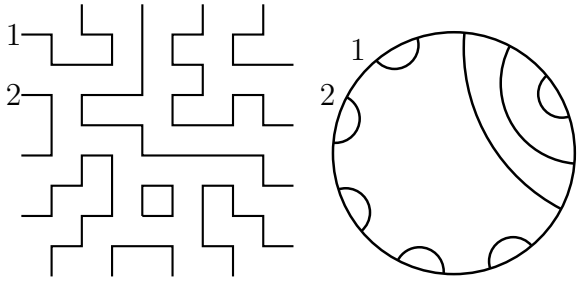

FIGURE 1. An FPL of size 8 and its coupling

Let us introduce a particular coupling, denoted $\pi_{0,n}$ (or π_0 if there is no ambiguity), defined as:

$$\pi_{0,n} = \{\{i, 2n+1-i\}_{1\leq i\leq n}\}.$$

The coupling π_0 is, up to rotation the rarest one: $A(n, \pi_0) = 1$.

The $2N$ generators $e_1, \ldots, e_{2N}$ of the cyclic Temperley-Lieb algebra act on couplings of size N in the following way: if the coupling π contains pairs (i, j) and $(i+1, k)$, then $e_i\pi = \pi'$, where π' is obtained from π by replacing the pairs (i, j) and $(i+1, k)$ by $(i, i+1)$ and (j, k); if $(i, i+1) \in \pi$, then $\pi' = \pi$. An illustration of this action is given by Figure 2.

One can define a Markov chain on couplings where we choose at each time step one of the appropriate generators (uniformly at random) and apply it to the current state. The Markov chain defined in this way is easily checked to be irreducible and aperiodic, hence it has a unique stationary distribution. The celebrated Razumov-Stroganov conjecture [16], proven by Cantini and Sportiello [1], may be stated as follows.

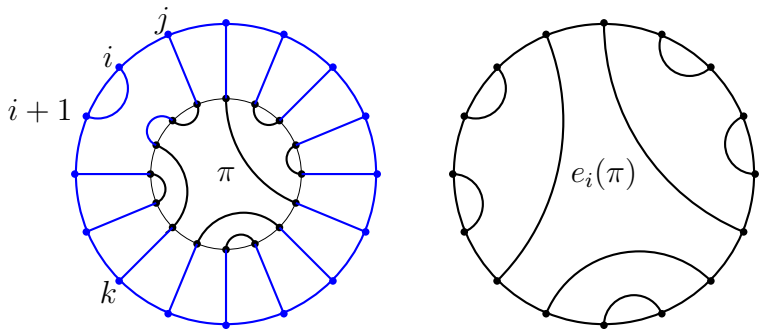

FIGURE 2. Temperley-Lieb action

Theorem 1.1. [Cantini, Sportiello] *The stationary distribution for couplings of size N is*

$$\mu(\pi) = \frac{A(N;\pi)}{A(N)}. \tag{2}$$

1.2. **HTFPLs – Punctured and slit couplings.** An FPL is said to be *half-turn symmetric* if it is invariant under the central symmetry of the square grid. It is easy to observe that such HTFPLs do exist whatever the parity of the size N. Let us denote by $A_{HT}(N)$ the number of HTFPLs of size N.

HTFPLs, be they of even or odd size, have couplings that are invariant under a half-turn rotation: if their size is L and the coupling contains an edge (i, j), it must also contain $(i + L, j + L)$.

For odd L, parity and planarity considerations immediately imply that the coupling must contain exactly one diameter edge of the form $(i, i + L)$, with the endpoints $i + 1$ to $i + L - 1$ organized into a normal coupling (and endpoints $L + i + 1$ to $i - 1$ organized into a translated version of the same). Such a coupling of size $2L$ can be represented more compactly as a "slit" coupling of (odd) size L, where the diameter edge becomes a singleton (i) and each pair of edges (j, k) and $(j + L, k + L)$ becomes a single (j, k) edge. Graphically, this corresponds to a classical coupling of size $L - 1$ with an added single vertex (which we represent by a half-edge leading inside the circle).

For even L, no diagonal edge can exist for parity reasons. Instead, HT-symmetric couplings of size $2L$ can be represented as classical plane couplings of size L drawn on a punctured disk (or half-cylinder) instead of a full disk.

Figure 3 shows examples of half-turn symmetric couplings respectively of odd (left) and even (right) size.

Let us denote by $A_{HT}(N;\pi)$ the number of HTFPLs which have π as coupling.

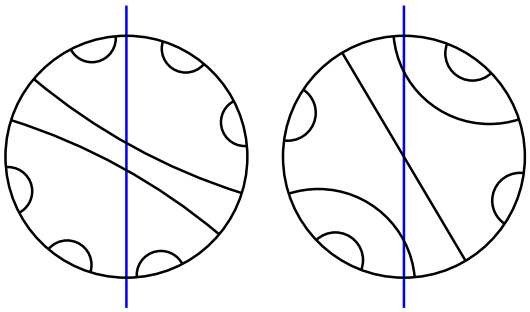

FIGURE 3. Half-turn symmetric couplings of odd (left) and even (right) size

Similarly to the asymmetric case, and for $N \geq 2$, we consider the N "symmetrized" operators

$$e'_i = eie_{i+N}. \tag{3}$$

These operators act on the couplings of HTFPLs of size N, we may define a Markov chain on the set of half-turn symmetric couplings. The assertion analogous to Theorem 1.1 is due to de Gier [9] and may be stated as follows.

Conjecture 1.2. [de Gier] *The stationary distribution for couplings of size N is*

$$\mu_{HT}(\pi) = \frac{A_{HT}(N;\pi)}{A_{HT}(N)}. \tag{4}$$

2. Even-sized HTFPLs with rare couplings

2.1. **A general formula.** When viewed as "punctured" plane couplings, the couplings of even-sized HTFPLs have a natural projection to "normal" plane couplings of half their size - the projection corresponds to simply forgetting the puncture. What is more important, this projection commutes with the e_i and e'_i operators: if π' is a punctured plane coupling and p is the projection from punctured to normal plane couplings, one has $p(e'_i(\pi')) = e_i(p(\pi'))$.

An immediate consequence is that the eigenvector for the H' Hamiltonian must project to the eigenvector for H. In terms of FPL and HTFPL enumerations, in light of (2) and assuming (4), this becomes, for any coupling π:

$$\frac{A(n;\pi)}{A(n)} = \sum_{\pi'} \frac{A_{HT}(2n;\pi')}{A_{HT}(2n)}, \tag{5}$$

where the sum in the right-hand side extends to all punctured couplings π' such that $p(\pi') = \pi$.

Now, it is known that $A_{HT}(2n) = P_{SC}(2n)A(n)$, where $P_{SC}(2n)$ denotes the number of cyclically symmetric plane partitions of size $2n$. Thus, (5) is equivalent to

$$\sum_{\pi'} A_{HT}(2n;\pi') = P_{SC}(2n)A(n;\pi) \tag{6}$$

with the same convention on the summation.

2.2. **The case of the rarest coupling.** When π is one of the rotated versions of the rarest coupling, one has $A(n;\pi) = 1$ and (6) simplifies accordingly. Our first result is a bijective proof of this special case of equation (6).

Theorem 2.1. *For any integer n, there exists a bijection between the set of HTFPLs of size $2n$ whose coupling is a punctured version $\pi_{0,n}$, and cyclically symmetric plane partitions of size $2n$.*

Proof. The first thing to do is identify exactly which punctured couplings project to $\pi_{0,n}$. As plane couplings of size $4n$, these must link 1 to either $2n$ or $4n$, and $2n+1$ with the other, and more generally, for each $1 \leq k \leq n$, k must be linked with either $2n+1-k$ or $4n+1-k$, and $2n+k$ must be linked with the other. If we add the noncrossing condition, we obtain a full description of the $n+1$ possible couplings:

$$\pi'_{k,n} = \{\{i, 4n+1-i\}_{1\leq i\leq k}\} \cup \{\{i, 2n+1-i\}_{k<i\leq n}\}$$

where k ranges from 0 to n.

Now, the important property of this set of plane couplings is that they are exactly *all* plane couplings of size $4n$ whose short links are among $\{1,4n\},\{n,n+1\},\{2n,2n+1\},\{3n,3n+1\}$ - in fact, except for $\pi'_{0,n}$ and $\pi'_{n,n}$, these are exactly all short links of each $\pi'_{k,n}$. On the corresponding (HT)FPLs, this translates into exactly the same set of *fixed edges* (we refer to [2] for the presentation of the fixed edges technique):

- all eastbound edges from odd vertices in the $(i > j, i+j < 2n-1)$ area;
- all edges obtained by rotations from the previous: northbound from even vertices in the $(i > j, i+j > 2n-1)$ area, westbound from odd vertices with $(i < j, i+j > 2n-1)$, and southbound from even vertices with $(i < j, i+j < 2n-1)$.

The fixed edges for size 12 are shown in Figure 4 (left).

It is easy to check that all HTFPLs with these edges will have one of the $\pi'_{k,n}$ as their coupling; more precisely, though it is not important for our purpose, any FPL, whether half-turn-symmetric or not, will have such a coupling.

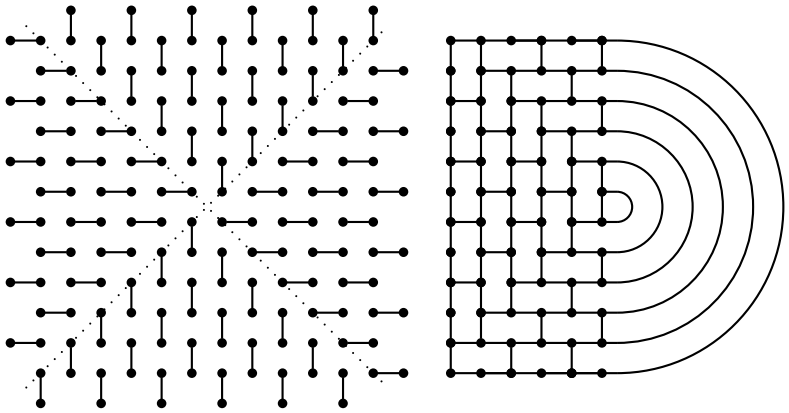

FIGURE 4. Fixed and non-fixed edges in even size

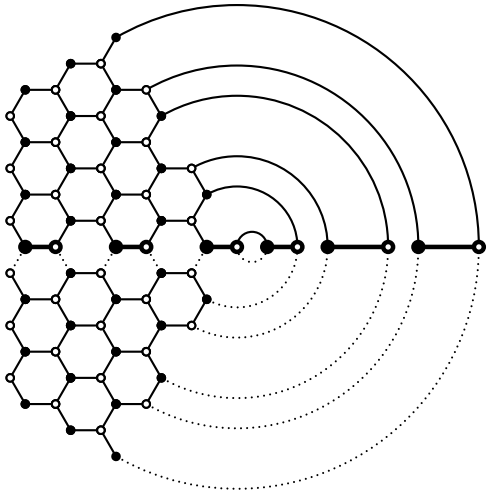

FIGURE 5. Honeycomb lattice version of the graph in Figure 4

Thus, our problem becomes that of finding a bijection between the set of HTFPLs with this set of fixed edges, and CSPPs of size $2n$. This is relatively straightforward: since each vertex in the grid is incident to one fixed edge, these (HT)FPLs are in a natural bijection with the (half-turn-symmetric) perfect matchings of the subgraph of non-fixed edges. Taking symmetry into account corresponds to taking the quotient of the graph under the half-turn-symmetry, and it is easy to check that this quotient graph is also the quotient under a $2\pi/3$-rotation of a hexagonal region (of size $2n$) of the honeycomb lattice. In other words, the HTFPLs with these fixed edges are in bijection with the perfect matchings of the honeycomb lattice that are invariant under a third-turn rotation - or, taking the dual, lozenge tilings of a regular hexagon

(of side $2n$) that are invariant under a rotation of order 3, that is, cyclically symmetric plane partitions of size $2n$. ∎

3. Rare couplings in odd size

3.1. **Factorization.** While there is an easy way to project couplings of HTFPLs of odd size $2n+1$ to those of FPLs of size n (by "unslitting" them), this projection, contrary to the even sized case, does not commute with the e_i and e'_i operators, so that (2) and (4) together do not have a "nice" consequences on the numbers $A_{HT}(2n+1;\pi')$ and $A(n;\pi)$. Still, applying the fixed edges technique to some sets of HTFPLs whose couplings are a slit version of the rarest coupling does lead to intriguing enumerative results.

The slit couplings we are looking for are all those with at most two short edges, *i.e.* of the form (for some $0 \leq k \leq n+1$)

$$\{\{i, 2n+2-i\}_{1\leq i\leq k}\} \cup \{\{k\}\} \cup \{\{i, 2n+3-i\}_{k+1\leq i\leq n+1}\}$$

or rotated from this form. In extended form, these are all HT-symmetric couplings with at most 4 short edges that are restricted to be in positions $(i, i+1)$, $(n+i+1, n+i+2)$, $(2n+i+1, 2n+i+2)$ and $(3n+i+2, 3n+i+3)$ for some i.

If we use the case $i=0$ above (that is, we allow short edges $(4n+2, 1)$, $(n+1, n+1)$, $(2n+1, 2n+2)$ and $(3n+2, 3n+3)$), we get a large set of fixed edges that is similar to what we got in the even-sized case:

- eastbound edges from odd vertices in the $(i > j, i+j < 2n)$ area, and westbound edges from odd vertices in the symmetric $(i < j, i+j > 2n)$ area;
- northbound edges from even vertices in the $(i > j, i+j > 2n)$ area, and southbound edges from even vertices in the symmetric $(i < j, i+j < 2n)$.

Figure 6 shows the fixed edges, and the fundamental domain of non-fixed edges, for size 13 ($n=6$), and Figure 7 shows the same graph of non-fixed edges as a region of the honeycomb lattice. In the latter figure, dotted edges are those that are "cut" by Ciucu's Factorization Theorem, and bold edges are those that are given a weight 1/2 by the same.

When we restrict our attention to the nonfixed edges, the corresponding HTFPLs are in bijection with the perfect matchings of a region G_n of the honeycomb lattice as shown on Figure 8 (where the sides of the region along the bold line must be glued together).

The region G_n can be deformed to have a reflexive symmetry as shown on Figure 9 size $4k+1$ and $4k+3$.

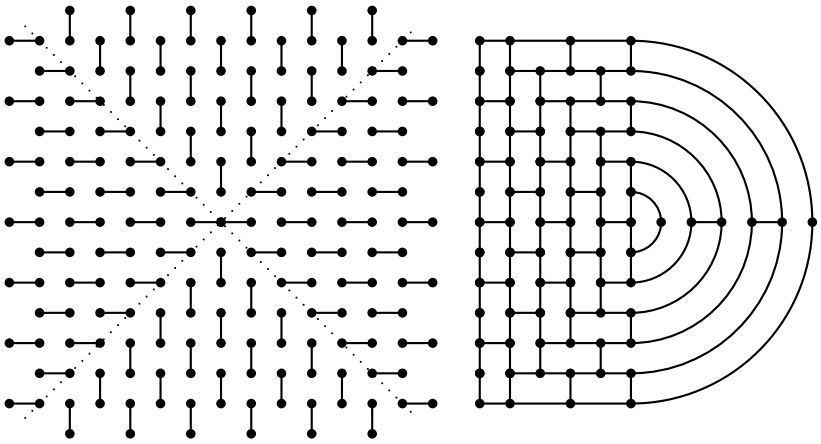

FIGURE 6. Fixed and non-fixed edges in odd size

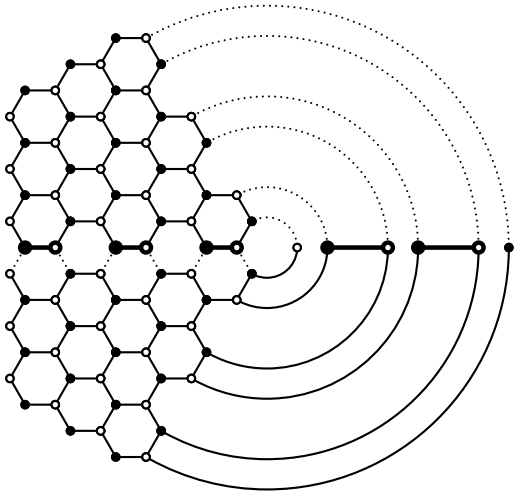

FIGURE 7. Honeycomb version of the graph in Figure 6

Figure 9 shows (in the triangular lattice) the result of applying Ciucu's Factorization Theorem [3]: for each size, we are to count the lozenge tilings of two regions of the trangular lattice. In the figure, grayed lozenges have a weight of 1/2 attached, and dashed lozenges are "fixed" in the sense that they must appear in all lozenge tilings of the corresponding region.

Ciucu's theorem thus implies that the number H_n of such HTFPLs of size n is given in odd size by:

- $H_{4k+1} = 2^{2k} R_k(1/2, 1) R'_{k-1}(1/2, 1)$,
- $H_{4k+3} = 2^{2k+1} R_k(1/2, 1) R'_k(1/2, 1)$

and in even size by:

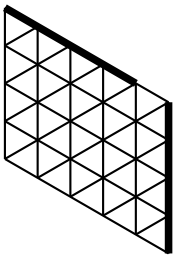

FIGURE 8. The region G_4.

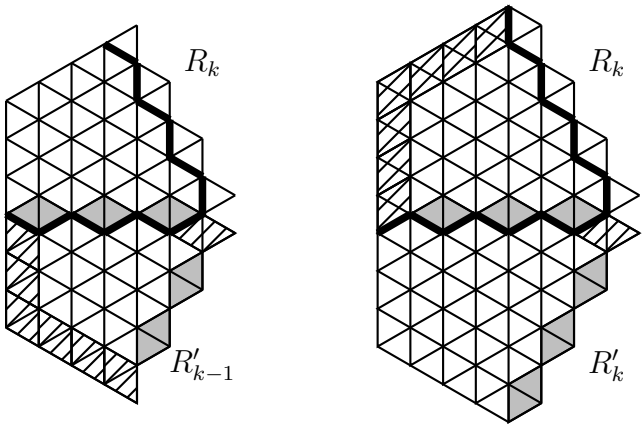

FIGURE 9. Decomposition by symmetry of G_6 and G_7.

- $H_{4k} = 2^{2k} R_k(1/2, 1/2) R_{k-1}(1, 1)$,
- $H_{4k+2} = 2^{2k+2} R_k(1/2, 1/2) R_{k-1}(1, 1)$.

To enumerate weighted tilings of regions R_k and R'_k, we may use Lindström-Gessel-Viennot's [11] determinants to get:

$$\begin{aligned} R_k(x,y) &= \det\left(m_{i,j,0}\right)_{1\le i,j\le k} \\ R'_k(x,y) &= \det\left(m_{i,j,1}\right)_{1\le i,j\le k} \\ m_{i,j,\ell} &= (1+xy)\binom{i+j+\ell-2}{2i-j-1} + x\binom{i+j+\ell-2}{2i-j-2} + y\binom{i+j+\ell-2}{2i-j} \end{aligned}$$

3.2. **Enumeration of certain tilings of hexagons.** When evaluating $R_k(x, y)$ and $R'_k(x, y)$ for x and y in $\{1/2, 1\}$, we are surprised to recover well-known sequences. More generally, we define:

$$\mathcal{R}_\ell(n; x, y) = \det\left(m_{i,j,\ell}\right)_{1\le i,j\le n}.$$

The aim of this subsection is to identify some specializations of the functions $\mathcal{R}_\ell(n; x, y)$ in terms of cardinality of some classes of alternating sign matrices.

The function $\mathcal{R}_\ell(n; x, y)$ counts the weighted lozenge tilings of the region shown in Figure 10, where grayed lozenges carry a multiplicative weight of x or y as indicated.

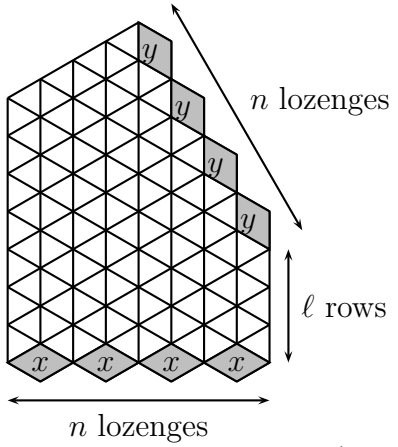

FIGURE 10. Interpretation of $\mathcal{R}_\ell(n; x, y)$

Proposition 3.1. *We have the following special values for the functions $\mathcal{R}$:*

$$\mathcal{R}_0(n; 1/2, 1) = A_{HT}(2n+1) \tag{7}$$

$$\mathcal{R}_1(n; 1/2, 1) = \frac{1}{2} A_{HT}(2n+2) \tag{8}$$

$$\mathcal{R}_1(n; 1, 1) = A_V(2n+3) \tag{9}$$

$$\mathcal{R}_1(n; 1, 1/2) = A(n)^2 \tag{10}$$

$$\mathcal{R}_2(n; 1/2, 1) = A(n)A(n+1) \tag{11}$$

where $A_{HT}(N)$, $A_V(N)$ and $A(N)$ stand respectively for the number of half-turn symmetric, vertically symmetric and unrestricted alternating sign matrices.

Proof – It appears that the three specializations we need to interpret, namely $\mathcal{R}_\ell(n; 1, 1/2)$, $\mathcal{R}_\ell(n; 1/2, 1)$ and $\mathcal{R}_\ell(n; 1, 1)$ are computed in [7, 12]. We denote by $(a)_i = a(a+1)\cdots(a+i-1)$ the shifted factorial.

Proof of (7) and (8). We may use [12] to write:

$$\mathcal{R}_\ell(n; 1/2, 1) = \prod_{i=0}^{n-1} (2\ell+3i)i! \frac{(\ell+i-1)!(2\ell+2i)_i(\ell+2i)_i}{(\ell+2i)!(2i)!}.$$

It is then a simple computation to check that

- for $\ell = 0$:

$$(3i)i!\frac{(i-1)!(2i)_i^2}{(2i)!^2} = \frac{4}{3}\frac{\binom{3n}{n}^2}{\binom{2n}{n}^2} = \frac{A_{HT}(2n+1)}{A_{HT}(2n-1)},$$

- for $\ell = 1$:

$$(2+3i)i!\frac{(i)!(2i+2)_i(2i+1)_i}{(2i+1)!(2i)!} = \frac{4}{3}\frac{\binom{3n+3}{n+1}\binom{3n}{n}}{\binom{2n+2}{n+1}\binom{2n}{n}} = \frac{A_{HT}(2n)}{A_{HT}(2n)}.$$

Now to conclude, we observe that $\mathcal{R}_0(1;1/2,1) = 3 = A_{HT}(3)$ and $\mathcal{R}_0(1;1/2,1) = 5 = 1/2.A_{HT}(4)$. This implies equations (7) and (8).

Proof of (9). We know from [7] that $\mathcal{R}_1(n;1,1)$ is equal to the number of cyclically symmetric transpose-complementary plane partitions (CSTCPP) in a hexagonal region with a triangular hole of size 2. We thus get:

$$\mathcal{R}_1(n;1,1) = P_{CSTC}(2n,2) = \frac{1}{2^n}\prod_{j=0}^{n-1}\frac{P_{CS}(2j+1,2)}{P_{CS}(2j,2)}$$

By using

$$\begin{aligned}
P_{CS}(2j+1,2) &= \frac{(-1/2)!(2j+3)_{j+1}}{(j+1/2)!} \\
&\quad\times\prod_{i=0}^{j}\frac{i!^2(2i+1)_i^2(i+1/2)!(2i+1/2)_{i+1}(2i+1+1/2)_i}{(2i)!^2(j+i+1+1/2)!} \\
P_{CS}(2j,2) &= \frac{(-1/2)!j!(2j+1/2)_{j+1}}{(2j)!(2j+1/2)!} \\
&\quad\times\prod_{i=0}^{j-1}\frac{i!^2(2i+3)_{i+1}^2(i+1/2)!(2i+1+1/2)_i(2i+1/2)_{i+1}}{(2i)!^2(j+i+1/2)!}
\end{aligned}$$

we get:

$$\mathcal{R}_1(n;1,1) = \frac{1}{2^n}\prod_{j=0}^{n-1}\frac{j!(2j+1+1/2)_j(2j)!^2(2j+1)_j}{(3j)!^2(j+1+1/2)_{j+1}}.$$

Thus, because of [15, 14]

$$A_V(2n+1) = (-3)^{n^2}\prod_{i,j\le 2n+1, j\equiv 1[2]}\frac{3(j-i)+1}{j-i+2n+1} = \prod_{j=1}^{n}\frac{\binom{6j-2}{2j}}{\binom{4j-1}{2j}},$$

we have to check that:

$$\frac{j!(2j+1+1/2)_j(2j)!^2(2j+1)_j}{(3j)!^2(j+1+1/2)_{j+1}} = \frac{\binom{6j+4}{2j+2}}{\binom{4j+3}{2j+2}}$$

which comes from a simple computation.

Proof of (10) and (11). For equation (10), we know from [7] that $\mathcal{R}_1(n;1,1/2)$ is equal to the number of cyclically symmetric self-complementary plane partitions (CSSCPP) in a hexagonal region, which is known [13] to be given by:

$$P_{CSSC}(2n) = \Big(\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}\Big)^2 = A(n)^2.$$

For equation (11), it has been shown in [10] that $\mathcal{R}_2(n;1/2,1)$ is the number of quasi-cyclically symmetric self-complementary plane partitions (qCSSCPP) in a hexagonal region, which is proved to be given by:

$$P_{qCSSC}(2n+1) = A(n)A(n+1).$$

■

Remark 3.2. *It appears that the specialization of the functions $\mathcal{R}_\ell(n;x,y)$ to $x=y=1/2$ may also have interesting values. In particular, it seems that $\mathcal{R}_0(n;1/2,1/2)$ corresponds to the development of the generating series for $A^{(2)}_{UU}(4n)$ (cf.* [14]*) and that:*

$$\mathcal{R}_2(n;1/2,1/2) = A_V(2n+3)\binom{2n+1}{n+1}$$

which needs an explanation.

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