Plünnecke and Kneser type theorems for dimension estimates

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Abstract

Given a division ring K containing the field k in its center and A, B two finite subsets of K^*, we give some analogues of Plünnecke and Kneser theorems for the dimension of the k-linear span of the Minkowski product AB in terms of the dimensions of the k-linear spans of A and B. These Plünnecke type estimates are then generalized to the case of associative algebras. We also obtain an analogue in the context of division rings of a theorem by Tao classifying the sets of small doubling in a group.

1 Introduction

A classical problem in additive number theory is to evaluate the cardinality of sumsets in $\mathbb{Z}$ in terms of the cardinality of their summands. Many results and methods used to obtain such evaluations are in fact also suited for studying the cardinality of any sumset in abelian additive groups (or any product set in abelian multiplicative groups). In this paper, we will write group operations multiplicatively. Among numerous interesting results in this area are the Plünnecke-Ruzsa and Kneser Theorems.
Plünnecke-Ruzsa's theorem gives an upper bound for the cardinality of A^n knowing such a bound for $\left|A^2\right|$.

Theorem 1.1 *Let A and B be two finite subsets in an abelian group G. Assume α is a positive real such that $|AB| \leq \alpha |A|$. Then there exists a subset X of A such that for any integer n, $|XB^n| \leq \alpha^n |X|$. In particular $\left|A^2\right| \leq \alpha |A|$ implies that $|A^n| \leq \alpha^n |A|$.*

Kneser's Theorem gives a lower bound for the cardinality of the product set AB where A and B are finite nonempty subsets in an abelian group G.

Theorem 1.2 *Let A, B be finite subsets of the abelian group G. Write H for the stabilizer of AB in G. Then*

$$|AB| \geq |A| + |B| - |H|.$$

It is then natural to ask for analogous results in the general case of groups possibly non abelian. The question of finding lower and upper bounds for product sets in non abelian groups is considerably more difficult than in the abelian case. Nevertheless there is now a growing literature on this subject due to Diderrich [1], Hamidoune [6], Kemperman [9], Olson [12], Ruzsa [14], Tao [17] and many others. Let us mention that Kneser's theorem does not hold for non abelian groups as noticed in [9]. Nevertheless, there exists in this case weaker versions due to Diderrich [1] and Olson [12].

It is also worth mentioning that many problems in additive or multiplicative combinatorics have a continuous counterpart. Here one can consider compact groups G and the cardinality of

subsets is replaced by the Haar measure. Very recently Tao [16] has also obtained analogues of Plünnecke-Ruzsa's theorem in abelian groups where the subsets are replaced by random variables on the group considered and the cardinality by the Shannon entropy of these variables.

The Plünnecke-Ruzsa and Kneser Theorems give subtle informations on the structure of general groups. In this paper, we establish analogous results in the context of division rings. According to the usual terminology, a field is a commutative division ring. It is very easy to produce fields as extensions of simpler ones. Recall that the representation theory of groups gives also a general procedure yielding division rings which are not necessarily fields. Indeed, given any field k, the Schur lemma implies that the endomorphism algebra K of any irreducible finite-dimensional $k[G]$-module ($k[G]$ is the ring algebra of G over k) is a division ring. When k is not algebraically closed, $K \supsetneqq k$ and is not commutative in general. So incidentally, we will obtain structural informations on the division ring of automorphisms of any finite-dimensional $k[G]$-module.

Let us precise our notation. In the sequel, K is a division ring containing the field k in its center. We address the question of finding upper and lower bounds for the dimension of the k-span $k\langle A_1 \cdots A_n \rangle$ of product sets $A_1 \cdots A_n$ where $A_1, \dots A_n$ are nonempty subsets of $K^* = K \setminus \{0\}$. Note that this problem also makes sense for any algebra $\mathcal{A}$ defined over k. The estimates we obtain can thus also be applied when $\mathcal{A}$ is contained in a division ring. In the commutative case, this happens in particular for any algebra $\mathcal{A} := k[\alpha_1, \dots, \alpha_n]$ where $\alpha_1, \dots, \alpha_n$ are elements of the field K or for any sub-algebra of a field of rational functions in several variables.

As far as we are aware this kind of problems have been considered for the first time by Hou, Leung and Xiang. In [4] they proved the following analogue of Kneser's theorem for fields.

Theorem 1.3 *Let K be a commutative extension of k. Assume every algebraic element in K is separable over k. Let A and B be two nonempty finite subsets of K^*. Then*

$$\dim_k(AB) \geq \dim_k(A) + \dim_k(B) - \dim_k(H)$$

where $H := \{x \in K \mid xk\langle AB \rangle = k\langle AB \rangle\}$.

Here H is an intermediate field containing k in its center. Remarkably, the authors showed that their theorem easily implies Kneser's theorem for abelian group. It essentially suffices to use the structure theorem of abelian groups and the Galois correspondence. In [2], we obtain an analogue of Olson's theorem for division rings without any separability hypothesis. In the sequel, we shall refer to these analogues as linear Kneser and linear Olson theorems. The combinatorial methods used in the linear setting (that is for fields or division rings) are often very similar to their counterparts in groups. Nevertheless, there are some restrictions and complications mostly due to the fact that

- $k\langle A \rangle$ and $k\langle A^{-1} \rangle$ have not the same dimension in general whereas A and A^{-1} have the same cardinality,
- a k-subspace V in K may admit infinitely many k-subspaces W such that $V \oplus W = K$ whereas a subset A in a group G has a unique complement,
- when K is finite-dimensional over k, there may exist infinitely many intermediate division rings H such that $k \subset H \subset K$ whereas a finite group G has only a finite number of subgroups,
- given H_1 and H_2 subfields of the (commutative field) K, $H_1 H_2$ is not a field in general whereas the product set of two subgroups of an abelian group is always a group.

So, to avoid gaps or ambiguities, we have completely written down the proofs of our linear statements. These proofs sometimes differ from their analogue in groups. For example, the possible existence of an infinite number of intermediate fields seems to impose a separability hypothesis in the previous linear Kneser theorem. It is nevertheless conjectured in [8], that this hypothesis can be relaxed as in the linear Olson theorem. Also, in order to adapt the arguments used to establish the estimates in groups, we often need in our division ring context, to carefully chose the decomposition in direct summands of the spaces we consider in our proofs.

The paper is organized as follows. In Section 2, we precise our notation, give equivalent forms of the linear Kneser Theorem and also recall for completion the linear Olson Theorem. Section 3 is devoted to some linear analogues of results by Ruzsa. In particular, we derive a Plünnecke-Ruzsa's type theorem for fields. The arguments we use here are adaptations to the context of division rings of some very elegant and elementary proofs recently obtained by Petridis in [13]. In Section 4, we establish different Kneser type estimates for division rings. More precisely, we first study the case where A is assumed commutative (that is the elements of A are pairwise commutative) and obtain linear analogues of Theorems by Diderrich [1]. Next, we adapt Hamidoune connectivity to the context of division rings and obtain a linear version of a theorem by Tao classifying the sets of small doubling in a group. Finally in Section 5, we generalize the Plünnecke-Ruzsa type theorems of Section 2 in the context of associative unital algebras.

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2 The division ring setting

2.1 Vector span in a division ring

Let K be a division ring and k a subfield (thus commutative) of K contained in its center. We denote by $K^* = K \setminus \{0\}$ the group of invertible elements in K.

For any subset A of K^*, let $k\langle A\rangle$ be the k-subspace of K generated by A. We write $|A|$ for the cardinality of A and $\dim_k(A)$ for the dimension of $k\langle A\rangle$ over k. When $|A|$ is finite, $\dim_k(A)$ is also finite and we have $\dim_k(A) \leq |A|$. We denote by $\mathbb{D}(A) \subset K$ the sub division ring generated by A in K.

Given A, B subsets of K, we thus have $k\langle A \cup B\rangle = k\langle A\rangle + k\langle B\rangle$ the sum of the two spaces $k\langle A\rangle$ and $k\langle B\rangle$. We have also $k\langle A \cap B\rangle \subset k\langle A\rangle \cap k\langle B\rangle$ and $k\langle AB\rangle = k\langle k\langle A\rangle k\langle B\rangle\rangle$. We write as usual

$$AB := \{ab \mid a \in A, b \in B\}$$

for the Minkowski product of the sets A and B. Given $A_1, \dots, A_n$ a family of nonempty subsets of K^*, we define $A_1 \cdots A_n$ similarly. We also set $A^{-1} := \{a^{-1} \mid a \in A\}$. Observe that any finite-dimension k-subspace V of K can be realized as $V = k\langle A\rangle$ where A is any finite subset of nonzero vectors spanning V. Also when V_1 and V_2 are two k-vector spaces in K, $V_1V_2 \subset k\langle V_1V_2\rangle$ but V_1V_2 is not a vector space in general.

In the sequel we aim to give some estimates of $\dim_k(AB)$ or more generally of $\dim_k(A_1 \cdots A_r)$ where $A_1, \dots, A_r$ are finite subsets of K^*. The following is straightforward

$$\max(\dim_k(A), \dim_k(B)) \leq \dim_k(AB) \leq \dim_k(A) \dim_k(B).$$

The methods we will use to estimate $\dim_k(AB)$ are quite analogue to the tools used to estimate the cardinality $|AB|$ of the product set AB where A and B are subsets of a given group. Many results on estimates of product sets have a linear analogue for the dimension of the space generated by such products. Nevertheless there are crucial differences due notably to the fact that $k\langle A\rangle$ and $k\langle A^{-1}\rangle$ have not the same dimension in general whereas A and A^{-1} have the same cardinality. This can be easily verified by taking $k=\mathbb{C}$, $K=\mathbb{C}(T)$ and $A_n := \{(T-k) \mid k=1,\dots,n\}$. Indeed $\dim_{\mathbb{C}}(A_n)=2$ but $\dim_{\mathbb{C}}(A_n^{-1})=n$. In particular the triangle Ruzsa inequality

$$|A|\left|BC^{-1}\right| \leq |AB|\,|AC|$$

for any finite subsets A, B, C in K^* does not have a linear analogue (take $A=B=\{1\}$ and $C=A_n$). Also observe that a finite abelian group have only a finite number of subgroups whereas a finite commutative extension of a commutative field have an infinite number of intermediate extensions when it is not separable.

The two following elementary lemmas will be useful in the sequel.

Lemma 2.1 *Let A be a finite subset of K^* containing 1.*

1. *Assume $\dim(A^2)=\dim(A)$. Then $k\langle A\rangle$ is a division ring.*
2. *Assume that $A^{-1}=A$ and xy^{-1} belongs to $k\langle A\rangle$ for any nonzero elements x and y in A. Then $k\langle A\rangle$ is a division ring.*

Proof. 1: It suffices to observe that $k\langle A^2\rangle = k\langle A\rangle$ so that $k\langle A\rangle$ is stable under multiplication. Then, for any nonzero $a\in k\langle A\rangle$, the map $\varphi_a : k\langle A\rangle \to k\langle A\rangle$ which sends $\alpha\in k\langle A\rangle$ on $\varphi_a(\alpha)=a\alpha$ is an k-linear automorphism of the space $k\langle A\rangle$. In particular, φ_a is surjective and since $1\in A$, a^{-1} belongs to $k\langle A\rangle$.

2: For any x and t in A, $xt=x(t^{-1})^{-1}$ belongs to $k\langle A\rangle$ since x and t^{-1} belong to A. This proves that $k\langle A\rangle$ is stable under multiplication. Since it contains 1, $k\langle A^2\rangle = k\langle A\rangle$ and we conclude as in 1. ■

Lemma 2.2 *Consider V a finite-dimension k-subspace of K and H a sub division ring of K containing k such $HV=V$. Then there exists a finite subset of K^* such that*

$$V=\bigoplus_{s\in S} Hs. \tag{1}$$

In particular V is a left H-module of dimension $|S|\dim_k(H)$ and (1) gives its decomposition into irreducible components.

Proof. For any nonzero vector v in V, Hv is a k-subspace of V. In particular $\dim_k(Hv)=\dim_k(H)$ is finite. Moreover if v' is a nonzero vector in V, $Hv\cap Hv'=\{0\}$ when $v\neq v'$. The lemma easily follows. ■

Remarks:

1. From the previous lemma applied to $V=K$, we deduce that $\dim_k(H)$ divides $\dim_k(K)$ when $\dim_k(K)$ is finite.

2. When $VH = V$ we obtain similarly $V = \bigoplus_{s\in S} sH$, a decomposition in right H-modules.

The previous remark can be made more precise when k is the center of K. Write H' for the commutator of H in K that is

$$H' := \{x \in K \mid xh = hx \text{ for any } h \in H\}.$$

Clearly H' est a division ring containing k. We have then the following classical proposition (see [10]).

Proposition 2.3 *Assume $Z = Z(K)$ is the center of K and $\dim_Z(K)$ is finite. Then for any division ring $H \subset K$*

1. *$\dim_Z(K)$ is a square,*
2. *$\dim_Z(H)\dim_Z(H') = \dim_Z(K)$ for any division ring H such that $Z \subset H \subset K$,*
3. *$(H')' = H$.*

For any subset X in K^*, we set

$$H_{k,l}(X) := \{h \in K \mid h\,k\langle X\rangle = k\langle X\rangle\} \text{ and } H_{k,r}(X) := \{h \in K \mid k\langle X\rangle\, h = k\langle X\rangle\}$$

the left and right stabilizers of $k\langle X\rangle$ in K. Clearly $H_{k,l}(X)$ and $H_{k,r}(X)$ are division rings containing k. In particular, when K is a field, $H_{k,l}(X) = H_{k,r}(X)$ is a commutative extension of k that we simply write $H_k(X)$. If $H_{k,l}(X)$ (resp. $H_{k,r}(X)$) is not reduced to k, we says that $k\langle X\rangle$ is *left periodic* (resp. *right periodic*). When $k\langle X\rangle$ is finite-dimensional, there exists by Lemma 2.2 a finite subset S in $k\langle X\rangle$ such that $k\langle X\rangle = \oplus_{s\in S} H_{k,l}(X)s$ (resp. $k\langle X\rangle = \oplus_{s\in S} sH_{k,r}(X)$).

2.2 The linear Kneser theorem

We now recall the linear Kneser theorem stated in [4] for fields.

Theorem 2.4 *Let K be a commutative extension of k. Assume every algebraic element in K is separable over k. Let A and B be two nonempty finite subsets of K^*. Then*

$$\dim_k(AB) \geq \dim_k(A) + \dim_k(B) - \dim_k(H_k(AB)).$$

In § 4.4, we will give a noncommutative version of the following corollary where the separability hypothesis can moreover be relaxed.

Corollary 2.5 *Let K be a commutative extension of k. Assume every algebraic element in K is separable over k. Let A be a nonempty finite subset of K^* such that $\dim_k(A^2) \leq (2-\varepsilon)\dim_k(A)$ for a real ε with $0 < \varepsilon < 2$. Then, there exists a field H finite-dimensional over k and a finite non empty subset X of K^* with $|X| \leq \frac{2}{\varepsilon} - 1$ such that $k\langle A^2\rangle \subset \bigoplus_{x\in X} xH$.*

Proof. By the previous theorem, we must have $\dim_k(H_k(A^2)) \geq 2\dim_k(A) - \dim_k(A^2) \geq \varepsilon\dim_k(A)$. By Lemma 2.2, there exists a finite subset X of K^* such that $k\langle A^2\rangle = \bigoplus_{x\in X} xH$. We thus have

$$|X| = \frac{\dim_k(A^2)}{\dim_k(H)} \leq \frac{(2-\varepsilon)\dim_k(A)}{\varepsilon\dim_k(A)} \leq \frac{2}{\varepsilon} - 1$$

as desired. ■

Remarks:

1. Theorem 2.4 can be regarded as a linear version of Kneser's theorem. Recall that this theorem establishes that for any nonempty finite subsets A and B in an abelian group G (written multiplicatively), we have $|AB| \geq |A| + |B| - |H|$ where H is the stabilizer of AB in G.

2. As proved in [4], the linear Kneser theorem implies easily the Kneser theorem for abelian groups.

3. The separability hypothesis is crucial in the proof of the theorem in which the finite extensions of k should have a finite number of intermediate extensions. Nevertheless, it is conjectured in [8], that the separability hypothesis can be relaxed. Also observe that the separability hypothesis is always satisfied in characteristic zero.

As the original Kneser Theorem, Theorem 2.4 can be generalized for Minkowski products of any finite number of finite subsets of K^*. The following Theorem is not explicitly stated in [4]. We give its proof below for completion. We first need the following easy lemma.

Lemma 2.6 *Let K be a commutative extension of k. Consider $n \geq 2$ and integer and $A_1, \ldots, A_n$ a collection of finite nonempty subsets of K^* such that*

$$\dim_k(\prod_{i=1}^{j} A_i) \geq \dim_k(\prod_{i=1}^{j-1} A_i) + \dim_k(A_j) - 1 \tag{2}$$

holds for any $j = 2, \ldots, n$. Then

$$\dim_k(\prod_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} \dim_k(A_i) - n + 1.$$

Proof. We proceed by induction on j. For $j = 2$, we have $\dim_k(A_1A_2) \geq \dim_k(A_1) + \dim_k(A_2) - 1$ by (2). Assume we have

$$\dim_k(\prod_{i=1}^{j} A_i) \geq \sum_{i=1}^{j} \dim_k(A_i) - j + 1. \tag{3}$$

Writing (2) with $j + 1$ gives

$$\dim_k(\prod_{i=1}^{j+1} A_i) \geq \dim_k(\prod_{i=1}^{j} A_i) + \dim_k(A_{j+1}) - 1.$$

Combining with (3), one obtains

$$\dim_k(\prod_{i=1}^{j+1} A_i) \geq \sum_{i=1}^{j} \dim_k(A_i) - j + \dim_k(A_{j+1})$$

as desired. ■

Theorem 2.7 *Let K be a commutative extension of k. Consider $A_1, \ldots, A_n$ a collection of finite nonempty subsets of K^* Set $H := H_k(A_1 \cdots A_n)$. The following statements are equivalent:*

1. $\dim_k(A_1 \cdots A_n) \geq \sum_{i=1}^n \dim_k(A_i H) - (n-1)\dim_k(H)$,

2. $\dim_k(A_1 \cdots A_n) \geq \sum_{i=1}^n \dim_k(A_i) - (n-1)\dim_k(H)$,

3. *either* $\dim_k(A_1 \cdots A_n) \geq \sum_{i=1}^n \dim_k(A_i) - (n-1)$ *or* $k\langle A_1 \cdots A_n\rangle$ *is periodic,*

4. *any one of the above four statements in the case* $n = 2$.

Proof. (I): We obtain $1 \Rightarrow 2$ by using $\dim_k(A_i H) \geq \dim_k(A_i)$ for any $i = 1, \ldots, n$. The implication $2 \Longrightarrow 3$ is immediate. To prove $3 \Longrightarrow 1$, we first observe the implication is true when $H = \{1\}$. When $k\langle A_1 \cdots A_n\rangle$ is periodic, $H(A_1 \cdots A_n)$ is not. Thus we can apply assertion 3 by considering K as a commutative extension of H. This gives $\dim_H(A_1 \cdots A_n) \geq \sum_{i=1}^n \dim_H(A_i) - (n-1)$. Multiplying this inequality by $\dim_k(H)$ yields

$$\dim_H(A_1 \cdots A_n)\dim_k(H) \geq \sum_{i=1}^n \dim_H(A_i)\dim_k(H) - (n-1)\dim_k(H)$$

which is equivalent to 1 since $\dim_H(A_1 \cdots A_n)\dim_k(H) = \dim_k(A_1 \cdots A_n)$ and $\dim_H(A_i)\dim_k(H) = \dim_k(A_i)$ for any $i = 1, \ldots, n$.

(II): It remains to prove that assertion 3 is equivalent to the following :

3' Given A and B two finite nonempty subsets of K^*, either $\dim_k(AB) \geq \dim_k(A) + \dim_k(B) - 1$ or $k\langle AB\rangle$ is periodic.

Clearly $3 \Longrightarrow 3'$. Assume now $3'$ holds and $\dim_k(A_1 \cdots A_n) < \sum_{i=1}^n \dim_k(A_i) - (n-1)$. Then by Lemma 2.6, there exists $j \in \{2, \ldots, n\}$ such that

$$\dim_k(\prod_{i=1}^{j} A_i) < \dim_k(\prod_{i=1}^{j-1} A_i) + \dim_k(A_j) - 1.$$

By applying $3'$ with $A = \prod_{i=1}^{j-1} A_i$ and $B = A_j$, we obtain that $k\langle AB\rangle = k\langle \prod_{i=1}^{j} A_i\rangle$ is periodic. Therefore $k\langle \prod_{i=1}^{n} A_i\rangle$ is also periodic. This shows that $3' \Longrightarrow 3$. ■

Remark: The four assertions of the Theorem are equivalent without the separability hypothesis of Theorem 2.4.

2.3 The linear Olson theorem

Kneser's theorem does not hold in general for non abelian groups. In [12], Olson gave an upper bound for the cardinality of AB where A and B are two finite subsets of a group G. A crucial ingredient of the proof is the Kemperman transformation introduced in [9]. A linear version of this theorem was obtained in [2].

Theorem 2.8 *Let k be a field and K a division ring containing k in its center. Consider A and B two finite nonempty subsets of K^*. Then there exists a k-vector subspace S of $k\langle AB\rangle$ and a division ring $H \subset K$ such that*

1. $k \subset H \subset K$,

2. $\dim_k(S) \geq \dim_k(A) + \dim_k(B) - \dim_k(H)$,

3. $HS = S$ *or* $SH = S$.

Remarks:

1. By assertion 2, we have in particular $\dim_k(AB) \geq \dim_k(A) + \dim_k(B) - \dim_k(H)$ since S is a subspace of $k\langle AB\rangle$.

2. Recall that Olson theorem establishes that for any nonempty finite subsets A and B in a group G, there exists a subset S of AB and a subgroup H of G such that $|S| \geq |A| + |B| - |H|$ with $HS = S$ or $SH = S$. This theorem and its linearization are weaker than Kneser type statements for commutative structures notably because S is not made explicit. Observe nevertheless that no separability hypotheses is required in Theorem 2.8 contrary to Theorem 2.4.

3. Theorem 2.8 implies the following weaker statement. The space $k\langle AB\rangle$ contains a (left or right) periodic subspace or $\dim_k(AB) \geq \dim_k(A) + \dim_k(B) - 1$.

3 Plünnecke-type estimates in division rings

Plünnecke-type estimates permits to bound the cardinality of sumsets in abelian groups.

Theorem 3.1 *Let A and B be two finite subsets in an abelian group G. Assume α is a positive real such that $|AB| \leq \alpha |A|$. Then there exists a subset X of A such that for any integer n, $|XB^n| \leq \alpha^n |X|$. In particular $\left|A^2\right| \leq \alpha |A|$ implies that $|A^n| \leq \alpha^n |A|$.*

Plünnecke result was first stated for $G = \mathbb{Z}$ but his proof based on a graph-theoretic method can be extended to arbitrary abelian groups. Very recently, Petridis gave a surprisingly elegant and short proof of Theorem 3.1. This proof can be adapted to the context of rings as explained in the sequel. In Section 5, we will consider the case of associative unital algebras.

3.1 Minimal growth under multiplication

Let K be a division ring containing the field k in its center. Consider A and B two finite subsets of K^*. For any k-subspace $V \neq \{0\}$ of $k\langle A\rangle$, we set

$$r(V) := \frac{\dim_k(VB)}{\dim_k(V)} \tag{4}$$

the growth of V under multiplication by B. Write $\rho := \min_{V \subset k\langle A\rangle, V \neq \{0\}} r(V)$. Since the image of the map r is contained in a discrete set of positive numbers, there exists a nonempty set $X \subset A$ is such that $r(k\langle X\rangle) = \rho$. We thus have $\dim_k(XB) = \rho \dim_k(X)$ and $\dim_k(XB)/\dim_k(X) \leq \dim_k(ZB)/\dim_k(Z)$ for any $Z \subset A$.

Proposition 3.2 *Under the previous hypotheses, we have for any finite set C in K^**

$$\dim_k(CXB) \leq \rho \dim_k(CX) = \frac{\dim_k(CX)\dim_k(XB)}{\dim_k(X)}.$$

Proof. Write $C = \{c_1, \dots, c_r\}$. Set $X_1 = X$ and for any $i = 1, \dots, r$, let X_i be a finite subset of $k\langle X\rangle$ such that

$$k\langle X\rangle = k\langle X_i\rangle \oplus c_i^{-1} \sum_{a=1}^{i-1} c_a k\langle X\rangle \cap k\langle X\rangle.$$

Such a subset does exists. It suffices to consider any finite subset of $k\langle X\rangle$ spanning a direct summand of $V_i := c_i^{-1} \sum_{a=1}^{i-1} c_a k\langle X\rangle \cap k\langle X\rangle$. We thus have

$$c_i k\langle X\rangle = c_i k\langle X_i\rangle \oplus \sum_{a=1}^{i-1} c_a k\langle X\rangle \cap c_i k\langle X\rangle.$$

We have in particular $c_i k\langle X_i\rangle \subset c_i k\langle X\rangle$ and

$$\sum_{i=1}^{j} c_i k\langle X_i\rangle = \bigoplus_{i=1}^{j} c_i k\langle X_i\rangle$$

for any $j = 1, \dots, r$. By induction on j, we have then

$$\sum_{i=1}^{j} c_i k\langle X\rangle = \bigoplus_{i=1}^{j} c_i k\langle X_i\rangle.$$

Therefore

$$\dim_k(\sum_{i=1}^{j} c_i k\langle X_i\rangle) = \sum_{i=1}^{j} \dim_k(c_i X_i) = \sum_{i=1}^{j} \dim_k(X_i) \tag{5}$$

for any $j = 1, \dots, r$.

As in the proof of Petridis, we now proceed by induction on r. When $r = 1$, $\dim_k(c_1 XB) = \dim_k(XB) = \rho \dim_k(X) = \rho \dim_k(c_1 X)$. Assume $r > 1$. Recall that $V_r := c_r^{-1} \sum_{a=1}^{r-1} c_a k\langle X\rangle \cap k\langle X\rangle$ and $k\langle X\rangle = k\langle X_r\rangle \oplus V_r$. We then have $c_r V_r \subset \sum_{a=1}^{r-1} c_a k\langle X\rangle$ and $c_r k\langle V_r B\rangle \subset \sum_{a=1}^{r-1} c_a k\langle XB\rangle$. Since $k\langle V_r B\rangle$ is a subspace of $k\langle XB\rangle$, this gives

$$k\langle CXB\rangle = \sum_{a=1}^{r} c_a k\langle XB\rangle = \sum_{a=1}^{r-1} c_a k\langle XB\rangle + c_r k\langle XB\rangle = \sum_{a=1}^{r-1} c_a k\langle XB\rangle + c_r W$$

where W is a k-subspace of $k\langle XB\rangle$ such that $k\langle XB\rangle = W \oplus k\langle V_r B\rangle$. We have in particular $\dim_k(c_r W) = \dim_k(W) = \dim_k(XB) - \dim_k(V_r B)$. We thus obtain

$$\dim_k(CXB) \leq \dim_k(\sum_{a=1}^{r-1} c_a k\langle XB\rangle) + \dim_k(XB) - \dim_k(V_r B). \tag{6}$$

By the induction hypothesis, we have

$$\dim_k(\sum_{a=1}^{r-1} c_a k\langle XB\rangle) = \dim_k(C'XB) \leq \rho \dim_k(C'X) = \rho \dim_k(\sum_{a=1}^{r-1} c_a k\langle X\rangle)$$

with $C' = C \setminus \{c_r\}$. Thus by using (5) with $j = r - 1$, one gets

$$\dim_k(\sum_{a=1}^{r-1} c_a k\langle XB\rangle) \leq \rho \sum_{a=1}^{r-1} \dim_k(c_a X_a).$$

Since $V_r \subset k\langle X\rangle$, we have $\dim_k(V_rB) \geq \rho \dim_k(V_r)$. By definition of X, we have $\dim_k(XB) = \rho \dim_k(X)$. Therefore

$$\dim_k(XB) - \dim_k(V_rB) \leq \rho \dim_k(X) - \rho \dim_k(V_r) \leq \rho \dim_k(X_r).$$

Combining the two previous inequalities with (6), we finally obtain

$$\dim_k(CXB) \leq \rho \sum_{a=1}^{r-1} \dim_k(c_aX_a) + \rho \dim_k(c_rX_r) \leq$$
$$\rho \sum_{a=1}^{r} \dim_k(c_aX_a) = \rho \dim_k(\sum_{a=1}^{r} c_ak\langle X\rangle) = \rho \dim_k(CX).$$

where the last equality is obtained by (5) with $j = r$. ■

Corollary 3.3 *Let K be a division ring containing the field k in its center. Consider A and B two finite subsets of K^*. Assume α is a positive real such that $\dim_k(AB) \leq \alpha \dim_k(A)$. Then there exists a subset $X \subset A$ such that for any finite subset C of K^* $\dim_k(CXB) \leq \alpha \dim_k(CX)$.*

Proof. Let $X \subset A$ such that $\rho = r(X)$. We have

$$\rho = \frac{\dim_k(XB)}{\dim_k(X)} \leq \frac{\dim_k(AB)}{\dim_k(A)} \leq \alpha$$

by definition of ρ and with $Z = A$. We then apply Proposition 3.2 which yields $\dim_k(CXB) \leq \alpha \dim_k(CX)$. ■

3.2 Plünnecke upper bounds for $k\langle AB\rangle$

We assume in this paragraph that K is a *commutative extension* of k. The following theorem can be regarded as a linear version of Theorem 3.1. In fact, it is a linear version of the slightly stronger result obtained by Petridis where X is the same for any positive integer n.

Theorem 3.4 *Let A and B be nonempty finite subsets in K^*. Assume that $\dim_k(AB) \leq \alpha \dim_k(A)$ where α is a positive real. Then, there exists a subset $X \subset A$ such that for any positive integer n*

$$\dim_k(XB^n) \leq \alpha^n \dim_k(X).$$

In particular $\dim_k(A^2) \leq \alpha \dim_k(A)$ implies that $\dim(A^n) \leq \alpha^n \dim(A)$.

Proof. The proof is by induction on n. Let X be such that $\rho = r(X)$. For $n = 1$, we have

$$\dim_k(XB) \leq \dim_k(X)\frac{\dim_k(AB)}{\dim_k(A)} \leq \alpha \dim_k(X).$$

For any $n > 1$, we set $C = B^{n-1}$. By applying Proposition 3.2 with $C = B^{n-1}$, we have

$$\dim_k(XB^n) = \dim_k(B^{n-1}XB) \leq \frac{\dim_k(B^{n-1}X)\dim_k(XB)}{\dim_k(X)}.$$

Next by the induction hypothesis $\dim_k(B^{n-1}X) = \dim_k(XB^{n-1}) \leq \alpha^{n-1} \dim(X)$. Therefore

$$\dim_k(XB^n) \leq \alpha^{n-1} \dim_k(XB) \leq \alpha^n \dim_k(B)$$

where the last inequality follows from the case $n = 1$. ■

Remark: Observe that commutativity is crucial in the previous proof.

3.3 Double and triple product

The following theorem shows how to estimate in a field K the dimension of the vector span generated by a triple product set in terms of the dimensions of the vector spans obtained from the corresponding double product sets. This is a linear version of Theorem 9.2 in [14].

Theorem 3.5 *Consider K a division ring containing the field k in its center. Let A, B, C be finite nonempty subsets of K^*. Then*

$$\dim_k(ABC)^2 \leq \dim_k(AB)\dim_k(BC)\max_{b\in B}\{\dim_k(AbC)\}. \tag{7}$$

In particular, when K is a field, we have

$$\dim_k(ABC)^2 \leq \dim_k(AB)\dim_k(BC)\dim_k(AC).$$

Proof. We proceed by induction on $\dim_k(B)$. When $B = \{b\}$, we obtain

$$\dim_k(AbC)^2 \leq \dim_k(Ab)\dim_k(bC)\dim_k(AbC)$$

by observing that $\dim_k(AbC) \leq \dim_k(Ab)\dim_k(C)$ and $\dim_k(C) = \dim_k(bC)$. Now assume $|B| > 1$ and fix $b \in B$ such that $\max_{u\in B}\{\dim_k(AuC)\} = \dim_k(AbC)$. Set $m = \dim_k(AbC)$. Write $B = B' \cup \{b\}$. Set $A = \{a_1, \ldots, a_r\}$ and $C = \{c_1, \ldots, c_s\}$. We have $k\langle AB\rangle = k\langle AB'\rangle + \sum_{a\in A} k\langle ab\rangle$. Let $S_{AB'}$ be a basis of $k\langle AB'\rangle$. Since $S_{A'B} \cup Ab$ generates $k\langle AB\rangle$, there exists a subset $A^\flat$ of A such that

$$k\langle AB\rangle = k\langle AB'\rangle \oplus \bigoplus_{a\in A^\flat} k\langle ab\rangle.$$

Similarly, there exists a subset $C^\flat$ of C such that

$$k\langle BC) = k\langle B'C\rangle \oplus \bigoplus_{c\in C^\flat} k\langle bc\rangle.$$

We get

$$k\langle ABC\rangle = k\langle AB'C\rangle + \sum_{a\in A^\flat} k\langle abC\rangle = k\langle AB'C\rangle + \sum_{a\in A^\flat} k\langle aBC\rangle =$$
$$k\langle AB'C\rangle + \sum_{a\in A^\flat} k\langle aB'C\rangle + \sum_{a\in A^\flat}\sum_{c\in C^\flat} k\langle abc\rangle.$$

But $\sum_{a\in A^\flat} k\langle aB'C\rangle \subset k\langle AB'C\rangle$, thus

$$k\langle ABC\rangle = k\langle AB'C\rangle + \sum_{a\in A^\flat}\sum_{c\in C^\flat} k\langle abc\rangle.$$

Let $S_{AB'C}$ be a basis of $k\langle AB'C\rangle$. By the previous decomposition, there exists $X \subset A^\flat \times B^\flat$ such that

$$k\langle ABC\rangle = k\langle AB'C\rangle \oplus \bigoplus_{(a,c)\in X} k\langle abc\rangle.$$

Set $\alpha = |X|$, $\beta = \left|A^\flat\right|$ and $\gamma = \left|C^\flat\right|$. We have to prove (7), that is

$$(\dim_k(AB'C) + \alpha)^2 \leq (\dim_k(AB') + \beta)(\dim_k(B'C) + \gamma)m. \tag{8}$$

By the induction hypothesis, we have

$$\dim_k(AB'C)^2 \leq \dim_k(AB')\dim_k(B'C)m \tag{9}$$

because $\max_{u\in B'}\{\dim_k(AuC)\} \leq \max_{u\in B}\{\dim_k(AuC)\} = m$. We have $\bigoplus_{(a,c)\in X} k\langle abc\rangle \subset k\langle AbC\rangle$. So $\alpha \leq m$. Since $X \subset A^\flat \times B^\flat$, we have also $\alpha \leq \beta\gamma$. We get $\alpha^2 \leq m\beta\gamma$. By multiplying in (9), this gives

$$\alpha^2 \dim_k(AB'C)^2 \leq \beta\gamma \dim_k(AB')\dim_k(B'C)m^2.$$

Therefore

$$\alpha \dim_k(AB'C) \leq m\sqrt{\beta\gamma \dim_k(AB')\dim_k(B'C)} \leq m\frac{\gamma \dim_k(AB') + \beta \dim_k(B'C)}{2}.$$

So

$$2\alpha \dim_k(AB'C) \leq m(\gamma \dim_k(AB') + \beta \dim_k(B'C)).$$

Combining this last equality with $\alpha^2 \leq m\beta\gamma$ and (9), we finally get

$$\dim_k(AB'C)^2 + 2\alpha \dim_k(AB'C) + \alpha^2 \leq m(\dim_k(AB')\dim_k(B'C) + \gamma \dim_k(AB') + \beta \dim_k(B'C) + \beta\gamma)$$

as desired. ■

By using Theorems 3.4 and 3.5, we can obtain a bound for $\dim_k(A^3)$ knowing $\dim_k(A^2)$ and $\dim_k(A)$.

Corollary 3.6 *Consider K a field extension of k and A a nonempty finite subset of K^*. Assume $\dim_k(A) = m$ and $\dim_k(A^2) = n$, then*

$$\dim_k(A^3) \leq \min(n^{3/2}, \frac{n^3}{m^2}).$$

4 Kneser type theorems for division rings

In this section K is a division ring and k a field contained in the center of K.

4.1 Assuming A is commutative

Consider A a finite nonempty subset of K^*. We say that A is commutative when $aa' = a'a$ for any $a, a' \in A$. This then implies that the elements of $k\langle A\rangle$ are pairwise commutative. Moreover the division ring $\mathbb{D}(A)$ generated by A is a field. Typical examples of commutative sets are the geometric progressions $A = \{a^r, a^{r+1}, \ldots, a^{r+s}\}$ with r, s integers. The following theorem is the linearization of a theorem by Diderrich [1] extending Kneser's theorem for arbitrary groups when only the subset A is assumed commutative. Observe, it was shown by Hamidoune in [5] that Diderrich's result can also be derived from the original Kneser theorem in abelian group thanks to the isoperimetric method developed by the author. This method seems difficult to adapt to the context of division rings. So we will prove our theorem without using Theorem 2.4. Also we will also assume that k is infinite. When K is finite-dimensional over k and k is finite, K is a field so we can apply Theorem 2.4.

Theorem 4.1 *Assume k is infinite and every algebraic element of K is separable over k.*

1. *Let A and B be two finite nonempty subsets of K^* such that A is commutative. Then either $\dim_k(AB) \geq \dim_k(A) + \dim_k(B) - 1$, or $k\langle AB\rangle$ is left periodic.*

2. *Let $A_1, \ldots, A_n$ be a collection of finite nonempty subsets of K^* such that $A_1, \ldots, A_{n-1}$ are commutative. Then either $\dim_k(A_1 \cdots A_n) \geq \sum_{i=1}^n \dim_k(A_i) - (n-1)$ or $k\langle A_1 \cdots A_n\rangle$ is periodic.*

To prove the theorem, we need to adapt the arguments of [4] to our noncommutative situation. We begin with the following lemma based on the linear Dyson transform.

Lemma 4.2 *Let A and B be two finite nonempty subsets of K^* such that A is commutative. Then for each nonzero $a \in k\langle A\rangle$, there exists a (commutative) subfield H_a of K such that $k \subset H_a \subset \mathbb{D}(A)$ and a vector space $V_a \neq \{0\}$ contained in $k\langle AB\rangle$ such that $H_a V_a = V_a$, $k\langle aB\rangle \subset V_a$ and*

$$\dim_k(V_a) + \dim_k(H_a) \geq \dim_k(A) + \dim_k(B).$$

Proof. By replacing a by $A' = a^{-1}A$, we can assume $a = 1$. Indeed, if there exist a subfield $H \subset \mathbb{D}(A')$ and a vector space $V \neq \{0\}$ contained in $k\langle A'B\rangle$ such that $HV = V$ and $k\langle B\rangle \subset V$ with

$$\dim_k(V) + \dim_k(H) \geq \dim_k(A') + \dim_k(B),$$

it suffices to take $V_a = aV$ and $H_a = H \subset \mathbb{D}(A') \subset \mathbb{D}(A)$. Since $H \subset \mathbb{D}(A)$, we must have $Ha = aH$ for any $a \in A$ and $H(V_a) = H(aV) = aHV = aV = V_a$. Moreover $k\langle aB\rangle = ak\langle B\rangle \subset aV = V_a$ and $\dim_k(V_a) + \dim_k(H_a) \geq \dim_k(A) + \dim_k(B)$ because $\dim_k(V_a) = \dim_k(V)$ and $H_a = H$.

We can also assume that $1 \in B$ by replacing B by $B' = Bb^{-1}$. Indeed, if there exist a subfield $H' \subset \mathbb{D}(A)$ and a vector space $V' \neq \{0\}$ contained in $k\langle AB'\rangle$ such that $H'V' = V'$ and $k\langle B'\rangle \subset V'$ with

$$\dim_k(V') + \dim_k(H') \geq \dim_k(A) + \dim_k(B'),$$

it suffices to take $V = V'b$ and $H = H'$. We will have then $V = V'b \subset k\langle AB'\rangle b = k\langle AB\rangle$, $HV = H(V'b) = (HV')b = V'b = V$, $k\langle B\rangle = k\langle B'\rangle b \subset V'b = V$ and

$$\dim_k(V) + \dim_k(H) \geq \dim_k(A) + \dim_k(B)$$

since $\dim_k(B) = \dim_k(B')$ and $\dim_k(V) = \dim_k(V')$.

We thus assume in the sequel of the proof that $1 \in A \cap B$ and proceed by induction on $\dim_k(A)$. When $\dim_k(A) = 1$, we have $k\langle A\rangle = k = \mathbb{D}(A)$. It suffices to take $V_1 = V = k\langle B\rangle \neq \{0\}$ and $H_a = H = k = \mathbb{D}(A)$. Assume $\dim_k(A) > 1$. Given $e \in k\langle B\rangle$ such that $e \neq 0$, define $A(e)$ and $B(e)$ finite subsets of K^* such that

$$k\langle A(e)\rangle = k\langle A\rangle \cap k\langle B\rangle e^{-1} \text{ and } k\langle B(e)\rangle = k\langle B\rangle + k\langle A\rangle e.$$

Observe that $k\langle A(e)\rangle$ and $k\langle B(e)\rangle$ contain k since $1 \in A \cap B$. Thus we may and do assume that $1 \in A(e) \cap B(e)$. Moreover $k\langle A(e)\rangle k\langle B(e)\rangle$ is contained in $k\langle AB\rangle$. Indeed, for $v \in k\langle A\rangle \cap k\langle B\rangle e^{-1}$ and $w \in k\langle B\rangle$, we have $vw \in k\langle A\rangle k\langle B\rangle \subset k\langle AB\rangle$ because $v \in k\langle A\rangle$. If $w \in k\langle A\rangle e$, we have

$vw \in k\langle B\rangle e^{-1}k\langle A\rangle e$ because $v \in k\langle B\rangle e^{-1}$. But $e \in k\langle A\rangle$ and A is commutative. Therefore, $k\langle A\rangle e = ek(A)$ and $vw \in k\langle A\rangle k\langle B\rangle \subset k\langle AB\rangle$. We get

$$\dim_k(A(e)) + \dim_k(B(e)) = \dim_k(k\langle A\rangle \cap k\langle B\rangle e^{-1}) + \dim_k(k\langle B\rangle + k\langle A\rangle e) =$$
$$\dim_k(k\langle A\rangle e \cap k\langle B\rangle) + \dim_k(k\langle B\rangle + k\langle A\rangle e) = \dim_k(Ae) + \dim_k(B) = \dim_k(A) + \dim_k(B).$$

Also $A(e) \subset k\langle A\rangle$.

Assume $k\langle A(e)\rangle = k\langle A\rangle$ for any nonzero $e \in k\langle B\rangle$. Then $k\langle A\rangle e \subset k\langle B\rangle$ for any nonzero $e \in k\langle B\rangle$. Thus $k\langle AB\rangle \subset k\langle B\rangle$. Since $1 \in A$, we have in fact $k\langle AB\rangle = k\langle B\rangle$. The sub division ring $H = \mathbb{D}(A)$ is a field since A is commutative and it contains k since $1 \in A$. Take $V = k\langle B\rangle \neq \{0\}$. Then $HV = V$ since $AV = V$. We clearly have $V \subset k\langle AB\rangle$ and $B \subset V$ as desired.

Now assume $k\langle A(e)\rangle \neq k\langle A\rangle$ for at least one nonzero $e \in k\langle A\rangle$. Then $0 < \dim_k(A(e)) < \dim_k(A)$ and $1 \in A(e) \cap B(e)$. By our induction hypothesis, there exist a subfield H of $\mathbb{D}(A(e)) \subset \mathbb{D}(A)$ containing k and a k-vector space $V \subset k\langle A(e)B(e)\rangle \subset k\langle AB\rangle, V \neq \{0\}$ such that $HV = V$ and $k\langle B\rangle \subset k\langle B(e)\rangle \subset V$ with

$$\dim_k(V) + \dim_k(H) \geq \dim_k(A(e)) + \dim_k(B(e)) = \dim_k(A) + \dim_k(B).$$

The subfield $H \subset \mathbb{D}(A)$ and the nonzero space $V \supset k\langle B\rangle$ satisfy the statement of the lemma for the pair of subsets A, B which terminates the proof. ■

As in the proof of Theorem 2.4, we also need the following lemma which is an application of the Vandermonde determinant.

Lemma 4.3 *Let V be a n-dimensional vector space over the infinite field k. Assume $x_1, \ldots, x_n$ form a basis of V over k. Then any n vectors in the set*

$$\{x_1 + \alpha x_2 + \cdots + \alpha^{n-1}x_n \mid \alpha \in k\}$$

form a basis of V over k.

Proof. (of Theorem 4.1)

1: Let $\mathcal{B} = \{x_1, \ldots, x_n\}$ be a basis of $k\langle A\rangle$. For any $\alpha \in k$, set $x_\alpha = x_1 + \alpha x_2 + \cdots + \alpha^{n-1}x_n$. Observe that $x_\alpha \neq 0$. Thus, by Lemma 4.2, there exist a subfield H_α such that $k \subset H_\alpha \subset \mathbb{D}(A) \subset K$ and a k-vector space $V_\alpha \subset k\langle AB\rangle$ with $x_\alpha B \subset V_\alpha$, $H_\alpha V_\alpha = V_\alpha$ and $\dim_k(V_\alpha) + \dim_k(H_\alpha) \geq \dim_k(A) + \dim_k(B)$. Since $V_\alpha \neq \{0\}$ and H_α stabilizes $V_\alpha \subset k\langle AB\rangle$, there exists a nonzero vector $v \in V_\alpha$ such that $H_\alpha v \subset k\langle AB\rangle$. Hence $H_\alpha \subset v^{-1}k\langle AB\rangle$ and $\dim_k(H_\alpha)$ is finite. Therefore $H_\alpha \subset \mathbb{D}(A)$ is a finite field extension of k. Let F be the algebraic closure of k in $\mathbb{D}(A)$. The elements of H_α belong to $\mathbb{D}(A)$ and are algebraic over k since $\dim_k(H_\alpha)$ is finite. Therefore $H_\alpha \subset F$ for any $\alpha \in k$.

The field $\mathbb{D}(A)$ is finitely generated by $x_1, \ldots, x_n$. Therefore, if $F = \mathbb{D}(A)$, each x_i is algebraic over k and $\dim_k(F)$ is finite. If $F \subsetneqq \mathbb{D}(A)$, we can choose a family $y_1, \ldots, y_r$ in $\mathbb{D}(A)$ such that $k' = k(y_1, \ldots, y_r)$ is purely transcendental over k and $\mathbb{D}(A)$ is algebraic finitely generated over k'. Then $\dim_{k'}(\mathbb{D}(A))$ is finite. Thus $\dim_k(F) = \dim_{k'}(F(y_1, \ldots, y_r)) \leq \dim_{k'}(\mathbb{D}(A))$ is finite. So in both cases, we obtain that $\dim_k(F)$ is finite.

By the separability hypothesis, we obtain that the extension F is separable over k. Thus, it only admits a finite number of intermediate extensions. There should exists n distinct elements $\alpha_1, \ldots, \alpha_n$ in k such that

$$H_{\alpha_1} = H_{\alpha_2} = \cdots = H_{\alpha_n} = H.$$

By Lemma 4.3, $x_{\alpha_1}, \ldots, x_{\alpha_n}$ form a basis of $k\langle A\rangle$ over k. We thus have $k\langle AB\rangle = \sum_{i=1}^{n} x_{\alpha_i} k\langle B\rangle \subset \sum_{i=1}^{n} V_{\alpha_i}$ since $x_{\alpha_i} k\langle B\rangle \subset V_{\alpha_i}$ for any $i = 1, \ldots, n$. On the other hand, $V_{\alpha_i} \subset k\langle AB\rangle$ for any $i = 1, \ldots, n$. Hence $k\langle AB\rangle = \sum_{i=1}^{n} V_{\alpha_i}$ is stabilized by H. If $k \subsetneqq H$, $k\langle AB\rangle$ is periodic. Otherwise, $k = H$ and we have

$$\dim_k(A) + \dim_k(B) \leq \dim_k(V_{\alpha_i}) - 1$$

for any $i = 1, \ldots, n$. Since $V_{\alpha_i} \subset k\langle AB\rangle$, we obtain

$$\dim_k(A) + \dim_k(B) \leq \dim_k(AB) - 1$$

as desired.

2: In part (II) of the proof of Theorem 2.7, we does not use any commutativity hypothesis on K. So both assertions of Theorem 4.1 are equivalent by exactly the same arguments. ■

Remarks:

1. When B is assumed commutative, we have a similar statement by replacing left periodicity by right periodicity.

2. Observe also that the separability hypothesis is always satisfied when k has characteristic 0.

3. Theorem 4.1 means that when A is commutative and $\dim_k(AB) \leq \dim_k(A) + \dim_k(B) - 2$, $k\langle AB\rangle$ is an left H-module. When $A = B$ or $A^{-1} = B$, this suggests that spaces $k\langle A\rangle$ with $\dim_k(A^2) = O(\dim A)$ should have interesting properties related to some H-modules of K where H is a subdivision ring of K. We will precise this observation in the following paragraphs.

Theorem 4.1 also permits to construct k-subspaces in K containing subdivision rings. Assume $\dim_k(K)$ is finite and let $a_1, \ldots, a_n$ be a sequence of elements in K^* *distinct from* 1 (with repetition allowed). For any nonempty subset $S \subset \{1, \ldots, n\}$ write $a_S := \prod_{i \in S} a_i$. Denote by A_S the finite subset of K^* containing the elements a_S when S runs over the nonempty subsets of $\{1, \ldots, n\}$.

Corollary 4.4 *Assume k is infinite, $\dim_k(K) = n > 1$ and every element of K is separable over k. Then with the above notation, the space $V = k\langle A_S\rangle$ contains a sub division ring $H \supsetneqq k$.*

Proof. For any $i = 1, \ldots, n$, put $A_i = \{1, a_i\}$ and $V' = k\langle A_1 \cdots A_n\rangle$. Write $p_i = a_1 \cdots a_i$ for any $i = 1, \ldots, n$. If $p_1, \ldots, p_n$ are linearly independent, $V = K$ since $\dim_k(K) = n$. In particular $1 \in V$. Otherwise, there exist $i_0 \in \{1, \ldots, n\}$ and elements $\alpha_i, i = i_0, \ldots, n$ in k such that

$$\sum_{i=i_0}^{n} \alpha_i p_i = 0 \text{ and } \alpha_{i_0} \neq 0.$$

Dividing by $\alpha_{i_0} p_{i_0}$, we obtain

$$1 = -\sum_{i=i_0+1}^{n} \frac{\alpha_i}{\alpha_{i_0}} a_{i_0+1} \cdots a_i.$$

We thus also obtain that $1 \in V$.

We have proved that $1 \in V$. This implies that $V' = V$. If V is periodic, then $H(A_1 \cdots A_n) \subset V$ since $1 \in V$ and we are done. We can thus assume that $k\langle A_1 \cdots A_n \rangle$ is not periodic. As the sets $A_i, i = 1, \ldots n$ are commutative, we can apply 2 of Theorem 2.7 which gives

$$\dim_k(A_1 \cdots A_n) \geq \sum_{i=1}^{n} \dim_k(A_i) - (n-1) \geq 2n - (n-1) \geq n+1.$$

We thus obtain a contradiction with the hypothesis $\dim_k(K) = n$. ■

4.2 The 3/2 bound

We say that $V = k\langle A \rangle$ where A is a finite subset of K^* is a *space of small doubling* when $\dim_k(A^2) = O(\dim_k(A))$. Simplest examples of spaces of small doubling are the spaces $V = k\langle A \rangle$ containing 1 and such that $\dim_k(A^2) = \dim_k(A)$. Then by Lemma 2.1, V is a division ring containing k. In general, a space of small doubling $k\langle A \rangle$ is not a division ring, neither a left or right H-module for a division ring $k \subset H \subset K$. Nevertheless, the following elementary proposition shows that the spaces such that $\dim_k(A^2) < \frac{3}{2}\dim_k(A)$ are necessarily division rings. The situation is thus analogue to that of product sets in groups.

Proposition 4.5 *Let K be a division ring containing the field k in its center. Consider A a nonempty subset of K^*.*

1. *If $\dim_k(A^2) < 2\dim_k(A)$, then $k\langle AA^{-1} \rangle = k\langle A^{-1}A \rangle$.*
2. *If $\dim_k(A^{-1}A) < \frac{3}{2}\dim_k(A)$, then $k\langle AA^{-1} \rangle$ is a division ring containing k.*
3. *If $\dim_k(A^2) < \frac{3}{2}\dim_k(A)$, then $k\langle AA^{-1} \rangle = k\langle A^{-1}A \rangle$ is a division ring containing k.*

Proof. 1: Consider a_1 and a_2 in A. Then $\dim_k(Aa_1) = \dim_k(Aa_2) = \dim_k(A)$. Moreover $k\langle Aa_1 \rangle$ and $k\langle Aa_2 \rangle$ are k-subspaces of $k\langle A^2 \rangle$. Therefore $k\langle Aa_1 \rangle \cap k\langle Aa_2 \rangle \neq \{0\}$ and there exists α_1, α_2 in $k\langle A \rangle$ such that $\alpha_1 a_1 = \alpha_2 a_2$, that is $a_1 a_2^{-1} = \alpha_2^{-1}\alpha_1$. This shows that $AA^{-1} \subset k\langle A^{-1}A \rangle$ and therefore $k\langle AA^{-1} \rangle \subset k\langle A^{-1}A \rangle$. We obtain $k\langle A^{-1}A \rangle \subset k\langle AA^{-1} \rangle$ similarly by considering $k\langle a_1 A \rangle$ and $k\langle a_2 A \rangle$.

2: Set $V = k\langle A^{-1}A \rangle$. Since V is finite-dimensional over k and contains 1, it suffices by Lemma 2.1 to prove it is closed under multiplication. In fact V is generated by the elements of $A^{-1}A$ so we only need to prove that the product of two such elements remains in $k\langle A^{-1}A \rangle$. So consider x and y two elements in $A^{-1}A$. Write $x = a_1 a_2^{-1}$ and $y = a_3 a_4^{-1}$. The k-subspaces $k\langle a_1^{-1}A \rangle$ and $k\langle a_2^{-1}A \rangle$ have both dimension $\dim_k(A)$ and are contained in V with $\dim_k(V) < \frac{3}{2}\dim_k(A)$. Thus $\dim_k(k\langle a_1^{-1}A \rangle \cap k\langle a_2^{-1}A \rangle) > \frac{1}{2}\dim_k(A)$. By left multiplying by a_2, we obtain $\dim_k(k\langle x^{-1}A \rangle \cap k\langle A \rangle) > \frac{1}{2}\dim_k(A)$. Similarly $\dim_k(k\langle a_3^{-1}A \rangle \cap k\langle a_4^{-1}A \rangle) > \frac{1}{2}\dim_k(A)$ and after a left multiplication by a_3, we get $\dim_k(k\langle A \rangle \cap k\langle yA \rangle) > \frac{1}{2}\dim_k(A)$. Now $k\langle x^{-1}A \rangle \cap k\langle A \rangle$ and $k\langle A \rangle \cap k\langle yA \rangle$ are two subspaces of $k\langle A \rangle$ with dimensions at least $\frac{1}{2}\dim_k(A)$. Therefore they intersect and there exist

$\alpha, \alpha_1, \alpha_2$ in $k\langle A\rangle$ such that $\alpha = y\alpha_1 = x^{-1}\alpha_2$. In particular, $xy = \alpha_2\alpha_1^{-1} \in k\langle A\rangle k\langle A^{-1}\rangle \subset k\langle AA^{-1}\rangle$ as desired.

3: By 1, we have $V = k\langle AA^{-1}\rangle = k\langle A^{-1}A\rangle$. Observe that AA^{-1} contains 1 and $(AA^{-1})^{-1} = AA^{-1}$. By Lemma 2.1, it thus suffices to prove that for any x, y in AA^{-1}, $xy^{-1} \in V$. Consider x and y two elements in AA^{-1}. Write $x = a_1a_2^{-1}$ and $y = a_3a_4^{-1}$ with a_1, a_2, a_3, a_4 in A. We obtain $\dim_k(k\langle a_1A\rangle \cap k\langle a_2A\rangle) > \frac{1}{2}\dim_k(A)$ which gives $\dim_k(k\langle A\rangle \cap k\langle a_2^{-1}a_1A\rangle) = \dim_k(k\langle A\rangle \cap k\langle x^{-1}A\rangle) > \frac{1}{2}\dim_k(A)$. We have also $\dim_k(k\langle a_3A\rangle \cap k\langle a_4A\rangle) > \frac{1}{2}\dim_k(A)$ which gives $\dim_k(k\langle A\rangle \cap k\langle y^{-1}A\rangle) > \frac{1}{2}\dim_k(A)$. There thus exist $\alpha, \alpha_1, \alpha_2$ in $k\langle A\rangle$ such that $\alpha = x^{-1}\alpha_1 = y^{-1}\alpha_2$. Hence $xy^{-1} = \alpha_1\alpha_2^{-1} \in V$. ■

Remark: There exists some analogues of the previous proposition for subsets A in arbitrary groups. These analogues admit fewer interesting refinements. Unfortunately, their proofs implicitly use the equality $|A| = \left|A^{-1}\right|$ and a careful counting of the number of representations of elements z in AA^{-1} on the form $z = xy^{-1}$. We have seen that $\dim_k(A^{-1}) \neq \dim_k(A)$ in general. Moreover, a similar counting seems have no natural generalization in the space $k\langle AA^{-1}\rangle$.

4.3 Linear Hamidoune connectivity

The notion of connectivity for a subset S of a group G was developed by Hamidoune in [7]. As suggested by Tao in [15], it is interesting to generalize Hamidoune definition by introducing an additional parameter λ. The purpose of this paragraph is to define a natural linear version of this connectivity used in [15] suited for the k-subspaces V in K where K is a division ring containing k in its center. Assume V is a finite-dimensional fixed k-subspace of K and λ a real parameter. For any finite-dimensional k-subspace W of K, we define

$$c(W) := \dim_k(WV) - \lambda\dim_k(W). \tag{10}$$

For any $x \in K^*$, we have immediately that $c(xW) = c(W)$.

Lemma 4.6 *For any finite-dimensional subspaces W_1, W_2, V of K, we have*

$$c(W_1 + W_2) + c(W_1 \cap W_2) \leq c(W_1) + c(W_2).$$

Proof. We have

$$\dim_k(W_1 + W_2) + \dim_k(W_1 \cap W_2) = \dim_k(W_1) + \dim_k(W_2) \tag{11}$$

and

$$\dim_k(k\langle W_1V\rangle + k\langle W_2V\rangle) + \dim_k(k\langle W_1V\rangle \cap k\langle W_2V\rangle) = \dim_k(W_1V) + \dim_k(W_2V).$$

Observe that $k\langle (W_1 + W_2)\cdot V\rangle = k\langle W_1V\rangle + k\langle W_2V\rangle$ and $k\langle (W_1 \cap W_2)\cdot V\rangle \subset k\langle W_1V\rangle \cap k\langle W_2V\rangle$. This gives

$$\dim_k(k\langle (W_1 + W_2)\cdot V\rangle) + \dim_k(k\langle (W_1 \cap W_2)\cdot V\rangle) \leq \dim_k(k\langle W_1V\rangle) + \dim_k(k\langle W_2V\rangle). \tag{12}$$

We then obtain the desired equality by subtracting to (12), λ copies of (11). ■

Similarly to [7], we define the *connectivity* $\kappa = \kappa(V)$ as the infimum of $c(W)$ over all finite-dimensional k-subspaces of K. A *fragment* of V is a finite-dimensional k-subspace of K which

attains the infimum κ. An *atom* of V is a fragment of minimal dimension. Since $c(xW) = c(W)$, any left translate of a fragment is a fragment and any left translate of an atom is an atom. Since $\dim_k(WV) \geq \dim_k(W)$, we have

$$c(W) \geq (1-\lambda)\dim_k(W). \tag{13}$$

We observe that when $\lambda < 1$, $c(W)$ is always positive and takes a discrete set of values. Therefore, when $\lambda < 1$, there exists at least one fragment and at least one atom. Let W_1 and W_2 be two fragments with nonzero intersection. By the previous lemma, we derive

$$c(W_1 + W_2) + c(W_1 \cap W_2) \leq c(W_1) + c(W_2) \leq 2\kappa.$$

Since $W_1 + W_2$ and $W_1 \cap W_2$ are finite-dimensional and not reduced to $\{0\}$, we must have $c(W_1 + W_2) \geq \kappa$ and $c(W_1 \cap W_2) \geq \kappa$. Hence $c(W_1 + W_2) = c(W_1 \cap W_2) = \kappa$. This means that $W_1 + W_2$ and $W_1 \cap W_2$ are also fragments. If we assume now that W_1 and W_2 are atoms, we obtain that $W_1 = W_2$ or $W_1 \cap W_2 = \{0\}$.

Proposition 4.7 *Let V be a finite-dimensional k-subspace of K a division ring containing k in its center.*

1. *There exists a unique atom H for V containing 1.*
2. *This atom is a division ring containing k in its center.*
3. *Moreover the atoms of V are the right modules xH where x runs over K^*.*

Proof. Since there exists at least one atom and the left translate of any atom is an atom, there exists one atom H containing 1. Now, this atom must be unique because atoms are equal or with an intersection reduced to $\{0\}$. In particular, for any $x \in K$, $H = xH$ or $H \cap xH = \{0\}$. We claim that this implies that H is a division ring. Indeed, for any $h \in H$, $H \cap h^{-1}H$ contains 1 for $1 \in H$. Therefore $h^{-1}H = H$ and $H = hH$. So H is stable by multiplication. We then deduce that H is a division ring as in the proof of Lemma 2.1. Finally, given any atom W of V, we must have $w^{-1}W = H$ for any nonzero $w \in W$ since H is the unique atom containing 1 and $w^{-1}H$ is an atom. ■

4.4 Tao theorem for division rings

The following theorem is the linear version of Theorem 1.2 in [15].

Theorem 4.8 *Let K be a division ring containing the field k in its center. Consider V, W finite-dimensional k-subspaces of K such that $\dim_k(W) \geq \dim_k(V)$ and $\dim_k(WV) \leq (2-\varepsilon)\dim_k(V)$ for some real ε such that $0 < \varepsilon < 2$. Then one of the following statements holds :*

- *There exists a division ring H containing k such that $\dim_k(H) \leq (\frac{2}{\varepsilon} - 1)\dim_k(V)$ and V is contained in a left module Hx with $x \in K^*$.*
- *There exists a division ring H containing k such that $\dim_k(H) \leq (\frac{2}{\varepsilon} - 1)/(\frac{2}{\varepsilon} + 1)\dim_k(V)$ and a finite subset X of K^* with $|X| \leq \frac{2}{\varepsilon} - 1$ such that $V \subset \bigoplus_{x \in X} Hx$.*

Proof. We apply linear Hamidoune connectivity with $\lambda = 1 - \frac{\varepsilon}{2}$. We have by (13) $c(U) \geq \frac{\varepsilon}{2}\dim_k(U)$ for any k-subspace U. This can be rewritten

$$\dim_k(U) \leq \frac{2}{\varepsilon}c(U). \tag{14}$$

We also get

$$c(W) := \dim_k(WV) - (1-\frac{\varepsilon}{2})\dim_k(W) \leq (2-\varepsilon)\dim_k(V) - (1-\frac{\varepsilon}{2})\dim_k(V) = (1-\frac{\varepsilon}{2})\dim_k(V).$$

since $\dim_k(k\langle WV\rangle) \geq (2-\varepsilon)\dim_k(V)$ and $\dim_k(W) \geq \dim_k(V)$. By Proposition 4.7, the unique atom containing 1 is a division ring H. By definition of an atom, we should have

$$\kappa = c(H) \leq c(W) \leq (1-\frac{\varepsilon}{2})\dim_k(V).$$

We obtain therefore by using (14) with $U = H$

$$\dim_k(H) \leq \frac{2}{\varepsilon}c(H) \leq \frac{2}{\varepsilon}c(W) \leq (\frac{2}{\varepsilon}-1)\dim_k(V).$$

If V is contained in a left module Hx, we are done. Assume V intersects at least two such left H-modules. By using that $c(H) = \dim_k(HV) - (1-\frac{\varepsilon}{2})\dim_k(H)$ and the previous inequality $c(H) \leq (1-\frac{\varepsilon}{2})\dim_k(V)$, we get

$$\dim_k(HV) \leq (1-\frac{\varepsilon}{2})\dim_k(V) + (1-\frac{\varepsilon}{2})\dim_k(H). \tag{15}$$

Since V intersects at least two left H-modules, we must have $\dim_k(HV) \geq 2\dim_k(H)$. By using (15), this gives

$$\dim_k(H) \leq \frac{1-\frac{\varepsilon}{2}}{1+\frac{\varepsilon}{2}}\dim_k(V) < \dim_k(V).$$

We can also bound $\dim_k(V)$ by $\dim_k(HV)$ in (15). This yields

$$\dim_k(HV) \leq (\frac{2}{\varepsilon}-1)\dim_k(H). \tag{16}$$

Now $k\langle HV\rangle$ is left H-invariant and finite-dimensional because $\dim_k(HV) \leq \dim_k(V)^2$. There thus exists a finite subset X of K^* such that

$$k\langle HV\rangle = \bigoplus_{x\in X} Hx.$$

By (16), we should have $|X| \leq (\frac{2}{\varepsilon}-1)$. Moreover $V \subset k\langle HV\rangle$ which concludes the proof. ■

Remark: When $\dim_k(A^2) \leq (2-\varepsilon)\dim_k(A)$, we can apply Theorem 4.8 with $V = W = k\langle A\rangle$ and obtain precise informations on the covering of V by left H-modules. This can be interpreted as a classification of subspaces with small doubling similar to the classification of sets with small doubling obtained in [15].

5 Plünnecke-type estimates in associative algebras

The arguments we have used in the proofs of Section 3 to obtain Plünnecke-type estimates in division rings remains in fact valid in the more general context of associative unital algebras with suitable hypotheses on the subspaces considered. More precisely, let $\mathcal{A}$ be a unital associative algebra over the field k. Write $U(\mathcal{A})$ for the group of invertible elements in $\mathcal{A}$. As a classical example, we can consider any matrix algebra containing the identity matrix.

Given a subset A of $\mathcal{A}$ and $x \in \mathcal{A}$, $\dim_k(xA)$ and $\dim_k(Ax)$ do not necessarily coincide with $\dim_k(A)$. This is nevertheless true when $x \in U(\mathcal{A})$. Let A, B, C be nonempty finite subsets of $\mathcal{A}$ such that

- $B \cap U(\mathcal{A}) \neq \emptyset$,
- $C \subset U(\mathcal{A})$.

Then for any k-subspace $V \neq \{0\}$ of $k\langle A\rangle$,

$$\dim_k(VB) \geq \dim_k(V), \quad r(V) = \frac{\dim_k(VB)}{\dim_k(V)} > 0 \text{ and } \rho := \min_{V \subset k\langle A\rangle, V \neq \{0\}} r(V) > 0.$$

There thus also exists a nonempty set $X \subset A$ such that $r(k\langle X\rangle) = \rho$. One then easily verifies that the arguments used in the proof of Proposition 3.2 remain valid for the algebra $\mathcal{A}$ with the previous assumptions on A, B and C. We then obtain the following statements which generalize Corollary 3.3, Theorem 3.4 and Theorem 3.5.

Corollary 5.1 *Consider A and B two finite subsets of $\mathcal{A}$ with $B \cap U(\mathcal{A}) \neq \emptyset$. Assume α is a positive real such that $\dim_k(AB) \leq \alpha \dim_k(A)$. Then there exists a subset $X \subset A$ such that for any finite subset C of $U(\mathcal{A})$ $\dim_k(CXB) \leq \alpha \dim_k(CX)$.*

Theorem 5.2 *Assume $\mathcal{A}$ is commutative. Let A be a nonempty finite subset of $\mathcal{A}$ and B be nonempty finite subset of $U(\mathcal{A})$. Assume that $\dim_k(AB) \leq \alpha \dim_k(A)$ where α is a positive real. Then, there exists a subset $X \subset A$ such that for any positive integer n*

$$\dim_k(XB^n) \leq \alpha^n \dim_k(X).$$

In particular if $A \subset U(\mathcal{A})$, $\dim_k(A^2) \leq \alpha \dim_k(A)$ implies that $\dim(A^n) \leq \alpha^n \dim(A)$.

Theorem 5.3 *Let A, B, C be finite nonempty subsets of $\mathcal{A}$ such that $B \subset U(\mathcal{A})$. Then*

$$\dim_k(ABC)^2 \leq \dim_k(AB) \dim_k(BC) \max_{b \in B}\{\dim_k(AbC)\}.$$

In particular, when $\mathcal{A}$ is commutative, we have

$$\dim_k(ABC)^2 \leq \dim_k(AB) \dim_k(BC) \dim_k(AC).$$

Remark: The proof of the Kneser's type theorems we have obtained in Section 4 (and also that of the linear Olson theorem) cannot be so easily adapted to the case of associative algebras. Indeed, given A and B subsets of $U(\mathcal{A})$, $k\langle A\rangle \cap k\langle B\rangle$ may have an empty intersection with $U(\mathcal{A})$. In particular arguments based on the use of linear versions of Dyson or Kemperman transforms fail.

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