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Jérôme Le Rousseau, Matthieu Léautaud, Luc Robbiano. Controllability of a parabolic system with a diffusive interface. *Journal of the European Mathematical Society*, 2013, 15 (4), pp.1485-1574. 10.4171/JEMS/397. hal-00618234

**HAL Id: hal-00618234**

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Submitted on 1 Sep 2011

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# Controllability of a parabolic system with a diffusive interface\*

Jérôme Le Rousseau<sup>†</sup>, Matthieu Léautaud<sup>‡§</sup> and Luc Robbiano<sup>¶</sup>

June 8, 2011

## Abstract

We consider a linear parabolic transmission problem across an interface of codimension one in a bounded domain or on a Riemannian manifold, where the transmission conditions involve an additional parabolic operator on the interface. This system is an idealization of a three-layer model in which the central layer has a small thickness  $\delta$ . We prove a Carleman estimate in the neighborhood of the interface for an associated elliptic operator by means of partial estimates in several microlocal regions. In turn, from the Carleman estimate, we obtain a spectral inequality that yields the null-controllability of the parabolic system. These results are uniform with respect to the small parameter  $\delta$ .

## Keywords

Elliptic operator; parabolic system; transmission problem; controllability; spectral inequality;

**AMS 2000 subject classification:** 35J15; 35J57; 35S15; 35K05; 93B05;

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\*The authors were partially supported by l'Agence Nationale de la Recherche under grant ANR-07-JCJC-0139-01.

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# 1 Introduction

When considering elliptic and parabolic operators in  $\mathbb{R}^n$  with a diffusion coefficient that jumps across an interface of codimension one, say  $\{x_n = 0\}$ , we can interpret the associated equations as two equations with solutions that are coupled at the interface via transmission conditions at  $x_n = 0$ , viz. in the parabolic case,

$$\partial_t y_1 + \nabla_x c_1 \nabla_x y_1 = f_1 \quad \text{in } \{x_n < 0\}, \quad \partial_t y_2 + \nabla_x c_2 \nabla_x y_2 = f_2 \quad \text{in } \{x_n > 0\}, \quad (1.1)$$

and

$$y_1|_{x_n=0^-} = y_2|_{x_n=0^+}, \quad c_1 \partial_{x_n} y_1|_{x_n=0^-} = c_2 \partial_{x_n} y_2|_{x_n=0^+}. \quad (1.2)$$

Here, we are interested in parabolic/elliptic models in which part of the diffusion occurs along the interface. Then the transmission conditions are of higher order, involving differentiations in the direction of the interface. Such a model can be viewed as an idealization of two diffusive media separated by a thin membrane. We derive this model starting from three media and formally letting the thickness of the intermediate layer become very small. We introduce a small parameter  $\delta > 0$  that measures the thickness of this layer. Questions such as unique continuation, observation and controllability are natural for such a model. This is the main goal of the present article.

Most of the analysis that we shall carry concerns a related elliptic operator, including an additional variable. Our key result is the derivation of a Carleman estimate for this operator (see Theorem 1.2 below). The general form of Carleman estimates for a second-order elliptic operator  $P$  is (local form)

$$h \|e^{\varphi/h} w\|_{L^2}^2 + h^3 \|e^{\varphi/h} \nabla w\|_{L^2}^2 \leq Ch^4 \|e^{\varphi/h} Pw\|_{L^2}^2, \quad (1.3)$$

for  $h$  sufficiently small, an appropriately chosen weight function  $\varphi$ , and for smooth compactly supported functions  $w$ . We then deduce an interpolation inequality and a spectral inequality for the original operator in the spirit of the work [LR95]. This spectral inequality then yields the null controllability of the considered parabolic system. A important feature of the results we obtain here is their uniformity in the thickness parameter  $\delta$ . In particular this allows us to recover the earlier results obtained on (1.1)–(1.2) in [LR10]; this corresponds to the limit  $\delta \rightarrow 0$  in the model we consider here.

## 1.1 Setting

Let  $(\Omega, g)$  be a smooth compact  $n$ -dimensional ( $n \geq 2$ ) connected Riemannian manifold (with or without boundary), with  $g$  denoting the metric, and  $S$  a  $n-1$ -dimensional smooth submanifold of  $\Omega$  (without boundary). We assume<sup>1</sup> that  $\Omega \setminus S = \Omega_1 \cup \Omega_2$  with  $\Omega_1 \cap \Omega_2 = \emptyset$ , so that  $\Omega_1$  and  $\Omega_2$  are two smooth open subsets of  $\Omega$ . Endowed with the metric  $g|_{T(S)}$ ,  $S$  has a Riemannian structure. We denote by  $\partial_\eta$  a non vanishing vector field defined in a neighborhood of  $S$  and normal to  $S$  (for the Riemannian metric). We choose the vector field  $\partial_\eta$  outgoing from  $\Omega_1$ , incoming in  $\Omega_2$ . In local coordinates, we have

$$\partial_\eta = \sum_j \eta^j \partial_{x_j}, \quad \text{with } \eta^j = \lambda \sum_k n_k g^{jk}, \quad |\eta|_g = 1,$$

where  $g^{ij} g_{jk} = \delta_k^i$ ,  $\lambda^2 = (g^{ij} n_i n_j)^{-1}$ , and  $n$  is the normal to  $S$  for the Euclidean metric in the local coordinates, outgoing from  $\Omega_1$ , incoming in  $\Omega_2$ . In fact  $\lambda|_S^2 = \det(g)/\det(g|_{T(S)})$  at  $S$ .

The covariant gradient and the divergence operators are given in local coordinates by

$$\nabla_g = \sum_i g^{ij} \partial_{x_i}, \quad \text{div}_g v = \frac{1}{\sqrt{\det(g)}} \sum_i \partial_{x_i} (\sqrt{\det(g)} v_i),$$

---

<sup>1</sup>other geometrical situations can be dealt with because of the local nature of the estimates we prove here. See Section 1.3.2 below.

with similar definition for the gradient  $\nabla^s = \nabla_{g|_{T(S)}}$  and divergence  $\operatorname{div}^s = \operatorname{div}_{g|_{T(S)}}$  on the interface  $S$  with the metric  $g|_{T(S)}$ .

We consider a (scalar) diffusion coefficient  $c(x)$  with  $c|_{\Omega_i} \in \mathcal{C}^\infty(\overline{\Omega_i})$ ,  $i = 1, 2$ , yet discontinuous across  $S$  and satisfies  $c(x) \geq c_{\min} > 0$  uniformly for  $x \in \Omega_1 \cup \Omega_2$ . We set

$$\Delta_c = \operatorname{div}_g c(x) \nabla_g = \frac{1}{\sqrt{\det(g)}} \sum_{i,j} \partial_{x_i} (c g^{ij} \sqrt{\det(g)} \partial_{x_j}), \quad \text{in } \Omega_1 \cup \Omega_2,$$

in local coordinates. Let us denote  $c^s$  a smooth (scalar) diffusion coefficient on  $S$  satisfying  $c^s(x) \geq c_{\min}^s > 0$ . Similarly we define  $\Delta_{c^s} = \operatorname{div}^s c^s \nabla^s$  as a second-order elliptic differential operator on  $S$ .

In what follows, we shall use the notation  $z|_{S_j} = (z|_{\Omega_j})|_S$ ,  $j = 1, 2$ , for the traces of functions on  $S$ .

Given a time  $T > 0$ , we consider the following parabolic control problem

$$\begin{cases} \partial_t z - \Delta_c z = \mathbb{1}_\omega u & \text{in } (0, T) \times \Omega_1 \cup \Omega_2, \\ \partial_t z^s - \Delta_{c^s} z^s = \frac{1}{\delta} ((c \partial_\eta z)|_{S_2} - (c \partial_\eta z)|_{S_1}) & \text{in } (0, T) \times S, \\ z|_{S_1} = z^s = z|_{S_2} & \text{in } (0, T) \times S, \\ z|_{\partial\Omega} = 0; \end{cases} \quad (1.4)$$

with some initial data in  $L^2(\Omega_1 \cup \Omega_2) \times L^2(S)$ . Here,  $\delta$  denotes a *bounded* parameter,  $0 < \delta \leq \delta_0$ , and  $\omega$  is an open nonempty subset of  $\Omega_1 \cup \Omega_2$ . Let us suppose for instance that  $\omega \subset \Omega_2$ . The function  $u$  is a control function and the null-controllability problem concerns the ability to drive the solution  $(z, z^s)$  to zero at the final time  $T$ .

Such a coupling condition at the interface was considered in [KZ06] and [LZ10] for the associated hyperbolic system. In Appendix A, we briefly explain how this model can be formally derived. This model corresponds to two diffusive media separated by a thin layer in which diffusion also occurs. The parameter  $\delta$  is then a measure of the thickness of this intermediate layer. In the derivation of the model  $\delta$  is assumed small.

We present here some function spaces and operators and their basic properties to formulate Problem (1.4) in a more abstract way. The reader is referred to Section 2 for the details. We introduce the Hilbert space  $\mathcal{H}_\delta^0 = L^2(\Omega_1 \cup \Omega_2) \times L^2(S)$  with the inner product

$$(Z, \tilde{Z})_{\mathcal{H}_\delta^0} = (z, \tilde{z})_{L^2(\Omega_1 \cup \Omega_2)} + \delta (z^s, \tilde{z}^s)_{L^2(S)}, \quad Z = (z, z^s), \quad \tilde{Z} = (\tilde{z}, \tilde{z}^s),$$

where

$$(z, \tilde{z})_{L^2(\Omega_1 \cup \Omega_2)} = \int_{\Omega_1 \cup \Omega_2} z \tilde{z} \, d\nu, \quad (z^s, \tilde{z}^s)_{L^2(S)} = \int_S z^s \tilde{z}^s \, d\nu^s, \quad (1.5)$$

with  $d\nu = \sqrt{\det(g)} \, dx$  and  $d\nu^s = \sqrt{\det(g|_{T(S)})} \, dy$ . We also introduce the following Hilbert space

$$\mathcal{H}_\delta^1 = \{Z = (z, z^s) \in H^1(\Omega_1 \cup \Omega_2) \times H^1(S); z|_{\partial\Omega} = 0; z|_{S_1} = z^s = z|_{S_2}\}, \quad (1.6)$$

with the inner product

$$(Z, \tilde{Z})_{\mathcal{H}_\delta^1} = (Z, \tilde{Z})_{\mathcal{H}_\delta^0} + (c \nabla_g z, \nabla_g \tilde{z})_{L^2(\Omega_1 \cup \Omega_2)} + \delta (c^s \nabla^s z, \nabla^s \tilde{z}^s)_{L^2(S)}, \quad Z = (z, z^s), \quad \tilde{Z} = (\tilde{z}, \tilde{z}^s).$$

Problem (1.4) can be written as

$$\partial_t Z + A_\delta Z = Bu, \quad (1.7)$$

where the state is  $Z = (z, z^s) \in \mathcal{H}_\delta^0$  and the operator  $A_\delta$  reads

$$A_\delta Z = \begin{pmatrix} -\Delta_c z & -\Delta_c z \\ -\Delta_{c^s} z^s - \frac{1}{\delta} ((c \partial_\eta z)|_{S_2} - (c \partial_\eta z)|_{S_1}) & -\Delta_{c^s} z^s \end{pmatrix}, \quad (1.8)$$

with domain

$$D(A_\delta) = \{(z, z^s) \in \mathcal{H}_\delta^1; A_\delta(z, z^s) \in \mathcal{H}_\delta^0\}. \quad (1.9)$$

The operator  $(A_\delta, D(A_\delta))$  is nonnegative self-adjoint on  $\mathcal{H}_\delta^0$ . The control operator  $B$  is the bounded operator from  $L^2(\Omega_1 \cup \Omega_2)$  into  $L^2(\Omega_1 \cup \Omega_2) \times L^2(S)$  given by  $B : u \mapsto {}^t(\mathbb{1}_\omega u, 0)$ . We shall prove that System (1.7), i.e. System (1.4), is well-posed for an initial condition in  $\mathcal{H}_\delta^0$ .

**Remark 1.1.** In the limit  $\delta \rightarrow 0$ , from System (1.4), we obtain the following system (see Section 2.2 for a proof of convergence)

$$\begin{cases} \partial_t z - \Delta_c z = \mathbf{1}_\omega u & \text{in } (0, T) \times \Omega_1 \cup \Omega_2, \\ (c\partial_\eta z)|_{S_2} = (c\partial_\eta z)|_{S_1} \text{ and } z|_{S_1} = z|_{S_2} & \text{in } (0, T) \times S, \\ z|_{\partial\Omega} = 0; \end{cases} \quad (1.10)$$

which corresponds to the case studied in [LR10]. We also refer to the recent works [DOP02, BDL07a, Le 07, BDL07b, BGL07, LR11, LL11, BDL11] for the derivation of Carleman estimates for elliptic and parabolic operators with such coefficients with applications to controllability and inverse problems.

## 1.2 Statement of the main results

### 1.2.1 Carleman estimate

The Carleman estimate we prove concerns an augmented elliptic operator: we introduce an additional coordinate,  $x_0 \in (0, X_0) \subset \mathbb{R}$ , so that  $(x_0, x) \in (0, X_0) \times \Omega$ . This variable  $x_0$  was introduced in [LR95]; there it allowed to obtain the null-controllability of the heat equation. This approach was followed in several works [LZ98, JL99, LR10]. It was also used to prove stabilization properties of the wave equation [Leb96, LR97].

We consider the  $n + 1$ -dimensional partially determined elliptic problem

$$\begin{cases} -\partial_{x_0}^2 w - \Delta_c w + \nabla_a w + bw = f & \text{in } (0, X_0) \times (\Omega_1 \cup \Omega_2), \\ -\partial_{x_0}^2 w^s - \Delta_{c^s} w^s + \nabla_a^s w^s + b^s w^s \\ \quad = \frac{1}{\delta} ((c\partial_\eta w)|_{(0, X_0) \times S_2} - (c\partial_\eta w)|_{(0, X_0) \times S_1} + \theta^s) & \text{in } (0, X_0) \times S, \\ w|_{(0, X_0) \times S_1} = w^s + \theta^1 \text{ and } w|_{(0, X_0) \times S_2} = w^s + \theta^2 & \text{in } (0, X_0) \times S. \end{cases} \quad (1.11)$$

Note that we add lower-order terms to the elliptic operators here:  $\nabla_a$  (resp.  $\nabla_a^s$ ) denotes any *smooth* vector field on  $\Omega_1 \cup \Omega_2$  (resp.  $S$ ) and  $b$  (resp.  $b^s$ ) are some *bounded* functions on  $\Omega_1 \cup \Omega_2$  (resp.  $S$ ). Moreover, we include source terms  $\theta^j$ ,  $j = 1, 2$ ,  $\theta^s$  at the interface through the transmission conditions. This system is not fully determined as we do not prescribe any boundary condition on  $\{0\} \times \Omega$  and  $\{X_0\} \times \Omega$ .

In Section 3, we introduce a small neighborhood  $V_\varepsilon$  of  $S$  in  $\Omega$ , where we can use coordinates of the form  $(y, x_n)$  with  $y \in S$  and  $x_n \in [-2\varepsilon, 2\varepsilon]$ . We then set  $\mathcal{M} = (0, X_0) \times V_\varepsilon$  and  $\mathcal{M}_j = \mathcal{M} \cap ((0, X_0) \times \Omega_j)$ ,  $j = 1, 2$ .

For a properly chosen weight function  $\varphi$  (see Section 3.1), for some  $0 < \alpha_0 < X_0/2$ , and a cut-off function  $\zeta = \zeta(x_n) \in \mathcal{C}^\infty([0, 2\varepsilon])$ , with  $\zeta = 1$  on  $[0, \varepsilon)$ , we shall prove the following theorem.

**Theorem 1.2.** *For all  $\delta_0 > 0$ , there exist  $C > 0$ , and  $h_0 > 0$  such that*

$$\begin{aligned} h \|e^{\varphi/h} w\|_0^2 + h^3 \|e^{\varphi/h} \nabla_{x_0, x} w\|_0^2 + h \sum_{j=1,2} |e^{\varphi/h} w|_{S_j}|_0^2 + h^3 \sum_{j=1,2} |e^{\varphi/h} \nabla_{x_0, x} w|_{S_j}|_0^2 \\ \leq C \left( h^4 \|e^{\varphi/h} f|_{\mathcal{M}_1}\|_0^2 + h^4 \|e^{\varphi/h} f|_{\mathcal{M}_2}\|_0^2 + h^2 \delta^2 \|\zeta e^{\varphi/h} f|_{\mathcal{M}_2}\|_0^2 \right. \\ \left. + h |e^{\varphi/h} \theta^1|_0^2 + \left( h + \frac{\delta^2}{h} \right) |e^{\varphi/h} \theta^2|_0^2 + h^3 |e^{\varphi/h} \nabla_{x_0, S} \theta^1|_0^2 + h^3 |e^{\varphi/h} \nabla_{x_0, S} \theta^2|_0^2 + h^3 |e^{\varphi/h} \theta^s|_0^2 \right), \end{aligned} \quad (1.12)$$

for all  $0 < \delta < \delta_0$ ,  $0 < h \leq h_0$ , for  $(w, \theta^1, \theta^2, \theta^s, f)$  satisfying (1.11),  $w|_{\mathcal{M}_j} \in \mathcal{C}^\infty(\overline{\mathcal{M}_j})$ , and  $w^s \in \mathcal{C}^\infty((0, X_0) \times S)$  with

$$\text{supp}(w) \subset (\alpha_0, X_0 - \alpha_0) \times S \times (-2\varepsilon, 2\varepsilon), \quad \text{supp}(w^s) \subset (\alpha_0, X_0 - \alpha_0) \times S.$$

Here  $\nabla_{x_0, x} = (\partial_{x_0}, \nabla_g)^t$ ,  $\nabla_{x_0, S} = (\partial_{x_0}, \nabla^s)^t$  and  $\|\cdot\|_0, |\cdot|_0$  are  $L^2$ -norms on  $\mathcal{M}$  and  $(0, X_0) \times S$  respectively. The weight function  $\varphi$  will be chosen increasing when crossing  $S$  from  $\mathcal{M}_1$  to  $\mathcal{M}_2$ , which corresponds to an observation on the side  $(0, X_0) \times \Omega_2$ . Observe the non symmetric form of the r.h.s. of the estimate above. This originates from our choice of observing the solution  $w$  in  $(0, X_0) \times \Omega_2$ .

This type of Carleman estimate is well known away from the interface  $S$  (see [Hör63], and [LR95] for an estimate at the Dirichlet boundary  $\partial\Omega$ ).

**Remark 1.3.** The additional variable  $x_0$  is used here to obtain the spectral inequality of Theorem 1.5 below. The same Carleman inequality holds for the operator  $A_\delta$ . The proof can be adapted from that of Theorem 1.12. In fact, without the additional variable, the proof becomes less involved.

The Carleman estimate of Theorem 1.2 exhibits the loss of a half derivative apart from one term in the r.h.s. (see below). Usually, one proves such Carleman estimates locally in a neighborhood of a point, for instance using local coordinates, treating only the principal part of the operator. Next, one includes lower order terms in the operator, exploiting that the associated contributions can be absorbed thanks to the coefficients  $h^\alpha$  of the terms in the l.h.s. of the Carleman estimate<sup>2</sup>. Finally, one patches these estimates together if a global estimate

<sup>2</sup>Note that the powers of  $h$  in estimate (1.3) are in fact optimal.

is needed. This can be achieved again thanks to the precise powers of  $h$  in all the terms. For a review of these derivations see for example [LLar].

At the interface, for technical reasons, the powers of  $h$  obtained in the following terms in the r.h.s. of (1.12) are

$$h^2 \delta^2 \|\zeta e^{\varphi/h} f|_{\mathcal{M}_2}\|_0^2 + \frac{\delta^2}{h} |e^{\varphi/h} \theta^2|_0^2.$$

For the first term this corresponds in fact to a loss of one and a half derivative. We do not know if they are optimal or not. If we simply prove the Carleman estimate in the neighborhood of a point, because of the powers of  $h$  in these terms such local estimates cannot be patched together. The obstruction originates from the diffusion that occurs in the  $(n-1)$ -dimensional submanifold  $S$  through the operator  $\Delta_{c^s}$ . Note that this obstruction naturally disappears in the limit  $\delta \rightarrow 0$ .

Our strategy will thus differ from what is done classically. The estimate of Theorem 1.2 is of semi-global nature. It is global in the direction of the submanifold  $S$  and local in the other directions ( $x_0$  and a normal direction to  $S$  in  $\Omega$ ): we work in a neighborhood of the whole interface  $S$ . Thanks to the cut-off function  $\zeta$  that confines the term

$$h^2 \delta^2 \|\zeta e^{\varphi/h} f|_{\mathcal{M}_2}\|_0^2$$

in a neighborhood of  $S$ , estimate (1.12) can in turn be patched with Carleman estimates away from the interface to form a global estimate. Moreover for the same reasons we do not restrict our analysis to the principal part: in proof we consider also the first-order terms of the operator <sup>3</sup>.

Following [LR10] we shall introduce microlocal regions. Here, the regions are defined on the whole (cotangent bundle of)  $S$ . For each region we shall derive a partial Carleman estimate. The different estimates can then be patched together to yield (1.12). Our strategy requires us to work on  $S$  globally; we shall thus consider (pseudo-)differential operators on  $S$ . Yet, we shall often use their expression in local coordinates; this will allow us to use some of the results proven in [LR10].

For the purpose of proving the null controllability of the parabolic problem (1.4), a local Carleman estimate of the form of Theorem 1.2 in the neighborhood of any point at the interface would be sufficient. Yet, an important property of Carleman estimates resides in the possibility of patching them together to obtain a global estimate. Our result thus preserves this important feature.

### 1.2.2 Interpolation inequality

With the Carleman estimate of Theorem 1.2 we then prove an interpolation inequality of the form of that introduced in [LR95]. This type of interpolation inequality for elliptic operators has also been used in [Leb96, LR97] to address stabilization problems for the wave equation.

Let  $\alpha_1 \in [0, X_0/2)$ , we set  $\mathcal{K}_\delta^0(\alpha_1) = L^2((\alpha_1, X_0 - \alpha_1); \mathcal{H}_\delta^0)$  with also  $\mathcal{K}_\delta^0 = \mathcal{K}_\delta^0(0)$ , and the following Sobolev spaces

$$\mathcal{K}_\delta^1(\alpha_1) = L^2((\alpha_1, X_0 - \alpha_1); \mathcal{H}_\delta^1) \cap H^1((\alpha_1, X_0 - \alpha_1); \mathcal{H}_\delta^0), \quad \mathcal{K}_\delta^1 = \mathcal{K}_\delta^1(0),$$

and

$$\mathcal{K}_\delta^2 = L^2((0, X_0); D(A_\delta)) \cap H^1((0, X_0); \mathcal{H}_\delta^1) \cap H^2((0, X_0); \mathcal{H}_\delta^0).$$

**Theorem 1.4.** *For all  $\delta_0 > 0$ , there exist  $C \geq 0$  and  $\nu_0 \in (0, 1)$  such that for all  $\delta \in (0, \delta_0)$  we have*

$$\|U\|_{\mathcal{K}_\delta^1(\alpha_1)} \leq C \|U\|_{\mathcal{K}_\delta^1}^{1-\nu_0} \left( \left\| (-\partial_{x_0}^2 + A_\delta) U \right\|_{\mathcal{K}_\delta^0} + \|\partial_{x_0} u(0, x)\|_{L^2(\omega)} \right)^{\nu_0}, \quad (1.13)$$

for all  $U = (u, u^s) \in \mathcal{K}_\delta^2$  with  $u|_{x_0=0} = 0$  in  $\Omega_1 \cup \Omega_2$ .

An important consequence of this interpolation inequality is the spectral inequality that we present in the next section.

### 1.2.3 Spectral inequality and null-controllability result

From the above interpolation inequality we deduce a spectral inequality for the elliptic operator  $A_\delta$  defined in (1.8). We consider  $\mathcal{E}_{\delta,j} = (e_{\delta,j}, e_{\delta,j}^s)$ ,  $j \in \mathbb{N}$ , a Hilbert basis of  $\mathcal{H}_\delta^0$  composed of eigenfunctions of the operator  $A_\delta$  associated with the nonnegative eigenvalues  $\mu_{\delta,j} \in \mathbb{R}$ ,  $j \in \mathbb{N}$ , sorted in an increasing sequence (see Proposition 2.5).

<sup>3</sup>This technical point explains the regularity requirements we made above for  $\nabla_a$  and  $\nabla_a^s$ . Yet, we can treat bounded coefficients for the zero-order terms.

**Theorem 1.5.** For  $\delta_0 > 0$ , there exists  $C > 0$  such that for all  $0 < \delta \leq \delta_0$  and  $\mu \in \mathbb{R}$ , we have

$$\|Z\|_{\mathcal{H}_\delta^0} \leq C e^{C\sqrt{\mu}} \|z\|_{L^2(\omega)}, \quad Z = (z, z^s) \in \text{span}\{\mathcal{E}_{\delta,j}; \mu_{\delta,j} \leq \mu\}. \quad (1.14)$$

Following [LR95], this estimation then yields a construction of the control function  $u_\delta(t, x)$  in (1.4), by sequentially acting on a finite yet increasing number of eigenspaces, and we hence obtain the following  $\delta$ -uniform controllability theorem. The proof can be adapted to those in [LR95] or [LZ98, Section 5, Proposition 2] and the uniformity w.r.t. the parameter  $\delta > 0$  comes naturally. We refer also to [LLar] for an exposition of the method and to [Mil06, Léa10, Mil10, TT10] for further developments.

**Theorem 1.6.** Let  $\delta_0 > 0$ . For an arbitrary time  $T > 0$  and an arbitrary nonempty open subset  $\omega \subset \Omega$  there exists  $C > 0$  such that: for all initial conditions  $Z_0 = (z_0, z_0^s) \in \mathcal{H}_\delta^0$  and all  $0 < \delta \leq \delta_0$ , there exists  $u_\delta \in L^2((0, T) \times \omega)$  such that the solution  $(z, z^s)$  of (1.4) satisfies  $(z(T), z^s(T)) = (0, 0)$  and moreover

$$\|u_\delta\|_{L^2((0, T) \times \omega)} \leq C \|Z_0\|_{\mathcal{H}_\delta^0}.$$

An important feature of this result is that the control is uniformly bounded as  $\delta \rightarrow 0$ , so that we can extract a subsequence  $u_\delta$  weakly convergent in  $L^2((0, T) \times \omega)$ . In Corollary 2.9 we prove that the associated solution of Problem (1.4) converges towards a controlled solution of Problem (1.10). For this last control problem (previously treated in [LR10]), we hence construct a control function which is robust with respect to small viscous perturbations in the interface.

It is classical to deduce a boundary null controllability result from the previous distributed control result.

N.B. Here, for the sake of fixing the notation for the statement of the Carleman estimate above we chose the observation in  $\Omega_2$ . This corresponds to  $\omega \subset \Omega_2$  in the proofs of Theorems 1.4, 1.5 and 1.6. Yet,  $\omega$  can be chosen as any arbitrary open subset of  $\Omega$ .

### 1.3 Some additional results and remarks

#### 1.3.1 A stabilization result.

A second important consequence of the interpolation inequality of Theorem 1.4 concerns the stabilization properties of the hyperbolic system (studied in [KZ06, LZ10])

$$\begin{cases} \partial_{tt}z - \Delta_c z + a(x)\partial_t z = 0 & \text{in } (0, T) \times \Omega_1 \cup \Omega_2, \\ \partial_{tt}z^s - \Delta_{c^s} z^s = \frac{1}{\delta} ((c\partial_\eta z)|_{S_2} - (c\partial_\eta z)|_{S_1}) & \text{in } (0, T) \times S, \\ z|_{S_1} = z^s = z|_{S_2} & \text{in } (0, T) \times S, \\ z|_{\partial\Omega} = 0; \end{cases} \quad (1.15)$$

where  $a$  is a nonvanishing nonnegative smooth function on  $\Omega_1 \cup \Omega_2$ . According to [Leb96, LR97], a local version of (1.13) (see Lemma 5.1 below) allows one to produce resolvent estimates which in turn give a result of the following type: for all  $\delta_0 > 0$  and all  $k \in \mathbb{N}$  there exists  $C > 0$  such that for any  $0 < \delta < \delta_0$ , we have the energy decay estimate

$$\|(\partial_t z, \partial_t z^s)\|_{\mathcal{H}_\delta^0} + \|(z, z^s)\|_{\mathcal{H}_\delta^1} \leq \frac{C}{[\log(2+t)]^k} \left( \|(\partial_t z, \partial_t z^s)|_{t=0}\|_{D(A_\delta^{\frac{k}{2}})} + \|(z, z^s)|_{t=0}\|_{D(A_\delta^{\frac{k+1}{2}})} \right),$$

for all solutions of (1.15). In particular, this decay rate is uniform w.r.t.  $\delta$ . See [Bur98, Theorem 3] to obtain the power  $k$  exactly. The same properties can be obtained for this hyperbolic system with a boundary damping (see [LR97]).

#### 1.3.2 Other geometrical situations

Above we assumed that  $\Omega$  could be partitioned according to  $\Omega = \Omega_1 \cup \Omega_2 \cup S$ . More general situations can be treated (interpolation and spectral inequalities, and null controllability result) because of the local nature of the Carleman estimate of Theorem 1.2. If  $V$  is a neighborhood of  $S$ , we require  $V$  to be of the form  $V_1 \cup V_2 \cup S$  with  $V_1$  and  $V_2$  on both sides of  $S$ . Several non intersecting interfaces can be considered as well. For example, the geometrical situations in Figure 1 can be addressed as well. If needed the derivation of a global Carleman estimate can be done by combining Theorem 1.2 and the arguments of Section 5 in [LR11].

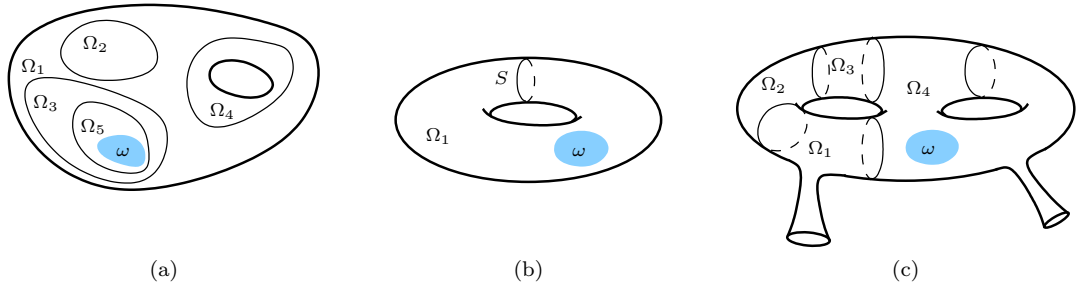


Figure 1: Other geometrical situations: (a)  $\Omega$  is an bounded open subset of  $\mathbb{R}^n$ ; (b) and (c)  $\Omega$  is a compact manifold with boundary.

### 1.3.3 Lack of controllability from the interface

It is important to note that the parabolic controllability result of Theorem 1.6 does not hold in general if the control function acts on the interface  $S$ . Let  $\omega^s$  be an open subset of  $S$  then in general there is no  $u \in L^2((0, T) \times S)$  that brings the solution of

$$\begin{cases} \partial_t z - \Delta_c z = 0 & \text{in } (0, T) \times \Omega_1 \cup \Omega_2, \\ \partial_t z^s - \Delta_{c^s} z^s = \frac{1}{\delta} ((c\partial_\eta z)|_{S_2} - (c\partial_\eta z)|_{S_1}) + \mathbb{1}_{\omega^s} u & \text{in } (0, T) \times S, \\ z|_{S_1} = z^s = z|_{S_2} & \text{in } (0, T) \times S, \\ z|_{\partial\Omega} = 0 & \end{cases} \quad (1.16)$$

to zero at time  $T$ .

Let us consider the following two-dimensional example :  $\Omega = \mathbb{R}/(2\pi\mathbb{Z}) \times (-\pi, \pi)$  is the cylinder endowed with a flat metric. For consistency with the notation of Section 3 we use  $(y, x_n)$  as the coordinates in  $\Omega$ , with periodic conditions in  $y$ . We define the interface as  $S = \{x_n = 0\} = \mathbb{R}/(2\pi\mathbb{Z}) \times \{0\}$ , so that  $\Omega_1 = \{x_n < 0\}$  and  $\Omega_2 = \{x_n > 0\}$ .

We take the diffusion coefficient  $c$  to be piecewise constant (i.e.  $c = c_j$  in  $\Omega_j$  for  $j = 1, 2$ ) and define the operator  $A_\delta$  as in (1.8) (with Dirichlet boundary conditions in the  $x_n$ -variable). In this geometrical context, we have the following result.

**Proposition 1.7.** *If  $\gamma := \sqrt{\frac{c_2}{c_1}} \in \mathbb{N}^*$ , then for all  $c^s > 0$ ,  $\delta > 0$ , and  $j \in \mathbb{Z}$ , the function*

$$\mathcal{E}_{\delta,j} := \begin{pmatrix} e_{\delta,j} \\ 0 \end{pmatrix}, \quad \text{with} \quad e_{\delta,j}(y, x_n) = \begin{cases} e^{ij\gamma y} \sin(\gamma^2 j x_n) & \text{for } x_n < 0, \\ e^{ij\gamma y} \sin(j x_n) & \text{for } x_n > 0, \end{cases}$$

*is an eigenfunction of the operator  $A_\delta$  associated with the eigenvalue  $c_2 j^2 (1 + \gamma^2)$ .*

As a consequence, the adjoint problem of (1.16) (which is of the same form as (1.16) without any control function) does not satisfy the unique continuation property when observed from any subset of  $S$ . More precisely, we notice that the set of “invisible” modes is of infinite dimension. As a consequence, System (1.16) is not approximately controllable in this case and moreover the set of non-controllable modes is of infinite dimension.

The phenomenon exhibited in this example is due to the high level of symmetry. However, in a general setting, if the Laplace operator has an eigenfunction which has a  $\mathcal{C}^\infty$  closed nodal curve, then the associated problem (1.16) with  $c_1 = c_2 = 1$  and  $S$  given by this nodal curve is not controllable from  $S$ . We hence see that this question is connected to properties of the eigenfunctions of the Laplace operator and of their nodal sets.

## 1.4 Notation: semi-classical operators and geometrical setting

### 1.4.1 Semi-classical operators on $\mathbb{R}^d$

We shall use of the notation  $\langle \eta \rangle := (1 + |\eta|^2)^{\frac{1}{2}}$ . For a parameter  $h \in (0, h_0]$  for some  $h_0 > 0$ , we denote by  $S^m(\mathbb{R}^d \times \mathbb{R}^d)$ ,  $S^m$  for short, the space of smooth functions  $a(z, \zeta, h)$  that satisfy the following property: for all  $\alpha, \beta$  multi-indices, there exists  $C_{\alpha,\beta} \geq 0$ , such that

$$\left| \partial_z^\alpha \partial_\zeta^\beta a(z, \zeta, h) \right| \leq C_{\alpha,\beta} \langle \zeta \rangle^{m-|\beta|}, \quad z \in \mathbb{R}^d, \quad \zeta \in \mathbb{R}^d, \quad h \in (0, h_0].$$



Then, for all sequences  $a_{m-j} \in S^{m-j}$ ,  $j \in \mathbb{N}$ , there exists a symbol  $a \in S^m$  such that  $a \sim \sum_j h^j a_{m-j}$ , in the sense that

$$a - \sum_{j < N} h^j a_{m-j} \in h^N S^{m-N} \quad (1.17)$$

(see for instance [Mar02, Proposition 2.3.2] or [Hör85a, Proposition 18.1.3]), with  $a_m$  as principal symbol. We define  $\Psi^m$  as the space of semi-classical operators  $A = \text{Op}(a)$ , for  $a \in S^m$ , formally defined by

$$A u(z) = (2\pi h)^{-d} \iint e^{i\langle z-t, \zeta \rangle / h} a(z, \zeta, h) u(t) dt d\zeta, \quad u \in \mathcal{S}'(\mathbb{R}^d).$$

We shall denote the principal symbol  $a_m$  by  $\sigma(A)$ . We shall use techniques of pseudo-differential calculus in this article, such as construction of parametrices, composition formula, formula for the symbol of the adjoint operator, etc. We refer the reader to [Tay81, Hör85a, Mar02]. We provide composition and change of variables formulæ in the case of tangential operators in Appendix B. Those formulæ can be adapted to the case of operators acting in the whole space  $\mathbb{R}^d$ . In the main text the variable  $z$  will be  $(x_0, x) \in \mathbb{R}^{n+1}$  and  $\zeta = (\xi_0, \xi) \in \mathbb{R}^{n+1}$ .

We set

$$\begin{aligned} S^{-\infty} &= \bigcap_{m>0} S^{-m}, & h^\infty S^{-\infty} &= \bigcap_{m>0} h^m S^{-m}, \\ \Psi^{-\infty} &= \bigcap_{m>0} \Psi^{-m}, & h^\infty \Psi^{-\infty} &= \bigcap_{m>0} h^m \Psi^{-m}. \end{aligned}$$

Note that if there exists a closed set  $F$  such that in the asymptotic expansion (1.17) we have  $\text{supp}(a_{m-j}) \subset F$ ,  $j \in \mathbb{N}$ , then a representative of  $a$  modulo  $h^\infty S^{-\infty}$  can be chosen supported in  $F$ .

We shall also denote by  $\mathcal{D}^m$  the space of semi-classical *differential* operators, i.e., the case where  $a(z, \zeta, h)$  is a polynomial function of order  $m$  in  $\zeta$ . In particular we set

$$D = \frac{h}{i} \partial, \quad \text{and we have } \sigma(D) = \xi.$$

We now introduce Sobolev spaces on  $\mathbb{R}^d$  and Sobolev norms which are adapted to the scaling parameter  $h$ . The natural norm on  $L^2(\mathbb{R}^d)$  is written as  $\|u\|_{L^2(\mathbb{R}^d)} = \|u\|_0 := (\int |u(x)|^2 dx)^{\frac{1}{2}}$ . Let  $r \in \mathbb{R}$ ; we then set

$$\|u\|_r = \|u\|_{\mathcal{H}^r(\mathbb{R}^d)} = \|\Lambda^r u\|_0, \quad \text{with } \Lambda^r := \text{Op}(\langle \xi \rangle^r) \quad \text{and } \mathcal{H}^r(\mathbb{R}^d) := \{u \in \mathcal{S}'(\mathbb{R}^d); \|u\|_r < \infty\}.$$

The space  $\mathcal{H}^r(\mathbb{R}^d)$  is algebraically equal to the classical Sobolev space  $H^r(\mathbb{R}^d)$ . For a fixed value of  $h$ , the norm  $\|\cdot\|_r$  is equivalent to the classical Sobolev norm that we write  $\|\cdot\|_{H^r(\mathbb{R}^d)}$ . However, these norms are not uniformly equivalent as  $h$  goes to 0.

### 1.4.2 Tangential semi-classical operators on $\mathbb{R}^d$ , $d \geq 2$

We set  $z = (z', z_d)$ ,  $z' = (z_1, \dots, z_{d-1})$  and  $\zeta' = (\zeta_1, \dots, \zeta_{d-1})$  accordingly. We denote by  $S_{\mathcal{T}}^m(\mathbb{R}^d \times \mathbb{R}^{d-1})$ ,  $S_{\mathcal{T}}^m$  for short, the space of smooth functions  $b(z, \zeta', h)$ , defined for  $h \in (0, h_0]$  for some  $h_0 > 0$ , that satisfy the following property: for all  $\alpha, \beta$  multi-indices, there exists  $C_{\alpha, \beta} \geq 0$ , such that

$$\left| \partial_z^\alpha \partial_{\zeta'}^\beta b(z, \zeta', h) \right| \leq C_{\alpha, \beta} \langle \zeta' \rangle^{m-|\beta|}, \quad z \in \mathbb{R}^d, \zeta' \in \mathbb{R}^{d-1}, h \in (0, h_0].$$

As above, for any sequence  $b_{m-j} \in S_{\mathcal{T}}^{m-j}$ ,  $j \in \mathbb{N}$ , there exists a symbol  $b \in S_{\mathcal{T}}^m$  such that  $b \sim \sum_j h^j b_{m-j}$ , in the sense that  $b - \sum_{j < N} h^j b_{m-j} \in h^N S_{\mathcal{T}}^{m-N}$ , with  $b_m$  as principal symbol. We define  $\Psi_{\mathcal{T}}^m$  as the space of tangential semi-classical operators  $B = \text{Op}_{\mathcal{T}}(b)$  (observe the notation we adopt is different from above to avoid confusion), for  $b \in S_{\mathcal{T}}^m$ , formally defined by

$$B u(z) = (2\pi h)^{-(d-1)} \iint e^{i\langle z'-t', \zeta' \rangle / h} b(z, \zeta', h) u(t', z_d) dt' d\zeta', \quad u \in \mathcal{S}'(\mathbb{R}^d).$$

In the main text the variable  $z$  will be  $(x_0, x', x_n) \in \mathbb{R}^{n+1}$  and  $\zeta' = (\xi_0, \xi') \in \mathbb{R}^n$ .

We shall also denote the principal symbol  $b_m$  by  $\sigma(B)$ . In the case where the symbol is polynomial in  $\zeta'$  and  $h$ , we shall denote the space of associated tangential *differential* operators by  $\mathcal{D}_{\mathcal{T}}^m$ . We shall denote by  $\Lambda_{\mathcal{T}}^s$  the tangential pseudo-differential operator whose symbol is  $\langle \zeta' \rangle^s$ . We set

$$\begin{aligned} S_{\mathcal{T}}^{-\infty} &= \bigcap_{m>0} S_{\mathcal{T}}^{-m}, & h^\infty S_{\mathcal{T}}^{-\infty} &= \bigcap_{m>0} h^m S_{\mathcal{T}}^{-m}, \\ \Psi_{\mathcal{T}}^{-\infty} &= \bigcap_{m>0} \Psi_{\mathcal{T}}^{-m}, & h^\infty \Psi_{\mathcal{T}}^{-\infty} &= \bigcap_{m>0} h^m \Psi_{\mathcal{T}}^{-m}. \end{aligned}$$

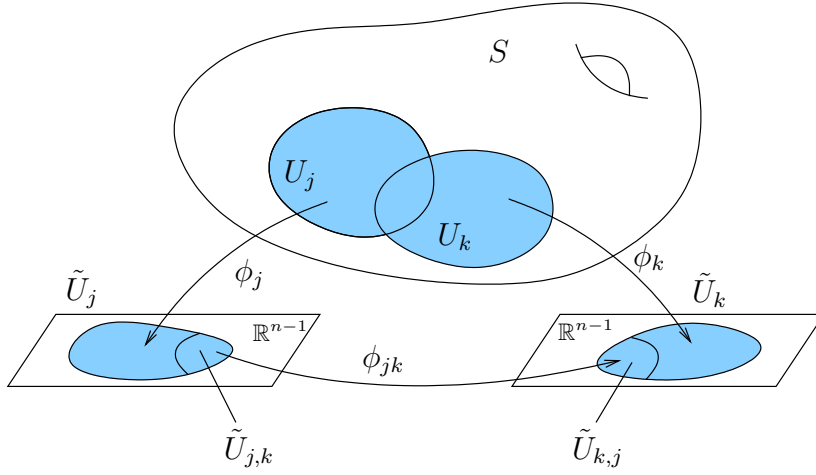


Figure 2: Local charts and diffeomorphisms for the submanifold  $S$ .

For function defined on  $z_d = 0$  or restricted to  $z_d = 0$ , following [LR95, LR97], we shall denote by  $(\cdot, \cdot)_0$  the inner product, i.e.,  $(f, g)_0 := \iint f(z') \bar{g}(z') dz'$ . The induced norm is denoted by  $|\cdot|_0$ , i.e.,  $|f|_0^2 = (f, f)_0$ . For  $r \in \mathbb{R}$  we introduce

$$|f|_{\mathcal{H}^r(\mathbb{R}^{d-1})} = |f|_r := |\Lambda_{\mathcal{T}}^r f|_0. \quad (1.18)$$

The composition Formula and the action of change of variables are given in Appendix B.1.

Note that we shall keep the notation  $\Psi_{\mathcal{T}}^m$  for operators with symbols independent of  $z_d$ , acting on  $\{z_d = 0\}$ . These operators are in fact in  $\Psi^m(\mathbb{R}^{d-1})$ . A similar notation will be used in the case of operators on a manifold.

### 1.4.3 Local charts, pullbacks, and Sobolev norms

The submanifold  $S$  is of dimension  $n - 1$  and is furnished with a *finite* atlas  $(U_j, \phi_j)$ ,  $j \in J$ . The maps  $\phi_j : U_j \rightarrow \tilde{U}_j \subset \mathbb{R}^{n-1}$  is a smooth diffeomorphism. If  $U_j \cap U_k \neq \emptyset$  we also set

$$\begin{aligned} \phi_{jk} : \phi_j(U_j \cap U_k) \subset \tilde{U}_j &\rightarrow \phi_k(U_j \cap U_k) \subset \tilde{U}_k, \\ y &\mapsto \phi_k \circ \phi_j^{-1}(y). \end{aligned}$$

The local charts and the diffeomorphisms we introduce are illustrated in Figure 2.

For a diffeomorphism  $\phi$  between two open sets,  $\phi : U_1 \rightarrow U_2$ , the associated pullback (here stated for continuous functions) is

$$\begin{aligned} \phi^* : \mathcal{C}(U_2) &\rightarrow \mathcal{C}(U_1), \\ u &\mapsto u \circ \phi. \end{aligned}$$

For a function defined on phase-space, e.g. a symbol, the pullback is given by

$$\phi^* u(y, \eta) = u(\phi(y), {}^t(\phi'(y))^{-1} \eta), \quad y \in U_1, \eta \in T_y^*(U_1), \quad u \in \mathcal{C}(T^*U_2). \quad (1.19)$$

We shall use semi-classical Sobolev norms over the manifold  $S$  together with a finite atlas  $(U_j, \phi_j)_j$ ,  $\phi_j : U_j \rightarrow \mathbb{R}^{n-1}$ , and a partition of unity  $(\psi_j)_j$  subordinated to this covering of  $S$ :

$$\psi_j \in \mathcal{C}^\infty(S), \quad \text{supp}(\psi_j) \subset U_j, \quad 0 \leq \psi_j \leq 1, \quad \sum_j \psi_j = 1.$$

We then set:

$$|u|_{\mathcal{H}^r(S)} = \sum_j |(\phi_j^{-1})^* \psi_j u|_{\mathcal{H}^r(\mathbb{R}^{n-1})}. \quad (1.20)$$

Note that the l.h.s. denotes a norm on the manifold and the r.h.s. is defined in (1.18). We shall need the following elementary result.

**Lemma 1.8.** *Let  $(f_j)_j$  be a family of smooth functions on  $S$  with  $\text{supp}(f_j) \subset U_j$  and  $\sum_j f_j = f \geq C > 0$  in  $S$ . We set  $N_r(u) = \sum_j |(\phi_j^{-1})^* f_j u|_{\mathcal{H}^r(\mathbb{R}^{n-1})}$ . Then  $N_r$  is an equivalent norm to  $|\cdot|_{\mathcal{H}^r(\mathbb{R}^{n-1})}$ , uniformly in  $h$ .*

For a proof see Appendix C.1. Note that the  $L^2$ -norm ( $r = 0$ ) defined in (1.20) is equivalent to the natural  $L^2$ -norm on the Riemannian manifold  $S$  given through the inner product in (1.5).

**Norms in codimension 1.** For a function  $u$  defined on  $(0, X_0) \times \mathbb{R}^{n-1}$  we set

$$|u|_0 = |u|_{L^2((0, X_0) \times \mathbb{R}^{n-1})}, \quad |u|_1^2 = |D_{x_0} u|_0^2 + \int_0^{X_0} |u|_{\mathcal{H}^1(\mathbb{R}^{n-1})}^2 dx_0.$$

Note that the latter norm is equivalent to  $|u|_{\mathcal{H}^1(\mathbb{R} \times \mathbb{R}^{n-1})}$  if moreover the function  $u$  is compactly supported in the  $x_0$  variable. For a function  $u$  defined on  $(0, X_0) \times S$ , we set

$$|u|_\ell = \sum_j |(\phi_j^{-1})^* \psi_j u|_\ell, \quad \ell = 0, 1, \quad (1.21)$$

where  $\phi_j$  stands for  $\text{Id} \otimes \phi_j$ .

**Norms in all dimensions.** For a function  $u$  defined on  $(0, X_0) \times \mathbb{R}^{n-1} \times \mathbb{R}$  we set

$$\|u\|_0 = \|u\|_{L^2((0, X_0) \times \mathbb{R}^{n-1} \times \mathbb{R})}, \quad \|u\|_1^2 = \|D_{x_0} u\|_0^2 + \int_0^{X_0} \int_{\mathbb{R}} \|u\|_{\mathcal{H}^1(\mathbb{R}^{n-1})}^2 dx_0 dx_n + \|D_{x_n} u\|_0^2.$$

Note that the latter norm is equivalent to  $\|u\|_{\mathcal{H}^1(\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R})}$  if moreover the function  $u$  is compactly supported in the  $x_0$  variable. For a function  $u$  defined on  $(0, X_0) \times S \times \mathbb{R}$ , we set

$$\|u\|_\ell = \sum_j \|(\phi_j^{-1})^* \psi_j u\|_\ell, \quad \ell = 0, 1, \quad (1.22)$$

where  $\phi_j$  stands for  $\text{Id} \otimes \phi_j \otimes \text{Id}$ .

The following lemma is a counterpart of Lemma 1.8 when working on a local chart of  $(0, X_0) \times S$  or  $(0, X_0) \times S \times \mathbb{R}$ .

**Lemma 1.9.** *Let  $u$  be such that  $\text{supp}(u) \subset K \subset (0, X_0) \times U_j$  (resp.  $(0, X_0) \times U_j \times \mathbb{R}$ ) with  $K$  compact. Then for some constant  $C_K$  we have*

$$C_K^{-1} |u|_\ell \leq |(\phi_j^{-1})^* u|_\ell \leq C_K |u|_\ell \quad (\text{resp. } C_K^{-1} \|u\|_\ell \leq \|(\phi_j^{-1})^* u\|_\ell \leq C_K \|u\|_\ell), \quad \ell = 0, 1.$$

*Proof.* We treat the case of a function defined in  $(0, X_0) \times U_j$ . Consider a partition of unity of  $S$ ,  $\sum_k \hat{\psi}_k = 1$ ,  $\hat{\psi}_k \in \mathcal{C}_c^\infty((0, X_0) \times U_k)$ , such that  $1 \otimes \hat{\psi}_j = 1$  in a neighborhood of  $K$ . Then the induced norms are equivalent to that given above by Lemma 1.8 and for the particular function  $u$  they are equal to  $|(\phi_j^{-1})^* u|_\ell$ ,  $\ell = 0, 1$ .  $\square$

**Tangential semi-classical operators on a manifold.** We can define tangential semi-classical operators on a manifold by means of local representations. This relies on the change of variables formula for semi-classical operators in  $\mathbb{R}^d$  presented in Appendix B.1. We provide details of this construction in Appendix B.2. In particular we define the local symbol of the operator in each chart and its principal symbol on the manifold. We also provide composition and Sobolev regularity results for such operators. In Section 3.6 below we introduce a particular class of tangential operators that will be important in the proof of the Carleman estimate as they will allow us to separate the analysis into microlocal regions.

**A trace formula.** In the sections below, we shall also use of the following trace formula [LR97, page 486] connecting the tangential and volume norms introduced above:

$$|\psi|_{x_n=0^+}|_0 \leq Ch^{-\frac{1}{2}} \|\psi\|_1, \quad (1.23)$$

for  $\psi$  defined on  $\mathbb{R}^{n+1}$ , as well as for  $\psi$  defined on  $(0, X_0) \times S \times [0, 2\varepsilon]$ .

## 2 Well-posedness and asymptotic behavior

We introduce a more general operator

$$\mathcal{A}_\delta Z = \left( \begin{array}{c} -\Delta_c z + \nabla_a z + bz \\ -\Delta_{c^s} z^s + \nabla_a^s z^s + b^s z^s - \frac{1}{\delta} ((c\partial_\eta z)|_{S_2} - (c\partial_\eta z)|_{S_1}) \end{array} \right),$$

with domain  $D(\mathcal{A}_\delta) = D(A_\delta)$  (see (1.9)), where  $\nabla_a$  (resp.  $\nabla_a^s$ ) denotes a smooth vector field  $a(x)\nabla_g$  (resp.  $a^s(x)\nabla^s$ ), and  $b$  (resp.  $b^s$ ) is a bounded function.

We start by considering the well-posedness of the evolution problem (1.4),  $\partial_t Z + \mathcal{A}_\delta Z = F$ . Note that the lower-order perturbations we add to  $A_\delta$  to form  $\mathcal{A}_\delta$  do not affect the well-posedness properties (compare with (1.8)).

## 2.1 Well-posedness

In this section we simply assume that  $a, a^s$  are bounded coefficients. For  $Z, \tilde{Z} \in D(\mathcal{A}_\delta)$ , an integration by parts gives

$$\begin{aligned} ((\mathcal{A}_\delta + \lambda \text{Id})Z, \tilde{Z})_{\mathcal{H}_\delta^0} &= (c\nabla_g z, \nabla_g \tilde{z})_{L^2(\Omega_1 \cup \Omega_2)} + (\nabla_a z + (b + \lambda)z, \tilde{z})_{L^2(\Omega_1 \cup \Omega_2)} \\ &\quad + \delta(c^s \nabla^s z^s, \nabla^s \tilde{z}^s)_{L^2(S)} + \delta(\nabla_a^s z^s + (b^s + \lambda)z^s, \tilde{z}^s)_{L^2(S)} \\ &=: a_\lambda(Z, \tilde{Z}). \end{aligned} \tag{2.1}$$

The bilinear form  $a_\lambda$  is in fact continuous on  $(\mathcal{H}_\delta^1)^2$ .

**Lemma 2.1.** *There exists  $\lambda_0 \geq 0$  sufficiently large such that the bilinear form  $a_\lambda$  is coercive, uniformly in  $\delta$ , if  $\lambda \geq \lambda_0$ .*

*Proof.* The result follows since we have

$$\begin{aligned} a_\lambda(Z, Z) &\geq \frac{c_{\min}}{2} \|\nabla_g z\|_{L^2(\Omega_1 \cup \Omega_2)}^2 + \left( \lambda - \frac{\|a\|_{L^\infty(\Omega_1 \cup \Omega_2)}^2}{2c_{\min}} - \|b\|_{L^\infty(\Omega_1 \cup \Omega_2)} \right) \|z\|_{L^2(\Omega_1 \cup \Omega_2)}^2 \\ &\quad + \delta \frac{c_{\min}^s}{2} \|\nabla^s z^s\|_{L^2(S)}^2 + \delta \left( \lambda - \frac{|a^s|_{L^\infty(S)}^2}{2c_{\min}^s} - |b^s|_{L^\infty(S)} \right) \|z^s\|_{L^2(S)}^2. \end{aligned}$$

□

The coercivity of  $a_\lambda$  shows that the problem  $(\mathcal{A}_\delta + \lambda \text{Id})Z = F$  for  $F \in \mathcal{H}_\delta^0$  is well-posed in a weak sense; for any continuous linear form  $L$  on  $\mathcal{H}_\delta^1$ , the Lax-Milgram theorem ensures the existence and uniqueness of  $Z \in \mathcal{H}_\delta^1$  satisfying

$$a_\lambda(Z, \tilde{Z}) = L(\tilde{Z}) \quad \text{for any } \tilde{Z} \in \mathcal{H}_\delta^1. \tag{2.2}$$

and  $\|Z\|_{\mathcal{H}_\delta^1} \leq C\|L\|_{(\mathcal{H}_\delta^1)'}'$  with the constant  $C$  uniform in  $\delta$ . If we take  $L(\tilde{Z}) = (F, \tilde{Z})_{\mathcal{H}_\delta^0}$  for some  $F \in \mathcal{H}_\delta^0$ , this linear form is continuous on  $\mathcal{H}_\delta^1$ . Then, for some constant  $C > 0$  uniform in  $\delta$  the solution satisfies

$$\|Z\|_{\mathcal{H}_\delta^1} \leq C\|F\|_{\mathcal{H}_\delta^0}. \tag{2.3}$$

Higher regularity can be obtained.

**Proposition 2.2.** *Let  $\lambda \geq \lambda_0$  and  $F \in \mathcal{H}_\delta^0$ . The unique weak solution  $Z = (z, z^s) \in \mathcal{H}_\delta^1$  to (2.2) with  $L(\tilde{Z}) = (F, \tilde{Z})_{\mathcal{H}_\delta^0}$  belongs to  $D(\mathcal{A}_\delta)$ . Hence, for all  $F \in \mathcal{H}_\delta^0$  there exists a unique  $Z \in D(\mathcal{A}_\delta)$  such that  $\mathcal{A}_\delta Z + \lambda Z = F$  and moreover for some positive constant  $C$  uniform in  $\delta$  we have*

$$\sum_{i=1,2} \|z|_{\Omega_i}\|_{H^2(\Omega_i)} + \delta^{\frac{1}{2}} \|z^s\|_{H^2(S)} \leq C\|F\|_{\mathcal{H}_\delta^0}. \tag{2.4}$$

**Proposition 2.3.** *Let  $\lambda \geq \lambda_0$  and  $F = (f, f^s) \in H^m(\Omega_1 \cup \Omega_2) \times H^m(S)$ . The unique weak solution  $Z = (z, z^s) \in \mathcal{H}_\delta^1$  to (2.2) with  $L(\tilde{Z}) = (F, \tilde{Z})_{\mathcal{H}_\delta^0}$  belongs to  $H^{m+2}(\Omega_1 \cup \Omega_2) \times H^{m+2}(S)$  with*

$$\sum_{i=1,2} \|z|_{\Omega_i}\|_{H^{m+2}(\Omega_i)} + \delta^{\frac{1}{2}} \|z^s\|_{H^{m+2}(S)} \leq C \left( \sum_{i=1,2} \|f|_{\Omega_i}\|_{H^m(\Omega_i)} + \delta^{\frac{1}{2}} \|f^s\|_{H^m(S)} \right) \tag{2.5}$$

We refer to Appendices C.2 and C.3 for proofs.

A consequence of the properties we have gathered on  $\mathcal{A}_\delta$  is the following well-posedness for the evolution problem.

**Proposition 2.4.** *Let  $a, b, a^s, b^s$  be bounded coefficients. Then, the operator  $(-\mathcal{A}_\delta, D(\mathcal{A}_\delta))$  generates a  $\mathcal{C}^0$ -semigroup on  $\mathcal{H}_\delta^0$ . If moreover  $a = 0, a^s = 0$  and  $b, b^s \in \mathbb{R}$ , then  $\mathcal{A}_\delta$  is self-adjoint on  $\mathcal{H}_\delta^0$ .*

*Proof.* Lemma 2.1 shows that  $\mathcal{A}_\delta + \lambda_0 \text{Id}$  is monotone and Proposition 2.2 shows that this operator maps its domain  $D(\mathcal{A}_\delta)$  onto  $\mathcal{H}_\delta^0$ . Hence  $\mathcal{A}_\delta + \lambda_0 \text{Id}$  is maximal monotone. The Lumer-Phillips theorem (see e.g. [Paz83]) then allows one to conclude that  $\mathcal{A}_\delta$  generates a strongly continuous semigroup on  $\mathcal{H}_\delta^0$ .

Note that if  $a = 0, a^s = 0$  and  $b, b^s \in \mathbb{R}$ , with (2.1) we see that the operator  $\mathcal{A}_\delta$  is symmetric. It is self-adjoint as the surjectivity of  $\mathcal{A}_\delta + \lambda_0 I$  implies  $D(\mathcal{A}_\delta^*) = D(\mathcal{A}_\delta) = D(\mathcal{A}_\delta)$  (see e.g. [Bre83, Proposition VII-6]). □

With the Rellich theorem we see that  $\mathcal{H}_\delta^1$  can be compactly injected in  $\mathcal{H}_\delta^0$ . It follows that the inverse  $(\mathcal{A}_\delta + \lambda_0 \text{Id})^{-1}$  that we constructed is a compact map from  $\mathcal{H}_\delta^0$  into itself. One then deduces the following spectral result.

**Proposition 2.5.** *There exists a Hilbert basis of  $\mathcal{H}_\delta^0$  formed of eigenfunctions  $\mathcal{E}_j = (e_{\delta,j}, e_{\delta,j}^s)$ ,  $j \in \mathbb{N}$ , of the self-adjoint operator  $A_\delta$  associated with the eigenvalues  $0 \leq \mu_{\delta,0} \leq \mu_{\delta,1} \leq \dots \leq \mu_{\delta,j} \leq \dots$ .*

Note that if  $\Omega$  is a manifold with no boundary then 0 is an eigenfunction for  $A_\delta$ . If  $\Omega$  has a boundary, the Dirichlet boundary condition that we prescribe yield the first eigenvalue to be positive.

**Corollary 2.6.** *The following space of functions*

$$\mathcal{F} = \{(z, z^s) \in \mathcal{H}_\delta^1; z|_{\Omega_i} \in \mathcal{C}^\infty(\overline{\Omega_i}), i = 1, 2\}$$

is dense in  $D(A_\delta)$ .

*Proof.* From Proposition 2.3 the eigenfunctions of  $A_\delta$  are in  $\mathcal{F}$ . The results follows as they generate a dense subset in  $D(A_\delta)$ .  $\square$

## 2.2 Asymptotic behavior of the solutions as $\delta \rightarrow 0$

### 2.2.1 Asymptotic behavior in the elliptic problem

Consider  $F_\delta = (f_\delta, f_\delta^s) \in \mathcal{H}_\delta^0$ . Let  $Z_\delta = (z_\delta, z_\delta^s)$  be the strong solution defined in the previous section for the elliptic equation  $(A_\delta + \lambda)Z_\delta = F_\delta$ .

We also consider the weak solution  $\mathbf{z} \in H_0^1$  of the elliptic problem

$$-\operatorname{div}_g(c\nabla_g \mathbf{z}) + \lambda \mathbf{z} = \mathbf{f} \quad \text{in } \Omega. \quad (2.6)$$

Arguing as in the previous section such a solution exists and is unique for  $\lambda \geq \lambda_0$  (the same value of  $\lambda_0$  as in Lemma 2.1 can be used). In particular we have  $\mathbf{z}|_{S_1} = \mathbf{z}|_{S_2}$ , i.e. the solution is continuous across the interface, and as  $c\nabla_g \mathbf{z}$  has its divergence in  $L^2(\Omega)$  we have  $c\partial_\eta \mathbf{z}|_{S_1} = c\partial_\eta \mathbf{z}|_{S_2}$ . Moreover  $\mathbf{z}|_{\Omega_i} \in H^2(\Omega_i)$  and

$$\sum_{i=1,2} \|\mathbf{z}|_{\Omega_i}\|_{H^2(\Omega_i)} \leq C \|\mathbf{f}\|_{L^2(\Omega_1 \cup \Omega_2)}. \quad (2.7)$$

**Proposition 2.7.** *Suppose that  $\|F_\delta\|_{\mathcal{H}_\delta^0} \leq C$  uniformly in  $\delta$  and that  $f_\delta \rightharpoonup \mathbf{f}$  in  $L^2(\Omega_1 \cup \Omega_2)$  as  $\delta \rightarrow 0$ . Then,  $z_{\delta|\Omega_j} \rightharpoonup \mathbf{z}|_{\Omega_j}$  in  $H^2(\Omega_j)$  for  $j = 1, 2$ .*

Note that the assumption  $\|F_\delta\|_{\mathcal{H}_\delta^0} \leq C$  implies that there always exists a sequence  $\delta_n \rightarrow 0$  such that  $f_{\delta_n} \rightharpoonup \mathbf{f}$ .

*Proof.* We set  $\zeta_\delta := z_\delta - \mathbf{z}$ . According to (2.4), the boundedness assumption on  $F_\delta$ , and (2.7), we have

$$\sum_{i=1,2} \|\zeta_\delta|_{\Omega_i}\|_{H^2(\Omega_i)} \leq C,$$

uniformly in  $\delta$ . Moreover,  $\zeta_\delta$  satisfies

$$\begin{cases} -\operatorname{div}_g(c\nabla_g \zeta_\delta) + \lambda \zeta_\delta = f_\delta - \mathbf{f} & \text{in } \Omega_1 \cup \Omega_2, \\ (c\partial_\eta \zeta_\delta)|_{S_2} - (c\partial_\eta \zeta_\delta)|_{S_1} = \delta(-\Delta_{c^s} z_\delta^s + \lambda z_\delta^s - f_\delta^s) & \text{in } S, \\ \zeta_\delta|_{S_1} = \zeta_\delta|_{S_2} & \text{in } S, \\ \zeta_\delta|_{\partial\Omega} = 0. \end{cases}$$

Taking the inner product of the first line of this system with  $\zeta_\delta$  and integrating by parts, we obtain

$$(c\nabla_g \zeta_\delta, \nabla_g \zeta_\delta)_{L^2(\Omega_1 \cup \Omega_2)} + ((c\partial_\eta \zeta_\delta)|_{S_2} - (c\partial_\eta \zeta_\delta)|_{S_1}, \zeta_\delta)_{L^2(S)} + \lambda (\zeta_\delta, \zeta_\delta)_{L^2(\Omega_1 \cup \Omega_2)} = (f_\delta - \mathbf{f}, \zeta_\delta)_{L^2(\Omega_1 \cup \Omega_2)}.$$

In this expression, we have

$$\begin{aligned} |((c\partial_\eta \zeta_\delta)|_{S_2} - (c\partial_\eta \zeta_\delta)|_{S_1}, \zeta_\delta)_{L^2(S)}| &= \delta^{\frac{1}{2}} \left| \left( \delta^{\frac{1}{2}} \Delta_{c^s} z_\delta^s - \delta^{\frac{1}{2}} \lambda z_\delta^s + \delta^{\frac{1}{2}} f_\delta^s, \zeta_\delta \right)_{L^2(S)} \right| \\ &\leq C \delta^{\frac{1}{2}} \left( \delta^{\frac{1}{2}} \|z_\delta^s\|_{H^2(S)} + \|F_\delta\|_{\mathcal{H}_\delta^0} \right) \|\zeta_\delta\|_{H^1(\Omega_1 \cup \Omega_2)} \\ &\leq C \delta^{\frac{1}{2}} \|\zeta_\delta\|_{H^2(\Omega_1 \cup \Omega_2)} \rightarrow 0, \end{aligned} \quad (2.8)$$

according to (2.4), the trace estimate and the boundedness assumption on  $F_\delta$ . Moreover, since  $\zeta_\delta$  is bounded in  $H^2(\Omega_1 \cup \Omega_2)$ , for all sequence  $\delta_n \rightarrow 0$ , we can extract a subsequence, also called  $\delta_n$ , such that  $\zeta_{\delta_n}$  converges strongly in  $L^2(\Omega_1 \cup \Omega_2)$ , and we have

$$(f_{\delta_n} - \mathbf{f}, \zeta_{\delta_n})_{L^2(\Omega_1 \cup \Omega_2)} \rightarrow 0.$$

As a consequence, we obtain

$$(c\nabla_g \zeta_{\delta_n}, \nabla_g \zeta_{\delta_n})_{L^2(\Omega_1 \cup \Omega_2)} + \lambda (\zeta_{\delta_n}, \zeta_{\delta_n})_{L^2(\Omega_1 \cup \Omega_2)} \rightarrow 0,$$

i.e.  $\zeta_{\delta_n} \rightarrow 0$  in  $H^1(\Omega_j)$ , for  $j = 1, 2$ . Because the limit is the same for any subsequence of  $\zeta_{\delta_n}$ , this implies that the whole  $\zeta_\delta$  converges to zero in  $H^1(\Omega_j)$ . Since  $\zeta_\delta|_{\Omega_j}$  is uniformly bounded in  $H^2(\Omega_j)$ , the result follows.  $\square$

### 2.2.2 Asymptotic behavior in the parabolic problem

Here, we discuss, for some  $\lambda > 0$  (one can take  $\lambda = 0$  if  $\partial\Omega \neq \emptyset$ ) the convergence properties of the solution  $Z_\delta = (z_\delta, z_\delta^s)$  of

$$\begin{cases} \partial_t z_\delta - \Delta_c z_\delta + \lambda z_\delta = f_\delta & \text{in } (0, T) \times \Omega_1 \cup \Omega_2, \\ \partial_t z_\delta^s - \Delta_{c^s} z_\delta^s + \lambda z_\delta^s = \frac{1}{\delta} ((c\partial_\eta z_\delta)|_{S_2} - (c\partial_\eta z_\delta)|_{S_1}) + f_\delta^s & \text{in } (0, T) \times S, \\ z_\delta|_{S_1} = z_\delta^s = z_\delta|_{S_2} & \text{in } (0, T) \times S, \\ z_\delta|_{\partial\Omega} = 0 & \text{in } (0, T), \\ z_\delta|_{t=0} = z_0 \text{ and } z_\delta^s|_{t=0} = z_0^s, & \end{cases} \quad (2.9)$$

towards the solution  $\mathbf{z}$  of

$$\begin{cases} \partial_t \mathbf{z} - \Delta_c \mathbf{z} + \lambda \mathbf{z} = \mathbf{f} & \text{in } (0, T) \times \Omega_1 \cup \Omega_2, \\ \mathbf{z}|_{S_1} = \mathbf{z}|_{S_2} \text{ and } (c\partial_\eta \mathbf{z})|_{S_2} = (c\partial_\eta \mathbf{z})|_{S_1} & \text{in } (0, T) \times S, \\ \mathbf{z}|_{\partial\Omega} = 0 & \text{in } (0, T), \\ \mathbf{z}|_{t=0} = z_0 & \text{in } \Omega. \end{cases} \quad (2.10)$$

**Proposition 2.8.** *Suppose that  $\|F_\delta\|_{L^2(0, T; \mathcal{H}_\delta^0)} \leq C$  uniformly in  $\delta$ , that  $f_\delta \rightharpoonup \mathbf{f}$  in  $L^2((0, T) \times \Omega_1 \cup \Omega_2)$  as  $\delta \rightarrow 0$  and that  $z_0 \in H_0^1(\Omega)$  and  $z_0^s \in H^1(S)$ . Then, we have,  $z_\delta|_{\Omega_j} \rightharpoonup z|_{\Omega_j}$  in  $L^2(0, T; H^2(\Omega_j)) \cap H^1(0, T; L^2(\Omega_j))$  and  $*$ -weak in  $L^\infty(0, T; H^1(\Omega_j))$ , and there exists  $C' > 0$  such that for all  $t \in [0, T]$ ,  $\|z_\delta|_{\Omega_j}(t)\|_{H^1(\Omega_j)} \leq C'$  for  $j = 1, 2$ .*

*Proof.* First, Problem (2.9) can be equivalently rewritten as  $\partial_t Z_\delta + (A_\delta + \lambda)Z_\delta = F_\delta$  with  $Z_\delta(0) = (z_0, z_0^s)$ . For  $Z_\delta(0) \in D(A_\delta)$  and  $F_\delta \in \mathcal{C}^0([0, T]; \mathcal{H}_\delta^0)$  the semigroup solution of this equation is in  $\mathcal{C}^0([0, T]; D(A_\delta)) \cap \mathcal{C}^1([0, T]; \mathcal{H}_\delta^0)$  (see [Paz83, Corollary 2.6 Chap 4] or [Bre83, Théorème VII.10]). As a consequence, we can form the square of the  $\mathcal{H}_\delta^0$ -norm of this equation and integrate on  $(0, T)$ . This yields

$$\int_0^T \frac{d}{dt} \|(A_\delta + \lambda)^{\frac{1}{2}} Z_\delta\|_{\mathcal{H}_\delta^0}^2 + \int_0^T \|(A_\delta + \lambda)Z_\delta(t)\|_{\mathcal{H}_\delta^0}^2 dt + \int_0^T \|\partial_t Z_\delta(t)\|_{\mathcal{H}_\delta^0}^2 dt = \int_0^T \|F_\delta(t)\|_{\mathcal{H}_\delta^0}^2 dt,$$

which, in turns gives the stability estimate for the solution of (2.9) :

$$\|Z_\delta(T)\|_{\mathcal{H}_\delta^1}^2 + \int_0^T \|(A_\delta + \lambda)Z_\delta(t)\|_{\mathcal{H}_\delta^0}^2 dt + \int_0^T \|\partial_t Z_\delta(t)\|_{\mathcal{H}_\delta^0}^2 dt \leq C \left( \int_0^T \|F_\delta(t)\|_{\mathcal{H}_\delta^0}^2 dt + \|Z_\delta(0)\|_{\mathcal{H}_\delta^1}^2 \right),$$

uniformly in  $\delta$ . With a density argument, this energy estimate remains valid if  $Z_\delta(0) \in \mathcal{H}_\delta^1$  and  $F_\delta(t) \in L^2(0, T; \mathcal{H}_\delta^0)$ .

According to (2.4), this yields

$$\begin{aligned} & \|z_\delta(T)\|_{H^1}^2 + \delta \|z_\delta^s(T)\|_{H^1(S)}^2 + \int_0^T \|z_\delta(t)\|_{H^2}^2 dt + \delta \int_0^T |z_\delta^s(t)|_{H^2(S)}^2 dt \\ & + \int_0^T \|\partial_t z_\delta(t)\|_{L^2}^2 dt + \delta \int_0^T \|\partial_t z_\delta^s(t)\|_{L^2(S)}^2 dt \leq C \left( \int_0^T \|F_\delta(t)\|_{\mathcal{H}_\delta^0}^2 dt + \|Z_\delta(0)\|_{\mathcal{H}_\delta^1}^2 \right) \leq C, \end{aligned} \quad (2.11)$$

uniformly in  $\delta$  (the volume norms are taken over  $\Omega_1 \cup \Omega_2$ ).

In addition, the solution of (2.10) also satisfies

$$\|\mathbf{z}(T)\|_{H^1}^2 + \int_0^T \|(-\Delta_c + \lambda)\mathbf{z}(t)\|_{L^2}^2 dt + \int_0^T \|\partial_t \mathbf{z}(t)\|_{L^2}^2 dt \leq C \left( \int_0^T \|\mathbf{f}(t)\|_{L^2}^2 dt + \|\mathbf{z}(0)\|_{H^1}^2 \right),$$

where all the norms are taken over  $\Omega_1 \cup \Omega_2$ . Using the additional regularity (2.7), this gives

$$\|\mathbf{z}(T)\|_{H^1}^2 + \int_0^T \|\mathbf{z}(t)\|_{H^2}^2 dt + \int_0^T \|\partial_t \mathbf{z}(t)\|_{L^2}^2 dt \leq C \left( \int_0^T \|\mathbf{f}(t)\|_{L^2}^2 dt + \|\mathbf{z}(0)\|_{H^1}^2 \right). \quad (2.12)$$

Now, we set  $\zeta_\delta = z_\delta - \mathbf{z}$ . According to (2.11)-(2.12), we have,

$$\sum_{j=1,2} \left( \|\zeta_\delta|_{\Omega_j}\|_{L^\infty(0,T;H^1(\Omega_j))} + \|\zeta_\delta|_{\Omega_j}\|_{L^2(0,T;H^2(\Omega_j))} + \|\zeta_\delta|_{\Omega_j}\|_{H^1(0,T;L^2(\Omega_j))} \right) \leq C, \quad (2.13)$$

uniformly in  $\delta$ . Moreover,  $\zeta_\delta$  satisfies

$$\begin{cases} \partial_t \zeta_\delta - \Delta_c \zeta_\delta + \lambda \zeta_\delta = f_\delta - \mathbf{f} & \text{in } (0, T) \times (\Omega_1 \cup \Omega_2), \\ (c\partial_\eta \zeta_\delta)|_{S_2} - (c\partial_\eta \zeta_\delta)|_{S_1} = \delta (\partial_t z_\delta^s - \Delta_{c^s} z_\delta^s + \lambda z_\delta^s - f_\delta^s) & \text{in } (0, T) \times S, \\ \zeta_\delta|_{S_1} = \zeta_\delta|_{S_2} & \text{in } (0, T) \times S, \\ \zeta_\delta|_{\partial\Omega} = 0 & \text{in } (0, T), \\ \zeta_\delta|_{t=0} = 0 & \text{in } \Omega. \end{cases} \quad (2.14)$$

Forming the inner product of the first line of this system with  $\zeta_\delta$  and integrating on  $(0, T)$ , we obtain

$$\begin{aligned} & \frac{1}{2} \|\zeta_\delta(T)\|_{L^2(\Omega)}^2 + \|\sqrt{c}\nabla_g \zeta_\delta\|_{L^2((0,T)\times\Omega)}^2 + \lambda \|\zeta_\delta\|_{L^2((0,T)\times\Omega)}^2 \\ & + ((c\partial_\eta \zeta_\delta)|_{S_2} - (c\partial_\eta \zeta_\delta)|_{S_1}, \zeta_\delta)_{L^2((0,T)\times S)} = (f_\delta - \mathbf{f}, \zeta_\delta)_{L^2((0,T)\times\Omega)}. \end{aligned}$$

In this expression, we have

$$\begin{aligned} \left| ((c\partial_\eta \zeta_\delta)|_{S_2} - (c\partial_\eta \zeta_\delta)|_{S_1}, \zeta_\delta)_{L^2((0,T)\times S)} \right| &= \delta^{\frac{1}{2}} \left| \left( \delta^{\frac{1}{2}} \partial_t z_\delta^s - \delta^{\frac{1}{2}} \Delta_{c^s} z_\delta^s + \delta^{\frac{1}{2}} \lambda z_\delta^s - \delta^{\frac{1}{2}} f_\delta^s, \zeta_\delta \right)_{L^2((0,T)\times S)} \right| \\ &\leq C \delta^{\frac{1}{2}} \|\zeta_\delta\|_{L^2(0,T;H^2(\Omega_1 \cup \Omega_2))} \rightarrow 0, \end{aligned}$$

according to (2.11) (proceeding as in (2.8)). Proceeding as in the proof of Proposition 2.7, we have

$$(f_\delta - \mathbf{f}, \zeta_\delta)_{L^2((0,T)\times(\Omega_1 \cup \Omega_2))} \rightarrow 0,$$

for a subsequence, and we obtain

$$\frac{1}{2} \|\zeta_\delta(T)\|_{L^2(\Omega)}^2 + \|\sqrt{c}\nabla_g \zeta_\delta\|_{L^2((0,T)\times\Omega)}^2 + \lambda \|\zeta_\delta\|_{L^2((0,T)\times\Omega)}^2 \rightarrow 0.$$

This, together with (2.13) concludes the proof of the proposition.  $\square$

As a consequence, we can obtain a convergence result for the control problem under view. We denote by  $u_\delta$  the control function given by Theorem 1.6, that satisfies

$$\begin{cases} \partial_t Z_\delta + A_\delta Z_\delta = B u_\delta \\ Z_\delta|_{t=0} = Z_0 \\ Z_\delta|_{t=T} = 0. \end{cases}$$

According to Theorem 1.6,  $u_\delta$  is uniformly bounded in  $L^2((0, T) \times \omega)$ , so that we can extract a subsequence (also denoted by  $u_\delta$ ) weakly converging in this space towards  $\mathbf{u}$ . We also consider the solution  $\tilde{Z}_\delta = (\tilde{z}_\delta, \tilde{z}_\delta^s)$  of

$$\begin{cases} \partial_t \tilde{Z}_\delta + A_\delta \tilde{Z}_\delta = B \mathbf{u} \\ \tilde{Z}_\delta|_{t=0} = Z_0. \end{cases} \quad (2.15)$$

The following result is a consequence of Proposition 2.8.

**Corollary 2.9.** *The limit  $\mathbf{u}$  is a null-control function for the limit system (1.10). Moreover,  $(\tilde{z}_\delta - z_\delta)|_{\Omega_j} \rightharpoonup 0$  in  $L^2(0, T; H^2(\Omega_j)) \cap H^1(0, T; L^2(\Omega_j))$  and  $*$ -weak in  $L^\infty(0, T; H^1(\Omega_j))$ , and there exists  $C > 0$  such that for all  $t \in [0, T]$ ,  $\|z_\delta|_{\Omega_j}(t) - \tilde{z}_\delta|_{\Omega_j}(t)\|_{H^1(\Omega_j)} \leq C$  for  $j = 1, 2$ .*

In particular, we have  $\tilde{z}_\delta(T) \rightharpoonup 0$  in  $H^1(\Omega)$ . This shows that the limit  $\mathbf{u}$  is a control function for the limit system (1.10) which is robust with respect to small viscous perturbations. Indeed, it realizes an approximate control for System (2.15).

### 3 Local setting in a neighborhood of the interface

In a sufficiently small neighborhood of  $S$ , say  $V_\varepsilon$ , we place ourselves in normal geodesic coordinates (w.r.t. to the spatial variables  $x$ ). More precisely (see [Hör85a, Appendix C.5]) for  $\varepsilon$  sufficiently small, there exists a diffeomorphism

$$\begin{aligned} \mathbf{F} : S \times [-2\varepsilon, 2\varepsilon] &\rightarrow V_\varepsilon \\ (y, x_n) &\mapsto \mathbf{F}(y, x_n), \end{aligned}$$

so that the differential operator  $-\partial_{x_0}^2 - \Delta_c + \nabla_a$  takes the form on both sides of the interface:

$$-\partial_{x_0}^2 - c(y, x_n) (\partial_{x_n}^2 - R_2(y, x_n)) + R_1(y, x_n),$$

and the differential operator  $-\partial_{x_0}^2 - \Delta_c^s + \nabla_a^s$  takes the form on the interface

$$-\partial_{x_0}^2 + c^s(y) R_2(y, x_n = 0) + R_1^s(y),$$

where  $R_2(y, x_n)$  is a  $x_n$ -family of second-order elliptic differential operators on  $S$ , i.e., a tangential operator, with principal symbol  $r(y, x_n, \eta)$ ,  $\eta \in T_y^*(S)$ , that satisfies

$$r(y, x_n, \eta) \in \mathbb{R}, \quad \text{and} \quad C_1 |\eta|_g^2 \leq r(y, x_n, \eta) \leq C_2 |\eta|_g^2, \quad (3.1)$$

for some  $0 < C_1 \leq C_2 < \infty$ , and  $R_1(y, x_n)$  is a  $x_n$ -family of first-order operators on  $S \times [-2\varepsilon, 2\varepsilon]$ ,  $R_1^s(y)$  is a first-order operator on  $S$ .

By abuse of notation we shall write  $V_\varepsilon$  in place of  $S \times [-2\varepsilon, 2\varepsilon]$ . In this setting, we have

$$V_\varepsilon^- = \mathbf{F}(S \times [-2\varepsilon, 0]) = V_\varepsilon \cap \Omega_1, \quad V_\varepsilon^+ = \mathbf{F}(S \times (0, 2\varepsilon]) = V_\varepsilon \cap \Omega_2,$$

and we recall that the observation region  $\omega$  is in  $\Omega_2$ .

In the sequel, we shall often write

$$x := (y, x_n), \quad \text{and} \quad \mathbf{x} := (x_0, x) = (x_0, y, x_n) \in [0, X_0] \times S \times [-2\varepsilon, 2\varepsilon].$$

We set

$$P = -\frac{1}{c} \partial_{x_0}^2 - (\partial_{x_n}^2 - R_2(x)) + \frac{1}{c} R_1(x), \quad P^s = -\frac{1}{c^s} \partial_{x_0}^2 + R_2(y, x_n = 0) + \frac{1}{c^s} R_1^s(y).$$

They both have smooth coefficients.

In this framework, in the neighborhood  $V_\varepsilon$  of  $S$ , System (1.11) becomes

$$\begin{cases} Pw = F, & \text{in } (0, X_0) \times S \times ([-2\varepsilon, 0] \cup (0, 2\varepsilon]), \\ P^s w^s = \frac{1}{c^s \delta} ((c \partial_{x_n} w)|_{x_n=0^+} - (c \partial_{x_n} w)|_{x_n=0^-} + \Theta^s) & \text{in } (0, X_0) \times S, \\ w|_{x_n=0^-} = w^s + \theta^1 \quad \text{and} \quad w|_{x_n=0^+} = w^s + \theta^2, & \text{in } (0, X_0) \times S, \end{cases} \quad (3.2)$$

with

$$F = \frac{1}{c} f + R_0 w, \quad \Theta^s = \theta^s + \delta R_0^s w^s, \quad (3.3)$$

where  $R_0$  and  $R_0^s$  are zero-order operators with *bounded* coefficients on  $S \times ([-2\varepsilon, 0] \cup (0, 2\varepsilon])$  and  $S$  respectively.

#### 3.1 Properties of the weight functions

We denote by  $\tilde{r}(x, \eta, \eta')$  the symmetric bilinear form associated with the quadratic principal symbol  $r(x, \eta)$ . We introduce the following symmetric bilinear form

$$\tilde{\beta}(x; \xi_0, \eta; \xi'_0, \eta') = \frac{1}{c(x)} \xi_0 \xi'_0 + \tilde{r}(x, \eta, \eta'). \quad (3.4)$$

and the associated positive definite quadratic form  $\beta(x; \xi_0, \eta)$ . We choose a positive bounded continuous function  $\gamma(x)$  in  $\overline{V_\varepsilon^+}$  such that

$$\beta(y, -x_n; \xi_0, \eta) - \gamma(y, x_n) \beta(y, x_n; \xi_0, \eta) \geq C |(\xi_0, \eta)|^2 > 0, \quad (\xi_0, \eta) \in \mathbb{R} \times T_y^*(S), \quad (3.5)$$

for  $x = (y, x_n) \in \overline{V_\varepsilon^+}$ .

We then choose a function  $\varphi = \varphi(\mathbf{x})$  on  $[0, X_0] \times V_\varepsilon$  that is smooth on both sides of the interface and simply continuous across the interface, that moreover satisfies the following properties.



1. For a function  $\gamma'$  such that  $0 < \gamma'(x) \leq \gamma(x) - \epsilon$  in  $\overline{V_\epsilon^+}$ , for some  $\epsilon > 0$ , we have

$$\gamma'(y, x_n)(\partial_{x_n}\varphi)^2(x_0, y, x_n) - (\partial_{x_n}\varphi)^2(x_0, y, -x_n) \geq C > 0, \quad (3.6)$$

for  $x_0 \in [0, X_0]$ , and  $x = (y, x_n) \in V_\epsilon^+$ .

2. For a given value of  $\nu > 0$  sufficiently small we have

$$|\partial_{x_0}\varphi(\mathbf{x})| + |\nabla^s\varphi(\mathbf{x})|_g \leq \nu \inf_{\overline{V_\epsilon}} |\partial_{x_n}\varphi|, \quad \mathbf{x} = (x_0, x) \in [0, X_0] \times V_\epsilon. \quad (3.7)$$

3. We have  $|\partial_{x_0}\varphi| + |\nabla^s\varphi|_g + |\partial_{x_n}\varphi| > 0$  in  $[0, X_0] \times V_\epsilon$  and Hörmander's sub-ellipticity condition is satisfied on both sides of the interface. This condition will be precisely stated below after the introduction of the conjugated operator (see (3.18)).

Note that we have  $\inf_{V_\epsilon^+} |\partial_{x_n}\varphi| \geq C > 0$ .

The first condition states the increase in the normal slope of the weight function when crossing the interface. The value of  $\nu$  in the second condition will be determined below (see (3.19)-(3.20) and the proof of Proposition 3.5). We thus ask the weight function to be relatively flat in the tangent directions to the interface as compared to its variations in the normal direction. We explain below how a weight function satisfying the sub-ellipticity condition can be built through a convexification procedure (see Remark 3.3).

**Remark 3.1.** Property (3.6) and  $|\partial_{x_0}\varphi| + |\nabla^s\varphi|_g + |\partial_{x_n}\varphi| > 0$  can be obtained by choosing  $\varphi$  such that  $(\partial_{x_n}\varphi)|_{[0, X_0] \times S} \geq C > 0$  and assuming that (3.6) only holds on  $[0, X_0] \times S$  and then shrinking the neighborhood  $V_\epsilon$  by choosing  $\epsilon$  sufficiently small.

An example of such a function will be given in the application of the Carleman estimate in Section 5.

**Remark 3.2.** Note that the conditions we impose on the weight function are much simpler than the conditions given in [LR10]. Such condition are proven sharp in [LL11] in the limiting case  $\delta \rightarrow 0$ . If (3.5) is not satisfied, i.e., the increase in the normal slope of the weight function is chosen too small, one can then build a quasi-mode that concentrates at the interface and shows that the Carleman estimate cannot hold.

## 3.2 A system formulation

Following [Bel03, LR10], we shall consider (3.2) as a system of two equations coupled at the boundary  $x_n = 0^+$ . Here, the coupling involves a tangential second-order elliptic operator. In  $[0, X_0] \times S \times [-2\epsilon, 0)$ , we make the change of variables  $x_n$  to  $-x_n$ . For a function  $\psi$  defined in  $V_\epsilon$ , we set

$$\psi^r(y, x_n) = \psi(y, x_n) \quad \text{and} \quad \psi^l(y, x_n) := \psi(y, -x_n), \quad \text{for } x_n \geq 0,$$

and similarly for symbols and operators, e.g.,

$$r^r(y, x_n, \eta) = r(y, x_n, \eta) \quad \text{and} \quad r^l(y, x_n, \eta) = r(y, -x_n, \eta), \quad \text{for } x_n \geq 0.$$

We set  $V_\epsilon^+ = S \times (0, 2\epsilon]$ . System (3.2) then takes the form

$$\begin{cases} P^{r/l} w^{r/l} = F^{r/l}, & \text{in } (0, X_0) \times V_\epsilon^+, \\ P^s w^s = \frac{1}{c^s \delta} ((c^r \partial_{x_n} w^r)|_{x_n=0^+} + (c^l \partial_{x_n} w^l)|_{x_n=0^+} + \Theta^s) & \text{in } (0, X_0) \times S, \\ w_{|x_n=0^+}^{r/l} = w^s + \theta^{r/l} & \text{in } (0, X_0) \times S, \end{cases} \quad (3.8)$$

## 3.3 Conjugation by the weight function

We now consider the weight functions  $\varphi^{r/l}$  built up as above from the *continuous* function  $\varphi$  defined on  $V_\epsilon$ . We introduce the following conjugated differential operators

$$P_\varphi^{r/l} = h^2 e^{\varphi^{r/l}/h} P^{r/l} e^{-\varphi^{r/l}/h}, \quad P_\varphi^s = h^2 e^{\varphi|_S/h} P^s e^{-\varphi|_S/h},$$

With the functions

$$\begin{aligned} v^{r/l} &= e^{\varphi^{r/l}/h} w^{r/l}, \quad v^s = e^{\varphi|_S/h} w^s, \\ F_\varphi^{r/l} &= h^2 e^{\varphi^{r/l}/h} F^{r/l}, \quad \Theta_\varphi^s = -i h e^{\varphi|_S/h} \Theta^s, \quad \theta_\varphi^{r/l} = e^{\varphi|_S/h} \theta^{r/l}, \end{aligned}$$

with  $0 < h < h_0$ , System (3.8) can be rewritten as

$$\begin{cases} P_\varphi^{7/h} v^{7/h} = F_\varphi^{7/h} & \text{in } (0, X_0) \times V_\varepsilon^+, \\ P_\varphi^s v^s = \frac{hi}{c^s \delta} (c^r (D_{x_n} + i \partial_{x_n} \varphi^r) v|_{x_n=0^+} \\ \quad + c^l (D_{x_n} + i \partial_{x_n} \varphi^l) v|_{x_n=0^+} + \Theta_\varphi^s) & \text{in } (0, X_0) \times S, \\ v|_{x_n=0^+}^{7/h} = v^s + \theta_\varphi^{7/h} & \text{in } (0, X_0) \times S, \end{cases} \quad (3.9)$$

Recall that  $D = h\partial/i$  here. We shall consider the operators  $P_\varphi^{7/h}$  and  $P_\varphi^s$  as semi-classical differential operators.

We separate the self- and anti-adjoint parts of the operators  $P_\varphi^{7/h}$ , viz.,

$$\tilde{Q}_2^{7/h} = \frac{1}{2} (P_\varphi^{7/h} + (P_\varphi^{7/h})^*), \quad \tilde{Q}_1^{7/h} = \frac{1}{2i} (P_\varphi^{7/h} - (P_\varphi^{7/h})^*),$$

The (semi-classical) principal symbols  $\tilde{q}_j$  of  $\tilde{Q}_j$ ,  $j = 1, 2$  are then

$$\begin{aligned} \tilde{q}_2^{7/h}(\mathbf{x}, \xi_0, \eta, \xi_n) &= \xi_n^2 + q_2^{7/h}(\mathbf{x}, \xi_0, \eta), \\ \tilde{q}_1^{7/h}(\mathbf{x}, \xi_0, \eta, \xi_n) &= 2\xi_n \partial_{x_n} \varphi^{7/h} + 2q_1^{7/h}(\mathbf{x}, \xi_0, \eta), \end{aligned}$$

for  $(y, \eta) \in T^*(S)$ , with

$$\begin{aligned} q_2^{7/h}(\mathbf{x}, \xi_0, \eta) &= \frac{\xi_0^2}{c^{7/h}} + r^{7/h}(x, \eta) - \left( \frac{(\partial_{x_0} \varphi^{7/h})^2}{c^{7/h}} + r^{7/h}(x, d_y \varphi^{7/h}) + (\partial_{x_n} \varphi^{7/h})^2 \right) \\ q_1^{7/h}(\mathbf{x}, \xi_0, \eta) &= \frac{\xi_0 \partial_{x_0} \varphi^{7/h}}{c^{7/h}} + \tilde{r}^{7/h}(x; \eta, d_y \varphi^{7/h}). \end{aligned}$$

Recall that  $\tilde{r}^{7/h}(x, \eta, \eta')$  stands for the symmetric bilinear form associated with the quadratic principal symbol  $r^{7/h}(x, \eta)$ . The principal symbol of  $P_\varphi^{7/h}$  is naturally

$$p_\varphi^{7/h} = \tilde{q}_2^{7/h} + i\tilde{q}_1^{7/h} = \xi_n^2 + 2i\xi_n \partial_{x_n} \varphi^{7/h} + q_2^{7/h} + 2iq_1^{7/h}. \quad (3.10)$$

For the sake of concision we have at places omitted some of the variable dependencies, e.g. writing  $\varphi^{7/h}$  in place of  $\varphi^{7/h}(\mathbf{x})$ .

Note also that the symbol of  $P_\varphi^s$  is given by

$$p_\varphi^s = \frac{\xi_0^2}{c^s} + r(x, \eta) - \left( \frac{(\partial_{x_0} \varphi)^2}{c^s} + r(x, d_y \varphi|_{x_n=0}) \right) \Big|_{x_n=0} + 2i \left( \frac{\xi_0 \partial_{x_0} \varphi}{c^s} + \tilde{r}(x; \eta, d_y \varphi|_{x_n=0}) \right) \Big|_{x_n=0}. \quad (3.11)$$

(Recall that  $r^l$  and  $r^r$  (resp.  $\varphi^l$  and  $\varphi^r$ ) coincide at  $x_n = 0^+$ .)

### 3.4 Phase-space regions

Following [LR97, LR10] we introduce the following quantity

$$\mu^{7/h}(\mathbf{x}, \xi_0, \eta) = q_2^{7/h}(\mathbf{x}, \xi_0, \eta) + \frac{(q_1^{7/h}(\mathbf{x}, \xi_0, \eta))^2}{(\partial_{x_n} \varphi^{7/h})^2}, \quad (3.12)$$

and the following sets in the (tangential) phase space:

$$E^{7/h, \pm} = \{(x_0, y, x_n; \xi_0, \eta) \in [0, X_0] \times S \times [0, 2\varepsilon] \times \mathbb{R} \times T_y^*(S); \mu^{7/h}(x_0, y, x_n; \xi_0, \eta) \gtrless 0\}, \quad (3.13)$$

$$Z^{7/h} = \{(x_0, y, x_n; \xi_0, \eta) \in [0, X_0] \times S \times [0, 2\varepsilon] \times \mathbb{R} \times T_y^*(S); \mu^{7/h}(x_0, y, x_n; \xi_0, \eta) = 0\}. \quad (3.14)$$

The analysis we carry on will make precise the behavior of the roots of  $p_\varphi^{7/h}$  (viewing  $p_\varphi^{7/h}$  as a second-order polynomial in the variable  $\xi_n$ , see (3.10)) as  $(\mathbf{x}, \xi_0, \eta)$  varies. In particular, we prove that  $(\mathbf{x}, \xi_0, \eta) \in Z^{7/h}$ , i.e.  $\mu^{7/h}(\mathbf{x}, \xi_0, \eta) = 0$ , if and only if there exists  $\xi_n \in \mathbb{R}$  such that  $(\mathbf{x}, \xi_0, \eta, \xi_n) \in \text{Char}(P_\varphi^{7/h})$ .

With the following symmetric bilinear forms,

$$\begin{aligned} \tilde{\beta}^{7/h}(x; \xi_0, \eta; \xi'_0, \eta') &= \frac{1}{c^{7/h}} \xi_0 \xi'_0 + \tilde{r}^{7/h}(x, \eta, \eta'), \\ \tilde{\alpha}^{7/h}(x; \xi_0, \eta, \xi_n; \xi'_0, \eta', \xi'_n) &= \tilde{\beta}^{7/h}(x; \xi_0, \eta; \xi'_0, \eta') + \xi_n \xi'_n, \end{aligned}$$

and the associated quadratic forms,  $\beta^{r/l}(x; \xi_0, \eta)$  and  $\alpha^{r/l}(x; \xi_0, \eta, \xi_n)$ , we have

$$\mu^{r/l}(\mathbf{x}, \xi_0, \eta) = \beta^{r/l}(x, \xi_0, \eta) + \frac{(\tilde{\beta}^{r/l}(x; \xi_0, \eta; \partial_{x_0}\varphi^{r/l}, d_y\varphi^{r/l}))^2}{(\partial_{x_n}\varphi^{r/l})^2} - \alpha^{r/l}(x; \partial_{x_0}\varphi^{r/l}, d_y\varphi^{r/l}, \partial_{x_n}\varphi^{r/l}),$$

We also set the following quadratic form

$$\beta_\varphi^{r/l}(\mathbf{x}; \xi_0, \eta) = (\tilde{\beta}^{r/l}(x; \xi_0, \eta; \partial_{x_0}\varphi^{r/l}, d_y\varphi^{r/l}))^2.$$

The quadratic forms  $\beta^{r/l}$  are positive definite. With the function  $\gamma(x)$  on  $\overline{V_\varepsilon^+}$  we chose in Section 3.1 we have

$$\beta^l(x; \xi_0, \eta) - \gamma(x)\beta^r(x; \xi_0, \eta) \geq C|(\xi_0, \eta)|^2 > 0. \quad (3.15)$$

From the properties of the weight function listed in Section 3.1 we have

$$\gamma'(\partial_{x_n}\varphi^r)^2 - (\partial_{x_n}\varphi^l)^2 \geq C > 0, \quad 0 < \gamma'(x) \leq \gamma(x) - \varepsilon, \quad \varepsilon > 0, \quad (3.16)$$

and

$$|\partial_{x_0}\varphi^{r/l}| + |d_y\varphi^{r/l}|_g \leq \nu \inf(|\partial_{x_n}\varphi^l|), \quad (3.17)$$

with  $\nu > 0$  sufficiently small. Furthermore  $|\partial_{x_0}\varphi^{r/l}| + |\nabla_g\varphi^{r/l}| + |\partial_{x_n}\varphi^{r/l}| > 0$  in  $[0, X_0] \times \overline{V_\varepsilon^+}$  and the following sub-ellipticity property is satisfied

$$\forall \mathbf{x} \in [0, X_0] \times \overline{V_\varepsilon^+}, (\xi_0, \eta, \xi_n) \in \mathbb{R} \times T_y^*(S) \times \mathbb{R}, \quad p_\varphi^{r/l}(\mathbf{x}, \xi_0, \eta, \xi_n) = 0 \Rightarrow \{\tilde{q}_2^{r/l}, \tilde{q}_1^{r/l}\}(\mathbf{x}, \xi_0, \eta, \xi_n) > 0. \quad (3.18)$$

The sub-ellipticity property (3.18) is necessary for the derivation of the Carleman estimate and is geometrically invariant (see e.g. [Hör63, Section 8.1, page 186], see also [LLar]).

**Remark 3.3.** A weight function  $\varphi$  that satisfies the properties of Section 3.1, or (3.16)–(3.18) equivalently, can be obtained in the following classical way. Choose a continuous function  $\psi$ , smooth on both sides of  $S$ , such that  $\psi^{r/l}$  satisfies conditions (3.16), (3.17) and  $|\partial_{x_0}\psi^{r/l}| + |\nabla^s\psi^{r/l}|_g + |\partial_{x_n}\psi^{r/l}| > 0$  on  $[0, X_0] \times \overline{V_\varepsilon^+}$ . These conditions are then also satisfied by  $\varphi = e^{\lambda\psi}$ ,  $\lambda \geq 1$ . For the parameter  $\lambda$  sufficiently large  $\varphi$  will also fulfill the sub-ellipticity condition (see e.g. Lemma 3 in [LR95, Section 3.B], Theorem 8.6.3 in [Hör63, Chapter 8], or Proposition 28.3.3 in [Hör85b, Chapter 28]).

Using (3.15), (3.16) and (3.17) for  $\nu$  sufficiently small, we obtain

$$\beta^l - \gamma(x)(\beta^r + \beta_\varphi^r/(\partial_{x_n}\varphi^r)^2) \geq C|(\xi_0, \eta)|^2 > 0, \quad (3.19)$$

and

$$\gamma(x)\alpha^r(x; \partial_{x_0}\varphi^r, d_y\varphi^r, \partial_{x_n}\varphi^r) - \alpha^l(x; \partial_{x_0}\varphi^l, d_y\varphi^l, \partial_{x_n}\varphi^l) \geq C > 0, \quad (3.20)$$

where we have used that  $\gamma \geq \gamma' + \varepsilon$ .

The assumption we have formulated yields the following *key* property.

**Proposition 3.4.** *There exists  $C_0 > 0$  such that in the neighborhood  $V_\varepsilon$  we have*

$$(\mu^l - \gamma(x)\mu^r)(\mathbf{x}, \xi_0, \eta) \geq C_0|(\xi_0, \eta)|^2 > 0, \quad \mathbf{x} = (x_0, x) = (x_0, y, x_n), \quad (\xi_0, \eta) \in \mathbb{R} \times T_y^*(S).$$

*In particular,  $E^{r,+} \cup Z^r \subset E^{l,+}$ .*

*Proof.* With the properties of the weight function of Section 3.1, and more precisely (3.19)–(3.20) that follow from them, we have

$$\begin{aligned} \mu^l &= \beta^l(x; \xi_0, \eta) - \gamma(x)(\beta^r(x; \xi_0, \eta) + \beta_\varphi^r(\mathbf{x}; \xi_0, \eta)/(\partial_{x_n}\varphi^r)^2) + \beta_\varphi^l(\mathbf{x}; \xi_0, \eta)/(\partial_{x_n}\varphi^l)^2 \\ &\quad + \gamma(x)\alpha^r(x; \varphi^{r'}) - \alpha^l(x; \varphi^{l'}) + \gamma(x)\mu^r \\ &\geq C|(\xi_0, \eta)|^2 + \gamma(x)\mu^r. \end{aligned}$$

□

**Proposition 3.5.** *With the properties of the weight function of Section 3.1 we have*

$$\text{Char}(p_\varphi^s) \subset \text{Char}(\text{Re } p_\varphi^s) \subset (E^{l,-} \cap \{x_n = 0\}).$$

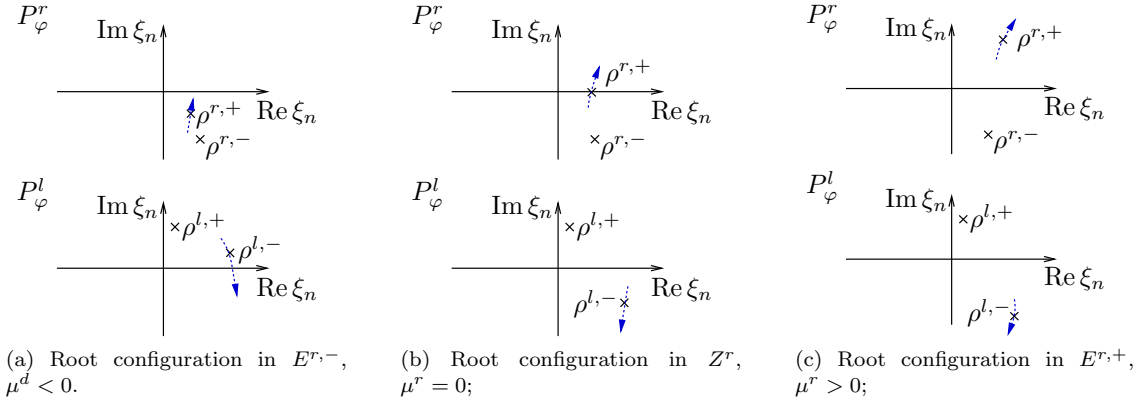


Figure 3: The root  $\rho^{r,+}$  crosses the real axis before the root  $\rho^{l,-}$  does, as  $\mu^r$  decreases.

*Proof.* From the form of (3.11) we see that  $\text{Re } p_\varphi^s = 0$  implies

$$|\eta|_g + |\xi_0| \leq C(|\partial_{x_0} \varphi^{\tau/l}| + |d_y \varphi^{\tau/l}|_g)_{x_n=0}, \quad (3.21)$$

and we find

$$\mu^l_{|x_n=0^+} = \left[ (\xi_0^2 - (\partial_{x_0} \varphi)^2) \left( \frac{1}{c^l} - \frac{1}{c^s} \right) - (\partial_{x_n} \varphi^l)^2 + \frac{1}{(\partial_{x_n} \varphi)^2} \left( \frac{\xi_0 \partial_{x_0} \varphi}{c^l} + \tilde{r}(x; \eta, d_y \varphi|_{x_n=0}) \right)^2 \right] \Big|_{x_n=0}.$$

Using (3.21) together with (3.17) in this expression gives

$$\mu^l_{|x_n=0^+} \leq \left[ -(\partial_{x_n} \varphi^l)^2 + C\nu \inf((\partial_{x_n} \varphi^l)^2) \right] \Big|_{x_n=0}.$$

The result thus follows when taking  $\nu$  sufficiently small.  $\square$

### 3.5 Root properties

The following lemma describes the position of the roots of  $p_\varphi^{\tau/l}$  of (3.10) viewed as a second-order polynomial in  $\xi_n$ . The proof is given in Appendix C.4.

**Lemma 3.6.** *We have the following root properties.*

1. In the region  $E^{\tau/l,+}$ , the polynomial  $p_\varphi^{\tau/l}$  defined in (3.10) has two distinct roots that satisfy  $\text{Im } \rho^{\tau/l,+} > 0$  and  $\text{Im } \rho^{\tau/l,-} < 0$ . Moreover we have

$$\mu^{\tau/l} \geq C > 0 \quad \Leftrightarrow \quad \text{Im } \rho^{\tau/l,+} \geq C' > 0 \quad \text{and} \quad \text{Im } \rho^{\tau/l,-} \leq -C' < 0,$$

2. In the region  $E^{\tau/l,-}$ , the imaginary parts of the two roots have the same sign as that of  $-\partial_{x_n} \varphi^{\tau/l}$ .

3. In the region  $Z^{\tau/l}$ , one of the roots is real.

Moreover, there exist  $C > 0$  and  $H > 0$  such that  $|\rho^{\tau/l,+} - \rho^{\tau/l,-}| \geq |\text{Im } \rho^{\tau/l,+} - \text{Im } \rho^{\tau/l,-}| \geq C > 0$  in the region  $\{\mu^{\tau/l} \geq -H\}$ .

**Remark 3.7.** Note that  $(\mathbf{x}, \xi_0, \eta) \in E^{\tau/l,+}$  for  $|\xi_0| + |\eta|_g$  sufficiently large, say  $|\xi_0| + |\eta|_g \geq R$ , uniformly in  $\mathbf{x} \in [0, X_0] \times \overline{V_\varepsilon^+}$  and for  $h$  bounded. Note also that in the region  $\{\mu^{\tau/l} \geq -H\}$ , the roots  $\rho^{\tau/l,\pm}$  are smooth since they do not cross.

For the polynomial  $p_\varphi^r$ , for  $|\xi_0| + |\eta|_g$  small, i.e. in the region  $E^{r,-}$ , the two roots  $\rho^{r,+}$  and  $\rho^{r,-}$  both have *negative* imaginary parts. As the value of  $\mu^r$  increases, the root  $\rho^{r,+}$  moves towards the real axis, and crosses it in the region  $Z^r$ . In the region  $E^{r,+}$  we have  $\text{Im } \rho^{r,+} > 0$  and  $\text{Im } \rho^{r,-} < 0$ .

For the polynomial  $p_\varphi^l$ , for  $|\xi_0| + |\eta|_g$  small, i.e. in the region  $E^{l,-}$ , the two roots  $\rho^{l,+}$  and  $\rho^{l,-}$  both have *positive* imaginary parts. As the value of  $\mu^l$  increases, the root  $\rho^{l,-}$  moves towards the real axis, and crosses it in the region  $Z^l$ . In the region  $E^{l,+}$  we have  $\text{Im } \rho^{l,+} > 0$  and  $\text{Im } \rho^{l,-} < 0$ . The “motion” of the roots of  $p_\varphi^l$  and  $p_\varphi^r$  is illustrated in Figure 3.

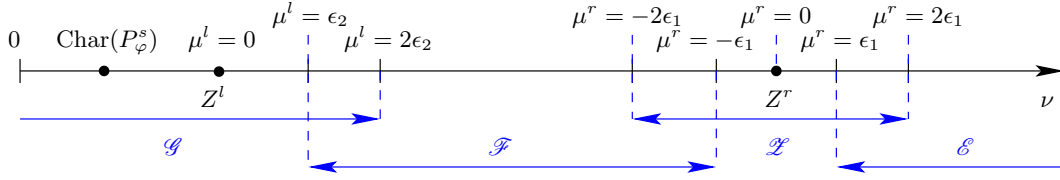


Figure 4: Sketch of the relative localization of the different phase-space regions. Here,  $(\mathbf{x}, \xi_0, \eta)$  is fixed and we plot the different zones for  $(\mathbf{x}, \nu\xi_0, \nu\eta)$  as  $\nu$  increases from 0 to  $\infty$ . Here,  $\nu$  represents the norm of the tangential frequencies. This situation can be represented under this form since for  $\mathbf{x}$  fixed, the sets  $E^{\gamma_l, \pm}$  and  $\{p_\varphi^s \leq 0\}$  are star-shaped with respect to 0 in the variables  $(\xi_0, \eta) \in T_{(x_0, y)}^*((0, X_0) \times S)$ .

We now call

$$\mathcal{M}_+ = (0, X_0) \times S \times [0, 2\varepsilon].$$

We also set

$$\mathcal{M}_+^* := \{(x_0, y, x_n, \xi_0, \eta) \in (0, X_0) \times S \times [0, 2\varepsilon] \times \mathbb{R} \times T_y^*(S)\} \simeq T^*((0, X_0) \times S) \times [0, 2\varepsilon].$$

With the symbols defined in Section B.2 (see Definition B.4) we obtain the following result.

**Lemma 3.8.** *Let  $H$  be as given in Lemma 3.6. Let  $\chi^{\gamma_l} \in S_T^0(\mathcal{M}_+^*)$  with support in  $\{\mu^{\gamma_l} \geq -H\}$ . Then  $\chi^{\gamma_l} \rho^{\gamma_l, \pm} \in S_T^1(\mathcal{M}_+^*)$ . Let  $C_0 > 0$ , there exists  $C > 0$  such that  $|\text{Im} \rho^{\gamma_l, \pm}| \geq C(1 + |\xi_0| + |\eta|_g)$  in  $\{\mu^{\gamma_l} \geq C_0\}$ . It follows that for some  $C' > 0$  we have*

$$|\rho^{\gamma_l, +} - \rho^{\gamma_l, -}| \geq |\text{Im}(\rho^{\gamma_l, +} - \rho^{\gamma_l, -})| \geq C'(1 + |\xi_0| + |\eta|_g), \quad \text{in } \{\mu^{\gamma_l} \geq C_0\}.$$

We refer to Appendix C.5 for a proof.

### 3.6 Microlocalisation operators

We define the following open sets in (tangential) phase-space:

$$\begin{aligned} \mathcal{E} &= \{(\mathbf{x}, \xi_0, \eta) \in \mathcal{M}_+^*; \varepsilon_1 < \mu^r(\mathbf{x}, \xi_0, \eta)\}, \\ \mathcal{Z} &= \{(\mathbf{x}, \xi_0, \eta) \in \mathcal{M}_+^*; -2\varepsilon_1 < \mu^r(\mathbf{x}, \xi_0, \eta) < 2\varepsilon_1\}, \\ \mathcal{F} &= \{(\mathbf{x}, \xi_0, \eta) \in \mathcal{M}_+^*; \varepsilon_2 < \mu^l(\mathbf{x}, \xi_0, \eta), \text{ and } \mu^r(\mathbf{x}, \xi_0, \eta) < -\varepsilon_1\}, \\ \mathcal{G} &= \{(\mathbf{x}, \xi_0, \eta) \in \mathcal{M}_+^*; \mu^l(\mathbf{x}, \xi_0, \eta) < 2\varepsilon_2\}. \end{aligned} \quad (3.22)$$

The constants  $\varepsilon_1$  and  $\varepsilon_2$  are taken such that  $\sup(\gamma)\varepsilon_1 + \varepsilon_2 < C_0/2$ , with  $C_0$  as in Proposition 3.4. Our analysis in the region  $\mathcal{Z}$  will require  $\varepsilon_1$  to be small (see Section 4.4 below). Recall that  $\gamma$  is defined in Section 3.1. This yields  $\overline{\mathcal{G}} \cap \overline{\mathcal{Z}} = \emptyset$ . As a consequence of Propositions 3.4 and 3.5, the localization of the different microlocal zones can be represented as in Figure 4. In particular, we have  $\text{Char}(p_\varphi^s) \subset (\mathcal{G} \setminus \mathcal{F}) \cap \{x_n = 0\}$ .

With the open covering of  $\mathcal{M}_+^*$  by  $\mathcal{E}$ ,  $\mathcal{Z}$ ,  $\mathcal{F}$  and  $\mathcal{G}$  we introduce a  $\mathcal{C}^\infty$  partition of unity,

$$\chi_{\mathcal{E}} + \chi_{\mathcal{Z}} + \chi_{\mathcal{F}} + \chi_{\mathcal{G}} = 1, \quad 0 \leq \chi_{\bullet} \leq 1, \quad \text{supp}(\chi_{\bullet}) \subset \bullet, \quad \bullet = \mathcal{E}, \mathcal{Z}, \mathcal{F}, \mathcal{G}.$$

The sets  $\mathcal{Z}$ ,  $\mathcal{F}$  and  $\mathcal{G}$  are relatively compact which gives  $\chi_{\mathcal{Z}}, \chi_{\mathcal{F}}, \chi_{\mathcal{G}} \in S_T^{-\infty}(\mathcal{M}_+^*)$  and consequently  $\chi_{\mathcal{E}} \in S_T^0(\mathcal{M}_+^*)$ . Associated with these symbols we now define tangential pseudo-differential operators on  $\mathcal{M}_+$ .

Given  $0 < \alpha_0 < X_0/2$ , we choose a function  $\zeta^1 \in \mathcal{C}_c^\infty(0, X_0)$  that satisfies  $\zeta^1 = 1$  on a neighborhood of  $(\alpha_0, X_0 - \alpha_0)$  and  $0 \leq \zeta^1 \leq 1$ . Setting

$$\zeta_j(x_0, y, x_n) = \zeta^1(x_0)\psi_j(y) \quad (3.23)$$

gives a partition of unity on  $(\alpha_0, X_0 - \alpha_0) \times S \times [0, 2\varepsilon]$ . Recall that  $(\psi_j)_{j \in J}$  is a partition of unity on  $S$  (see Section 1.4.3).

We define the following operators on  $\mathcal{M}_+$ :

$$\Xi_{\bullet} = \sum_{j \in J} \Xi_{\bullet, j}, \quad \text{with } \Xi_{\bullet, j} = \phi_j^* \text{Op}_T(\chi_{\bullet, j})(\phi_j^{-1})^* \zeta_j, \quad j \in J, \quad \bullet = \mathcal{E}, \mathcal{Z}, \mathcal{F}, \mathcal{G}, \quad (3.24)$$

where  $\phi_j^*$  denotes the pullback by the function  $\phi_j$  and

$$\chi_{\bullet, j} = \tilde{\zeta}_j(\phi_j^{-1})^* \chi_{\bullet}, \quad (3.25)$$

and  $\tilde{\zeta}_j$  denotes a function in  $\mathcal{C}_c^\infty((0, X_0) \times \tilde{U}_j)$  with  $\tilde{\zeta}_j = 1$  in a neighborhood of  $\text{supp}((\phi_j^{-1})^* \zeta_j)$ .

Proposition B.14 in Appendix B.3 shows that the operators  $\Xi_\bullet$  are zero-order tangential semi-classical operators on  $\mathcal{M}_+$ , with principal symbol  $\zeta^1(x_0)\chi_\bullet(\mathbf{x}, \xi_0, \eta)$ .

**Remark 3.9.** The role of the parameter  $\alpha_0$  introduced here is to avoid considering boundary problems on  $(\{0\} \cup \{X_0\}) \times S \times [0, 2\varepsilon]$ .

## 4 Proof of the Carleman estimate in a neighborhood of the interface

In this section, we prove Carleman estimates in the four microlocal regions described above, that is, for functions  $\Xi_\bullet v^{7l}$ , with  $v^{7l} \in \mathcal{C}_c^\infty((0, X_0) \times S \times [0, 2\varepsilon])$  and  $\bullet = \mathcal{E}, \mathcal{Z}, \mathcal{F}, \mathcal{G}$ . It will be more convenient to do this in local coordinates<sup>4</sup>, since we then can use the techniques and some of the results of [LR10].

Our strategy in each microlocal region  $\bullet$  (with  $\bullet = \mathcal{E}, \mathcal{Z}, \mathcal{F}, \mathcal{G}$ ) is hence the following: we first produce Carleman estimates in each local chart  $(0, X_0) \times \tilde{U}_j \times [0, 2\varepsilon]$  for the functions

$$u_{\bullet,j}^{7l} := \text{Op}_{\mathcal{T}}(\chi_{\bullet,j})v_j^{7l} \quad \text{and} \quad u_{\bullet,j}^s := \text{Op}_{\mathcal{T}}(\chi_{\bullet,j}|_{x_n=0^+})v_j^s, \quad (4.1)$$

where

$$v_j^{7l} := (\phi_j^{-1})^* \zeta_j v^{7l} \quad \text{and} \quad v_j^s := (\phi_j^{-1})^* \zeta_j v^s,$$

with  $\zeta_j$  defined in (3.23). Then, we pull the local estimates back to the manifold and patch them together to finally obtain a Carleman estimate for  $\Xi_\bullet v^{7l}$ , as

$$\Xi_\bullet v^{7l} = \sum_j \phi_j^* u_{\bullet,j}^{7l}. \quad (4.2)$$

Note that the functions  $v_j^{7l}$  (resp.  $v_j^s$ ) are defined in  $(0, X_0) \times \tilde{U}_j \times [0, 2\varepsilon]$  (resp.  $(0, X_0) \times \tilde{U}_j$ ). Yet, because of their compact support, we naturally extend them by zero to  $\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}^+$  (resp.  $\mathbb{R} \times \mathbb{R}^{n-1}$ ). In the sequel, functions with such a compact support will be extended similarly.

In what follows, we shall use the notation  $\lesssim$  for  $\leq C$ , with a constant  $C$  independent of  $\delta$  and  $h$  (but depending on  $\delta_0$  and  $h_0$ ).

### 4.1 Preliminary observations

In the local chart  $\tilde{U}_j$ , the differential operators  $P_\varphi^\alpha$ ,  $\alpha = r, l$  or  $s$ , are given by

$$P_{\varphi,j}^\alpha = (\phi_j^{-1})^* P_\varphi^\alpha \phi_j^*,$$

with principal symbol  $p_{\varphi,j}^\alpha = (\phi_j^{-1})^* p_\varphi^\alpha$ .

Observe that the definition of  $\mu^{7l}$  in (3.12), and of the associated microlocal regions  $Z^{7l}, E^{7l, \pm}$  in (3.13)–(3.14), and  $\mathcal{E}, \mathcal{Z}, \mathcal{F}$  and  $\mathcal{G}$  in (3.22), are geometrically invariant.

In local coordinates, System (3.9) becomes

$$\begin{cases} P_{\varphi,j}^{7l} v_j^{7l} = \tilde{F}_{\varphi,j}^{7l} & \text{in } (0, X_0) \times \tilde{U}_j \times [0, 2\varepsilon), \\ P_{\varphi,j}^s v_j^s = \frac{hi}{c_j^s \delta} (c_j^r (D_{x_n} + i\partial_{x_n} \varphi_j^r) v_j^r|_{x_n=0^+} \\ \quad + c_j^l (D_{x_n} + i\partial_{x_n} \varphi_j^l) v_j^l|_{x_n=0^+} + \tilde{\Theta}_{\varphi,j}^s) & \text{in } (0, X_0) \times \tilde{U}_j, \\ v_j^l|_{x_n=0^+} = v_j^s + \theta_{\varphi,j}^{7l} & \text{in } (0, X_0) \times \tilde{U}_j, \end{cases} \quad (4.3)$$

where we have set

$$\begin{cases} \tilde{F}_{\varphi,j}^{7l} = (\phi_j^{-1})^* \zeta_j F_\varphi^{7l} + (\phi_j^{-1})^* [P_\varphi^{7l}, \zeta_j] v^{7l}, \\ \theta_{\varphi,j}^{7l} = (\phi_j^{-1})^* \zeta_j \theta_\varphi^{7l}, \\ \Theta_{\varphi,j}^s = (\phi_j^{-1})^* \zeta_j \Theta_\varphi^s, \quad \tilde{\Theta}_{\varphi,j}^s = \Theta_{\varphi,j}^s + \frac{c_j^s \delta}{hi} (\phi_j^{-1})^* [P_\varphi^s, \zeta_j] v^s, \\ \varphi_j^{7l} = (\phi_j^{-1})^* \varphi^{7l}, \quad c_j^{7l} = (\phi_j^{-1})^* c^{7l}, \quad c_j^s = (\phi_j^{-1})^* c^s, \end{cases} \quad (4.4)$$

with  $[P_\varphi^{7l}, \zeta_j] \in h\mathcal{D}^1(\mathcal{M}_+)$  and  $[P_\varphi^s, \zeta_j] \in h\mathcal{D}_{\mathcal{T}}^1(\mathcal{M}_+)$ .

<sup>4</sup>However, note that it would be interesting to obtain the results of [LR10] directly in a global setting.

We now formulate System (4.3) in terms of  $u_{\bullet,j}$  in preparation for the estimations in the four different microlocal zones. First, we have

$$P_{\varphi,j}^{7l} u_{\bullet,j}^{7l} = \text{Op}\mathcal{T}(\chi_{\bullet,j}) P_{\varphi,j}^{7l} v_j^{7l} + \underbrace{[P_{\varphi,j}^{7l}, \text{Op}\mathcal{T}(\chi_{\bullet,j})]}_{\in h(\Psi_{\mathcal{T}}^0 D_{x_n} + \Psi_{\mathcal{T}}^1)} v_j^{7l}.$$

In particular, this gives

$$\|P_{\varphi,j}^{7l} u_{\bullet,j}^{7l}\|_0 \lesssim \|P_{\varphi,j}^{7l} v_j^{7l}\|_0 + h \|v_j^{7l}\|_1. \quad (4.5)$$

Second, as a consequence of (4.3), the transmission conditions satisfied by  $u_{\bullet,j}^{7l}$  and  $u_{\bullet,j}^s$  read

$$\begin{cases} \frac{\delta c_j^s}{hi} P_{\varphi,j}^s u_{\bullet,j}^s = (c_j^l(D_{x_n} + i\partial_{x_n} \varphi_j^l) u_{\bullet,j}^l)_{|x_n=0^+} + (c_j^r(D_{x_n} + i\partial_{x_n} \varphi_j^r) u_{\bullet,j}^r)_{|x_n=0^+} + G_1, \\ u_{\bullet,j}^{7l}|_{x_n=0^+} = u_{\bullet,j}^s + \theta_{\bullet,j}^{7l}, \end{cases} \quad (4.6)$$

with  $\theta_{\bullet,j}^{7l} = \text{Op}\mathcal{T}(\chi_{\bullet,j}|_{x_n=0^+}) \theta_{\varphi,j}^{7l}$  and

$$\begin{aligned} G_1 &= \frac{\delta c_j^s}{hi} \underbrace{[P_{\varphi,j}^s, \text{Op}\mathcal{T}(\chi_{\bullet,j}|_{x_n=0^+})]}_{\in h\Psi_{\mathcal{T}}^1} v_j^s + \underbrace{[\text{Op}\mathcal{T}(\chi_{\bullet,j}|_{x_n=0^+}), c_j^l(D_{x_n} + i\partial_{x_n} \varphi_j^l)]}_{\in h\Psi_{\mathcal{T}}^0} v_j^l|_{x_n=0^+} \\ &\quad + \underbrace{[\text{Op}\mathcal{T}(\chi_{\bullet,j}|_{x_n=0^+}), c_j^r(D_{x_n} + i\partial_{x_n} \varphi_j^r)]}_{\in h\Psi_{\mathcal{T}}^0} v_j^r|_{x_n=0^+} + \text{Op}\mathcal{T}(\chi_{\bullet,j}|_{x_n=0^+}) \tilde{\Theta}_{\varphi,j}^s. \end{aligned}$$

We have the following estimate

$$\begin{aligned} |G_1|_0 &\lesssim \delta |v_j^s|_1 + h |v_j^l|_{|x_n=0^+}|_0 + h |v_j^r|_{|x_n=0^+}|_0 + |\tilde{\Theta}_{\varphi,j}^s|_0 \\ &\lesssim (\delta + h) |v_j^s|_1 + h |\theta_{\varphi,j}^l|_0 + h |\theta_{\varphi,j}^r|_0 + |\Theta_{\varphi,j}^s|_0, \end{aligned} \quad (4.7)$$

by (4.3) and (4.4). We set

$$\gamma_0(u_{\bullet,j}^{7l}) = u_{\bullet,j}^{7l}|_{x_n=0^+}, \quad \gamma_1(u_{\bullet,j}^{7l}) = (D_{x_n} u_{\bullet,j}^{7l})|_{x_n=0^+}. \quad (4.8)$$

In this local setting we also introduce

$$\beta = (c_j^r/c_j^l)|_{x_n=0^+}, \quad \tilde{G}_1 = i\partial_{x_n} \varphi_j^l (\theta_{\bullet,j}^l - \theta_{\bullet,j}^r) + \frac{1}{c_j^l|_{x_n=0^+}} G_1, \quad (4.9)$$

$$k = -i(\partial_{x_n} \varphi_j^l|_{x_n=0^+} + \beta \partial_{x_n} \varphi_j^r|_{x_n=0^+}). \quad (4.10)$$

Transmission conditions (4.6) can be written as

$$\begin{cases} \frac{\delta c_j^s}{h i c_j^l} P_{\varphi,j}^s u_{\bullet,j}^s = \gamma_1(u_{\bullet,j}^l) + \beta \gamma_1(u_{\bullet,j}^r) - k \gamma_0(u_{\bullet,j}^r) + \tilde{G}_1, \\ \gamma_0(u_{\bullet,j}^{7l}) = u_{\bullet,j}^s + \theta_{\bullet,j}^{7l}. \end{cases} \quad (\text{TC}_{\bullet,j})$$

where the remainder  $\tilde{G}_1$  can be estimated thanks to (4.7) by

$$|\tilde{G}_1|_0 \lesssim (\delta + h) |v_j^s|_1 + |\Theta_{\varphi,j}^s|_0 + |\theta_{\varphi,j}^l|_0 + |\theta_{\varphi,j}^r|_0. \quad (4.11)$$

We are now prepared to prove the different Carleman estimates in the four microlocal regions.

## 4.2 Estimate in the region $\mathcal{G}$

Here, we place ourselves in the region  $\mathcal{G}$ , and prove a Carleman estimate for  $u_{\mathcal{G},j}$ , and consequently for  $\Xi_{\mathcal{G}} v$ .

We introduce a microlocal cut-off function  $\chi_{\mathcal{G},\mathcal{F}} \in \mathcal{C}_c^\infty(\mathcal{M}_+^*)$ ,  $0 \leq \chi_{\mathcal{G},\mathcal{F}} \leq 1$ , satisfying

$$\begin{aligned} \chi_{\mathcal{G},\mathcal{F}} &= 1 \text{ on a neighborhood of } \text{supp}(\chi_{\mathcal{G}}), \\ \chi_{\mathcal{G}} + \chi_{\mathcal{F}} &= 1 \text{ on a neighborhood of } \text{supp}(\chi_{\mathcal{G},\mathcal{F}}). \end{aligned} \quad (4.12)$$

We choose  $\zeta^2 \in \mathcal{C}_c^\infty(0, X_0)$  such that  $0 \leq \zeta^2 \leq 1$ ,  $\zeta^2 = 1$  on a neighborhood of  $\text{supp}(\zeta^1)$  (with  $\zeta^1$  defined in (3.23)), and such that  $\tilde{\zeta}_j = 1$  on  $\text{supp}((\phi_j^{-1})^* \zeta_j^2)$  where  $\zeta_j^2(x_0, y) = \zeta^2(x_0) \psi_j(y)$ . As in (3.25) we set

$$\chi_{\mathcal{GF},j} = \tilde{\zeta}_j (\phi_j^{-1})^* \chi_{\mathcal{GF}},$$

and we define the associated tangential pseudo-differential operator  $\Xi_{\mathcal{GF}}$  by

$$\Xi_{\mathcal{GF}} = \sum_{j \in J} \Xi_{\mathcal{GF},j}, \quad \text{with} \quad \Xi_{\mathcal{GF},j} = \phi_j^* \text{Op}_{\mathcal{T}}(\chi_{\mathcal{GF},j}) (\phi_j^{-1})^* \zeta_j^2, \quad j \in J,$$

Note that the local symbol (see Proposition B.7) of  $\Xi_{\mathcal{GF}}$  in each chart is equal to one in the support of that of  $\Xi_{\mathcal{G}}$ .

We recall that the function  $\zeta = \zeta(x_n) \in \mathcal{C}_c^\infty([0, 2\varepsilon])$  satisfies  $\zeta(0) = 1$  on  $[0, \varepsilon]$ .

Making use of the Calderón projector technique for  $P_{\varphi,j}^r$  and of the standard Carleman techniques for  $P_{\varphi,j}^l$ , we obtain the following partial estimate.

**Proposition 4.1.** *Suppose that the weight function  $\varphi$  satisfies the properties listed in Section 3.1. Then, for all  $\delta_0 > 0$ , there exist  $C > 0$  and  $h_0 > 0$  such that, for all  $0 < \delta \leq \delta_0$  and  $0 < h \leq h_0$ ,  $v^r \in \mathcal{C}_c^\infty((0, X_0) \times S \times [0, 2\varepsilon])$  and  $v^s \in \mathcal{C}_c^\infty((0, X_0) \times S)$  satisfying (3.9), we have*

$$\|\Xi_{\mathcal{G}} v^r\|_1^2 + h |\Xi_{\mathcal{G}} v^r|_{x_n=0^+}|_1^2 + h |D_{x_n} \Xi_{\mathcal{G}} v^r|_{x_n=0^+}|_0^2 \leq C \left( \|P_{\varphi}^r v^r\|_0^2 + h^2 \|v^r\|_1^2 + h^4 |D_{x_n} v^r|_{x_n=0^+}|_0^2 \right), \quad (4.13)$$

and

$$\begin{aligned} & h \|\Xi_{\mathcal{G}} v^l\|_1^2 + h |\Xi_{\mathcal{G}} v^l|_{x_n=0^+}|_1^2 + h |D_{x_n} \Xi_{\mathcal{G}} v^l|_{x_n=0^+}|_0^2 \\ & \leq C \left( 1 + \frac{\delta^2}{h^2} \right) \left( \|\zeta P_{\varphi}^r v^r\|_0^2 + h^2 \|\Xi_{\mathcal{GF}} v^r\|_1^2 + h^4 |D_{x_n} v^r|_{x_n=0^+}|_0^2 + h^4 \|v^r\|_1^2 + h^3 |v^s|_1^2 \right) \\ & \quad + C \left( \|P_{\varphi}^l v^l\|_0^2 + h^2 \|v^l\|_1^2 + h |\theta_{\varphi}^l|_1^2 + \frac{\delta^2}{h} |\theta_{\varphi}^r|_0^2 + h |\theta_{\varphi}^r|_1^2 + h |\Theta_{\varphi}^s|_0^2 \right). \end{aligned} \quad (4.14)$$

*Proof.* The function  $u_{\mathcal{G},j}$ , defined in (4.1), satisfies (TC $_{\bullet,j}$ ), with  $\bullet = \mathcal{G}$ . On the “r” side, the root configuration described in Lemma 3.6 (and represented in Figure 3) allows us to apply the Calderón projector technique used in [LR97, LR10]. According to [LR10, Remark 2.5] and using Eqs. (2.59), (2.60), and (2.61) therein, applied with  $v^d$  replaced here by  $v_j^r$ , we have

$$\|u_{\mathcal{G},j}^r\|_1 + h^{\frac{1}{2}} |\gamma_0(u_{\mathcal{G},j}^r)|_1 + h^{\frac{1}{2}} |\gamma_1(u_{\mathcal{G},j}^r)|_0 \lesssim \|P_{\varphi,j}^r v_j^r\|_0 + h \|v_j^r\|_1 + h^2 |D_{x_n} v_j^r|_{x_n=0^+}|_0, \quad (4.15)$$

which is a local version of (4.13).

Let us now explain how such local estimates can be patched together to yield (4.13). Concerning the first term on the left hand-side of (4.15), and using the definition of Sobolev norms given in (1.20)–(1.22), we have

$$\|\Xi_{\mathcal{G}} v^r\|_1 \lesssim \sum_{j \in J} \|u_{\mathcal{G},j}^r\|_1, \quad |\Xi_{\mathcal{G}} v^r|_{x_n=0^+}|_1 \lesssim \sum_{j \in J} |\gamma_0(u_{\mathcal{G},j}^r)|_1 \quad (4.16)$$

by (4.2) and Lemma B.15. Similarly we have  $D_{x_n} \Xi_{\mathcal{G}} v^r|_{x_n=0^+} = \sum_j \phi_j^* \gamma_1(u_{\mathcal{G},j}^r)$  since  $\phi_j^*$  does not depend on the  $x_n$ -variable. As a consequence, we obtain

$$|(D_{x_n} \Xi_{\mathcal{G}} v^r)|_{x_n=0^+}|_0 \leq \sum_{j \in J} |\phi_j^* \gamma_1(u_{\mathcal{G},j}^r)|_0 \lesssim \sum_{j \in J} |\gamma_1(u_{\mathcal{G},j}^r)|_0, \quad (4.17)$$

by Lemma 1.9.

Now concerning the right hand-side of (4.15), we directly have

$$\|v_j^r\|_1 = \|(\phi_j^{-1})^* \zeta_j v^r\|_1 = \|\zeta^1 (\phi_j^{-1})^* \psi_j v^r\|_1 \lesssim \|(\phi_j^{-1})^* \psi_j v^r\|_1 \lesssim \|v^r\|_1, \quad (4.18)$$

by the definition of  $\|\cdot\|_1$  on  $\mathcal{M}_+$ , as well as

$$|D_{x_n} v_j^r|_{x_n=0^+}|_0 \lesssim |D_{x_n} v^r|_{x_n=0^+}|_0. \quad (4.19)$$

Finally, we compute  $P_{\varphi,j}^r v_j^r = (\phi_j^{-1})^* P_{\varphi}^r \phi_j^* (\phi_j^{-1})^* \zeta_j v^r = (\phi_j^{-1})^* \zeta_j P_{\varphi}^r v^r + (\phi_j^{-1})^* [P_{\varphi}^r, \zeta_j] v^r$ . We have

$$\|(\phi_j^{-1})^* \zeta_j P_{\varphi}^r v^r\|_0 = \|\zeta^1 (\phi_j^{-1})^* \psi_j P_{\varphi}^r v^r\|_0 \leq \|P_{\varphi}^r v^r\|_0, \quad (4.20)$$



and, using Lemma 1.9,

$$\|(\phi_j^{-1})^*[P_\varphi^r, \zeta_j]v^r\|_0 \lesssim \|[P_\varphi^r, \zeta_j]v^r\|_0 \lesssim h\|v^r\|_1, \quad (4.21)$$

since  $[P_\varphi^r, \zeta_j] \in h\mathcal{D}^1(\mathcal{M}_+)$ . Finally combining all the estimates (4.16)-(4.21), together with the local inequalities (4.15) summed over  $j \in J$ , we obtain the sought global estimate (4.13) on  $\mathcal{M}_+$ .

To obtain Estimate (4.14) on the “ $l$ ” side we first need a more precise estimate for the “ $r$ ” side. For this, we introduce another microlocal cut-off function  $\tilde{\chi}_{\mathcal{G}\mathcal{F}}$  satisfying the same requirements (4.12) as  $\chi_{\mathcal{G}\mathcal{F}}$ , and such that  $\chi_{\mathcal{G}\mathcal{F}} = 1$  on a neighborhood of  $\text{supp}(\tilde{\chi}_{\mathcal{G}\mathcal{F}})$ . We choose  $\zeta^3 \in \mathcal{C}_c^\infty(0, X_0)$  such that  $0 \leq \zeta^3 \leq 1$ ,  $\zeta^3 = 1$  on a neighborhood of  $\text{supp}(\zeta^1)$ , and such that  $\zeta^2 = 1$  on a neighborhood of  $\text{supp}(\zeta^3)$ . As in (3.25) we set

$$\tilde{\chi}_{\mathcal{G}\mathcal{F},j} = \tilde{\zeta}_j(\phi_j^{-1})^* \tilde{\chi}_{\mathcal{G}\mathcal{F}},$$

and we define the associated tangential pseudo-differential operator  $\tilde{\Xi}_{\mathcal{G}\mathcal{F}}$  by

$$\tilde{\Xi}_{\mathcal{G}\mathcal{F}} = \sum_{j \in J} \tilde{\Xi}_{\mathcal{G}\mathcal{F},j}, \quad \text{with} \quad \tilde{\Xi}_{\mathcal{G}\mathcal{F},j} = \phi_j^* \text{Op}_{\mathcal{T}}(\tilde{\chi}_{\mathcal{G}\mathcal{F},j})(\phi_j^{-1})^* \zeta_j^3, \quad \zeta_j^3 = \zeta^3 \psi_j, \quad j \in J,$$

According to [LR10, Remark 2.5] and using (2.60) and (2.61) therein, applied with  $v^d$  replaced by  $\zeta(x_n)(\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r$ , we have

$$\begin{aligned} & h^{\frac{1}{2}} |\gamma_0(\text{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j})(\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r)|_1 + h^{\frac{1}{2}} |\gamma_1(\text{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j})(\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r)|_0 \\ & \lesssim \|P_{\varphi,j}^r \zeta(\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r\|_0 + h \|\zeta(\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r\|_1 + h^2 |\gamma_1((\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r)|_0. \end{aligned} \quad (4.22)$$

We notice that the right hand-side of this inequality can directly be bounded by global quantities. First, we have

$$\|\zeta(\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r\|_1 \lesssim \|\tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r\|_1 \quad (4.23)$$

Second, we estimate

$$|\gamma_1((\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r)|_0 \leq |(D_{x_n} \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r)|_{x_n=0^+}|_0,$$

where

$$(D_{x_n} \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r)|_{x_n=0^+} = (\tilde{\Xi}_{\mathcal{G}\mathcal{F}} D_{x_n} v^r)|_{x_n=0^+} + \underbrace{([D_{x_n}, \tilde{\Xi}_{\mathcal{G}\mathcal{F}}] v^r)}_{\in h\Psi_{\mathcal{T}}^0(\mathcal{M}_+)}|_{x_n=0^+}.$$

Using Proposition B.12 and the trace formula (1.23), we have the estimate

$$h^2 |\gamma_1((\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r)|_0 \lesssim h^2 |D_{x_n} v^r|_{x_n=0^+}|_0 + h^{\frac{5}{2}} \|v^r\|_1. \quad (4.24)$$

Concerning the term with  $P_{\varphi,j}^r$  in the right hand-side of (4.22), we can proceed as in (4.20)-(4.21) to obtain

$$\|P_{\varphi,j}^r \zeta(\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r\|_0 = \|(\phi_j^{-1})^* P_\varphi^r \zeta \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r\|_0 \lesssim \|P_\varphi^r \zeta \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r\|_0 + h \|\tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r\|_1 \quad (4.25)$$

Moreover, using Proposition B.10, we have  $\tilde{\Xi}_{\mathcal{G}\mathcal{F}}(1 - \Xi_{\mathcal{G}\mathcal{F}}) \in h^\infty \Psi_{\mathcal{T}}^{-\infty}(\mathcal{M}_+)$ , as their local symbols in every chart have disjoint supports by Proposition B.14, because of the supports of  $\zeta^3$  and  $\tilde{\chi}_{\mathcal{G}\mathcal{F}}$ . We then obtain with Proposition B.12

$$h \|\tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r\|_1 \lesssim h \|\tilde{\Xi}_{\mathcal{G}\mathcal{F}} \Xi_{\mathcal{G}\mathcal{F}} v^r\|_1 + h \|\tilde{\Xi}_{\mathcal{G}\mathcal{F}}(1 - \Xi_{\mathcal{G}\mathcal{F}}) v^r\|_1 \lesssim h \|\Xi_{\mathcal{G}\mathcal{F}} v^r\|_1 + h^2 \|v^r\|_1. \quad (4.26)$$

We also have

$$\|P_\varphi^r \zeta \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r\|_0 \lesssim \|\tilde{\Xi}_{\mathcal{G}\mathcal{F}} \zeta P_\varphi^r v^r\|_0 + \|[P_\varphi^r, \tilde{\Xi}_{\mathcal{G}\mathcal{F}} \zeta] v^r\|_0. \quad (4.27)$$

Arguing as above with Propositions B.10 and B.14, and also Corollary B.11, we have

$$[P_\varphi^r, \tilde{\Xi}_{\mathcal{G}\mathcal{F}} \zeta] = \underbrace{[P_\varphi^r, \tilde{\Xi}_{\mathcal{G}\mathcal{F}} \zeta] \Xi_{\mathcal{G}\mathcal{F}}}_{\in h\Psi^1(\mathcal{M}_+)} + \underbrace{[P_\varphi^r, \tilde{\Xi}_{\mathcal{G}\mathcal{F}} \zeta](1 - \Xi_{\mathcal{G}\mathcal{F}})}_{\in h^\infty \Psi^{-\infty}(\mathcal{M}_+)}$$

so that (4.27) now reads with Proposition B.12

$$\|P_\varphi^r \zeta \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r\|_0 \lesssim \|\zeta P_\varphi^r v^r\|_0 + h \|\Xi_{\mathcal{G}\mathcal{F}} v^r\|_1 + h^2 \|v^r\|_1. \quad (4.28)$$

The three estimates (4.25), (4.26) and (4.28) give

$$\|P_{\varphi,j}^r \zeta (\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r\|_0 \lesssim \|\zeta P_\varphi^r v^r\|_0 + h \|\Xi_{\mathcal{G}\mathcal{F}} v^r\|_1 + h^2 \|v^r\|_1. \quad (4.29)$$

Combining (4.22) together with (4.23)–(4.24), (4.26) and (4.29), we finally have

$$\begin{aligned} h^{\frac{1}{2}} |\gamma_0(\text{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j})(\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r)|_1 + h^{\frac{1}{2}} |\gamma_1(\text{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j})(\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r)|_0 \\ \lesssim \|\zeta P_\varphi^r v^r\|_0 + h \|\Xi_{\mathcal{G}\mathcal{F}} v^r\|_1 + h^2 |D_{x_n} v^r|_{|x_n=0^+}|_0 + h^2 \|v^r\|_1. \end{aligned} \quad (4.30)$$

Then, we need the following lemma to come back to the variable  $u_{\mathcal{G},j}^r = \text{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j})(\phi_j^{-1})^* \zeta_j v^r$  on the left hand-side of (4.30).

**Lemma 4.2.** *There exists  $R \in h^\infty \Psi_{\mathcal{T}}^{-\infty}(\mathcal{M}_+)$ , such that*

$$\text{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j})(\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r = u_{\mathcal{G},j}^r + (\phi_j^{-1})^* R v^r.$$

This lemma is proven in Appendix C.6. As a consequence we have

$$\begin{aligned} h^{\frac{1}{2}} |\gamma_0(u_{\mathcal{G},j}^r)|_1 &\lesssim h^{\frac{1}{2}} |\gamma_0(\text{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j})(\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r)|_1 + h^{\frac{1}{2}} |\gamma_0((\phi_j^{-1})^* R v^r)|_0 \\ &\lesssim h^{\frac{1}{2}} |\gamma_0(\text{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j})(\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r)|_1 + h^2 \|v^r\|_1 \end{aligned}$$

with the trace formula (1.23). This, together with Estimate (4.30) give

$$h^{\frac{1}{2}} |\gamma_0(u_{\mathcal{G},j}^r)|_1 \lesssim \|\zeta P_\varphi^r v^r\|_0 + h \|\Xi_{\mathcal{G}\mathcal{F}} v^r\|_1 + h^2 \|v^r\|_1 + h^2 |D_{x_n} v^r|_{|x_n=0^+}|_0. \quad (4.31)$$

Lemma 4.2 also yields

$$\begin{aligned} h^{\frac{1}{2}} |\gamma_1(u_{\mathcal{G},j}^r)|_0 &\lesssim h^{\frac{1}{2}} |\gamma_1(\text{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j})(\phi_j^{-1})^* \psi_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r)|_0 + h^{\frac{1}{2}} |\gamma_1((\phi_j^{-1})^* R v^r)|_0, \\ &\lesssim h^{\frac{1}{2}} |\gamma_1(\text{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j})(\phi_j^{-1})^* \psi_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r)|_0 + h^2 \|v^r\|_1 + h^2 |D_{x_n} v^r|_{|x_n=0^+}|_0, \end{aligned} \quad (4.32)$$

Combining (4.30) together with (4.32), we finally obtain

$$h^{\frac{1}{2}} |\gamma_1(u_{\mathcal{G},j}^r)|_0 \lesssim \|\zeta P_\varphi^r v^r\|_0 + h \|\Xi_{\mathcal{G}\mathcal{F}} v^r\|_1 + h^2 \|v^r\|_1 + h^2 |D_{x_n} v^r|_{|x_n=0^+}|_0. \quad (4.33)$$

On the “ $l$ ” side, we apply the Carleman method. With the properties of the weight function of Section 3.1 and in particular by (3.18), and by Lemma 2 in [LR95], we then have

$$h \|u_{\mathcal{G},j}^l\|_1^2 + \text{Re} \left( h \mathcal{B}^l(u_{\mathcal{G},j}^l) + h^2 \left( (D_n u_{\mathcal{G},j}^l + L_1^l u_{\mathcal{G},j}^l)|_{x_n=0^+}, L_0^l u_{\mathcal{G},j}^l|_{x_n=0^+} \right)_0 \right) \lesssim \|P_{\varphi,j} u_{\mathcal{G},j}^l\|_0^2, \quad (4.34)$$

for  $0 < h \leq h_0$ ,  $h_0$  sufficiently small, where  $L_1^l \in \mathcal{D}_{\mathcal{T}}^1$ ,  $L_0^l \in \Psi_{\mathcal{T}}^0$ . The quadratic form  $\mathcal{B}^l$  is given by

$$\mathcal{B}^l(\psi) = \left( \begin{pmatrix} 2\partial_{x_n} \varphi_j^l|_{x_n=0^+} & B_1^l \\ B_1^{l'} & B_2^l \end{pmatrix} \begin{pmatrix} \gamma_1(\psi) \\ \gamma_0(\psi) \end{pmatrix}, \begin{pmatrix} \gamma_1(\psi) \\ \gamma_0(\psi) \end{pmatrix} \right)_0, \quad \text{supp}(\psi) \subset (0, X_0) \times \tilde{U}_j \times [0, 2\varepsilon), \quad (4.35)$$

where  $B_1^l, B_1^{l'} \in \mathcal{D}_{\mathcal{T}}^1$ , with principal symbols  $\sigma(B_1^l) = \sigma(B_1^{l'}) = 2(\phi_j^{-1})^* q_1^l|_{x_n=0^+}$  and  $B_2^l \in \mathcal{D}_{\mathcal{T}}^2$ , with  $\sigma(B_2^l) = -2\partial_{x_n} \varphi_j^l (\phi_j^{-1})^* q_2^l|_{x_n=0^+}$ .

Observe that we have

$$\left| \left( (D_n u_{\mathcal{G},j}^l + L_1^l u_{\mathcal{G},j}^l)|_{x_n=0^+}, L_0^l u_{\mathcal{G},j}^l|_{x_n=0^+} \right)_0 \right| \lesssim |\gamma_1(u_{\mathcal{G},j}^l)|_0^2 + |\gamma_0(u_{\mathcal{G},j}^l)|_1^2. \quad (4.36)$$

and

$$|\mathcal{B}^l(u_{\mathcal{G},j}^l)| \lesssim |\gamma_0(u_{\mathcal{G},j}^l)|_1^2 + |\gamma_1(u_{\mathcal{G},j}^l)|_0^2. \quad (4.37)$$

Now, using (4.34), together with the estimates (4.36) and (4.37), we have,

$$h \|u_{\mathcal{G},j}^l\|_1^2 \lesssim \|P_{\varphi,j}^l u_{\mathcal{G},j}^l\|_0^2 + h |\gamma_0(u_{\mathcal{G},j}^l)|_1^2 + h |\gamma_1(u_{\mathcal{G},j}^l)|_0^2. \quad (4.38)$$

It remains to estimate the traces on the “ $l$ ” side by the traces on the “ $r$ ” side, through the transmission conditions (TC $_{\bullet,j}$ ):

$$\begin{cases} \gamma_0(u_{\mathcal{G},j}^l) = \gamma_0(u_{\mathcal{G},j}^r) + \theta_{\mathcal{G},j}^l - \theta_{\mathcal{G},j}^r, \\ \gamma_1(u_{\mathcal{G},j}^l) = \frac{\delta c_j^s}{h i c_j^l} P_{\varphi,j}^s (\gamma_0(u_{\mathcal{G},j}^r) - \theta_{\mathcal{G},j}^r) - \beta \gamma_1(u_{\mathcal{G},j}^r) + k \gamma_0(u_{\mathcal{G},j}^r) - \tilde{G}_1, \\ u_{\mathcal{G},j}^s = \gamma_0(u_{\mathcal{G},j}^r) - \theta_{\mathcal{G},j}^r. \end{cases}$$

As a consequence,  $\gamma_0(u_{\mathcal{G},j}^l)$  and  $\gamma_1(u_{\mathcal{G},j}^l)$  can be estimated as follows

$$\begin{cases} |\gamma_0(u_{\mathcal{G},j}^l)|_1 \leq |\gamma_0(u_{\mathcal{G},j}^r)|_1 + |\theta_{\mathcal{G},j}^l|_1 + |\theta_{\mathcal{G},j}^r|_1, \\ |\gamma_1(u_{\mathcal{G},j}^l)|_0 \lesssim |\gamma_1(u_{\mathcal{G},j}^r)|_0 + \frac{\delta}{h} |P_{\varphi,j}^s \gamma_0(u_{\mathcal{G},j}^r)|_0 + \frac{\delta}{h} |P_{\varphi,j}^s \theta_{\mathcal{G},j}^r|_0 + |\gamma_0(u_{\mathcal{G},j}^r)|_0 + |\tilde{G}_1|_0. \end{cases} \quad (4.39)$$

We now prove that, on the support of  $\chi_{\mathcal{G},j}$ , the operator  $P_{\varphi,j}^s$  is of order 0. For this, let  $\tilde{\chi} \in \mathcal{C}_c^\infty(T^*(\mathbb{R}^n))$ , be equal to one on a neighborhood of the  $\text{supp}(\chi_{\mathcal{G},j}|_{x_n=0+})$ . We then have

$$\gamma_0(u_{\mathcal{G},j}^r) = \text{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j}) v_j^r|_{x_n=0+} = \text{Op}_{\mathcal{T}}(\tilde{\chi}) \text{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j}) v_j^r|_{x_n=0+} + \underbrace{\text{Op}_{\mathcal{T}}(1 - \tilde{\chi}) \text{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j})}_{\in h^\infty \Psi_{\mathcal{T}}^{-\infty}} v_j^r|_{x_n=0+},$$

which yields

$$P_{\varphi,j}^s \gamma_0(u_{\mathcal{G},j}^r) = \underbrace{(P_{\varphi,j}^s \text{Op}_{\mathcal{T}}(\tilde{\chi}))}_{\in \Psi_{\mathcal{T}}^0} \gamma_0(u_{\mathcal{G},j}^r) + \underbrace{P_{\varphi,j}^s \text{Op}_{\mathcal{T}}(1 - \tilde{\chi}) \text{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j})}_{\in h^\infty \Psi_{\mathcal{T}}^{-\infty}} v_j^r|_{x_n=0+}.$$

This, together with the trace formula (1.23) gives the estimate,

$$\frac{\delta}{h} |P_{\varphi,j}^s \gamma_0(u_{\mathcal{G},j}^r)|_0 \leq C \frac{\delta}{h} |\gamma_0(u_{\mathcal{G},j}^r)|_0 + C_N \delta h^N \|v_j^r\|_1, \quad N \in \mathbb{N}.$$

Similarly, we have the estimate

$$\frac{\delta}{h} |P_{\varphi,j}^s \theta_{\mathcal{G},j}^r|_0 \leq C \frac{\delta}{h} |\theta_{\mathcal{G},j}^r|_0 + C_N \delta h^N |\theta_{\varphi,j}^r|_0 \lesssim \frac{\delta}{h} |\theta_{\varphi,j}^r|_0.$$

The last two estimates and the second equation of (4.39) yield,

$$|\gamma_1(u_{\mathcal{G},j}^l)|_0 \lesssim |\gamma_1(u_{\mathcal{G},j}^r)|_0 + (1 + \frac{\delta}{h}) |\gamma_0(u_{\mathcal{G},j}^r)|_0 + \frac{\delta}{h} |\theta_{\varphi,j}^r|_0 + |\tilde{G}_1|_0 + C_N \delta h^N \|v_j^r\|_1, \quad N \in \mathbb{N}.$$

Using estimates (4.31) and (4.33) to bound the traces on the “ $r$ ” side, we obtain

$$h^{\frac{1}{2}} |\gamma_1(u_{\mathcal{G},j}^l)|_0 \lesssim (1 + \frac{\delta}{h}) \left( \|\zeta P_{\varphi}^r v^r\|_0 + h \|\Xi_{\mathcal{G}\mathcal{F}} v^r\|_1 + h^2 \|v^r\|_1 + h^2 |D_{x_n} v^r|_{x_n=0+}|_0 \right) + \frac{\delta}{h^{\frac{1}{2}}} |\theta_{\varphi,j}^r|_0 + h^{\frac{1}{2}} |\tilde{G}_1|_0,$$

for  $0 < h \leq h_0$ , and, using (4.11) to estimate the remainder, we have

$$\begin{aligned} h^{\frac{1}{2}} |\gamma_1(u_{\mathcal{G},j}^l)|_0 &\lesssim (1 + \frac{\delta}{h}) \left( \|\zeta P_{\varphi}^r v^r\|_0 + h \|\Xi_{\mathcal{G}\mathcal{F}} v^r\|_1 + h^2 \|v^r\|_1 + h^2 |D_{x_n} v^r|_{x_n=0+}|_0 \right) \\ &\quad + h^{\frac{3}{2}} |v_j^s|_1 + h^{\frac{1}{2}} |\theta_{\varphi,j}^r|_0 + h^{\frac{1}{2}} |\Theta_{\varphi,j}^s|_0 + h^{\frac{1}{2}} |\theta_{\varphi,j}^l|_0, \end{aligned} \quad (4.40)$$

We observe now that the first line of (4.39) together with (4.31) yields

$$h^{\frac{1}{2}} |\gamma_0(u_{\mathcal{G},j}^l)|_1 \lesssim \|\zeta P_{\varphi}^r v^r\|_0 + h \|\Xi_{\mathcal{G}\mathcal{F}} v^r\|_1 + h^2 \|v^r\|_1 + h^2 |D_{x_n} v^r|_{x_n=0+}|_0 + h^{\frac{1}{2}} |\theta_{\varphi,j}^l|_1 + h^{\frac{1}{2}} |\theta_{\varphi,j}^r|_1. \quad (4.41)$$

Combining (4.5), with (4.38), (4.40) and (4.41) we obtain

$$\begin{aligned} &h \|u_{\mathcal{G},j}^l\|_1^2 + h |\gamma_0(u_{\mathcal{G},j}^l)|_1^2 + h |\gamma_1(u_{\mathcal{G},j}^l)|_0^2 \\ &\lesssim (1 + \frac{\delta^2}{h^2}) \left( \|\zeta P_{\varphi}^r v^r\|_0^2 + h^2 \|\Xi_{\mathcal{G}\mathcal{F}} v^r\|_1^2 + h^4 \|v^r\|_1^2 + h^4 |D_{x_n} v^r|_{x_n=0+}|_0^2 + h^3 |v_j^s|_1^2 \right) \\ &\quad + h |\theta_{\varphi,j}^l|_1^2 + \frac{\delta^2}{h} |\theta_{\varphi,j}^r|_0^2 + h |\theta_{\varphi,j}^r|_1^2 + h |\Theta_{\varphi,j}^s|_0^2 + \|P_{\varphi,j}^l v_j^l\|_0^2 + h^2 \|v_j^l\|_1^2. \end{aligned} \quad (4.42)$$

This is a local version of (4.14). Patching together on  $\mathcal{M}_+$  the local Carleman estimates (4.42) as we did in (4.16)-(4.21) yields (4.14). This concludes the proof of Proposition 4.1.  $\square$

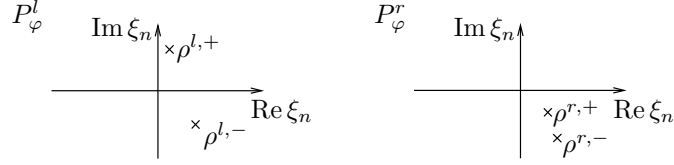


Figure 5: Root configuration in the region  $\mathcal{F}$ .

### 4.3 Estimate in the region $\mathcal{F}$

Here, we place ourselves in the region  $\mathcal{F}$ , and prove a Carleman estimate for  $u_{\mathcal{F},j}$ , and consequently for  $\Xi_{\mathcal{F}}v$ . Making use of the Calderón projector technique for both  $P_{\varphi,j}^r$  and  $P_{\varphi,j}^l$ , we obtain the following partial estimate.

**Proposition 4.3.** *Suppose that the weight function  $\varphi$  satisfies the properties listed in Section 3.1. Then, for all  $\delta_0 > 0$ , there exist  $C > 0$  and  $h_0 > 0$  such that, for all  $0 < \delta \leq \delta_0$  and  $0 < h \leq h_0$ ,  $v^{7l} \in \mathcal{C}_c^\infty((0, X_0) \times S \times [0, 2\varepsilon])$  and  $v^s \in \mathcal{C}_c^\infty((0, X_0) \times S)$  satisfying (3.9), we have*

$$\|\Xi_{\mathcal{F}}v^r\|_1^2 + h|\Xi_{\mathcal{F}}v_{|x_n=0+}^r|_1^2 + h|D_{x_n}\Xi_{\mathcal{F}}v_{|x_n=0+}^r|_0^2 \leq C\left(\|P_{\varphi}^rv^r\|_0^2 + h^2\|v^r\|_1^2 + h^4|D_{x_n}v_{|x_n=0+}^r|_0^2\right), \quad (4.43)$$

and

$$\begin{aligned} \|\Xi_{\mathcal{F}}v^l\|_1^2 + h|\Xi_{\mathcal{F}}v_{|x_n=0+}^l|_1^2 + h|D_{x_n}\Xi_{\mathcal{F}}v_{|x_n=0+}^l|_0^2 \leq C\left(\|P_{\varphi}^lv^l\|_0^2 + h^2\|v^l\|_1^2 + h^4|D_{x_n}v_{|x_n=0+}^l|_0^2 + \|P_{\varphi}^rv^r\|_0^2 \right. \\ \left. + h^2\|v^r\|_1^2 + h^4|D_{x_n}v_{|x_n=0+}^r|_0^2 + h|\theta_{\varphi}^l|_1^2 + h|\theta_{\varphi}^r|_1^2\right). \end{aligned} \quad (4.44)$$

*Proof.* Here, the functions  $u_{\mathcal{F},j}^{7l}$ ,  $j \in J$  satisfy (TC $_{\bullet,j}$ ), with  $\bullet = \mathcal{F}$ . On both the “r” and “l” sides, the roots configuration described in Lemma 3.6 (and represented in Figure 5) allows us to use the Calderón projector technique used in [LR97, LR10]. According to [LR10, Remark 2.5] and using Eqs. (2.59), (2.60), and (2.61) therein, applied with  $v^d$  replaced here by  $v_j^r$ , we have

$$\|u_{\mathcal{F},j}^r\|_1 + h^{\frac{1}{2}}|\gamma_0(u_{\mathcal{F},j}^r)|_1 + h^{\frac{1}{2}}|\gamma_1(u_{\mathcal{F},j}^r)|_0 \lesssim \|P_{\varphi,j}^rv_j^r\|_0 + h\|v_j^r\|_1 + h^2|D_{x_n}v_j^r|_{|x_n=0+}|_0. \quad (4.45)$$

This is a local version of (4.43). Patching together on  $\mathcal{M}_+$  the local Carleman estimates (4.45) as we did in (4.16)-(4.21) yields (4.43).

On the “l” side, since both roots are separated by the real axis (see Figure 5) we only obtain one relation between the two traces at the interface: according to [LR10, Eq. (2.67)], we have

$$\|u_{\mathcal{F},j}^l\|_1 \lesssim \|P_{\varphi,j}^lv_j^l\|_0 + h\|v_j^l\|_1 + h^{\frac{1}{2}}|\gamma_0(u_{\mathcal{F},j}^l)|_1 + h^{\frac{1}{2}}|\gamma_1(u_{\mathcal{F},j}^l)|_0 + h^2|D_{x_n}v_j^l|_{|x_n=0+}|_0, \quad (4.46)$$

together with the following relation between the two traces [LR10, Eq. (2.68)]:

$$(1 - \text{Op}_{\mathcal{T}}(a^l))\gamma_0(u_{\mathcal{F},j}^l) = \text{Op}_{\mathcal{T}}(b^l)\gamma_1(u_{\mathcal{F},j}^l) + G_2^l, \quad (4.47)$$

where  $a^l \in S_{\mathcal{T}}^0$  and  $b^l \in S_{\mathcal{T}}^{-1}$  have for principal part respectively

$$a_0^l = -\left(\tilde{\chi}\frac{\rho_j^{l,-}}{\rho_j^{l,+} - \rho_j^{l,-}}\right)\Big|_{x_n=0+}, \quad \text{and} \quad b_{-1}^l = \left(\tilde{\chi}\frac{1}{\rho_j^{l,+} - \rho_j^{l,-}}\right)\Big|_{x_n=0+},$$

where  $\rho_j^{l,\pm}$  are the roots of  $p_{\varphi,j}^l$  (i.e.  $\rho_j^{l,\pm} = (\phi_j^{-1})^*\rho^{l,\pm}$  with  $\rho^{l,\pm}$  described in Lemma 3.6) and  $\tilde{\chi} \in \mathcal{C}_c^\infty(T^*(\mathbb{R}^n))$  is compactly supported and equal to one on a neighborhood of the support of  $\chi_{\mathcal{F},j}|_{x_n=0+}$ . The remainder  $G_2^l$  (coming from the Calderón projector method) satisfies [LR10, Eq. (2.69)]:

$$|G_2^l|_1 \lesssim h^{-\frac{1}{2}}\left(\|P_{\varphi,j}^lv_j^l\|_0 + h\|v_j^l\|_1 + h^2|D_{x_n}v_j^l|_{|x_n=0+}|_0\right). \quad (4.48)$$

Let  $\hat{\chi} \in \mathcal{C}^\infty(T^*(\mathbb{R}^n))$  satisfy the same requirements as  $\tilde{\chi}$  with  $\tilde{\chi}$  equal to one a neighborhood the support of  $\hat{\chi}$ . Since  $b_{-1}^l$  does not vanish in a neighborhood of  $\text{supp}(\hat{\chi})$ , one can introduce a parametrix for  $\text{Op}_{\mathcal{T}}(b^l)$ , say  $\text{Op}_{\mathcal{T}}(e)$ , with  $e \in S_{\mathcal{T}}^1$ , satisfying

$$\text{Op}_{\mathcal{T}}(e) \text{Op}_{\mathcal{T}}(b^l) = \text{Op}_{\mathcal{T}}(\hat{\chi}) + R, \quad R \in h^\infty \Psi_{\mathcal{T}}^{-\infty}.$$

Applying this parametrix to (4.47) gives the estimate

$$|\gamma_1(u_{\mathcal{F},j}^l)|_0 \lesssim |\gamma_0(u_{\mathcal{F},j}^l)|_1 + |G_2^l|_1 + C_N h^N \left( \|v_j^l\|_1 + |D_{x_n} v_j^l|_{|x_n=0^+}|_0 \right), \quad N \in \mathbb{N}. \quad (4.49)$$

Here, we have used the trace formula (1.23) together with

$$\begin{aligned} \gamma_1(u_{\mathcal{F},j}^l) &= \text{Op}_{\mathcal{T}}(\hat{\chi})\gamma_1(u_{\mathcal{F},j}^l) + \underbrace{\left( (1 - \text{Op}_{\mathcal{T}}(\hat{\chi})) \text{Op}_{\mathcal{T}}(\chi_{\mathcal{F},j})|_{|x_n=0^+} \right)}_{\in h^\infty \Psi_{\mathcal{T}}^{-\infty}} D_{x_n} v_j^l|_{|x_n=0^+} \\ &\quad + \underbrace{\left( (1 - \text{Op}_{\mathcal{T}}(\hat{\chi})) [D_{x_n}, \text{Op}_{\mathcal{T}}(\chi_{\mathcal{F},j})]|_{|x_n=0^+} \right)}_{\in h^\infty \Psi_{\mathcal{T}}^{-\infty}} v_j^l|_{|x_n=0^+}. \end{aligned} \quad (4.50)$$

We now use the second equation in the transmission conditions (TC $_{\bullet,j}$ ), which with (4.45) yields

$$\begin{aligned} h^{\frac{1}{2}} |\gamma_0(u_{\mathcal{F},j}^l)|_1 &\leq h^{\frac{1}{2}} |\gamma_0(u_{\mathcal{F},j}^r)|_1 + h^{\frac{1}{2}} |\theta_{\mathcal{F},j}^l|_1 + h^{\frac{1}{2}} |\theta_{\mathcal{F},j}^r|_1 \\ &\lesssim \|P_{\varphi,j}^r v_j^r\|_0 + h \|v_j^r\|_1 + h^2 |D_{x_n} v_j^r|_{|x_n=0^+}|_0 + h^{\frac{1}{2}} |\theta_{\mathcal{F},j}^l|_1 + h^{\frac{1}{2}} |\theta_{\mathcal{F},j}^r|_1. \end{aligned}$$

This estimate together with (4.48) and (4.49) provides an estimate for  $|\gamma_1(u_{\mathcal{F},j}^l)|_0$ , which, summed with (4.46) yields

$$\begin{aligned} \|u_{\mathcal{F},j}^l\|_1 + h^{\frac{1}{2}} |\gamma_0(u_{\mathcal{F},j}^l)|_1 + h^{\frac{1}{2}} |\gamma_1(u_{\mathcal{F},j}^l)|_0 &\lesssim \|P_{\varphi,j}^l v_j^l\|_0 + h \|v_j^l\|_1 + h^2 |D_{x_n} v_j^l|_{|x_n=0^+}|_0 \\ &\quad + \|P_{\varphi,j}^r v_j^r\|_0 + h \|v_j^r\|_1 + h^2 |D_{x_n} v_j^r|_{|x_n=0^+}|_0 + h^{\frac{1}{2}} |\theta_{\mathcal{F},j}^l|_1 + h^{\frac{1}{2}} |\theta_{\mathcal{F},j}^r|_1. \end{aligned}$$

This is a local version of (4.44). Patching together on  $\mathcal{M}_+$  such local estimates as we did in (4.16)-(4.21) yields (4.44). This concludes the proof of Proposition 4.3.  $\square$

#### 4.4 Estimate in the region $\mathcal{Z}$

Here, we place ourselves in the region  $\mathcal{Z}$ , and prove a Carleman estimate for  $u_{\mathcal{Z},j}$ , and consequently for  $\Xi_{\mathcal{Z}} v$ .

As a consequence of property (3.6) of the weight function (see also (3.16)) and the compactness of  $[0, X_0] \times S \times [0, 2\varepsilon]$ , we remark that in the region  $\mathcal{Z}$ , there exists  $K_1 > 0$  such that

$$(\partial_{x_n} \varphi^r)^2 - \mu^r \geq \min (\partial_{x_n} \varphi^r)^2 - 2\epsilon_1 \geq K_1 > 0 \quad (4.51)$$

for  $\epsilon_1$  sufficiently small (the constant  $\epsilon_1$  is used in the definition of the microlocal regions in (3.22)).

Making use of the Calderón projector technique for  $P_{\varphi,j}^l$ , and standard techniques to prove Carleman estimates for  $P_{\varphi,j}^r$ , we obtain the following partial estimate.

**Proposition 4.4.** *Suppose that the weight function  $\varphi$  satisfies the properties listed in Section 3.1. Then, for all  $\delta_0 > 0$ , there exist  $C > 0$  and  $h_0 > 0$  such that, for all  $0 < \delta \leq \delta_0$  and  $0 < h \leq h_0$ ,  $v^l \in \mathcal{C}_c^\infty((0, X_0) \times S \times [0, 2\varepsilon])$  and  $v^s \in \mathcal{C}_c^\infty((0, X_0) \times S)$  satisfying (3.9), we have*

$$\begin{aligned} &h \|\Xi_{\mathcal{Z}} v^r\|_1^2 + h \left( 1 + \frac{\delta^2}{h^2} \right) |\Xi_{\mathcal{Z}} v^r|_{|x_n=0^+}|_1^2 + h |D_{x_n} \Xi_{\mathcal{Z}} v^r|_{|x_n=0^+}|_0^2 \\ &\leq C \left( \|P_{\varphi}^r v^r\|_0^2 + h^2 \|v^r\|_1^2 + h(\delta^2 + h^2) |v^s|_1^2 + \|P_{\varphi}^l v^l\|_0^2 + h^2 \|v^l\|_1^2 + h^4 |D_{x_n} v^l|_{|x_n=0^+}|_0^2 \right. \\ &\quad \left. + \frac{\delta^2}{h} |\theta_{\varphi}^r|_0^2 + h |\theta_{\varphi}^l|_1^2 + h |\theta_{\varphi}^r|_1^2 + h |\Theta_{\varphi}^s|_0^2 \right), \end{aligned} \quad (4.52)$$

and

$$\begin{aligned} &\|\Xi_{\mathcal{Z}} v^l\|_1^2 + h |\Xi_{\mathcal{Z}} v^l|_{|x_n=0^+}|_1^2 + h |D_{x_n} \Xi_{\mathcal{Z}} v^l|_{|x_n=0^+}|_0^2 \\ &\leq C \left( \|P_{\varphi}^l v^l\|_0^2 + h^2 \|v^l\|_1^2 + h^4 |D_{x_n} v^l|_{|x_n=0^+}|_0^2 + h^3 |v^s|_1^2 + \frac{h^2}{\delta^2 + h^2} (\|P_{\varphi}^r v^r\|_0^2 + h^2 \|v^r\|_1^2) \right. \\ &\quad \left. + h |\theta_{\varphi}^l|_1^2 + h |\theta_{\varphi}^r|_1^2 + \frac{h^3}{\delta^2 + h^2} |\Theta_{\varphi}^s|_0^2 \right). \end{aligned} \quad (4.53)$$

*Proof.* The function  $u_{\mathcal{Z},j}$  satisfies  $(\text{TC}_{\bullet,j})$ , with  $\bullet = \mathcal{Z}$ . On the “ $l$ ” side, the roots configuration described in Lemma 3.6 (and represented in Figure 3b) allows us to apply the Calderón projector technique as in [LR97, LR10]. Since both roots are separated by the real axis we only obtain one relation between the two traces at the interface: according to [LR10, Eq. (2.67)], we have

$$\|u_{\mathcal{Z},j}^l\|_1 \lesssim \|P_{\varphi,j}^l v_j^l\|_0 + h\|v_j^l\|_1 + h^{\frac{1}{2}}|\gamma_0(u_{\mathcal{Z},j}^l)|_1 + h^{\frac{1}{2}}|\gamma_1(u_{\mathcal{Z},j}^l)|_0 + h^2|D_{x_n} v_j^l|_{x_n=0^+}|_0, \quad (4.54)$$

together with the following relation between the two traces [LR10, Eq. (2.68)]:

$$(1 - \text{Op}_{\mathcal{T}}(a^l)) \gamma_0(u_{\mathcal{Z},j}^l) = \text{Op}_{\mathcal{T}}(b^l) \gamma_1(u_{\mathcal{Z},j}^l) + G_2^l, \quad (4.55)$$

where  $a^l \in S_{\mathcal{T}}^0$  and  $b^l \in S_{\mathcal{T}}^{-1}$  have for principal part respectively

$$a_0^l = - \left( \tilde{\chi} \frac{\rho_j^{l,-}}{\rho_j^{l,+} - \rho_j^{l,-}} \right) \Big|_{x_n=0^+}, \quad \text{and} \quad b_{-1}^l = \left( \tilde{\chi} \frac{1}{\rho_j^{l,+} - \rho_j^{l,-}} \right) \Big|_{x_n=0^+}, \quad (4.56)$$

where  $\rho_j^{l,\pm}$  are the roots of  $p_{\varphi,j}^l$  (i.e.  $\rho_j^{l,\pm} = (\phi_j^{-1})^* \rho^{l,\pm}$  with  $\rho^{l,\pm}$  described in Lemma 3.6) and  $\tilde{\chi} \in \mathcal{C}_c^\infty(T^*(\mathbb{R}^n))$  is equal to one on a neighborhood of the support of  $\chi_{\mathcal{Z},j}|_{x_n=0^+}$  and equal to zero in a neighborhood of

$$\left( (\phi_j^{-1})^* \mathcal{G} \right) \cap \{x_n = 0\} = \left\{ (x_0, \phi_j(y); \xi_0, {}^t d\phi_j^{-1}(\phi_j(y))\eta); (x_0, y, 0; \xi_0, \eta) \in \mathcal{G} \right\}.$$

The remainder  $G_2^l$  (coming from the Calderón projector method) satisfies [LR10, Eq. (2.69)]:

$$|G_2^l|_1 \lesssim h^{-\frac{1}{2}} \left( \|P_{\varphi,j}^l v_j^l\|_0 + h\|v_j^l\|_1 + h^2|D_{x_n} v_j^l|_{x_n=0^+}|_0 \right). \quad (4.57)$$

On the “ $r$ ” side, we apply the Carleman method to the operators  $P_{\varphi,j}^r$ . With the properties of the weight function of Section 3.1, and in particular by (3.18), and by Lemma 2 in [LR95], we then have

$$h\|u_{\mathcal{Z},j}^r\|_1^2 + \text{Re} \left( h\mathcal{B}^r(u_{\mathcal{Z},j}^r) + h^2 \left( (D_n u_{\mathcal{Z},j}^r + L_1^r u_{\mathcal{Z},j}^r)|_{x_n=0^+}, L_0^r u_{\mathcal{Z},j}^r|_{x_n=0^+} \right)_0 \right) \lesssim \|P_{\varphi,j}^r u_{\mathcal{Z},j}^r\|_0^2, \quad (4.58)$$

for  $h$  sufficiently small, where  $L_1^r \in \mathcal{D}_{\mathcal{T}}^1$ ,  $L_0^r \in \Psi_{\mathcal{T}}^0$ . The quadratic form  $\mathcal{B}^r$  is given by

$$\mathcal{B}^r(\psi) = \left( \begin{array}{cc} \left( 2\partial_{x_n} \varphi_j^r|_{x_n=0^+} & B_1^r \right) \\ B_1^{r'} & B_2^r \end{array} \right) \begin{pmatrix} \gamma_1(\psi) \\ \gamma_0(\psi) \end{pmatrix}, \quad \text{supp}(\psi) \subset (0, X_0) \times \tilde{U}_j \times [0, 2\varepsilon), \quad (4.59)$$

where  $B_1^r, B_1^{r'} \in \mathcal{D}_{\mathcal{T}}^1$ ,  $B_2^r \in \mathcal{D}_{\mathcal{T}}^2$ , with principal symbols  $\sigma(B_1^r) = \sigma(B_1^{r'}) = 2q_{1,j}^r|_{x_n=0^+}$  and  $\sigma(B_2^r) = -2\partial_{x_n} \varphi_j^r q_{2,j}^r|_{x_n=0^+}$  with  $q_{k,j}^r = (\phi_j^{-1})^* q_k^r$ ,  $k = 1, 2$ .

Observe that we have

$$\left| \left( (D_n u_{\mathcal{Z},j}^r + L_1^r u_{\mathcal{Z},j}^r)|_{x_n=0^+}, L_0^r u_{\mathcal{Z},j}^r|_{x_n=0^+} \right)_0 \right| \lesssim |\gamma_1(u_{\mathcal{Z},j}^r)|_0^2 + |\gamma_0(u_{\mathcal{Z},j}^r)|_1^2. \quad (4.60)$$

Thanks to the transmission conditions  $(\text{TC}_{\bullet,j})$  at the interface and the trace relation (4.55) on the “ $l$ ” side, we shall be able to express  $\gamma_1(u_{\mathcal{Z},j}^r)$  from  $\gamma_0(u_{\mathcal{Z},j}^r)$  on the “ $r$ ” side. This will allow us to turn  $\mathcal{B}^r$  into a quadratic form operating on  $\gamma_0(u_{\mathcal{Z},j}^r)$  only. We first formulate  $(\text{TC}_{\bullet,j})$  in the following manner:

$$\begin{cases} \gamma_0(u_{\mathcal{Z},j}^l) = \gamma_0(u_{\mathcal{Z},j}^r) + \theta_{\mathcal{Z},j}^l - \theta_{\mathcal{Z},j}^r \\ \gamma_1(u_{\mathcal{Z},j}^l) = \frac{\delta c_j^s}{h i c_j^l} P_{\varphi,j}^s (\gamma_0(u_{\mathcal{Z},j}^r) - \theta_{\mathcal{Z},j}^r) - \beta \gamma_1(u_{\mathcal{Z},j}^r) + k \gamma_0(u_{\mathcal{Z},j}^r) - \tilde{G}_1, \\ u_{\mathcal{Z},j}^s = \gamma_0(u_{\mathcal{Z},j}^r) - \theta_{\mathcal{Z},j}^r. \end{cases} \quad (4.61)$$

Let  $\hat{\chi} \in \mathcal{C}^\infty(T^*(\mathbb{R}^n))$  satisfy the same requirements as  $\tilde{\chi}$  with  $\hat{\chi}$  equal to one a neighborhood the support of  $\hat{\chi}$ . Since the principal part  $b_{-1}^l$  does not vanish in a neighborhood of  $\text{supp}(\hat{\chi})$  (see (4.56)) one can introduce a parametrix for  $\text{Op}_{\mathcal{T}}(b^l)$ , say  $\text{Op}_{\mathcal{T}}(e)$ , with  $e \in S_{\mathcal{T}}^1$ , satisfying

$$\text{Op}_{\mathcal{T}}(e) \text{Op}_{\mathcal{T}}(b^l) = \text{Op}_{\mathcal{T}}(\hat{\chi}) + R, \quad R \in h^\infty \Psi_{\mathcal{T}}^{-\infty}.$$

Note that the principal part of the parametrix  $e$  is given by  $\sigma(e) = \hat{\chi} \left( \rho_j^{l,+} - \rho_j^{l,-} \right) \Big|_{x_n=0^+}$ . Applying this parametrix to (4.55) gives

$$\begin{aligned} \text{Op}_{\mathcal{T}}(e) (1 - \text{Op}_{\mathcal{T}}(a^l)) \gamma_0(u_{\mathcal{Z},j}^l) &= \text{Op}_{\mathcal{T}}(\hat{\chi}) \gamma_1(u_{\mathcal{Z},j}^l) + R \gamma_1(u_{\mathcal{Z},j}^l) + \text{Op}_{\mathcal{T}}(e) G_2^l \\ &= \gamma_1(u_{\mathcal{Z},j}^l) + R_1 D_{x_n} v_j^l \Big|_{x_n=0^+} + R_0 v_j^l \Big|_{x_n=0^+} + \text{Op}_{\mathcal{T}}(e) G_2^l, \end{aligned} \quad (4.62)$$

with  $R_1 \in h^\infty \Psi_{\mathcal{T}}^{-\infty}$  and  $R_0 \in h^\infty \Psi_{\mathcal{T}}^{-\infty}$ , since

$$\begin{aligned} \gamma_1(u_{\mathcal{Z},j}^l) &= \text{Op}_{\mathcal{T}}(\hat{\chi}) \gamma_1(u_{\mathcal{Z},j}^l) + \underbrace{(1 - \text{Op}_{\mathcal{T}}(\hat{\chi})) \text{Op}_{\mathcal{T}}(\chi_{\mathcal{Z},j})}_{\in h^\infty \Psi_{\mathcal{T}}^{-\infty}} D_{x_n} v_j^l \Big|_{x_n=0^+} \\ &\quad + \underbrace{(1 - \text{Op}_{\mathcal{T}}(\hat{\chi})) [D_{x_n}, \text{Op}_{\mathcal{T}}(\chi_{\mathcal{Z},j})]}_{\in h^\infty \Psi_{\mathcal{T}}^{-\infty}} v_j^l \Big|_{x_n=0^+}, \end{aligned}$$

and

$$R \gamma_1(u_{\mathcal{Z},j}^l) = \underbrace{R \text{Op}_{\mathcal{T}}(\chi_{\mathcal{Z},j})}_{\in h^\infty \Psi_{\mathcal{T}}^{-\infty}} D_{x_n} v_j^l \Big|_{x_n=0^+} + \underbrace{R [D_{x_n}, \text{Op}_{\mathcal{T}}(\chi_{\mathcal{Z},j})]}_{\in h^\infty \Psi_{\mathcal{T}}^{-\infty}} v_j^l \Big|_{x_n=0^+}.$$

Using the first relation of (4.61) to replace  $\gamma_0(u_{\mathcal{Z},j}^l)$  by  $\gamma_0(u_{\mathcal{Z},j}^r)$  in (4.62), we obtain

$$\begin{aligned} \text{Op}_{\mathcal{T}}(e) (1 - \text{Op}_{\mathcal{T}}(a^l)) (\gamma_0(u_{\mathcal{Z},j}^r) + \theta_{\mathcal{Z},j}^l - \theta_{\mathcal{Z},j}^r) \\ = \gamma_1(u_{\mathcal{Z},j}^l) + R_1 D_{x_n} v_j^l \Big|_{x_n=0^+} + R_0 v_j^l \Big|_{x_n=0^+} + \text{Op}_{\mathcal{T}}(e) G_2^l. \end{aligned} \quad (4.63)$$

Now, replacing (4.63) in the second equation of (4.61) yields the following relation between the two traces of  $u_{\mathcal{Z},j}^r$ :

$$\begin{aligned} \beta \gamma_1(u_{\mathcal{Z},j}^r) &= \left( \frac{\delta c_j^s}{h i c_j^l} P_{\varphi,j}^s - \text{Op}_{\mathcal{T}}(e) (1 - \text{Op}_{\mathcal{T}}(a^l)) + k \right) \gamma_0(u_{\mathcal{Z},j}^r) - \frac{\delta c_j^s}{h i c_j^l} P_{\varphi,j}^s \theta_{\mathcal{Z},j}^r \\ &\quad - \text{Op}_{\mathcal{T}}(e) (1 - \text{Op}_{\mathcal{T}}(a^l)) (\theta_{\mathcal{Z},j}^l - \theta_{\mathcal{Z},j}^r) - \tilde{G}_1 + \text{Op}_{\mathcal{T}}(e) G_2^l \\ &\quad + R_1 D_{x_n} v_j^l \Big|_{x_n=0^+} + R_0 v_j^l \Big|_{x_n=0^+}. \end{aligned}$$

This equation can be written under the form

$$\gamma_1(u_{\mathcal{Z},j}^r) = \Sigma_\delta \gamma_0(u_{\mathcal{Z},j}^r) + G_3 \quad (4.64)$$

where

$$\Sigma_\delta = \frac{1}{\beta} \left( \frac{\delta c_j^s}{h i c_j^l} P_{\varphi,j}^s - \text{Op}_{\mathcal{T}}(e) (1 - \text{Op}_{\mathcal{T}}(a^l)) + k \right), \quad (4.65)$$

and with (4.11) and (4.57) the term  $G_3$  can be estimated as

$$\begin{aligned} |G_3|_0 &\lesssim \frac{\delta}{h} |\theta_{\varphi,j}^r|_0 + |\theta_{\varphi,j}^l|_1 + |\theta_{\varphi,j}^r|_1 + (\delta + h) |v_j^s|_1 + |\Theta_{\varphi,j}^s|_0 \\ &\quad + h^{-\frac{1}{2}} \left( \|P_{\varphi,j}^l v_j^l\|_0 + h \|v_j^l\|_1 + h^2 |D_{x_n} v_j^l \Big|_{x_n=0^+}|_0 \right), \end{aligned} \quad (4.66)$$

where we have used the trace formula (1.23) and

$$P_{\varphi,j}^s \theta_{\mathcal{Z},j}^r = \underbrace{P_{\varphi,j}^s \text{Op}_{\mathcal{T}}(\chi_{\mathcal{Z},j})}_{\in \Psi_{\mathcal{T}}^0} \theta_{\varphi,j}^r.$$

In  $\text{supp}(\hat{\chi})$ , from (4.56) the symbol  $\sigma_\delta$  of  $\Sigma_\delta$  reads

$$\sigma_\delta = \beta^{-1} \left( -i \frac{\delta c_j^s}{h c_j^l} P_{\varphi,j}^s - \rho_j^{l,+} + k \right) + r, \quad \text{with } r \in \delta S_{\mathcal{T}}^1 + h S_{\mathcal{T}}^0. \quad (4.67)$$

where functions are evaluated at the interface, i.e.  $x_n = 0^+$ .

Using (4.64) in (4.59), we can now write  $\mathcal{B}^r(u_{\mathcal{Z},j}^r)$  as

$$\begin{aligned}\mathcal{B}^r(u_{\mathcal{Z},j}^r) &= \left( \begin{pmatrix} 2\partial_{x_n}\varphi_j^r|_{x_n=0^+} & B_1^r \\ B_1^{r'} & B_2^r \end{pmatrix} \begin{pmatrix} \Sigma_\delta\gamma_0(u_{\mathcal{Z},j}^r) + G_3 \\ \gamma_0(u_{\mathcal{Z},j}^r) \end{pmatrix}, \begin{pmatrix} \Sigma_\delta\gamma_0(u_{\mathcal{Z},j}^r) + G_3 \\ \gamma_0(u_{\mathcal{Z},j}^r) \end{pmatrix} \right)_0 \\ &= \left( \tilde{\Sigma}_\delta\gamma_0(u_{\mathcal{Z},j}^r), \gamma_0(u_{\mathcal{Z},j}^r) \right)_0 + 4 \operatorname{Re} \left( \partial_{x_n}\varphi_j^r|_{x_n=0^+} \Sigma_\delta\gamma_0(u_{\mathcal{Z},j}^r), G_3 \right)_0 \\ &\quad + \left( B_1^r\gamma_0(u_{\mathcal{Z},j}^r), G_3 \right)_0 + \left( B_1^{r'}G_3, \gamma_0(u_{\mathcal{Z},j}^r) \right)_0 + 2 \left( \partial_{x_n}\varphi_j^r|_{x_n=0^+} G_3, G_3 \right)_0,\end{aligned}\tag{4.68}$$

with

$$\tilde{\Sigma}_\delta = 2\Sigma_\delta^*\partial_{x_n}\varphi_j^r|_{x_n=0^+} + \Sigma_\delta + \Sigma_\delta^*B_1^r + B_1^{r'}\Sigma_\delta + B_2^r.\tag{4.69}$$

The following lemma makes use of condition (4.51) that describes the smallness of the region  $\mathcal{Z}$ .

**Lemma 4.5.** *Let  $\tilde{\sigma}_\delta$  be the symbol of  $\tilde{\Sigma}_\delta$ . We have  $\hat{\chi}\frac{h^2}{h^2+\delta^2}\tilde{\sigma}_\delta \in S_{\mathcal{T}}^0$ . Moreover, in  $\operatorname{supp}(\hat{\chi})$ , for  $h_0 > 0$  sufficiently small, we have*

$$\frac{h^2}{h^2+\delta^2}\tilde{\sigma}_\delta \geq C_0 > 0, \quad 0 < h \leq h_0.$$

We refer to Appendix C.7 for a proof.

Let  $\tilde{\chi} \in \mathcal{C}_c^\infty(T^*(\mathbb{R}^n))$ , be equal to one on a neighborhood of  $\operatorname{supp}(\chi_{\mathcal{Z},j}|_{x_n=0^+})$  and such that  $\hat{\chi}$  is equal to one on a neighborhood of  $\operatorname{supp}(\tilde{\chi})$ . We then write

$$\frac{h^2}{h^2+\delta^2}\tilde{\sigma}_\delta = s_\delta + r_\delta, \quad s_\delta = \frac{h^2}{h^2+\delta^2}\tilde{\sigma}_\delta\tilde{\chi} + C_0\langle(\xi_0, \xi')\rangle^2(1-\tilde{\chi}), \quad r_\delta = \left( \frac{h^2}{h^2+\delta^2}\tilde{\sigma}_\delta - C_0\langle(\xi_0, \xi')\rangle^2 \right)(1-\tilde{\chi}).$$

With Lemma 4.5, we have  $s_\delta \geq C_0\langle(\xi_0, \xi')\rangle^2$  and observe that  $s_\delta \in S_{\mathcal{T}}^2$ . The Gårding inequality yields, for  $h_0$  sufficiently small and  $0 < h \leq h_0$ ,

$$\left( \tilde{\Sigma}_\delta\gamma_0(u_{\mathcal{Z},j}^r), \gamma_0(u_{\mathcal{Z},j}^r) \right)_0 \geq C \left( 1 + \frac{\delta^2}{h^2} \right) |\gamma_0(u_{\mathcal{Z},j}^r)|_1^2 - C_N h^N |v_j^r|_{x_n=0^+}|_0^2,\tag{4.70}$$

as  $\operatorname{supp}(r_\delta) \cap \operatorname{supp}(\chi_{\mathcal{Z},j}) = \emptyset$ .

We now estimate the other terms in the expression (4.68). Using the Young inequality, we have, for all  $\varepsilon > 0$ ,

$$\begin{aligned}& \left| (B_1^r\gamma_0(u_{\mathcal{Z},j}^r), G_3)_0 \right| + \left| (B_1^{r'}G_3, \gamma_0(u_{\mathcal{Z},j}^r))_0 \right| + 2 \left| (\partial_{x_n}\varphi_j^r|_{x_n=0^+} G_3, G_3)_0 \right| \\ & \lesssim \varepsilon \left( 1 + \frac{\delta^2}{h^2} \right) |\gamma_0(u_{\mathcal{Z},j}^r)|_1^2 + \left( 1 + \frac{h^2}{\varepsilon(h^2+\delta^2)} \right) |G_3|_0^2.\end{aligned}\tag{4.71}$$

For the remaining term in (4.68), we have

$$4 \left| \operatorname{Re} \left( \partial_{x_n}\varphi_j^r|_{x_n=0^+} \Sigma_\delta\gamma_0(u_{\mathcal{Z},j}^r), G_3 \right)_0 \right| \lesssim \left( \frac{\delta}{h} |P_{\varphi,j}^s\gamma_0(u_{\mathcal{Z},j}^r)|_0 + |\gamma_0(u_{\mathcal{Z},j}^r)|_1 \right) |G_3|_0,$$

according to (4.65) and (4.67). Taking  $\tilde{\chi}$  as above, we can write

$$P_{\varphi,j}^s\gamma_0(u_{\mathcal{Z},j}^r) = \underbrace{P_{\varphi,j}^s \operatorname{Op}_{\mathcal{T}}(\tilde{\chi})}_{\in \Psi_{\mathcal{T}}^0} \gamma_0(u_{\mathcal{Z},j}^r) + \underbrace{P_{\varphi,j}^s (1 - \operatorname{Op}_{\mathcal{T}}(\tilde{\chi})) \operatorname{Op}_{\mathcal{T}}(\chi_{\mathcal{Z},j})}_{\in h^\infty \Psi_{\mathcal{T}}^{-\infty}} v_j^r|_{x_n=0^+}.\tag{4.72}$$

Using the Young inequality, for all  $\varepsilon > 0$ ,  $N \in \mathbb{N}$  we obtain

$$4 \left| \operatorname{Re} \left( \partial_{x_n}\varphi_j^r|_{x_n=0^+} \Sigma_\delta\gamma_0(u_{\mathcal{Z},j}^r), G_3 \right)_0 \right| \lesssim \varepsilon \left( 1 + \frac{\delta^2}{h^2} \right) |\gamma_0(u_{\mathcal{Z},j}^r)|_1^2 + \frac{1}{\varepsilon} |G_3|_0^2 + \varepsilon C_N \delta^2 h^N |v_j^r|_{x_n=0^+}|_0^2.\tag{4.73}$$

Combining (4.70) and (4.68) together with (4.71) and (4.73) gives, for  $\varepsilon$  sufficiently small and  $\delta \leq \delta_0$ ,

$$\left( 1 + \frac{\delta^2}{h^2} \right) |\gamma_0(u_{\mathcal{Z},j}^r)|_1^2 \lesssim \mathcal{B}^r(u_{\mathcal{Z},j}^r) + |G_3|_0^2 + C_N h^N |v_j^r|_{x_n=0^+}|_0^2.$$

Finally, turning back to the Carleman form at the boundary (4.58), and using (4.60), we obtain, for all  $N \in \mathbb{N}$ , for  $h_0$  sufficiently small and  $0 < h \leq h_0$ ,

$$h \|u_{\mathcal{Z},j}^r\|_1^2 + h \left( 1 + \frac{\delta^2}{h^2} \right) |\gamma_0(u_{\mathcal{Z},j}^r)|_1^2 \lesssim \|P_{\varphi,j}^r v_j^r\|_0^2 + h |G_3|_0^2 + C_N h^N |v_j^r|_{x_n=0^+}|_0^2 + h^2 |\gamma_1(u_{\mathcal{Z},j}^r)|_0^2.$$



Using (4.64), (4.65), (4.67) and (4.72) to estimate  $|\gamma_1(u_{\mathcal{E},j}^r)|_0$  in terms of  $|\gamma_0(u_{\mathcal{E},j}^r)|_1$ , we obtain

$$|\gamma_1(u_{\mathcal{E},j}^r)|_0 \lesssim \left(1 + \frac{\delta}{h}\right) |\gamma_0(u_{\mathcal{E},j}^r)|_1 + |G_3|_0 + C_N h^N |v_j^r|_{x_n=0^+}|_0.$$

Then, replacing  $|G_3|_0$  by its estimate (4.66) gives, for  $h_0$  sufficiently small and  $0 < h \leq h_0$ ,

$$\begin{aligned} & h \|u_{\mathcal{E},j}^r\|_1^2 + h |\gamma_1(u_{\mathcal{E},j}^r)|_0^2 + h \left(1 + \frac{\delta^2}{h^2}\right) |\gamma_0(u_{\mathcal{E},j}^r)|_1^2 \\ & \lesssim \|P_{\varphi,j}^r v_j^r\|_0^2 + h^2 \|v_j^r\|_1^2 + h(\delta^2 + h^2) |v_j^s|_1^2 + \|P_{\varphi,j}^l v_j^l\|_0^2 + h^2 \|v_j^l\|_1^2 + h^4 |D_{x_n} v_j^l|_{x_n=0^+}|_0^2 \\ & \quad + \frac{\delta^2}{h} |\theta_{\varphi,j}^r|_0^2 + h |\theta_{\varphi,j}^l|_1^2 + h |\theta_{\varphi,j}^r|_1^2 + h |\Theta_{\varphi,j}^s|_0^2, \end{aligned} \quad (4.74)$$

using the trace formula (1.23). This is a local version of (4.52). Patching together on  $\mathcal{M}_+$  such local estimates as we did in (4.16)-(4.21) yields (4.52).

Let us now conclude the proof on the “l” side. The trace equation (4.62) yields

$$\begin{aligned} |\gamma_1(u_{\mathcal{E},j}^l)|_0 & \leq |\gamma_0(u_{\mathcal{E},j}^l)|_1 + |G_2^l|_1 + C_N h^N \left( |D_{x_n} v_j^l|_{x_n=0^+}|_0 + |v_j^l|_{x_n=0^+}|_0 \right), \\ & \leq |\gamma_0(u_{\mathcal{E},j}^r)|_1 + |\theta_{\varphi,j}^r|_1 + |\theta_{\varphi,j}^l|_1 + |G_2^l|_1 + C_N h^N \left( |D_{x_n} v_j^l|_{x_n=0^+}|_0 + |v_j^l|_{x_n=0^+}|_0 \right), \quad N \in \mathbb{N}, \end{aligned}$$

after using the first relation of (4.61).

Using this last inequality, together with Estimates (4.74) on  $|\gamma_0(u_{\mathcal{E},j}^r)|_1$ , Estimate (4.57) on  $|G_2^l|_1$ , (4.54), and the first transmission condition in (4.61), we finally obtain, for  $h_0$  sufficiently small, and  $0 < h \leq h_0$ ,

$$\begin{aligned} & \|u_{\mathcal{E},j}^l\|_1^2 + h |\gamma_0(u_{\mathcal{E},j}^l)|_1^2 + h |\gamma_1(u_{\mathcal{E},j}^l)|_0^2 \\ & \lesssim \|P_{\varphi,j}^l v_j^l\|_0^2 + h^2 \|v_j^l\|_1^2 + h^4 |D_{x_n} v_j^l|_{x_n=0^+}|_0^2 + \frac{h^2}{\delta^2 + h^2} (\|P_{\varphi,j}^r v_j^r\|_0^2 + h^2 \|v_j^r\|_1^2) + h^3 |v_j^s|_1^2 \\ & \quad + \frac{h\delta^2}{\delta^2 + h^2} |\theta_{\varphi,j}^r|_0^2 + h |\theta_{\varphi,j}^r|_1^2 + h |\theta_{\varphi,j}^l|_1^2 + \frac{h^3}{\delta^2 + h^2} |\Theta_{\varphi,j}^s|_0^2. \end{aligned}$$

This is a local version of (4.53). Patching together on  $\mathcal{M}_+$  such local estimates as we did in (4.16)-(4.21) yields (4.53).  $\square$

## 4.5 Estimate in the region $\mathcal{E}$

Here, we place ourselves in the region  $\mathcal{E}$  (high frequencies), and prove a Carleman estimate for  $u_{\mathcal{E},j}$ , and consequently for  $\Xi_{\mathcal{E}} v$ . Using in this region the ellipticity of  $P_{\varphi,j}^s$  and the Calderón projector technique for both  $P_{\varphi,j}^r$  and  $P_{\varphi,j}^l$ , we obtain the following partial estimate.

**Proposition 4.6.** *Suppose that the weight function  $\varphi$  satisfies the properties listed in Section 3.1. Then, for all  $\delta_0 > 0$ , there exist  $C > 0$  and  $h_0 > 0$  such that, for all  $0 < \delta \leq \delta_0$  and  $0 < h \leq h_0$ ,  $v^{7l} \in \mathcal{C}_c^\infty((0, X_0) \times S \times [0, 2\varepsilon])$  and  $v^s \in \mathcal{C}_c^\infty((0, X_0) \times S)$  satisfying (3.9), we have*

$$\begin{aligned} & \|\Xi_{\mathcal{E}} v^{7l}\|_1^2 + h |\Xi_{\mathcal{E}} v^{7l}|_{x_n=0^+}|_1^2 + h |D_{x_n} \Xi_{\mathcal{E}} v^{7l}|_{x_n=0^+}|_0^2 \\ & \leq C \left( \|P_{\varphi}^r v^r\|_0^2 + h^2 \|v^r\|_1^2 + h^4 |D_{x_n} v^r|_{x_n=0^+}|_0^2 + \|P_{\varphi}^l v^l\|_0^2 + h^2 \|v^l\|_1^2 + h^4 |D_{x_n} v^l|_{x_n=0^+}|_0^2 \right. \\ & \quad \left. + h^3 |v^s|_1^2 + h |\Theta_{\varphi}^s|_0^2 + h |\theta_{\varphi}^r|_1^2 + h |\theta_{\varphi}^l|_1^2 \right). \end{aligned} \quad (4.75)$$

*Proof.* The function  $u_{\mathcal{E},j}$  satisfies  $(\text{TC}_{\bullet,j})$ , with  $\bullet = \mathcal{E}$ . On each side, the roots configuration described in Lemma 3.6 (and represented in Figure 3c) allows us to apply the Calderón projector technique as in [LR97, LR10]. Since both roots are separated by the real axis we only obtain one relation between the two traces at the interface: according to [LR10, Eq. (2.37)], we have

$$\|u_{\mathcal{E},j}^{7l}\|_1 \lesssim \|P_{\varphi,j}^{7l} v_j^{7l}\|_0 + h \|v_j^{7l}\|_1 + h^{\frac{1}{2}} |\gamma_0(u_{\mathcal{E},j}^{7l})|_1 + h^{\frac{1}{2}} |\gamma_1(u_{\mathcal{E},j}^{7l})|_0 + h^2 |D_{x_n} v_j^{7l}|_{x_n=0^+}|_0, \quad (4.76)$$

together with one relation between the two traces [LR10, Eq. (2.38)]:

$$\left(1 - \text{Op}_{\mathcal{T}}(a^{7l})\right) \gamma_0(u_{\mathcal{E},j}^{7l}) = \text{Op}_{\mathcal{T}}(b^{7l}) \gamma_1(u_{\mathcal{E},j}^{7l}) + G_2^{7l}. \quad (4.77)$$

In this last expression,  $a^{7l} \in S_{\mathcal{T}}^0$  and  $b^{7l} \in S_{\mathcal{T}}^{-1}$  have for principal part respectively

$$a_0^{7l} = - \left( \tilde{\chi} \frac{\rho_j^{7l,-}}{\rho_j^{7l,+} - \rho_j^{7l,-}} \right) \Big|_{x_n=0^+}, \quad \text{and} \quad b_{-1}^{7l} = \left( \tilde{\chi} \frac{1}{\rho_j^{7l,+} - \rho_j^{7l,-}} \right) \Big|_{x_n=0^+}, \quad (4.78)$$

where  $\rho_j^{l,\pm}$  are the roots of  $p_{\varphi,j}^l$  (i.e.  $\rho_j^{l,\pm} = (\phi_j^{-1})^* \rho^{l,\pm}$  with  $\rho^{l,\pm}$  described in Lemma 3.6) and  $\tilde{\chi} \in \mathcal{C}^\infty(T^*(\mathbb{R}^n))$  is equal to one in the neighborhood of the support of  $\chi_{\mathcal{E},j}|_{x_n=0^+}$ , with support in

$$\left( (\phi_j^{-1})^* \mathcal{E} \right) \cap \{x_n = 0\} = \{(x_0, \phi_j(y); \xi_0, {}^t d\phi_j^{-1}(\phi_j(y))\eta); (x_0, y, 0; \xi_0, \eta) \in \mathcal{E}\}.$$

The remainder  $G_2^{7l}$  satisfies

$$|G_2^{7l}|_1 \lesssim h^{-\frac{1}{2}} \left( \|P_{\varphi,j}^{7l} v_j^{7l}\|_0 + h \|v_j^{7l}\|_1 + h^2 |D_{x_n} v_j^{7l}|_{x_n=0^+} |0 \right). \quad (4.79)$$

The principal part of  $b^{7l}$  satisfies

$$b_{-1}^{7l} \geq C \langle (\xi_0, \xi') \rangle^{-1} \quad \text{in } \text{supp}(\tilde{\chi}),$$

as  $\rho_j^{7l,+}$  and  $\rho_j^{7l,-}$  are tangential symbol of order one such that  $\rho_j^{7l,+} - \rho_j^{7l,-}$  does not vanish in a neighborhood of  $\text{supp}(\tilde{\chi})$ . Let  $\hat{\chi} \in \mathcal{C}^\infty(T^*(\mathbb{R}^n))$  satisfy the same requirements as  $\tilde{\chi}$  with  $\tilde{\chi}$  equal to one a neighborhood the support of  $\hat{\chi}$ . We can introduce parametrices for  $\text{Op}_{\mathcal{T}}(b^{7l})$ , say  $\text{Op}_{\mathcal{T}}(e^{7l})$ , with  $e^{7l} \in S_{\mathcal{T}}^{-1}$ , satisfying

$$\text{Op}_{\mathcal{T}}(e^{7l}) \text{Op}_{\mathcal{T}}(b^{7l}) = \text{Op}_{\mathcal{T}}(\hat{\chi}) + R^{7l}, \quad R^{7l} \in h^\infty \Psi_{\mathcal{T}}^{-\infty}.$$

Note that the principal parts of the parametrices  $e^{7l}$  are given by  $\sigma(e^{7l}) = \hat{\chi} \left( \rho_j^{7l,+} - \rho_j^{7l,-} \right) \Big|_{x_n=0^+}$ .

Applying these parametrices to (4.77) and arguing as in (4.50) give

$$\text{Op}_{\mathcal{T}}(e^{7l}) \left( 1 - \text{Op}_{\mathcal{T}}(a^{7l}) \right) \gamma_0(u_{\mathcal{E},j}^{7l}) = \gamma_1(u_{\mathcal{E},j}^{7l}) + R_1^{7l} D_{x_n} v_j^{7l} \Big|_{x_n=0^+} + R_0^{7l} v_j^{7l} \Big|_{x_n=0^+} + \text{Op}_{\mathcal{T}}(e^{7l}) G_2^{7l}, \quad (4.80)$$

with  $R_0^{7l}, R_1^{7l} \in h^\infty \Psi_{\mathcal{T}}^{-\infty}$ . This yields the following estimate of  $\gamma_1(u_{\mathcal{E},j}^{7l})$ , in terms of  $\gamma_0(u_{\mathcal{E},j}^{7l})$ :

$$|\gamma_1(u_{\mathcal{E},j}^{7l})|_0 \lesssim |\gamma_0(u_{\mathcal{E},j}^{7l})|_1 + |G_2^{7l}|_1 + C_N h^N \left( |D_{x_n} v_j^{7l}|_{x_n=0^+} |0 + |v_j^{7l}|_{x_n=0^+} |0 \right), \quad N \in \mathbb{N}. \quad (4.81)$$

On the other hand, replacing  $u_{\mathcal{E},j}^s$  in the first equation of  $(\text{TC}_{\bullet,j})$  by its expression in the second equation of  $(\text{TC}_{\bullet,j})$  gives

$$\delta \frac{c_j^s}{i c_j^l} P_{\varphi,j}^s (\gamma_0(u_{\mathcal{E},j}^r) - \theta_{\mathcal{E},j}^r) = h \left( \gamma_1(u_{\mathcal{E},j}^l) + \beta \gamma_1(u_{\mathcal{E},j}^r) - k \gamma_0(u_{\mathcal{E},j}^r) + \tilde{G}_1 \right).$$

Using (4.80) and the first equation of  $(\text{TC}_{\bullet,j})$ , this yields

$$\Omega_\delta \gamma_0(u_{\mathcal{E},j}^r) = G_3, \quad (4.82)$$

with

$$\Omega_\delta = \delta \frac{c_j^s}{i c_j^l} P_{\varphi,j}^s + h \left( k - \beta \text{Op}_{\mathcal{T}}(e^r) (1 - \text{Op}_{\mathcal{T}}(a^r)) - \text{Op}_{\mathcal{T}}(e^l) (1 - \text{Op}_{\mathcal{T}}(a^l)) \right), \quad (4.83)$$

and

$$\begin{aligned} G_3 = & \delta \frac{c_j^s}{i c_j^l} P_{\varphi,j}^s \theta_{\mathcal{E},j}^r + h \tilde{G}_1 - h \beta \left( R_1^r D_{x_n} v_j^r \Big|_{x_n=0^+} + R_0^r v_j^r \Big|_{x_n=0^+} + \text{Op}_{\mathcal{T}}(e^r) G_2^r \right) \\ & - h \left( R_1^l D_{x_n} v_j^l \Big|_{x_n=0^+} + R_0^l v_j^l \Big|_{x_n=0^+} + \text{Op}_{\mathcal{T}}(e^l) G_2^l + \text{Op}_{\mathcal{T}}(e^l) (1 - \text{Op}_{\mathcal{T}}(a^l)) (\theta_{\mathcal{E},j}^r - \theta_{\mathcal{E},j}^l) \right). \end{aligned} \quad (4.84)$$

Here, we introduce a class of pseudo-differential operators adapted to the operator  $\Omega_\delta$  in order to perform uniform estimates in the singular limit  $\delta \rightarrow 0^+$ . On the tangential phase-space  $W = T^*(\mathbb{R}^n)$ , we define the order function

$$\Lambda^2 := \frac{\delta}{\delta + h} \langle (\xi_0, \xi') \rangle^2 + \frac{h}{\delta + h} \langle (\xi_0, \xi') \rangle,$$

associated with the metric,

$$g_W = |d(x_0, x')|^2 + \frac{|d(\xi_0, \xi')|^2}{\langle (\xi_0, \xi') \rangle^2}.$$

**Lemma 4.7.** *The order function  $\Lambda$  is admissible, i.e., slowly varying and temperate.*

We refer to Appendix C.8 for a proof. For a review of these notions see [Hör79] or [Hör85a, Sec. 18.4–5] or the recent monograph [Ler10, def. 2.2.4 and 2.2.15]. Thanks to the previous lemma, we can define a proper Hörmander-class calculus. We now prove that  $\Omega_\delta$  is elliptic in this class.

We set

$$\omega_\delta = \delta \frac{c_j^s}{ic_j^t} p_{\varphi,j}^s + h \left( k - \beta \hat{\chi} \rho_j^{r,+} - \hat{\chi} \rho_j^{l,+} \right).$$

We have  $(\delta + h)^{-1} \omega_\delta \in S_{\mathcal{T}}(\Lambda^2, g_W)$ . With (4.78) we see that

$$\Omega_\delta - \text{Op}_{\mathcal{T}}(\omega_\delta) \in h\delta\Psi_{\mathcal{T}}^1 + h^2\Psi_{\mathcal{T}}^0 \subset (h + \delta)\Psi_{\mathcal{T}}(h\Lambda^2/\langle(\xi_0, \xi')\rangle, g_W). \quad (4.85)$$

From the definition of  $k$  in (4.10) this gives

$$\text{Im}(\omega_\delta) = -\delta \frac{c_j^s}{c_j^t} \text{Re}(p_{\varphi,j}^s) - h(\partial_{x_n} \varphi_j^l|_{x_n=0^+} + \beta \partial_{x_n} \varphi_j^r|_{x_n=0^+} + \hat{\chi} \text{Im}(\rho_j^{l,+} + \beta \rho_j^{r,+})).$$

In this expression, we have

$$\text{Re}(p_{\varphi,j}^s) \geq C\langle(\xi_0, \xi')\rangle^2 \text{ on } \text{supp } \hat{\chi}, \quad (4.86)$$

by Proposition 3.5 (see also the localization of  $\text{Char}(P_{\varphi,j}^s)$  on Figure 4). Next, in the region where  $\hat{\chi} = 1$  we have

$$\begin{aligned} \partial_{x_n} \varphi_j^l|_{x_n=0^+} + \beta \partial_{x_n} \varphi_j^r|_{x_n=0^+} + \text{Im}(\rho_j^{l,+} + \beta \rho_j^{r,+}) &= \frac{1}{2} \text{Im}(\rho_j^{l,+} - \rho_j^{l,-}) + \frac{\beta}{2} \text{Im}(\rho_j^{r,+} - \rho_j^{r,-}) \\ &\geq C\langle(\xi_0, \xi')\rangle, \end{aligned} \quad (4.87)$$

as  $\partial_{x_n} \varphi_j^{7l} = -\frac{1}{2} \text{Im}(\rho_j^{l,+} + \rho_j^{l,-})$  and with Lemma 3.8. Estimates (4.86) and (4.87) yield

$$|\omega_\delta| \geq C(\delta + h)\Lambda^2,$$

in the region where  $\hat{\chi} = 1$ . There, the symbol  $(\delta + h)^{-1} \omega_\delta$  is elliptic in the class  $S_{\mathcal{T}}(\Lambda^2, g_W)$ . Hence, there exists  $l \in S_{\mathcal{T}}(\Lambda^{-2}, g_W)$  (with principal part  $\check{\chi} \omega_\delta^{-1}$ ) such that

$$\text{Op}_{\mathcal{T}}(l)(\delta + h)^{-1} \text{Op}_{\mathcal{T}}(\Omega_\delta) = \text{Op}_{\mathcal{T}}(\check{\chi}) + R, \quad R \in h^\infty \Psi_{\mathcal{T}}^{-\infty},$$

by (4.85), for some  $\check{\chi} \in \mathcal{C}^\infty(T^*(\mathbb{R}^n))$  equal to one on a neighborhood of  $\text{supp}(\chi_{\mathcal{E},j}|_{x_n=0^+})$  and such that  $\hat{\chi}$  is equal to one on a neighborhood of  $\text{supp}(\check{\chi})$ .

Applying this parametrix to Equation (4.82) gives

$$\gamma_0(u_{\mathcal{E},j}^r) + R\gamma_0(u_{\mathcal{E},j}^r) + \tilde{R}v_j^r|_{x_n=0^+} = \text{Op}_{\mathcal{T}}(l)(\delta + h)^{-1}G_3, \quad (4.88)$$

with  $R \in h^\infty \Psi_{\mathcal{T}}^{-\infty}$  and  $\tilde{R} = \text{Op}_{\mathcal{T}}(\check{\chi} - 1) \text{Op}_{\mathcal{T}}(\chi_{\mathcal{E},j}|_{x_n=0^+}) \in h^\infty \Psi_{\mathcal{T}}^{-\infty}$ .

We estimate

$$|\text{Op}_{\mathcal{T}}(l)(\delta + h)^{-1}G_3|_1 = |\text{Op}_{\mathcal{T}}(\langle(\xi_0, \xi')\rangle) \text{Op}_{\mathcal{T}}(l)(\delta + h)^{-1}G_3|_0,$$

with

$$\text{Op}_{\mathcal{T}}(\langle(\xi_0, \xi')\rangle) \text{Op}_{\mathcal{T}}(l)(\delta + h)^{-1} \in \Psi_{\mathcal{T}}\left(\frac{\langle(\xi_0, \xi')\rangle}{(\delta + h)\Lambda^2}, g_W\right) = \Psi_{\mathcal{T}}\left(\frac{1}{\delta\langle(\xi_0, \xi')\rangle + h}, g_W\right).$$

We thus obtain, as  $\text{Op}_{\mathcal{T}}\left(\frac{1}{\delta\langle(\xi_0, \xi')\rangle + h}\right)$  is a Fourier multiplier,

$$|\text{Op}_{\mathcal{T}}(l)(\delta + h)^{-1}G_3|_1 \lesssim \left| \text{Op}_{\mathcal{T}}\left(\frac{1}{\delta\langle(\xi_0, \xi')\rangle + h}\right)G_3 \right|_0.$$

With (4.84), this yields

$$\begin{aligned} &|\text{Op}_{\mathcal{T}}(l)(\delta + h)^{-1}G_3|_1 \\ &\lesssim |P_{\varphi,j}^s \theta_{\mathcal{E},j}^r|_{-1} + \left| \text{Op}_{\mathcal{T}}\left(\frac{h}{\delta\langle(\xi_0, \xi')\rangle + h}\right) \tilde{G}_1 \right|_0 + |R_1^r D_{x_n} v_j^r|_{x_n=0^+}|_0 + |R_0^r v_j^r|_{x_n=0^+}|_0 \\ &\quad + |R_1^l D_{x_n} v_j^l|_{x_n=0^+}|_0 + |R_0^l v_j^l|_{x_n=0^+}|_0 + |G_2^r|_1 + |G_2^l|_1 + |\theta_{\mathcal{E},j}^r|_1 + |\theta_{\mathcal{E},j}^l|_1. \end{aligned}$$

As we have

$$\left| \text{Op}_{\mathcal{T}} \left( \frac{h}{\delta \langle (\xi_0, \xi') \rangle + h} \right) \tilde{G}_1 \right|_0 \leq \frac{h}{\delta + h} |\tilde{G}_1|_0,$$

and with (4.11), (4.79), and the trace formula (1.23), using also  $P_{\varphi, j}^s \in \Psi_{\mathcal{T}}^2$ , gives

$$\begin{aligned} |\text{Op}_{\mathcal{T}}(l)(\delta + h)^{-1} G_3|_1 &\lesssim h |v_j^s|_1 + h^{-\frac{1}{2}} \left( \|P_{\varphi, j}^r v_j^r\|_0 + h \|v_j^r\|_1 + h^2 |D_{x_n} v_j^r|_{x_n=0^+} |0 \right) \\ &\quad + h^{-\frac{1}{2}} \left( \|P_{\varphi, j}^l v_j^l\|_0 + h \|v_j^l\|_1 + h^2 |D_{x_n} v_j^l|_{x_n=0^+} |0 \right) + |\theta_{\varphi, j}^r|_1 + |\theta_{\varphi, j}^l|_1 + |\Theta_{\varphi, j}^s|_0. \end{aligned}$$

With (4.88), the transmission conditions (TC $_{\bullet, j}$ ), and (4.80) that gives  $\gamma_1(u_{\mathcal{E}, j}^{7l})$  as a function of  $\gamma_0(u_{\mathcal{E}, j}^{7l})$  (thanks to the Calderón projectors), we obtain

$$\begin{aligned} h^{\frac{1}{2}} |\gamma_0(u_{\mathcal{E}, j}^{7l})|_1 + h^{\frac{1}{2}} |\gamma_1(u_{\mathcal{E}, j}^{7l})|_0 &\lesssim h^{\frac{3}{2}} |v_j^s|_1 + \|P_{\varphi, j}^r v_j^r\|_0 + h \|v_j^r\|_1 + h^2 |D_{x_n} v_j^r|_{x_n=0^+} |0 \\ &\quad + \|P_{\varphi, j}^l v_j^l\|_0 + h \|v_j^l\|_1 + h^2 |D_{x_n} v_j^l|_{x_n=0^+} |0 + h^{\frac{1}{2}} |\theta_{\varphi, j}^r|_1 + h^{\frac{1}{2}} |\theta_{\varphi, j}^l|_1 + h^{\frac{1}{2}} |\Theta_{\varphi, j}^s|_0. \end{aligned}$$

Injecting these estimates in (4.76) we obtain a local version of (4.75). Patching together on  $\mathcal{M}_+$  such local estimates as we did in (4.16)-(4.21) yields the result.  $\square$

## 4.6 A semi-global Carleman estimate: proof of Theorem 1.2

In this section, we explain how we can patch together the four microlocal estimate of Propositions 4.1, 4.3, 4.4 and 4.6, to obtain a global Carleman estimate in a neighborhood of  $S$ , and prove Theorem 1.2.

First, let us introduce some notation. We set

$$\begin{aligned} \text{BT}(w) &:= h |w|_{x_n=0^+} |1^2 + h |D_{x_n} w|_{x_n=0^+} |0^2, \\ \text{RHS}^{7l}(w) &:= \|P_{\varphi}^{7l} w\|_0^2 + h^2 \|w\|_1^2 + h^4 |D_{x_n} w|_{x_n=0^+} |0^2, \\ R_{\theta} &:= h |\Theta_{\varphi}^s|_0^2 + h |\theta_{\varphi}^r|_1^2 + h |\theta_{\varphi}^l|_1^2. \end{aligned}$$

This allows us to formulate concisely the four microlocal estimates of Propositions 4.1, 4.3, 4.4 and 4.6.

$$\|\Xi_{\mathcal{G}} v^r\|_1^2 + \text{BT}(\Xi_{\mathcal{G}} v^r) \lesssim \text{RHS}^r(v^r), \quad (4.89)$$

$$\begin{aligned} \varepsilon h \|\Xi_{\mathcal{G}} v^l\|_1^2 + \varepsilon \text{BT}(\Xi_{\mathcal{G}} v^l) &\lesssim \left( 1 + \frac{\delta^2}{h^2} \right) \left( \varepsilon \|\zeta P_{\varphi}^r v^r\|_0^2 + \varepsilon h^4 |D_{x_n} v^r|_{x_n=0^+} |0^2 + \varepsilon h^4 \|v^r\|_1^2 \right) \\ &\quad + \varepsilon \text{RHS}^l(v^l) + \boxed{\varepsilon (h^2 + \delta^2) \|\Xi_{\mathcal{G}} v^r\|_1^2} + \boxed{\varepsilon h (h^2 + \delta^2) |v^s|_1^2} + \varepsilon R_{\theta} + \varepsilon \frac{\delta^2}{h} |\theta_{\varphi}^r|_0^2. \end{aligned} \quad (4.90)$$

$$\|\Xi_{\mathcal{F}} v^r\|_1^2 + \text{BT}(\Xi_{\mathcal{F}} v^r) \lesssim \text{RHS}^r(v^r), \quad (4.91)$$

$$\|\Xi_{\mathcal{F}} v^l\|_1^2 + \text{BT}(\Xi_{\mathcal{F}} v^l) \lesssim \text{RHS}^l(v^l) + \text{RHS}^r(v^r) + R_{\theta}. \quad (4.92)$$

$$\varepsilon h \|\Xi_{\mathcal{F}} v^r\|_1^2 + \varepsilon \text{BT}(\Xi_{\mathcal{F}} v^r) \lesssim \varepsilon \text{RHS}^r(v^r) + \varepsilon \text{RHS}^l(v^l) + \boxed{\varepsilon h (h^2 + \delta^2) |v^s|_1^2} + \varepsilon \frac{\delta^2}{h} |\theta_{\varphi}^r|_0^2 + \varepsilon R_{\theta}, \quad (4.93)$$

$$\|\Xi_{\mathcal{E}} v^l\|_1^2 + \text{BT}(\Xi_{\mathcal{E}} v^l) \lesssim \text{RHS}^l(v^l) + h^3 |v^s|_1^2 + \frac{h^2}{\delta^2 + h^2} \text{RHS}^r(v^r) + R_{\theta}. \quad (4.94)$$

$$\|\Xi_{\mathcal{E}} v^{7l}\|_1^2 + \text{BT}(\Xi_{\mathcal{E}} v^{7l}) \lesssim \text{RHS}^l(v^l) + \text{RHS}^r(v^r) + h^3 |v^s|_1^2 + R_{\theta}. \quad (4.95)$$

To derive the final Carleman estimate we need to sum together these microlocal estimate and many terms in the r.h.s. need to be “absorbed” by those in the l.h.s.. This is a standard procedure usually making use of the powers of the parameter  $h$  in front of these terms and by choosing  $h$  sufficiently small. Note, however, that some powers of  $h$  are critical here so that the related terms (in frames) in the right hand-sides cannot be “absorbed” directly. To overcome this problem, we have multiplied the two concerned equations by a small parameter  $\varepsilon > 0$  whose value is independent of  $h$  and  $\delta$ .

Note that these three atypical terms are the reason for the introduction of the microlocal region  $\mathcal{F}$  (compare with the microlocal regions used in [LR10]). In fact, the microlocal region  $\mathcal{F}$  acts as a buffer: as  $\mathcal{F}$  is an elliptic region for both the operators  $P_{\varphi}^{7l}$ , it provides terms in the l.h.s. of the associated microlocal estimates of better

quality than those obtained in the regions  $\mathcal{G}$  and  $\mathcal{L}$  (compare the powers of  $h$  in the l.h.s. terms of these estimates).

Observe that the property  $\chi_{\mathcal{E}} + \chi_{\mathcal{L}} + \chi_{\mathcal{F}} + \chi_{\mathcal{G}} = 1$  implies, see Section 3.6,

$$\Xi_{\mathcal{G},j} + \Xi_{\mathcal{F},j} + \Xi_{\mathcal{L},j} + \Xi_{\mathcal{E},j} = \zeta_j(x_0, y).$$

As a consequence of the definition of the operators  $\Xi_{\bullet}$ ,  $\bullet = \mathcal{E}, \mathcal{L}, \mathcal{F}, \mathcal{G}$ , given in (3.24)-(3.25), this yields

$$\Xi_{\mathcal{E}} + \Xi_{\mathcal{L}} + \Xi_{\mathcal{F}} + \Xi_{\mathcal{G}} = \zeta^1(x_0). \quad (4.96)$$

We now treat the three atypical terms and use the small parameter  $\varepsilon$ .

As  $\text{supp}(v^s) \subset (\alpha_0, X_0 - \alpha_0) \times S$  (see the statement of Theorem 1.2 and Section 3.3), with (4.96), and using the transmission conditions (3.9), we have

$$\begin{aligned} v^s &= \zeta^1 v^s = \Xi_{\mathcal{E}} v^s + \Xi_{\mathcal{L}} v^s + \Xi_{\mathcal{F}} v^s + \Xi_{\mathcal{G}} v^s \\ &= \Xi_{\mathcal{E}} v^l - \Xi_{\mathcal{E}} \theta_{\varphi}^l + \Xi_{\mathcal{L}} v^l - \Xi_{\mathcal{L}} \theta_{\varphi}^l + \Xi_{\mathcal{F}} v^l - \Xi_{\mathcal{F}} \theta_{\varphi}^l + \Xi_{\mathcal{G}} v^r - \Xi_{\mathcal{G}} \theta_{\varphi}^r, \quad \text{at } x_n = 0^+. \end{aligned}$$

Hence, for  $\delta \leq \delta_0$  and  $h \leq h_0$  we can estimate the two atypical terms concerning  $v^s$  in (4.90) and (4.93) as

$$\varepsilon h \delta^2 |v^s|_1^2 \lesssim \varepsilon h |\Xi_{\mathcal{E}} v^l|_1^2 + \varepsilon h |\Xi_{\mathcal{L}} v^l|_1^2 + \varepsilon h |\Xi_{\mathcal{F}} v^l|_1^2 + \varepsilon h |\Xi_{\mathcal{G}} v^r|_1^2 + \varepsilon R_{\theta}.$$

When summing all the estimates (4.89)-(4.95) together and taking  $\varepsilon$  sufficiently small, the four terms  $\varepsilon h |\Xi_{\mathcal{E}} v^l|_1^2$ ,  $\varepsilon h |\Xi_{\mathcal{L}} v^l|_1^2$ ,  $\varepsilon h |\Xi_{\mathcal{F}} v^l|_1^2$ ,  $\varepsilon h |\Xi_{\mathcal{G}} v^r|_1^2$  can be ‘‘absorbed’’ by the l.h.s. of (4.95), (4.94), (4.92), and (4.89) respectively.

The remaining atypical term is in (4.90):

$$\varepsilon (h^2 + \delta^2) \|\Xi_{\mathcal{G}\mathcal{F}} v^r\|_1^2 \lesssim \varepsilon \|\Xi_{\mathcal{G}\mathcal{F}} v^r\|_1^2.$$

We choose a function  $\zeta^4 \in \mathcal{C}_c^\infty(0, X_0)$  such that  $\zeta^4 = 1$  on a neighborhood of  $(\alpha_0, X_0 - \alpha_0)$ ,  $\zeta^4 = 1$  on a neighborhood of  $\text{supp}(\zeta^4)$  and  $0 \leq \zeta^4 \leq 1$ . Since  $\text{supp}(v^r) \subset (\alpha_0, X_0 - \alpha_0) \times S \times [0, 2\varepsilon]$ , we have

$$\Xi_{\mathcal{G}\mathcal{F}} v^r = \Xi_{\mathcal{G}\mathcal{F}} (\Xi_{\mathcal{G}} + \Xi_{\mathcal{F}}) v^r + \Xi_{\mathcal{G}\mathcal{F}} (1 - \Xi_{\mathcal{G}} - \Xi_{\mathcal{F}}) \zeta^4 v^r. \quad (4.97)$$

From Proposition B.14 and Proposition B.10, the principal symbol of the operator  $\Xi_{\mathcal{G}\mathcal{F}} (1 - \Xi_{\mathcal{G}} - \Xi_{\mathcal{F}}) \zeta^4$  is

$$\zeta^2 \chi_{\mathcal{G}\mathcal{F}} (1 - \zeta^1 (\chi_{\mathcal{G}} + \chi_{\mathcal{F}})) \zeta^4 = \zeta^2 \chi_{\mathcal{G}\mathcal{F}} (1 - (\chi_{\mathcal{G}} + \chi_{\mathcal{F}})) \zeta^4 = 0$$

since  $\chi_{\mathcal{G}} + \chi_{\mathcal{F}} = 1$  on  $\text{supp}(\chi_{\mathcal{G}\mathcal{F}})$  by (4.12). We thus have  $\Xi_{\mathcal{G}\mathcal{F}} (1 - \Xi_{\mathcal{G}} - \Xi_{\mathcal{F}}) \zeta^4 \in h \Psi_{\mathcal{T}}^{-1}(\mathcal{M}_+)$ , so that (4.97) gives

$$\varepsilon (h^2 + \delta^2) \|\Xi_{\mathcal{G}\mathcal{F}} v^r\|_1^2 \lesssim \varepsilon \|\Xi_{\mathcal{G}} v^r\|_1^2 + \varepsilon \|\Xi_{\mathcal{F}} v^r\|_1^2 + \varepsilon h^2 \|v^r\|_1^2.$$

When summing all the estimates (4.89)-(4.95) together and taking  $\varepsilon$  sufficiently small, the two terms  $\varepsilon \|\Xi_{\mathcal{G}} v^r\|_1^2$ ,  $\varepsilon \|\Xi_{\mathcal{F}} v^r\|_1^2$  in this expression can be absorbed by the l.h.s. of (4.89) and (4.91), respectively. This is possible since in these two estimates are obtained in elliptic regions yielding better powers in  $h$ .

Now, if we sum all the partial estimates (4.89)-(4.95), and handle the atypical terms as explained above, we obtain

$$\begin{aligned} &\|\Xi_{\mathcal{G}} v^r\|_1^2 + \text{BT}(\Xi_{\mathcal{G}} v^r) + h \|\Xi_{\mathcal{G}} v^l\|_1^2 + \text{BT}(\Xi_{\mathcal{G}} v^l) + \|\Xi_{\mathcal{F}} v^r\|_1^2 + \text{BT}(\Xi_{\mathcal{F}} v^r) + \|\Xi_{\mathcal{F}} v^l\|_1^2 \\ &\quad + \text{BT}(\Xi_{\mathcal{F}} v^l) + h \|\Xi_{\mathcal{L}} v^r\|_1^2 + \text{BT}(\Xi_{\mathcal{L}} v^r) + \|\Xi_{\mathcal{L}} v^l\|_1^2 + \text{BT}(\Xi_{\mathcal{L}} v^l) + \|\Xi_{\mathcal{E}} v^{7l}\|_1^2 + \text{BT}(\Xi_{\mathcal{E}} v^{7l}) \\ &\lesssim \text{RHS}^r(v^r) + \text{RHS}^l(v^l) + \left(1 + \frac{\delta^2}{h^2}\right) \|\zeta P_{\varphi}^r v^r\|_0^2 + h^3 |v^s|_1^2 + h^2 |D_{x_n} v|_{x_n=0^+}^2 + R_{\theta} + \frac{\delta^2}{h} |\theta_{\varphi}^r|_0^2, \end{aligned} \quad (4.98)$$

Using  $\text{supp}(v^{7l}) \subset (\alpha_0, X_0 - \alpha_0) \times S \times [0, 2\varepsilon]$  and (4.96), we can write

$$\|v^{7l}\|_1 \leq \|\Xi_{\mathcal{G}} v^{7l}\|_1 + \|\Xi_{\mathcal{F}} v^{7l}\|_1 + \|\Xi_{\mathcal{L}} v^{7l}\|_1 + \|\Xi_{\mathcal{E}} v^{7l}\|_1,$$

together with

$$|v|_{x_n=0^+}^{7l} \leq |\Xi_{\mathcal{G}} v|_{x_n=0^+}^{7l} + |\Xi_{\mathcal{F}} v|_{x_n=0^+}^{7l} + |\Xi_{\mathcal{L}} v|_{x_n=0^+}^{7l} + |\Xi_{\mathcal{E}} v|_{x_n=0^+}^{7l},$$

and

$$|D_{x_n} v|_{x_n=0^+}^{7l} \leq |D_{x_n} \Xi_{\mathcal{G}} v|_{x_n=0^+}^{7l} + |D_{x_n} \Xi_{\mathcal{F}} v|_{x_n=0^+}^{7l} + |D_{x_n} \Xi_{\mathcal{L}} v|_{x_n=0^+}^{7l} + |D_{x_n} \Xi_{\mathcal{E}} v|_{x_n=0^+}^{7l}.$$

These three inequalities together with (4.98) give

$$\begin{aligned} & h\|v^{7l}\|_1^2 + h|v_{|x_n=0^+}^{7l}|_1^2 + h|D_{x_n}v_{|x_n=0^+}^{7l}|_0^2 \\ & \lesssim \|P_\varphi^l v^l\|_0^2 + h^2\|v^l\|_1^2 + h^4|D_{x_n}v_{|x_n=0^+}^l|_0^2 + \left(1 + \frac{\delta^2}{h^2}\right)\|\zeta P_\varphi^r v^r\|_0^2 + \|P_\varphi^r v^r\|_0^2 \\ & \quad + h^2\|v^r\|_1^2 + h^2|D_{x_n}v_{|x_n=0^+}^r|_0^2 + h^3|v^s|_1^2 + R_\theta + \frac{\delta^2}{h}|\theta_\varphi^r|_0^2, \end{aligned}$$

Taking  $0 < h \leq h_0$  with  $h_0$  sufficiently small in this expression gives

$$h\|v^{7l}\|_1^2 + h|v_{|x_n=0^+}^{7l}|_1^2 + h|D_{x_n}v_{|x_n=0^+}^{7l}|_0^2 \lesssim \|P_\varphi^l v^l\|_0^2 + \|P_\varphi^r v^r\|_0^2 + \left(1 + \frac{\delta^2}{h^2}\right)\|\zeta P_\varphi^r v^r\|_0^2 + R_\theta + \frac{\delta^2}{h}|\theta_\varphi^r|_0^2.$$

Recalling the definitions of  $v^{7l} = e^{\varphi^{7l}/h} w^{7l}$ ,  $F_\varphi^{7l}$ ,  $\theta_\varphi^{7l}$ ,  $\Theta_\varphi^s$  (see Section 3.3 and Equation (3.3)), and observing that we have

$$\|e^{\varphi^{7l}/h} D_{x_k} w^{7l}\|_0 \leq \|D_{x_k}(e^{\varphi^{7l}/h} w^{7l})\|_0 + \|(\partial_{x_k} \varphi^{7l}) e^{\varphi^{7l}/h} w^{7l}\|_0,$$

and similar inequalities for the norms at the interface  $\{x_n = 0^+\}$ , we can “absorb” the zero-order terms in (3.3), which concludes the proof of Theorem 1.2.  $\square$

## 5 Interpolation and spectral inequalities

### 5.1 Interpolation inequality

Here, we prove the result of Theorem 1.4. We shall start by proving a local version of the interpolation inequality at the interface. In fact, the inequality we prove is local in  $(x_0, x_n)$  but global on  $S$ . Here, we closely follow the geometrical setting of [LR10]. As in Section 3 we use local coordinates where the interface is given by  $\{x_n = 0\}$ , in a small neighborhood  $[0, X_0] \times V_\varepsilon$ . We choose a point  $z_0 \in (\alpha_1, X_0 - \alpha_1)$ . We also pick  $\alpha_0$  such that  $0 < \alpha_0 < \alpha_1$  to be used for the application of the Carleman estimate of Theorem 1.2.

We define the following anisotropic distance in  $\mathbb{R}^2$ :

$$\text{dist}_\alpha((a_0, a_n), (b_0, b_n)) = (\alpha|a_0 - b_0|^2 + |a_n - b_n|^2)^{\frac{1}{2}}, \quad \alpha > 0.$$

We fix  $z_n \in \mathbb{R}_+^*$ . Then, for  $(x_0, x_n) \in [0, X_0] \times \mathbb{R}$  and  $\kappa > 0$ , we set

$$\psi(x_0, x_n) = \begin{cases} -\text{dist}_\alpha((x_0, x_n), (z_0, z_n)) & \text{if } x_n \geq 0, \\ -\text{dist}_\alpha((x_0, \kappa x_n), (z_0, z_n)) & \text{if } x_n < 0. \end{cases}$$

We shall also consider  $\psi$  as a function on  $V_{z_0} \times S \times \mathbb{R}$ . We note that  $\psi$  is continuous across the interface  $\{x_n = 0\}$  and that

$$\begin{aligned} \partial_{x_n} \psi(x_0, x_n) &= (x_n - z_n)(\psi(x_0, x_n))^{-1} \quad \text{if } x_n \geq 0, \\ \partial_{x_n} \psi(x_0, x_n) &= \kappa(\kappa x_n - z_n)(\psi(x_0, x_n))^{-1} \quad \text{if } x_n \leq 0, \end{aligned}$$

which yields  $\partial_{x_n} \psi|_{x_n=0^-} = \kappa \partial_{x_n} \psi|_{x_n=0^+}$ . We also have

$$\partial_{x_0} \psi(x_0, x_n) = \alpha(x_0 - z_0)(\psi(x_0, x_n))^{-1}. \quad (5.1)$$

Let us check that the associated weight function  $\varphi = e^{\lambda\psi}$  satisfies the properties listed in Section 3.1.

According to Remark 3.3, it suffices to check that  $\varphi$  satisfies properties (3.6) and (3.7) possibly with different constants. In fact, we work in a sufficiently small neighborhood  $V = V_{z_0} \times V_{\varepsilon'}$  of  $\{z_0\} \times S \times \{0\}$  which does not contain  $(z_0, y, z_n)$  for all  $y \in S$ , where  $V_{z_0}$  is a neighborhood of  $z_0$  in  $(\alpha_0, X_0 - \alpha_0)$  and  $0 < \varepsilon' < \varepsilon$ , so that  $\nabla\psi$  does not vanish in  $V$ . First fixing  $\kappa$  sufficiently small, we see that Property (3.6) is satisfied. Second, note that  $|x_0 - z_0|$  is bounded. Hence, from (5.1), we can choose the parameter  $\alpha$  sufficiently small to have  $|\partial_{x_0} \psi|$  small as compared to  $\inf |\partial_{x_n} \psi|$ , so that (3.7) is satisfied. Level sets for the function  $\psi$  are represented in Figure 6.

The Carleman estimate of Theorem 1.2 then follows, with the weight function  $\varphi$ .

We choose  $0 < s_1 < s'_1$  and  $0 < \sigma < \sigma'$  such that

$$U' = \{(x_0, y, x_n); |x_0 - z_0| < s'_1, y \in S, |x_n| < \sigma'\} \subset V.$$

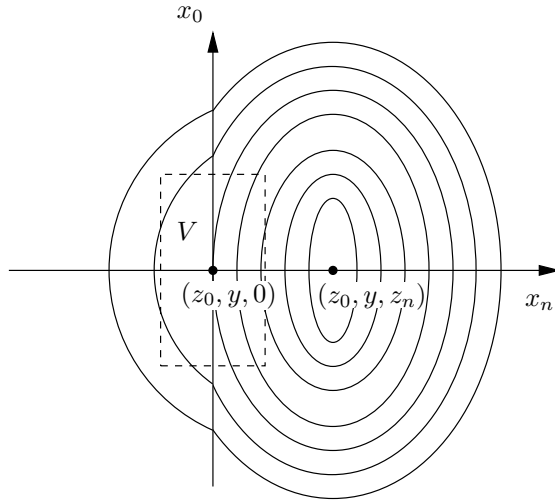


Figure 6: Level sets for the weight functions  $\psi$  and  $\varphi = e^{\lambda\psi}$  in  $(x_0, x_n)$  coordinates. The manifold  $S \ni y$  can be represented normal to the drawing. The Carleman estimate of Theorem 1.2 can be applied in a region  $V$  close to  $\{z_0\} \times S \times \{0\}$  (represented with a dashed line).

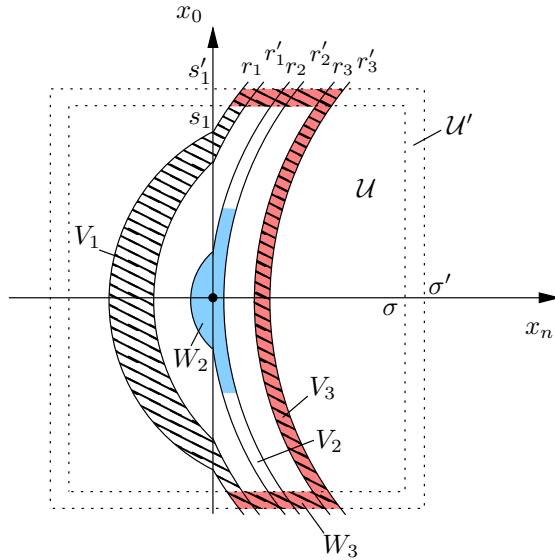


Figure 7: Neighborhoods around the point of interest for the proof of the interpolation inequality.

We also set

$$\mathcal{U} = \{(x_0, y, x_n); |x_0 - z_0| < s_1, y \in S, |x_n| < \sigma\} \subset \mathcal{U}'.$$

We now choose  $r_1 < r'_1 < r_2 < \psi(z_0, 0) < r'_2 < r_3 < r'_3$ , such that

$$\begin{aligned} \mathcal{C}_1 &= \{(x_0, y, x_n) \in \mathbb{R} \times S \times \mathbb{R}; \psi(x_0, x_n) = r_1\} \\ \text{and } \mathcal{C}'_3 &= \{(x_0, y, x_n) \in \mathbb{R} \times S \times \mathbb{R}; \psi(x_0, x_n) = r'_3\} \end{aligned}$$

satisfy  $\mathcal{C}_1 \cap \{x_n < 0\} \subset \mathcal{U}$ ,  $\mathcal{C}_1 \cap \{x_n > 0\} \cap \mathcal{U} \neq \emptyset$ , which is equivalent to having

$$\psi(z_0 \pm s_1, 0) = -(\alpha s_1^2 + z_n^2)^{\frac{1}{2}} < r_1,$$

and finally  $\mathcal{C}'_3 \cap \mathcal{U}' \subset \{x_n \leq \sigma\}$ . We illustrate these choices in Figure 7. We set  $R_j = e^{\lambda r_j}$ ,  $R'_j = e^{\lambda r'_j}$ ,  $j = 1, 2, 3$ .

Following [LR95], we introduce

$$V_j := \{(x_0, y, x_n) \in \mathcal{U}'; r_j < \psi(x_0, x_n) < r'_j\}, \quad j = 1, 2, 3.$$

and we further set

$$\begin{aligned} V_{1' \rightarrow 3} &:= \{(x_0, y, x_n) \in \mathcal{U}; r'_1 < \psi(x_0, x_n) < r_3\}, \quad V'_{1 \rightarrow 3'} := \{(x_0, y, x_n) \in \mathcal{U}'; r_1 < \psi(x_0, x_n) < r'_3\} \\ W_3 &= V_3 \cup (V'_{1 \rightarrow 3'} \setminus \mathcal{U}). \end{aligned}$$

The region  $W_3$  is represented shaded and stripped in Figure 7. With the choices we have made above, the region  $W_3$  is contained in  $\{x_n > 0\}$  and is finitely away from the interface  $\mathbb{R}_{x_0} \times S = \{x_n = 0\}$ . For  $s_0 \in (0, s_1)$  we also choose  $W_2 = V_2 \cap \{(x_0, y, x_n); |x_0 - z_0| < s_0, y \in S\} \Subset \mathcal{U}$ . The region  $W_2$  contains  $\{z_0\} \times S \times \{0\}$  and is represented shaded in Figure 7.

Now that the geometrical context is set, we can state a local interpolation inequality in the neighborhood of  $\{z_0\} \times S \times \{0\}$ .

**Lemma 5.1.** *For all  $\delta_0 > 0$ , there exist  $C \geq 0$  and  $\nu_0 \in (0, 1)$  such that for all  $\delta \in (0, \delta_0)$  we have*

$$\|u\|_{H^1(W_2)} + \delta^{\frac{1}{2}} |u^s|_{H^1(W_2 \cap \{x_n=0\})} \leq C \|U\|_{\mathcal{K}_\delta^1}^{1-\nu} \left( \|u\|_{H^1(W_3)} + \left\| (-\partial_{x_0}^2 + A_\delta)U \right\|_{\mathcal{K}_\delta^0} \right)^\nu \quad (5.2)$$

for all  $0 < \nu \leq \nu_0$  and  $U = (u, u^s) \in \mathcal{K}_\delta^2$ .

This inequality can be read as the ‘‘observation’’ of the local  $\mathcal{K}_\delta^1$  norm of  $U$  in the neighborhood  $W_2$  of any strip  $\{z_0\} \times S \times \{0\}$  by the  $H^1$  norm of  $u$  in a neighborhood away from the interface and the  $\mathcal{K}_\delta^0$  norm of  $(-\partial_{x_0}^2 + A_\delta)U$ .

*Proof.* We choose  $\chi \in \mathcal{C}_c^\infty(\mathcal{U}')$  independent of  $y \in S$  such that  $\chi$  is equal to one on  $V_{1' \rightarrow 3}$  and vanishes outside  $V'_{1 \rightarrow 3'}$ . Then  $\nabla_{x_0, x_n} \chi$  vanishes outside  $V'_{1 \rightarrow 3'} \setminus V_{1' \rightarrow 3}$  which is the stripped region in Figure 7.

For  $U = (u, u^s) \in \mathcal{K}_\delta^2$ , we set

$$\begin{cases} Bu := -(\partial_{x_0}^2 + \Delta_c)u \in L^2((0, X_0) \times \Omega_1 \cup \Omega_2) \\ B^s U := -(\partial_{x_0}^2 + \Delta_{c^s})u^s - \frac{1}{\delta} ((c \partial_{x_n} u)|_{x_n=0^+} - (c \partial_{x_n} u)|_{x_n=0^-}) \in L^2((0, X_0) \times S), \end{cases}$$

and recall that  $u|_{x_n=0^-} = u^s = u|_{x_n=0^+}$ . Setting  $W = (w, w^s)$  with  $w = \chi u$  and  $w^s = \chi|_{x_n=0} u^s$ , we have

$$\begin{cases} Bw = \chi Bu + F, & \text{in } \mathcal{U} \\ B^s W = \frac{1}{\delta} (\delta \chi B^s U + \Theta) & \text{in } \mathcal{U} \cap S, \\ w|_{x_n=0^-} = w^s = w|_{x_n=0^+} & \text{in } \mathcal{U} \cap S, \end{cases}$$

where

$$\begin{cases} F = [-(\partial_{x_0}^2 + \Delta_c), \chi]u \\ \Theta = \delta [-(\partial_{x_0}^2 + \Delta_{c^s}), \chi]u^s - (c|_{x_n=0^+} - c|_{x_n=0^-}) \partial_{x_n} \chi|_{x_n=0} u^s. \end{cases}$$

Using the density result of Corollary 2.6, the Carleman estimate of Theorem 1.2 can be applied to  $W = (w, w^s)$ :

$$\begin{aligned} h \|e^{\varphi/h} w\|_0^2 + h^3 \|e^{\varphi/h} \nabla_{x_0, x} w\|_0^2 + h \|e^{\varphi/h} w^s\|_0^2 + h^3 \|e^{\varphi/h} \nabla_{x_0, S} w^s\|_0^2 \\ \lesssim h^2 (h^2 + \delta^2) \|e^{\varphi/h} \chi Bu\|_0^2 + h^2 (h^2 + \delta^2) \|e^{\varphi/h} F\|_0^2 + h^3 \delta^2 \|e^{\varphi/h} \chi B^s U\|_0^2 + h^3 \|e^{\varphi/h} \Theta\|_0^2. \end{aligned} \quad (5.3)$$



Note that  $\Theta$  is supported in  $V_1 \cap \{x_n = 0\}$  and in this set  $e^{\varphi/h} \leq e^{R'_1/h}$ . Similarly,  $F$  is supported in  $V'_{1 \rightarrow 3} \setminus V_{1' \rightarrow 3}$  and in this set  $e^{\varphi/h} \leq e^{R'_3/h}$ . Moreover, the operators  $[-(\partial_{x_0}^2 + \Delta_c), \chi]$  and  $[-(\partial_{x_0}^2 + \Delta_{c^s}), \chi]$  are of order one. We thus have

$$\|e^{\varphi/h} F\|_0 \lesssim e^{R'_3/h} \|u\|_{H^1(W_3)} + e^{R'_1/h} \|u\|_{H^1(V_1)} \lesssim e^{R'_3/h} \|u\|_{H^1(W_3)} + e^{R'_1/h} \|U\|_{\mathcal{K}_\delta^1}, \quad (5.4)$$

$$|e^{\varphi/h} \Theta|_0 \lesssim e^{R'_1/h} (\delta |u^s|_{H^1(V_1 \cap \{x_n=0\})} + |u^s|_{L^2(V_1 \cap \{x_n=0\})}).$$

Using the trace formula together with  $\delta \leq \delta^{\frac{1}{2}} \delta_0^{\frac{1}{2}}$  in this last inequality, we obtain

$$|e^{\varphi/h} \Theta|_0 \lesssim e^{R'_1/h} (\delta^{\frac{1}{2}} |u^s|_{H^1((0, X_0) \times S)} + \|u\|_{H^1(\mathcal{U}')} ) \lesssim e^{R'_1/h} \|U\|_{\mathcal{K}_\delta^1}. \quad (5.5)$$

We also have

$$\|e^{\varphi/h} \chi B u\|_0 \lesssim e^{R'_3/h} \|B u\|_{L^2(\mathcal{U}')} \lesssim e^{R'_3/h} \|(-\partial_{x_0}^2 + A_\delta) U\|_{\mathcal{K}_\delta^0}, \quad (5.6)$$

and

$$\delta^{\frac{1}{2}} |e^{\varphi/h} \chi B^s U|_0 \lesssim \delta^{\frac{1}{2}} e^{R'_3/h} |B^s U|_{L^2(\mathcal{U}' \cap \{x_n=0\})} \lesssim e^{R'_3/h} \|(-\partial_{x_0}^2 + A_\delta) U\|_{\mathcal{K}_\delta^0}. \quad (5.7)$$

Concerning the l.h.s. of (5.3), we have  $e^{\varphi/h} \geq e^{R_2/h}$  and  $\chi = 1$  on  $W_2$ , so that, using  $\delta \leq \delta_0$ ,

$$\begin{aligned} h \|e^{\varphi/h} w\|_0^2 + h^3 \|e^{\varphi/h} \nabla_{x_0, y, x_n} w\|_0^2 + h |e^{\varphi/h} w^s|_0^2 + h^3 |e^{\varphi/h} \nabla_{x_0, y} w^s|_0^2 \\ \gtrsim h^3 e^{2R_2/h} \|u\|_{H^1(W_2)}^2 + h^3 \delta e^{2R_2/h} |u^s|_{H^1(W_2 \cap \{x_n=0\})}^2. \end{aligned} \quad (5.8)$$

Using (5.4)-(5.8) in (5.3), we thus obtain,

$$\begin{aligned} h^{\frac{3}{2}} e^{R_2/h} (\|u\|_{H^1(W_2)} + \delta^{\frac{1}{2}} |u^s|_{H^1(W_2 \cap \{x_n=0\})}) \\ \lesssim h (e^{R'_1/h} \|U\|_{\mathcal{K}_\delta^1} + e^{R'_3/h} (\|(-\partial_{x_0}^2 + A_\delta) U\|_{\mathcal{K}_\delta^0} + \|u\|_{H^1(W_3)})). \end{aligned} \quad (5.9)$$

Fixing some  $\tilde{R}_2 \in (R'_1, R_2)$ , we have  $h^{\frac{1}{2}} e^{R_2/h} \gtrsim e^{\tilde{R}_2/h}$  for all  $0 < h < h_0$ . Thus, (5.9) becomes

$$e^{\tilde{R}_2/h} (\|u\|_{H^1(W_2)} + \delta^{\frac{1}{2}} |u^s|_{H^1(W_2 \cap \{x_n=0\})}) \lesssim e^{R'_1/h} \|U\|_{\mathcal{K}_\delta^1} + e^{R'_3/h} (\|(-\partial_{x_0}^2 + A_\delta) U\|_{\mathcal{K}_\delta^0} + \|u\|_{H^1(W_3)}).$$

Finally, optimizing w.r.t. to  $h$  as in [Rob95] we obtain the sought local interpolation inequality.  $\square$

Away from the interface, the  $\mathcal{K}_\delta^s$  norms,  $s = 0, 1$  coincide with the usual  $H^s$  norm, and similar local interpolation inequalities as (5.2) are proven in [LR95, Lemme 3 page 352]. Now that we have obtained the interpolation inequality (5.2) at the interface, we can apply the procedure described in [LR95, pages 353-356] (propagation of smallness) and prove the sought global interpolation inequality (1.13). See [LZ98, Proof of Theorem 3] to obtain the term  $\|\partial_{x_0} u(0, x)\|_{L^2(\omega)}$  in the r.h.s. of (1.13). This concludes the proof of Theorem 1.4.  $\square$

## 5.2 Spectral inequality

From the interpolation inequality proven in Theorem 1.4, we now deduce the uniform spectral inequality of Theorem 1.5. Recall that  $\mathcal{E}_{\delta, j} = (e_{\delta, j}, e_{\delta, j}^s)$ ,  $j \in \mathbb{N}$ , denotes a Hilbert basis of  $\mathcal{H}_\delta^0$  composed of eigenfunctions of the operator  $A_\delta$  associated with the positive eigenvalues  $\mu_{\delta, j} \in \mathbb{R}$ ,  $j \in \mathbb{N}$ , sorted in an increasing sequence. We denote by  $\Pi_{\delta, \mu}$  the spectral projector over the eigenfunctions associated with eigenvalues lower than  $\mu$ , i.e.,

$$\Pi_{\delta, \mu} Y = \sum_{\mu_{\delta, j} \leq \mu} (Y, \mathcal{E}_{\delta, j})_{\mathcal{H}_\delta^0} \mathcal{E}_{\delta, j}, \quad Y \in \mathcal{H}_\delta^0.$$

The proof of Theorem 1.5 is classical. Yet, we have to make sure that all the constants involved do not depend upon the parameter  $\delta$ .

First we take some  $Y_\delta = (y_\delta, y_\delta^s) \in \Pi_{\delta, \mu} \mathcal{H}_\delta^0$ , and apply the interpolation inequality (1.13) of Theorem 1.4 to

$$U_\delta = (u_\delta, u_\delta^s) = A_\delta^{-\frac{1}{2}} \sinh(x_0 A_\delta^{\frac{1}{2}}) Y_\delta,$$

defined with the classical functional calculus for self-adjoint operators<sup>5</sup>.

We notice that we have  $(-\partial_{x_0}^2 + A_\delta)U_\delta = 0$ ,  $U_\delta(0, x) = 0$  and  $\|\partial_{x_0}u_\delta(0, x)\|_{L^2(\omega)} = \|y_\delta\|_{L^2(\omega)}$ . Concerning the l.h.s. of the interpolation inequality (1.13), we have

$$\begin{aligned} \|U_\delta\|_{\mathcal{K}_\delta^1(\alpha_1)}^2 &\geq \|U_\delta\|_{\mathcal{K}_\delta^0(\alpha_1)}^2 = \|U_\delta\|_{L^2(\alpha_1, X_0 - \alpha_1; \mathcal{H}_\delta^0)}^2 = \int_{\alpha_1}^{X_0 - \alpha_1} \|A_\delta^{-\frac{1}{2}} \sinh(x_0 A_\delta^{\frac{1}{2}}) Y_\delta\|_{\mathcal{H}_\delta^0}^2 dx_0 \\ &\geq \int_{\alpha_1}^{X_0 - \alpha_1} \left\| \left[ A_\delta^{-\frac{1}{2}} \sinh(x_0 A_\delta^{\frac{1}{2}}) \Pi_{\delta, \mu} \right]^{-1} \right\|_{\mathcal{L}(\mathcal{H}_\delta^0)}^{-2} \|Y_\delta\|_{\mathcal{H}_\delta^0}^2 dx_0 \\ &\geq \int_{\alpha_1}^{X_0 - \alpha_1} x_0^2 dx_0 \|Y_\delta\|_{\mathcal{H}_\delta^0}^2 \geq C(X_0, \alpha_1) \|Y_\delta\|_{\mathcal{H}_\delta^0}^2, \end{aligned} \quad (5.10)$$

since  $t^{-\frac{1}{2}} \sinh(x_0 t^{\frac{1}{2}}) \geq x_0$  for  $t > 0$ . Now, concerning the r.h.s. of the interpolation inequality (1.13) we have

$$\|U_\delta\|_{\mathcal{K}_\delta^1}^2 = \|U_\delta\|_{\mathcal{K}_\delta^0}^2 + \|A_\delta^{\frac{1}{2}} U_\delta\|_{\mathcal{K}_\delta^0}^2 + \|\partial_{x_0} U_\delta\|_{\mathcal{K}_\delta^0}^2 = \int_0^{X_0} \left( \|U_\delta\|_{\mathcal{H}_\delta^0}^2 + \|A_\delta^{\frac{1}{2}} U_\delta\|_{\mathcal{H}_\delta^0}^2 + \|\partial_{x_0} U_\delta\|_{\mathcal{H}_\delta^0}^2 \right) dx_0. \quad (5.11)$$

Let us estimate the three terms in this expression. First, we have,

$$\begin{aligned} \int_0^{X_0} \|U_\delta\|_{\mathcal{H}_\delta^0}^2 dx_0 &\leq \int_0^{X_0} \|A_\delta^{-\frac{1}{2}} \sinh(x_0 A_\delta^{\frac{1}{2}}) \Pi_{\delta, \mu}\|_{\mathcal{L}(\mathcal{H}_\delta^0)}^2 \|Y_\delta\|_{\mathcal{H}_\delta^0}^2 dx_0 \\ &\leq \int_0^{X_0} (x_0 e^{x_0 \sqrt{\mu}})^2 dx_0 \|Y_\delta\|_{\mathcal{H}_\delta^0}^2 \\ &\leq X_0^3 e^{2X_0 \sqrt{\mu}} \|Y_\delta\|_{\mathcal{H}_\delta^0}^2, \end{aligned}$$

since  $t^{-\frac{1}{2}} \sinh(x_0 t^{\frac{1}{2}}) \leq x_0 e^{x_0 \sqrt{\mu}}$  for  $0 \leq t \leq \mu$ . Second, We have

$$\begin{aligned} \int_0^{X_0} \|A_\delta^{\frac{1}{2}} U_\delta\|_{\mathcal{H}_\delta^0}^2 dx_0 &\leq \int_0^{X_0} \|\sinh(x_0 A_\delta^{\frac{1}{2}}) \Pi_{\delta, \mu}\|_{\mathcal{L}(\mathcal{H}_\delta^0)}^2 \|Y_\delta\|_{\mathcal{H}_\delta^0}^2 dx_0 \\ &\leq \int_0^{X_0} e^{2x_0 \sqrt{\mu}} dx_0 \|Y_\delta\|_{\mathcal{H}_\delta^0}^2 \\ &\leq X_0 e^{2X_0 \sqrt{\mu}} \|Y_\delta\|_{\mathcal{H}_\delta^0}^2, \end{aligned}$$

together with

$$\int_0^{X_0} \|\partial_{x_0} U_\delta\|_{\mathcal{H}_\delta^0}^2 dx_0 = \int_0^{X_0} \|\cosh(x_0 A_\delta^{\frac{1}{2}}) Y_\delta\|_{\mathcal{H}_\delta^0}^2 dx_0 \leq \int_0^{X_0} e^{2x_0 \sqrt{\mu}} dx_0 \|Y_\delta\|_{\mathcal{H}_\delta^0}^2 \leq X_0 e^{2X_0 \sqrt{\mu}} \|Y_\delta\|_{\mathcal{H}_\delta^0}^2.$$

Using the last three estimates in (5.11), together with (5.10), the interpolation inequality (1.13) yields

$$\|Y_\delta\|_{\mathcal{H}_\delta^0} \leq C(X_0, \alpha_1) \left( e^{X_0 \sqrt{\mu}} \|Y_\delta\|_{\mathcal{H}_\delta^0} \right)^{1-\nu_0} \|y_\delta\|_{L^2(\omega)}^{\nu_0}.$$

Finally, for  $\delta_0 > 0$ , there exists  $C > 0$  such that for all  $0 < \delta \leq \delta_0$  and  $\mu \in \mathbb{R}$ , we have

$$\|Y_\delta\|_{\mathcal{H}_\delta^0} \leq C e^{X_0 \frac{1-\nu_0}{\nu_0} \sqrt{\mu}} \|y_\delta\|_{L^2(\omega)}, \quad Y_\delta = (y_\delta, y_\delta^s) \in \Pi_{\delta, \mu} \mathcal{H}_\delta^0.$$

This concludes the proof of Theorem 1.5. □

## A Derivation of the model

Here, we (formally) derive the model (1.4) studied in the main part of this article. We use the notation of the beginning of Section 3. In a small neighborhood of the interface  $S$  we use normal geodesic coordinates

$$\begin{aligned} \mathbf{F} : S \times [-2\varepsilon, 2\varepsilon] &\rightarrow V_\varepsilon \\ (y, x_n) &\mapsto \mathbf{F}(y, x_n). \end{aligned}$$

<sup>5</sup>Note that if  $A_\delta$  is not invertible, i.e.  $0 \in \text{Sp}(A_\delta)$  (this occurs if  $\Omega$  has no boundary), the following analysis can be done with  $A_\delta + \text{Id}$  in place of  $A_\delta$ . Theorem 1.2 and Theorem 1.4 remains valid for this operator. The spectral inequality proven for  $A_\delta + \text{Id}$  implies the same inequality for  $A_\delta$ .

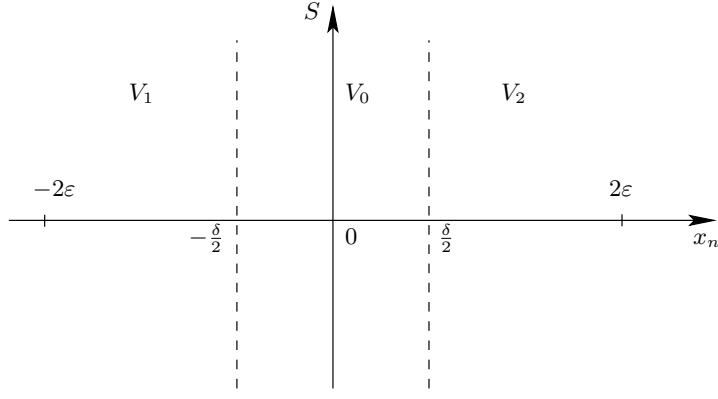


Figure 8: Local geometry of a three-layer model near the interface  $S = \{x_n = 0\}$ . The inner layer,  $V_0$ , shrinks to zero as  $\delta$  goes to zero.

In such coordinates the metric reads

$$g = \begin{pmatrix} g|_{T(S)} & 0 \\ 0 & 1 \end{pmatrix},$$

and the type of elliptic operators we consider,  $-\operatorname{div}_g c \nabla_g$ , take the form  $-\partial_{x_n} c \partial_{x_n} - \operatorname{div}^s c \nabla^s$ . The interface  $S$  is given by  $\{x_n = 0\}$ .

Let  $\delta \in (0, 4\varepsilon)$ . We consider three regions in  $V_\varepsilon$  as represented in Figure 8.

$$V_1 = \{-2\varepsilon \leq x_n \leq -\delta/2\}, \quad V_0 = \{-\delta/2 \leq x_n \leq \delta/2\}, \quad V_2 = \{\delta/2 \leq x_n \leq 2\varepsilon\}.$$

With three coefficients  $c^0, c^1, c^2$  we have in mind the following parabolic problem:

$$\partial_t z^j - \operatorname{div}_g(c^j \nabla_g z^j) = f^j \quad \text{in } (0, T) \times V_j, \quad j = 1, 0, 2, \quad (\text{A.1})$$

along with the natural transmission conditions at  $x_n = \frac{\delta}{2}$  and  $x_n = -\frac{\delta}{2}$ , given by the continuity of the solution and the continuity of the flux:

$$z^1|_{x_n=-\frac{\delta}{2}} = z^0|_{x_n=-\frac{\delta}{2}}, \quad z^0|_{x_n=\frac{\delta}{2}} = z^2|_{x_n=\frac{\delta}{2}}, \quad (\text{A.2})$$

and

$$(c^1 \partial_{x_n} z^1)|_{x_n=-\frac{\delta}{2}} = (c^0 \partial_{x_n} z^0)|_{x_n=-\frac{\delta}{2}}, \quad (c^0 \partial_{x_n} z^0)|_{x_n=\frac{\delta}{2}} = (c^2 \partial_{x_n} z^2)|_{x_n=\frac{\delta}{2}}. \quad (\text{A.3})$$

We now wish to describe the present three-region model as the thickness  $\delta$  of the inner region,  $V_0$ , becomes asymptotically small. This implies some approximation. Resulting approximate models can be very useful in practice as one is in need of effective models.

We introduce the mean values of  $z^0$  and  $f^0$  in the normal direction  $x_n$

$$z^s(y) := \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} z^0(y, x_n) dx_n \quad \text{and} \quad f^s(y) := \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} f^0(y, x_n) dx_n, \quad y \in S.$$

Keeping in mind that  $\delta$  is meant to be asymptotically small, we first make the following approximation.

**Assumption A.1.** *The diffusion coefficient  $c^0$  does not depend on the normal variable  $x_n$ . We set  $c^s(y) = c^0(y, x_n)$ .*

Under this assumption, using the transmission conditions (A.3), we have

$$\begin{aligned} f^s(y) &= \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \left( \partial_t z^0 - \operatorname{div}_g(c^0 \nabla_g z^0) \right) dx_n = \partial_t z^s - \operatorname{div}^s(c^s \nabla^s z^s) - \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \partial_{x_n} c^0 \partial_{x_n} z^0 dx_n \\ &= \partial_t z^s - \operatorname{div}^s(c^s \nabla^s z^s) - \frac{1}{\delta} \left( (c^0 \partial_{x_n} z^0)|_{x_n=\frac{\delta}{2}} - (c^0 \partial_{x_n} z^0)|_{x_n=-\frac{\delta}{2}} \right) \\ &= \partial_t z^s - \operatorname{div}^s(c^s \nabla^s z^s) - \frac{1}{\delta} \left( (c^2 \partial_{x_n} z^2)|_{x_n=\frac{\delta}{2}} - (c^1 \partial_{x_n} z^1)|_{x_n=-\frac{\delta}{2}} \right). \end{aligned} \quad (\text{A.4})$$

This provides a first transmission condition between  $z^1$  and  $z^2$  that involves the function  $z^s$ . For the problem to be closed, we need two additional transmission conditions.

We begin with a first-order approximation of the system. Yet we show that it cannot be used for the purpose of modeling controllability properties of the original system. We then lower the degree of our approximations and obtain the model studied in the main part of this article.

## A.1 A first-order model

Using the transmission conditions (A.2)–(A.3) we write

$$\begin{aligned} z^2(y, \delta/2) - z^1(y, -\delta/2) &= z^0(y, \delta/2) - z^0(y, -\delta/2) = \int_{-\delta/2}^{\delta/2} \partial_{x_n} z^0(y, x_n) dx_n \\ &= \left[ x_n \partial_{x_n} z^0(y, x_n) \right]_{-\delta/2}^{\delta/2} + R_1 \\ &= \frac{\delta}{2} (\partial_{x_n} z^0(y, \delta/2) + \partial_{x_n} z^0(y, -\delta/2)) + R_1 \\ &= \frac{\delta}{2c^s(y)} (c^2(y, \delta/2) \partial_{x_n} z^2(y, \delta/2) + c^1(y, -\delta/2) \partial_{x_n} z^1(y, -\delta/2)) + R_1, \end{aligned}$$

with  $R_1 = -\int_{-\delta/2}^{\delta/2} x_n \partial_{x_n}^2 z^0(y, x_n) dx_n$ .

A second set of transmission conditions is needed. With two integrations by parts we write

$$\begin{aligned} z^s(y) &= \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} z^0(y, x_n) dx_n \\ &= \frac{1}{\delta} [x_n z^0(y, x_n)]_{-\delta/2}^{\delta/2} - \frac{1}{\delta} \left[ \frac{x_n^2}{2} \partial_{x_n} z^0(y, x_n) \right]_{-\delta/2}^{\delta/2} + \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \frac{x_n^2}{2} \partial_{x_n}^2 z^0(y, x_n) dx_n \\ &= \frac{1}{2} (z^0(y, \delta/2) + z^0(y, -\delta/2)) - \frac{\delta}{8} (\partial_{x_n} z^0(y, \delta/2) - \partial_{x_n} z^0(y, -\delta/2)) + \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \frac{x_n^2}{2} \partial_{x_n}^2 z^0(y, x_n) dx_n \\ &= \frac{1}{2} (z^2(y, \delta/2) + z^1(y, -\delta/2)) + R_2, \end{aligned}$$

with

$$R_2 = -\frac{\delta}{8} (\partial_{x_n} z^0(y, \delta/2) - \partial_{x_n} z^0(y, -\delta/2)) + \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \frac{x_n^2}{2} \partial_{x_n}^2 z^0(y, x_n) dx_n$$

We now make the following assumption on the variations of  $z^0$  with respect to  $x_n$ .

**Assumption A.2.** *We have  $|\partial_{x_n}^2 z^0(y, x_n)| \leq C$  uniformly in  $\delta$ , (and  $x_n$ , and  $y \in S$ ).*

We then find that  $R_1 = O(\delta^2)$ . Observe that we have

$$\partial_{x_n} z^0(y, \delta/2) - \partial_{x_n} z^0(y, -\delta/2) = \int_{-\delta/2}^{\delta/2} \partial_{x_n}^2 z^0(y, x_n) dx_n = O(\delta).$$

We then find that  $R_2 = O(\delta^2)$ .

At first order in  $\delta$  we thus obtain

$$\begin{cases} z^2(y, \delta/2) - z^1(y, -\delta/2) = \frac{\delta}{2c^s(y)} (c^2(y, \delta/2) \partial_{x_n} z^2(y, \delta/2) + c^1(y, -\delta/2) \partial_{x_n} z^1(y, -\delta/2)) \\ z^s(y) = \frac{1}{2} (z^2(y, \delta/2) + z^1(y, -\delta/2)). \end{cases} \quad (\text{A.5})$$

As  $\delta$  is small we consider that  $z^1$  and  $z^2$  are defined on  $\{x_n < 0\}$  and  $\{x_n > 0\}$  respectively. We thus write  $z^2|_{x_n=0^+}$  and  $c^2 \partial_{x_n} z^2|_{x_n=0^+}$  in place of  $z^2(y, \delta/2)$  and  $(c^2 \partial_{x_n} z^2)(y, \delta/2)$  and similarly  $z^1|_{x_n=0^-}$  and  $c^1 \partial_{x_n} z^1|_{x_n=0^-}$  in place of  $z^1(y, -\delta/2)$  and  $(c^1 \partial_{x_n} z^1)(y, -\delta/2)$ . We obtain the following model:

$$\partial_t z^j - \operatorname{div}_g c^j \nabla_g z^j = f^j \quad \text{in } (0, T) \times \Omega_j, \quad j = 1, 2, \quad (\text{A.6})$$

and

$$\begin{cases} \partial_t z^s - \operatorname{div}^s (c^s \nabla^s z^s) = f^s + \frac{1}{\delta} ((c^2 \partial_{x_n} z^2)|_{x_n=0^+} - (c^1 \partial_{x_n} z^1)|_{x_n=0^-}) \\ z^s = \frac{1}{2} (z^2|_{x_n=0^+} + z^1|_{x_n=0^-}) \\ z^2|_{x_n=0^+} - z^1|_{x_n=0^-} = \frac{\delta}{2c^s} ((c^2 \partial_{x_n} z^2)|_{x_n=0^+} + (c^1 \partial_{x_n} z^1)|_{x_n=0^-}) \end{cases} \quad (\text{A.7})$$

in  $(0, T) \times S$ .

For the study of the controllability of such a parabolic model we wish to investigate the unique continuation properties of the associated elliptic problem:

$$-\operatorname{div}_g c^j \nabla_g z^j = f^j \quad \text{in } (0, T) \times \Omega_j, \quad j = 1, 2, \quad (\text{A.8})$$

and

$$\begin{cases} -\operatorname{div}^s(c^s \nabla^s z^s) = f^s + \frac{1}{\delta} ((c^2 \partial_{x_n} z^2)|_{x_n=0^+} - (c^1 \partial_{x_n} z^1)|_{x_n=0^-}) \\ z^s = \frac{1}{2} (z^2|_{x_n=0^+} + z^1|_{x_n=0^-}) \\ z^2|_{x_n=0^+} - z^1|_{x_n=0^-} = \frac{\delta}{2c^s} ((c^2 \partial_{x_n} z^2)|_{x_n=0^+} + (c^1 \partial_{x_n} z^1)|_{x_n=0^-}) \end{cases} \quad (\text{A.9})$$

in  $(0, T) \times S$ .

Note that unique continuation holds for the original problem. This is an important property that we wish to see preserved in this approximation process. Here, we show that there are instances for which eigenfunctions of the elliptic operator in the approximate model (A.8)–(A.9) vanish on one side of the interface. These eigenmodes are then *invisible* for the observability of the parabolic system (A.6)–(A.7) ruining any hope of controllability. This is similar to the situation described in Section 1.3.3.

Let us consider the following two-dimensional example :  $\Omega = \mathbb{R}/(2\pi\mathbb{Z}) \times (-\pi, \pi)$  is the cylinder endowed with a flat metric. For consistency with the notation of Section 3 we use  $(y, x_n)$  as the coordinates in  $\Omega$ , with periodic conditions in  $y$ . We define the interface as  $S = \{x_n = 0\} = \mathbb{R}/(2\pi\mathbb{Z}) \times \{0\}$ , so that  $\Omega_1 = \{x_n < 0\}$  and  $\Omega_2 = \{x_n > 0\}$ .

**Proposition A.3.** *Let  $c^s$  and  $c^1$  be constant functions such that  $c^s = rc^1$  with  $r > 1$ . For any  $\delta_0 > 0$ , there exist  $0 < \delta \leq \delta_0$ ,  $e^1 \in \mathcal{C}^\infty(\Omega_1)$ ,  $e^s \in \mathcal{C}^\infty(S)$ ,  $\lambda > 0$  such that*

$$-\operatorname{div}_g c^1 \nabla_g e^1 = \lambda e^1 \quad \text{in } \Omega_1, \quad -\operatorname{div}^s(c^s \nabla^s e^s) + \frac{1}{\delta} (c^1 \partial_{x_n} e^1)|_{x_n=0^-} = \lambda e^s \quad \text{in } S, \quad (\text{A.10})$$

and

$$e^s = \frac{1}{2} e^1|_{x_n=0^-}, \quad -e^1|_{x_n=0^-} = \frac{\delta}{2c^s} (c^1 \partial_{x_n} e^1)|_{x_n=0^-}, \quad \text{in } S, \quad (\text{A.11})$$

and  $e^1|_{x_n=-\pi} = 0$ . Hence  $(e^1, e^s, 0)$  is an eigenfunction of the elliptic operator in (A.8)–(A.9) associated with the eigenvalue  $\lambda$ , for Dirichlet boundary conditions (in  $x_n$ ).

*Proof.* We choose  $k \in \mathbb{N}$  such that  $(r-1)k > 1$ . For  $\mu \in (0, 1)$  we set

$$g(\mu) = \left( \frac{1}{r((r-1)k^2 - \mu^2)} \right)^{\frac{1}{2}} \frac{\mu \cos(\mu\pi)}{\sin(\mu\pi)}.$$

As  $g$  vanishes for  $\mu = 1/2$  and  $\lim_{\mu \rightarrow 1^-} g(\mu) = -\infty$ , there exists  $\mu_0 \in (1/2, 1)$  such that  $g(\mu_0) = -1$ . We then set

$$\delta = 2 \left( \frac{r}{((r-1)k^2 - \mu_0^2)} \right)^{\frac{1}{2}}, \quad \alpha = \frac{2}{\sin(\mu_0\pi)}.$$

For any given  $\delta_0$  we can have  $0 < \delta \leq \delta_0$  by choosing  $k$  sufficiently large. We have

$$\frac{\delta \mu_0 \cos(\mu_0\pi)}{2r \sin(\mu_0\pi)} = -1. \quad (\text{A.12})$$

We now set

$$e^s(y) = e^{iky}, \quad e^1(y, x_n) = \alpha \sin(\mu_0(x_n + \pi)) e^s(y), \quad -\pi \leq x_n \leq 0.$$

We have  $e^1|_{x_n=-\pi} = 0$ . Hence the Dirichlet boundary condition is satisfied at  $x_n = -\pi$ .

We have  $-c^1(\partial_y^2 + \partial_{x_n}^2)e^1 = \lambda e^1$  with  $\lambda = c^1(k^2 + \mu_0^2)$ . Observing that  $\partial_{x_n} e^1|_{x_n=0^-} = \alpha \mu_0 \cos(\mu_0\pi) e^s$  we find

$$\begin{aligned} -c^s \partial_y^2 e^s + \frac{1}{\delta} c \partial_{x_n} e^1|_{x_n=0^-} &= c^1 \left( rk^2 + \frac{\alpha \mu_0}{\delta} \cos(\mu_0\pi) \right) e^s = c^1 \left( rk^2 + \frac{2\mu_0}{\delta \sin(\mu_0\pi)} \cos(\mu_0\pi) \right) e^s \\ &= c^1 \left( rk^2 - \frac{4r}{\delta^2} \right) e^s = c^1 \left( rk^2 - ((r-1)k^2 - \mu_0) \right) e^s = \lambda e^s, \end{aligned}$$

by (A.12) and the value we have assigned to  $\delta$ . We have thus obtained (A.10).

We now compute, using (A.12) and the value we have assigned to  $\alpha$ ,

$$\frac{1}{2}e^1_{|x_n=0^-} = \frac{\alpha}{2} \sin(\mu_0\pi)e^s = e^s.$$

Using (A.12) we also compute

$$e^1_{|x_n=0^-} + \frac{\delta}{2c^s} (c^1 \partial_{x_n} e^1)_{|x_n=0^-} = \alpha \sin(\mu_0\pi) \left(1 + \frac{\delta}{2r} \mu_0 \frac{\cos(\mu_0\pi)}{\sin(\mu_0\pi)}\right) e^s = 0.$$

We have thus obtained (A.11).  $\square$

## A.2 A zero-order model

The lack of unique continuation of the previous (elliptic) model makes us consider a simpler model. We make a lower-order approximation and we show how to formally obtain the model studied in the main text of this article.

Neglecting the first-order terms in  $\delta$  in (A.5) we find

$$z^2(y, \delta/2) = z^1(y, -\delta/2) = z^s(y).$$

As  $\delta^{-1}(\partial_{x_n} z^0(y, \delta/2) - \partial_{x_n} z^0(y, -\delta/2)) = O(1)$  we cannot neglect this term in (A.4). Proceeding as above we thus obtain the following model

$$\partial_t z^j - \operatorname{div}_g c^j \nabla_g z^j = f^j \quad \text{in } (0, T) \times \Omega_j, \quad j = 1, 2,$$

and

$$\begin{cases} \partial_t z^s - \operatorname{div}^s(c^s \nabla^s z^s) = f^s + \frac{1}{\delta} ((c^2 \partial_{x_n} z^2)_{|x_n=0^+} - (c^1 \partial_{x_n} z^1)_{|x_n=0^-}) \\ z^2_{|x_n=0^+} = z^s = z^1_{|x_n=0^-}, \end{cases}$$

in  $(0, T) \times S$ .

## B Facts on semi-classical operators

### B.1 Results for tangential semi-classical operators on $\mathbb{R}^d$ , $d \geq 2$

Semi-classical operators are defined in Section 1.4. Here, we provide the properties that we need in the main text.

The composition formula for tangential symbols,  $b \in S_{\mathcal{T}}^m$ ,  $b' \in S_{\mathcal{T}}^{m'}$ , is given by

$$\begin{aligned} (b \# b')(z, \zeta') &= (2\pi h)^{-(d-1)} \iint e^{-i\langle t', \tau' \rangle / h} b(z, \zeta' + \tau', h) b'(z' + t', z_d, \zeta', h) dt' d\tau' \\ &= \sum_{|\alpha| < N} \frac{(-ih)^{|\alpha|}}{\alpha!} \partial_{\zeta'}^\alpha b(z, \zeta', h) \partial_{z'}^\alpha b'(z, \zeta', h) + r_N, \quad r_N \in h^N S_{\mathcal{T}}^{m+m'-N}, \end{aligned} \quad (\text{B.1})$$

where

$$r_N = \frac{(-ih)^N}{(2\pi h)^{(d-1)}} \sum_{|\alpha|=N} \int_0^1 \frac{(N)(1-s)^{N-1}}{\alpha!} \iint e^{-i\langle t', \tau' \rangle / h} \partial_{\zeta'}^\alpha b(z, \zeta' + \tau', h) \partial_{z'}^\alpha b'(z' + st', z_d, \zeta', h) dt' d\tau' ds,$$

and yields a tangential symbol in  $S_{\mathcal{T}}^{m+m'}$ .

If  $s, m \in \mathbb{R}$  and  $b \in S_{\mathcal{T}}^m$  we then have the following regularity result:

$$\|\Lambda_{\mathcal{T}}^s \operatorname{Op}_{\mathcal{T}}(b)u\|_{L^2(\mathbb{R}^d)} \leq \|\Lambda_{\mathcal{T}}^{s+m}u\|_{L^2(\mathbb{R}^d)}, \quad u \in \mathcal{S}(\mathbb{R}^d).$$

We now consider the effect of change of variables.

**Theorem B.1.** *Let  $Z'$  and  $Z'_\kappa$  be open subsets of  $\mathbb{R}^{d-1}$  and let  $\kappa : Z' \rightarrow Z'_\kappa$  be a diffeomorphism. If  $b(z, \zeta', h) \in S_{\mathcal{T}}^m$ , and the kernel of  $\operatorname{Op}_{\mathcal{T}}(b)$  has its support contained in  $K \times \mathbb{R} \times K \times \mathbb{R}$ , with  $K$  compact of  $Z'$  then the function*

$$b_\kappa(\hat{z}', z_d, \zeta', h) = \begin{cases} e^{-i\langle \kappa(z'), \zeta' \rangle / h} \operatorname{Op}_{\mathcal{T}}(b) e^{i\langle \kappa(z'), \zeta' \rangle / h} & \text{if } \hat{z}' = \kappa(z') \in Z'_\kappa, \\ 0 & \text{if } \hat{z}' \notin Z'_\kappa, \end{cases} \quad (\text{B.2})$$

is in  $S_{\mathcal{T}}^m$ , and the kernel of  $\text{Op}_{\mathcal{T}}(b_{\kappa})$  has its support contained in  $\kappa(K) \times \mathbb{R} \times \kappa(K) \times \mathbb{R}$ , and

$$(\kappa \otimes \text{Id})^* \text{Op}_{\mathcal{T}}(b_{\kappa})u = \text{Op}_{\mathcal{T}}(b)((\kappa \otimes \text{Id})^*u), \quad u \in \mathcal{S}'(\mathbb{R}^d). \quad (\text{B.3})$$

For  $b_{\kappa}$  we have the following asymptotic expansion

$$b_{\kappa}(\kappa(z'), z_d, \zeta', h) - T_{\kappa, N}(b)(\kappa(z'), z_d, \zeta', h) \in h^{N/2} S_{\mathcal{T}}^{m-N/2}, \quad (\text{B.4})$$

with

$$T_{\kappa, N}(b)(\kappa(z'), z_d, \zeta', h) = \sum_{\alpha < N} \frac{(-ih)^{|\alpha|}}{\alpha!} \partial_{\zeta'}^{\alpha} b(z', z_d, {}^t\kappa(z')' \zeta', h) \partial_{t'}^{\alpha} e^{i\langle \rho_{z'}(t'), \zeta' \rangle / h} \Big|_{t'=z'} \quad (\text{B.5})$$

where  $\rho_{z'}(t') = \kappa(z') - \kappa(t') - \kappa'(z')(t' - z')$ .

A proof is provided in Appendix C.9. In particular we find that

$$b_{\kappa}(\kappa(z'), z_d, \zeta, h) = b(z', z_d, {}^t\kappa(z')' \zeta', h) + hr(z', z_d, \zeta', h), \quad \text{with } r \in hS_{\mathcal{T}}^{m-1}, \quad (\text{B.6})$$

The principal symbol thus transforms as the regular pullback of a function defined in phase-space (see Section 1.4.3).

**Lemma B.2.** *Let  $a \in S_{\mathcal{T}}^m$  be such that the kernel  $K_h(z, t) = \mathcal{K}_{h, z_d}(z', t') \otimes \delta(z_d - t_d)$  of  $\text{Op}_{\mathcal{T}}(a)$  is such that  $\mathcal{K}_{h, z_d}(z', t')$  vanishes if  $|z' - t'| \leq \eta$  for some  $\eta > 0$ . Then  $a \in h^{\infty} S_{\mathcal{T}}^{-\infty}$ .*

*Proof.* We write, as an oscillatory integral,

$$\mathcal{K}_{h, z_d}(z', t') = \frac{1}{(2\pi h)^{d-1}} \int e^{i\langle z' - t', \zeta' \rangle / h} a(z, \zeta', h) d\zeta'.$$

Let  $\chi \in \mathcal{C}^{\infty}(\mathbb{R}^{d-1})$  be such that  $\chi(z') = 0$  if  $|z'| \leq \frac{\eta}{2}$  and  $\chi(z') = 1$  if  $|z'| \geq \eta$ . Then  $\mathcal{K}_{h, z_d}(z', t') = \chi(z' - t') \mathcal{K}_{h, z_d}(z', t')$ . Hence,  $\chi(z' - t')a(z, \zeta', h)$  is an amplitude for  $\text{Op}_{\mathcal{T}}(a)$ . The asymptotic series providing the associated symbol, which is in fact  $a(z, \zeta', h)$ , is [GS94]

$$a(z, \zeta', h) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|} h^{|\alpha|}}{\alpha!} \partial_{t'}^{\alpha} \partial_{\zeta'}^{\alpha} (\chi(z' - t')a(z, \zeta', h)) \Big|_{t'=z'}.$$

Because of the support of  $\chi$  the result follows.  $\square$

## B.2 Semi-classical (tangential) operators on a manifold

In the present article, we consider semi-classical operators that act on both the  $x_0$  and  $y$  variables,  $x_0 \in (0, X_0)$  and  $y \in S$ .

Let  $\mathcal{X}$  be a manifold of the form  $(0, X_0) \times S \times \mathbb{R}$ . We denote by  $(x_0, y, x_n)$  a typical element. We also set  $\mathcal{X}' = (0, X_0) \times S$ . By abuse of notation we shall also call  $\phi_j$  the map  $\text{Id} \otimes \phi_j \otimes \text{Id}$  (resp.  $\text{Id} \otimes \phi_j$ ) on  $\mathbb{R} \times U_j \times \mathbb{R}$  (resp.  $\mathbb{R} \times U_j$ ); see Section 1.4.3 where the diffeomorphisms  $\phi_j$ ,  $j \in J$ , are defined.

We recall the definition of a tangential semi-classical symbol in an open set  $O \subset \mathbb{R}^d$ .

**Definition B.3.** We say that  $a(z, \zeta', h) \in S_{\mathcal{T}}^m(O \times \mathbb{R}^{d-1})$  if, for any  $\chi \in \mathcal{C}_c^{\infty}(O)$ ,  $\chi a \in S_{\mathcal{T}}^m(\mathbb{R}^d \times \mathbb{R}^{d-1})$ .

We also recall the definition of tangential semi-classical symbols and operators on a manifold.

**Definition B.4.** 1. Let  $m \in \mathbb{R}$ ,  $j \in J$ , and  $a \in \mathcal{C}^{\infty}(T^*((0, X_0) \times U_j) \times \mathbb{R})$ . We say that  $a \in S_{\mathcal{T}}^m(T^*((0, X_0) \times U_j) \times \mathbb{R})$  if,  $(\phi_j^{-1})^* a \in S_{\mathcal{T}}^m((0, X_0) \times \tilde{U}_j \times \mathbb{R} \times \mathbb{R}^n)$ .

2. Let  $a \in \mathcal{C}^{\infty}(T^*(\mathcal{X}') \times \mathbb{R})$ . We say that  $a \in S_{\mathcal{T}}^m(T^*(\mathcal{X}') \times \mathbb{R})$  if, for all  $j \in J$ ,  $a|_{T^*((0, X_0) \times U_j) \times \mathbb{R}} \in S_{\mathcal{T}}^m(T^*((0, X_0) \times U_j) \times \mathbb{R})$ .

**Definition B.5.** An operator  $A : \mathcal{C}_c^{\infty}(\mathcal{X}) \rightarrow \mathcal{C}^{\infty}(\mathcal{X})$  is said to be tangential semi-classical on  $\mathcal{X}$  of order  $m \in \mathbb{R}$  if:

1. Its kernel is of the form

$$K_h(x_0, y, x_n; \hat{x}_0, \hat{y}, \hat{x}_n) = \mathcal{K}_{h, x_n}(x_0, y; \hat{x}_0, \hat{y}) \otimes \delta(x_n - \hat{x}_n).$$

2. Its kernel is regularizing outside  $\text{diag}(\mathcal{X} \times \mathcal{X})$  in the semi-classical sense: for all  $\chi, \hat{\chi} \in \mathcal{C}_c^\infty(\mathcal{X}')$  such that  $\text{supp}(\chi) \cap \text{supp}(\hat{\chi}) = \emptyset$  we have

$$\chi(x_0, y) \hat{\chi}(\hat{x}_0, \hat{y}) \mathcal{K}_{h, x_n}(x_0, y; \hat{x}_0, \hat{y}) \in \mathcal{C}^\infty(\mathcal{X}' \times \mathcal{X}'),$$

and for all  $N, \alpha \in \mathbb{N}$ , and for any semi-norm  $q$  on  $\mathcal{C}^\infty(\mathcal{X}' \times \mathcal{X}')$  there exists  $C = C_{\chi, \hat{\chi}, N, \alpha, q} > 0$  such that

$$\sup_{x_n \in \mathbb{R}} q(\chi(x_0, y) \hat{\chi}(\hat{x}_0, \hat{y}) \partial_{x_n}^\alpha \mathcal{K}_{h, x_n}(x_0, y; \hat{x}_0, \hat{y})) \leq Ch^N. \quad (\text{B.7})$$

3. For all  $j \in J$  and all  $\lambda \in \mathcal{C}_c^\infty((0, X_0) \times U_j)$ ,  $\tilde{\lambda} \in \mathcal{C}_c^\infty((0, X_0) \times \tilde{U}_j)$ , we have

$$\mathcal{S}'(\mathbb{R}^{n+1}) \ni u \mapsto (\phi_j^{-1})^* (\lambda \otimes \text{Id}) A \phi_j^* (\tilde{\lambda} \otimes \text{Id}) u$$

in  $\Psi_{\mathcal{T}}^m(\mathbb{R}^{n+1})$ .

In this case, we write  $A \in \Psi_{\mathcal{T}}^m(\mathcal{X})$ .

Note that we shall often write  $\lambda$  and  $\tilde{\lambda}$  in place of  $\lambda \otimes \text{Id}$  and  $\tilde{\lambda} \otimes \text{Id}$  respectively.

We set

$$\begin{aligned} h^\infty S_{\mathcal{T}}^{-\infty}((0, X_0) \times \tilde{U}_j \times \mathbb{R} \times \mathbb{R}^n) &= \bigcap_{N \in \mathbb{N}} h^N S_{\mathcal{T}}^{-N}((0, X_0) \times \tilde{U}_j \times \mathbb{R} \times \mathbb{R}^n), \\ h^\infty \Psi_{\mathcal{T}}^{-\infty}((0, X_0) \times \tilde{U}_j \times \mathbb{R}) &= \bigcap_{N \in \mathbb{N}} h^N \Psi_{\mathcal{T}}^{-N}((0, X_0) \times \tilde{U}_j \times \mathbb{R}), \end{aligned}$$

**Remark B.6.** The first two points of Definition B.5 in fact state that the semi-classical wave front of the kernel of the operator is confined in the conormal bundle of the diagonal of  $\mathcal{X}$ . As a consequence,  $A$  maps  $\mathcal{E}'(\mathcal{X})$  into  $\mathcal{D}'(\mathcal{X})$  [Hör90, Theorem 8.2.13]. We also note that the same properties hold for the transpose (resp. adjoint) operator. If moreover  $A$  is properly supported then

$$A : \mathcal{C}_c^\infty(\mathcal{X}) \rightarrow \mathcal{C}_c^\infty(\mathcal{X}), \quad \mathcal{C}^\infty(\mathcal{X}) \rightarrow \mathcal{C}^\infty(\mathcal{X}), \quad \mathcal{E}'(\mathcal{X}) \rightarrow \mathcal{E}'(\mathcal{X}), \quad \mathcal{D}'(\mathcal{X}) \rightarrow \mathcal{D}'(\mathcal{X}), \quad (\text{B.8})$$

continuously, and the same holds for  ${}^t A$ .

Observe that tangential semi-classical *differential* operators naturally satisfy all the properties listed above.

**Proposition B.7.** *If  $A \in \Psi_{\mathcal{T}}^m(\mathcal{X})$ , for all  $j \in J$ , there exists  $a_j(x_0, x', x_n; \xi_0, \xi')$  in  $S_{\mathcal{T}}^m((0, X_0) \times \tilde{U}_j \times \mathbb{R} \times \mathbb{R}^n)$  such that for all  $\lambda \in \mathcal{C}_c^\infty((0, X_0) \times U_j)$ ,  $\tilde{\lambda} \in \mathcal{C}_c^\infty((0, X_0) \times \tilde{U}_j)$  we have*

$$(\phi_j^{-1})^* \lambda A \phi_j^* \tilde{\lambda} - \text{Op}_{\mathcal{T}} \left( ((\phi_j^{-1})^* \lambda) a_j \right) \tilde{\lambda} \in h^\infty \Psi_{\mathcal{T}}^{-\infty}(\mathbb{R}^n \times \mathbb{R}).$$

Moreover,  $a_j$  is uniquely defined up to  $h^\infty S_{\mathcal{T}}^{-\infty}((0, X_0) \times \tilde{U}_j \times \mathbb{R} \times \mathbb{R}^n)$ .

We refer to Appendix C.10 for a proof. We say that  $a_j$  is the (representative of the) local symbol of  $A$  (modulo  $h^\infty S_{\mathcal{T}}^{-\infty}$ ) in the chart  $(0, X_0) \times \tilde{U}_j \times \mathbb{R}$ . We find that the symbol of  $(\phi_j^{-1})^* \lambda A \phi_j^* \tilde{\lambda}$  is given by  $((\phi_j^{-1})^* \lambda) a_j \# \tilde{\lambda}$  modulo  $h^\infty S_{\mathcal{T}}^{-\infty}(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ , from the previous proposition. The symbols  $(a_j)_{j \in J}$  follow the natural transformations when going from one chart to another.

**Proposition B.8.** *If  $U_j \cap U_k \neq \emptyset$ , we introduce*

$$\tilde{U}_{j,k} = \phi_j(U_j \cap U_k) \subset \tilde{U}_j \quad \text{and} \quad \tilde{U}_{k,j} = \phi_k(U_j \cap U_k) \subset \tilde{U}_k.$$

Let  $A \in \Psi_{\mathcal{T}}^m(\mathcal{X})$  with  $a_j$  as given in Proposition B.7, we have

$$a_{k|(0, X_0) \times \tilde{U}_{k,j} \times \mathbb{R}} - T_{\phi_{jk}, N}(a_{j|(0, X_0) \times \tilde{U}_{j,k} \times \mathbb{R}}) \in h^N S_{\mathcal{T}}^{m-N/2}((0, X_0) \times \tilde{U}_{k,j} \times \mathbb{R} \times \mathbb{R}^n).$$

We refer to Appendix C.11 for a proof. The notation  $T_{\phi_{jk}, N}$  is defined in (B.5). The open sets  $\tilde{U}_{j,k}$  and  $\tilde{U}_{k,j}$  are represented in Figure 2.

As a consequence, only considering the first term in the sum defining  $T_{\phi_{jk}, N}(a_j)$ , we observe that the principle part of  $a_j$  defined on  $(0, X_0) \times \tilde{U}_j \times \mathbb{R} \times \mathbb{R}^n$  transforms as a function on  $T^*(\mathcal{X}') \times \mathbb{R}$  through a change of variables.

Let  $A \in \Psi_{\mathcal{T}}^m(\mathcal{X})$  and let  $a_j$ ,  $j \in J$ , be representatives of the local symbol (class) given in the local chart by Proposition B.7. We set  $a = \sum_{j \in J} \psi_j \phi_j^* a_j$  and find

$$a - \phi_j^* a_j \in h \Psi_{\mathcal{T}}^{m-1}(T^*((0, X_0) \times U_j) \times \mathbb{R}).$$

This defines  $a$  modulo  $h S_{\mathcal{T}}^{m-1}(T^*(\mathcal{X}') \times \mathbb{R})$ .



**Definition B.9.** We define the principal symbol of  $A$  as the class of  $a$  in  $S_{\mathcal{T}}^m(T^*(\mathcal{X}') \times \mathbb{R})/hS_{\mathcal{T}}^{m-1}(T^*(\mathcal{X}') \times \mathbb{R})$  and we denote it by  $\sigma(A)$ .

**Proposition B.10.** Let  $A \in \Psi_{\mathcal{T}}^m(\mathcal{X})$ ,  $B \in \Psi_{\mathcal{T}}^{m'}(\mathcal{X})$  be both properly supported. Then  $AB \in \Psi_{\mathcal{T}}^{m+m'}(\mathcal{X})$  and (a representative of) its local symbol in any chart  $(U_j, \phi_j)$  is given by  $a_j \# b_j$  with the notation of Proposition B.7. In particular, we have  $\sigma(AB) = \sigma(A)\sigma(B)$ .

We refer to Appendix C.12 for a proof.

The following natural result is a consequence of what precedes.

**Corollary B.11.** If  $A \in \Psi_{\mathcal{T}}^m(\mathcal{X})$  and  $B \in \Psi_{\mathcal{T}}^{m'}(\mathcal{X})$  are both properly supported then the commutator  $[A, B] \in h\Psi_{\mathcal{T}}^{m+m'-1}(\mathcal{X})$  and  $\frac{h}{i}\{\sigma(A), \sigma(B)\}$  is (a representative of) its principal symbol.

With the Sobolev norms defined in Section 1.4.3 we have the following result.

**Proposition B.12.** Let  $A \in \Psi_{\mathcal{T}}^{\ell}(\mathcal{X})$  be properly supported,  $\ell = 0, 1$ . Let  $K$  be a compact set of  $\mathcal{X}'$ . Then there exist  $L$ , a compact of  $\mathcal{X}'$ , and  $C > 0$  such that for all  $u \in \mathcal{C}_c^{\infty}(\mathcal{X}')$  with  $\text{supp}(u) \subset K$  we have

$$\text{supp}((Au)|_{x_n=0}) \subset L \quad \text{and} \quad |(Au)|_{x_n=0}|_k \leq C|u|_{\ell+k} \quad \text{with } k = \begin{cases} 0 \text{ or } 1 & \text{if } \ell = 0, \\ 0 & \text{if } \ell = 1. \end{cases}$$

We refer to Appendix C.13 for a proof. The norms in the proposition are those defined in (1.21).

### B.3 A particular class of semi-classical operators on $\mathcal{M}_+$

In this section, we prove that the operators  $\Xi_{\bullet}$  defined in (3.24),  $\bullet = \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{Z}$ , are tangential semi-classical pseudo-differential operators on  $\mathcal{M}_+$ . We also establish some properties of their symbols.

Let  $\zeta^0 \in \mathcal{C}_c^{\infty}(0, X_0)$  that satisfies  $\zeta^0 = 1$  on a neighborhood of  $(\alpha_0, X_0 - \alpha_0)$  and  $0 \leq \zeta^0 \leq 1$ . We set

$$\zeta_j^0(x_0, y, x_n) = \zeta^0(x_0)\psi_j(y).$$

For all  $j \in J$ , we choose  $\tilde{\zeta}_j^0 \in \mathcal{C}_c^{\infty}((0, X_0) \times \tilde{U}_j)$  with  $\tilde{\zeta}_j^0 = 1$  in a neighborhood of  $\text{supp}((\phi_j^{-1})^* \zeta_j^0)$ .

Let  $p \in S_{\mathcal{T}}^m(\mathcal{M}_+)$ . We define, for some  $j \in J$ ,  $p_j = \tilde{\zeta}_j^0(\phi_j^{-1})^* p$  and  $Q = \phi_j^* \text{Op}_{\mathcal{T}}(p_j)(\phi_j^{-1})^* \zeta_j^0$ .

**Lemma B.13.** We have  $Q \in \Psi_{\mathcal{T}}^m(\mathcal{M}_+)$ . Moreover, denoting by  $q_k$  (a representative of) the local symbol of  $Q$  in the chart  $\tilde{U}_k$ , we have

1.  $q_j = p_j \# \left( (\phi_j^{-1})^* \zeta_j^0 \right) \bmod h^{\infty} S_{\mathcal{T}}^{-\infty}(\mathbb{R}^n \times [0, 2\varepsilon] \times \mathbb{R}^n)$  and  $q_j$  can be chosen such that  $\text{supp}(q_j) \subset \text{supp}(\tilde{\zeta}_j^0) \times \mathbb{R}^n \times [0, 2\varepsilon] \subset \tilde{U}_j \times \mathbb{R}^n \times [0, 2\varepsilon]$ ;
2.  $q_k = 0$  if  $U_j \cap U_k = \emptyset$ ;
3.  $q_k = T_{\phi_{j_k, N}}(q_j) \bmod h^{N/2} S_{\mathcal{T}}^{m-N/2}(\mathbb{R}^n \times [0, 2\varepsilon] \times \mathbb{R}^n)$  for all  $N \in \mathbb{N}$  and  $\text{supp}(q_k) \subset \phi_k(U_j \cap U_k) \times \mathbb{R}^n \times \mathbb{R}$  if  $k \neq j$  and  $U_j \cap U_k \neq \emptyset$ .

*Proof.* Let us first check that  $Q \in \Psi_{\mathcal{T}}^m(\mathcal{M}_+)$ . The definition of  $Q$  first yields  $\text{supp}(K_{Q, h}) \subset ((0, X_0) \times U_j \times [0, 2\varepsilon])^2$ . Then, for  $\lambda \in \mathcal{C}_c^{\infty}((0, X_0) \times U_j)$ ,  $\tilde{\lambda} \in \mathcal{C}_c^{\infty}((0, X_0) \times \tilde{U}_j)$ , we have

$$(\phi_j^{-1})^* \lambda Q \phi_j^* \tilde{\lambda} = \left( (\phi_j^{-1})^* \lambda \right) \text{Op}_{\mathcal{T}}(p_j) \left( (\phi_j^{-1})^* \zeta_j^0 \right) \tilde{\lambda} \in \Psi_{\mathcal{T}}^m(\mathbb{R}^n \times [0, 2\varepsilon]),$$

and the symbol of this operator is  $\left( (\phi_j^{-1})^* \lambda \right) \# p_j \# \left( (\phi_j^{-1})^* \zeta_j^0 \right) \tilde{\lambda}$ . According to Proposition B.7, this yields  $q_j = p_j \# (\phi_j^{-1})^* \zeta_j^0 \bmod h^{\infty} S_{\mathcal{T}}^{-\infty}(\mathbb{R}^n \times [0, 2\varepsilon] \times \mathbb{R}^n)$ . The local representation  $q_j$  can be chosen with compact support in  $\tilde{U}_j$  since  $p_j = \tilde{\zeta}_j^0(\phi_j^{-1})^* p$  and  $\text{supp}(\tilde{\zeta}_j^0) \subset \tilde{U}_j$ . As a consequence, the first point is fulfilled.

Taking now  $\lambda$  and  $\tilde{\lambda}$  such that  $\text{supp}(\lambda) \cap \text{supp}(\phi_j^* \tilde{\lambda}) = \emptyset$ , we find  $(\phi_j^{-1})^* \lambda Q \phi_j^* \tilde{\lambda} \in h^{\infty} \Psi_{\mathcal{T}}^{-\infty}(\mathbb{R}^n \times [0, 2\varepsilon])$ , so that the kernel of  $Q$  satisfies (B.7). Next, we take  $k \in J$ ,  $k \neq j$  and  $\lambda \in \mathcal{C}_c^{\infty}((0, X_0) \times U_k)$ ,  $\tilde{\lambda} \in \mathcal{C}_c^{\infty}((0, X_0) \times \tilde{U}_k)$  and compute  $(\phi_k^{-1})^* \lambda Q \phi_k^* \tilde{\lambda}$ .

If  $U_j \cap U_k = \emptyset$ , this is the null operator and the second point is satisfied. If  $U_j \cap U_k \neq \emptyset$ , we take

- $\hat{\lambda}_j \in \mathcal{C}_c^{\infty}((0, X_0) \times (U_j \cap U_k))$  such that  $\hat{\lambda}_j = 1$  on  $\text{supp}(\phi_j^* \tilde{\zeta}_j^0) \cap \text{supp}(\lambda)$

- $\lambda_j^{(1)}, \lambda_j^{(2)} \in \mathcal{C}_c^\infty((0, X_0) \times (U_j \cap U_k))$  such that  $\lambda_j^{(1)} = \lambda_j^{(2)} = 1$  on  $\text{supp}(\zeta_j^0) \cap \text{supp}(\phi_k^* \tilde{\lambda})$ .

We have

$$(\phi_k^{-1})^* \lambda Q \phi_k^* \tilde{\lambda} = \left( (\phi_k^{-1})^* \lambda \right) (\phi_{jk}^{-1})^* \tilde{Q} (\phi_{jk})^* \left( (\phi_k^{-1})^* \lambda_j^{(1)} \right) \tilde{\lambda}$$

where

$$\tilde{Q} = \left( (\phi_j^{-1})^* \hat{\lambda}_j \right) \text{Op}_{\mathcal{T}}(p_j) \left( (\phi_j^{-1})^* \zeta_j^0 \lambda_j^{(2)} \right)$$

The kernel of the operator  $\tilde{Q}$  has a compact support and we can hence apply the change of variables Theorem B.1. According to Formula (B.4) the symbol of the operator  $(\phi_k^{-1})^* \lambda Q \phi_k^* \tilde{\lambda}$  is given by

$$\begin{aligned} (\phi_k^{-1})^* \lambda \# T_{\phi_{jk}, N} \left( \left( (\phi_j^{-1})^* \hat{\lambda}_j \right) \# p_j \# \left( (\phi_j^{-1})^* \zeta_j^0 \lambda_j^{(2)} \right) \right) \# \left( (\phi_k^{-1})^* \lambda_j^{(1)} \right) \tilde{\lambda} \\ \text{mod } h^{N/2} S_{\mathcal{T}}^{m-N/2}(\mathbb{R}^n \times [0, 2\varepsilon] \times \mathbb{R}^n) \end{aligned}$$

Combining the definition of  $T_{\phi_{jk}, N}$ , the composition formula, and the definition of  $q_j$  we find this symbol to be

$$\begin{aligned} (\phi_k^{-1})^* \lambda \# T_{\phi_{jk}, N} \left( p_j \# (\phi_j^{-1})^* \zeta_j^0 \right) \# \tilde{\lambda} \text{ mod } h^{N/2} S_{\mathcal{T}}^{m-N/2}(\mathbb{R}^n \times [0, 2\varepsilon] \times \mathbb{R}^n) \\ = (\phi_k^{-1})^* \lambda \# T_{\phi_{jk}, N}(q_j) \# \tilde{\lambda} \text{ mod } h^{N/2} S_{\mathcal{T}}^{m-N/2}(\mathbb{R}^n \times [0, 2\varepsilon] \times \mathbb{R}^n), \end{aligned}$$

because of the supports of  $\hat{\lambda}_j$ ,  $\lambda_j^{(1)}$  and  $\lambda_j^{(2)}$ . This proves the third point. Finally, we obtain  $Q \in \Psi_{\mathcal{T}}^m(\mathcal{M}_+)$ , which concludes the proof of the lemma.  $\square$

**Proposition B.14.** *Let  $P = \sum_{j \in J} \phi_j^* \text{Op}_{\mathcal{T}}(p_j) (\phi_j^{-1})^* \zeta_j^0$  with  $p_j = \tilde{\zeta}_j^0 (\phi_j^{-1})^* p$ . Then, we have  $P \in \Psi_{\mathcal{T}}^m(\mathcal{M}_+)$  and its principal symbol is  $\sigma(P)(\mathbf{x}, \xi_0, \eta) = \zeta^0(x_0) p(\mathbf{x}, \xi_0, \eta)$ . Moreover, in each chart  $\tilde{U}_k$ , there exists a (representative of the) local symbol of  $P$  supported in  $\text{supp}(\zeta^0 \phi_k^* p)$ .*

*Proof.* According to Lemma B.13, in the chart  $\tilde{U}_k$ , the local symbol of  $P$  is

$$p_k \# (\phi_k^{-1})^* \zeta_k^0 + \sum_{j \neq k} T_{\phi_{jk}, N} \left( p_j \# (\phi_j^{-1})^* \zeta_j^0 \right) \text{ mod } h^{N/2} S_{\mathcal{T}}^{m-N/2}(\mathbb{R}^n \times [0, 2\varepsilon] \times \mathbb{R}^n) \quad (\text{B.9})$$

for all  $N \in \mathbb{N}$ . According to the composition formula (B.1) and the definition of  $T_{\phi_{jk}, N}$  (B.5), the principal part of this local representation is

$$\begin{aligned} p_k (\phi_k^{-1})^* \zeta_k^0 + \sum_{j \neq k} (\phi_{jk}^{-1})^* \left( p_j (\phi_j^{-1})^* \zeta_j^0 \right) \\ = \tilde{\zeta}_k^0 (\phi_k^{-1})^* p \zeta_k^0 + \sum_{j \neq k} (\phi_{jk}^{-1})^* \tilde{\zeta}_j^0 (\phi_j^{-1})^* p \zeta_j^0 \\ = \left( (\phi_k^{-1})^* p \right) \sum_{j \in J} (\phi_k^{-1})^* \zeta_j^0 = \zeta^0 (\phi_k^{-1})^* p. \end{aligned}$$

since  $\sum_{j \in J} \zeta_j^0 = \zeta^0$ , defined in Section 3.6. Moreover, for every  $N \in \mathbb{N}$ , the expression (B.9) is supported in the support of  $(\phi_k^{-1})^* p$ . This property can be preserved by a representative of the asymptotic series  $N \rightarrow +\infty$ . This concludes the proof of the proposition.  $\square$

With the Sobolev norms introduced in Section 1.4.3 we have the following natural result.

**Lemma B.15.** *Let  $P$  be as in Proposition B.14 and let  $v \in \mathcal{C}^\infty(\mathcal{M}_+)$  and set  $u_j = \text{Op}_{\mathcal{T}}(p_j) (\phi_j^{-1})^* \zeta_j^0 v$ . Then we have*

$$\|Pv\|_\ell \lesssim \sum_{j \in J} \|u_j\|_\ell, \quad |(Pv)|_{x_n=0^+}|_\ell \lesssim \sum_{j \in J} |u_j|_{x_n=0^+}|_\ell, \quad \ell = 0, 1.$$

*Proof.* We treat the case of norms in all dimensions. We have  $Pv = \sum_{j \in J} \phi_j^* \phi_j^* u_j$ . Then

$$\|Pv\|_\ell \leq \sum_{j \in J} \|\phi_j^* u_j\|_\ell.$$

We then conclude with Lemma 1.9.  $\square$

## C Proofs of some technical results

### C.1 Proof of Lemma 1.8

Let  $(g_j)_j$  be a family of smooth functions on  $S$  with  $\text{supp}(g_j) \subset U_j$  and  $\sum_j g_j = g \geq C > 0$  in  $S$ . We set  $M_r(u) = \sum_j |(\phi_j^{-1})^* g_j u|_{\mathcal{H}^r(\mathbb{R}^{n-1})}$ . It is sufficient to prove that  $|(\phi_j^{-1})^* f_j u|_{\mathcal{H}^r(\mathbb{R}^{n-1})} \leq C M_r(u)$  for some constant  $C > 0$ .

We set  $\tilde{g}_j = g_j/g$  which forms a partition of unity. We have

$$|(\phi_j^{-1})^* f_j u|_{\mathcal{H}^r(\mathbb{R}^{n-1})} \leq \sum_k |(\phi_j^{-1})^* f_j \tilde{g}_k u|_{\mathcal{H}^r(\mathbb{R}^{n-1})}$$

Next we write

$$|(\phi_j^{-1})^* f_j \tilde{g}_k u|_{\mathcal{H}^r(\mathbb{R}^{n-1})} \leq C |(\phi_k^{-1})^* f_j \tilde{g}_k u|_{\mathcal{H}^r(\mathbb{R}^{n-1})}$$

as  $\phi_{jk}$  is a  $\mathcal{C}^\infty$ -diffeomorphism between  $\phi_j(U_j \cap U_k)$  and  $\phi_k(U_j \cap U_k)$ . Introducing  $\hat{g}_k \in \mathcal{C}_c^\infty(U_k)$  such that  $\hat{g}_k = 1$  on  $\text{supp}(g_k)$  we find

$$\begin{aligned} |(\phi_j^{-1})^* f_j \tilde{g}_k u|_{\mathcal{H}^r(\mathbb{R}^{n-1})} &\leq C |(\phi_k^{-1})^* f_j \hat{g}_k g_k/g u|_{\mathcal{H}^r(\mathbb{R}^{n-1})} \\ &= C |(\phi_k^{-1})^* (f_j \hat{g}_k/g) (\phi_k^{-1})^* (g_k u)|_{\mathcal{H}^r(\mathbb{R}^{n-1})} \\ &\leq C' |(\phi_k^{-1})^* (g_k u)|_{\mathcal{H}^r(\mathbb{R}^{n-1})} \leq C'' M_r(u), \end{aligned}$$

as  $v \mapsto v(\phi_k^{-1})^* (f_j \hat{g}_k/g)$  is continuous in  $\mathcal{H}^r(\mathbb{R}^{n-1})$ . The proof is complete.  $\square$

### C.2 Proof of Proposition 2.2

First we note that in the proof it suffices to consider the operator  $A_\delta + \lambda \text{Id}$  for  $\lambda$  sufficiently large, in place of  $\mathcal{A}_\delta + \lambda \text{Id}$ . An inspection of the proof that follows also shows that a piecewise  $\mathcal{C}^1$  regularity of the coefficients  $c$  and a  $\mathcal{C}^1$  regularity of  $c^s$  is sufficient to prove the result.

We consider a finite open covering  $(O_j)_j$  of  $\bar{\Omega}$  together with a subordinated partition of unity  $\sum_j \theta_j = 1$  that satisfies moreover, if  $O_j \cap S \neq \emptyset$ ,

1. we can choose local coordinates in  $O_j$  such that  $S$  is given by  $\{x_n = 0\}$ .
2.  $\partial_\eta \theta_j|_S = 0$ , i.e.  $\theta_j$  is flat at  $S$  in the normal direction to  $S$ .

The result of Proposition 2.2 is clear away from  $S$  by standard elliptic regularity theory. We thus place ourselves in  $O = O_j$  such that  $O_j \cap S \neq \emptyset$ . With  $\theta = \theta_j$  we set  $v = \theta z$  and  $v^s = \theta z^s$  and  $V = (v, v^s)$ . From (2.3) we have

$$\|V\|_{\mathcal{H}_\delta^1} \lesssim \|F\|_{\mathcal{H}_\delta^0}. \quad (\text{C.1})$$

The result will be achieved if we prove

$$\sum_{i=1,2} |v|_{\Omega_i} |_{H^2(O \cap \Omega_i)} + \delta^{\frac{1}{2}} |v^s|_{H^2(O \cap S)} \lesssim \|F\|_{\mathcal{H}_\delta^0}, \quad (\text{C.2})$$

uniformly in  $\delta$ .

We write  $c \nabla_g v = c(\nabla_g \theta)z + c\theta(\nabla_g z)$ . If  $\psi \in H_0^1(O)$  we then have

$$\begin{aligned} (c \nabla_g v, \nabla_g \psi)_{L^2(O)} &= (c(\nabla_g \theta)z, \nabla_g \psi)_{L^2(O)} + (c(\nabla_g z)\theta, \nabla_g \psi)_{L^2(O)} \\ &= - \sum_{i=1,2} (\nabla_g(c(\nabla_g \theta)z), \psi)_{L^2(O \cap \Omega_i)} + (c \nabla_g z, \nabla_g(\theta \psi))_{L^2(O)} - (c(\nabla_g z \nabla_g \theta), \psi)_{L^2(O)}, \end{aligned}$$

with an integration by parts using that  $\partial_\eta \theta|_S = 0$ .

Similarly for  $\psi^s \in H_0^1(O \cap S)$  we have

$$\begin{aligned} (c^s \nabla^s v^s, \nabla^s \psi^s)_{L^2(O \cap S)} &= - (\nabla^s(c^s(\nabla^s \theta)z^s), \psi^s)_{L^2(O \cap S)} + (c^s \nabla^s z^s, \nabla^s(\theta \psi^s))_{L^2(O \cap S)} \\ &\quad - (c^s(\nabla^s z^s \nabla^s \theta), \psi^s)_{L^2(O \cap S)}. \end{aligned}$$

Considering the weak problem (2.2) satisfied by  $Z$  we thus obtain

$$(c\nabla_g v, \nabla_g \psi)_{L^2(O)} + \delta (c^s \nabla^s v^s, \nabla^s \psi^s)_{L^2(O \cap S)} + \lambda (v, \psi)_{L^2(O)} + \delta \lambda (v^s, \psi^s)_{L^2(O \cap S)} \\ = (\phi, \psi)_{L^2(O)} + \delta (\phi^s, \psi^s)_{L^2(O \cap S)}, \quad (\text{C.3})$$

for  $\Psi = (\psi, \psi^s) \in \mathcal{H}_\delta^1$ , where  $\Phi = (\phi, \phi^s) \in L^2 \times L^2$  with

$$\phi|_{\Omega_i} = (-\nabla_g(c(\nabla_g \theta)z) - c(\nabla_g z \nabla_g \theta) + \theta f)|_{\Omega_i}, \quad i = 1, 2 \\ \phi^s = -\nabla^s(c^s(\nabla^s \theta)z^s) - c^s(\nabla^s z^s \nabla^s \theta) + \theta f^s,$$

and by (2.3) we have

$$\|\Phi\|_{\mathcal{H}_\delta^0} \lesssim \|F\|_{\mathcal{H}_\delta^0}.$$

We now make a local change of variable in  $O$  such that  $S$  becomes  $\{x_n = 0\}$ . The weak problem that  $(v, v^s)$  satisfies takes the form

$$\sum_{k,l} \int_O \underline{c}_{k,l} \partial_{x_k} v \partial_{x_l} \bar{\psi} dx + \delta \sum'_{k,l} \int_{O \cap S} \underline{c}_{k,l}^s \partial_{x_k} v^s \partial_{x_l} \bar{\psi}^s dx + \lambda \int_O \alpha v \bar{\psi} dx + \lambda \delta \int_{O \cap S} \beta v^s \bar{\psi}^s dx \\ = \int_O \alpha \phi \bar{\psi} dx + \delta \int_{O \cap S} \beta \phi^s \bar{\psi}^s dx, \quad \Psi = (\psi, \psi^s) \in \mathcal{H}_\delta^1, \quad (\text{C.4})$$

where  $\sum_{k,l}$  is a sum with  $k, l$  running over  $\{1, \dots, n\}$  and  $\sum'_{k,l}$  is a sum with  $k, l$  running over  $\{1, \dots, n-1\}$ . The functions  $\alpha$  and  $\beta$  originate from the Jacobians. The functions  $\underline{c}_{k,l}$  are piecewise  $\mathcal{C}^1$  with a discontinuity across the interface  $S$  and the functions  $\underline{c}_{k,l}^s$  are  $\mathcal{C}^1$ . Note that  $v \in H^1(O)$  and  $v^s \in H^1(O \cap S)$  with their supports finitely away from  $\partial O$ .

We now use the Nirenberg translation method. Let  $h$  be parallel to  $S$ . Define  $D_h$  by  $D_h(\rho) = (\rho(x+h) - \rho(x))/|h|$ . Observe that  $D_{-h}(D_h v) \in H_0^1(O)$  and  $D_{-h}(D_h v^s) \in H_0^1(O \cap S)$  for  $|h|$  sufficiently small and set  $\psi = D_{-h}(D_h v)$  and  $\psi^s = D_{-h}(D_h v^s)$ . As  $(D_h f_1 f_2) = f_1(x+h) D_h f_2 + (D_h f_1) f_2$  this yields

$$\sum_{k,l} \int_O \underline{c}_{k,l}(x+h) \partial_{x_k} D_h v \partial_{x_l} D_h \bar{v} dx + \delta \sum'_{k,l} \int_{O \cap S} \underline{c}_{k,l}^s(x+h) \partial_{x_k} D_h v^s \partial_{x_l} D_h \bar{v}^s dx \\ + \lambda \int_O \alpha(x+h) |D_h v|^2 dx + \lambda \delta \int_{O \cap S} \beta(x+h) |D_h v^s|^2 dx \\ + \sum_{k,l} \int_O (D_h \underline{c}_{k,l}) \partial_{x_k} v \partial_{x_l} D_h \bar{v} dx + \delta \sum'_{k,l} \int_{O \cap S} (D_h \underline{c}_{k,l}^s) \partial_{x_k} v^s \partial_{x_l} D_h \bar{v}^s dx \\ + \lambda \int_O (D_h \alpha) v D_h \bar{v} dx + \delta \lambda \int_{O \cap S} (D_h \beta) v^s D_h \bar{v}^s dx \\ = \int_O \alpha \phi D_{-h}(D_h \bar{v}) dx + \delta \int_{O \cap S} \beta \phi^s D_{-h}(D_h \bar{v}^s) dx.$$

We note that

$$\left| \sum_{k,l} \int_O (D_h \underline{c}_{k,l}) \partial_{x_k} v \partial_{x_l} D_h \bar{v} dx + \delta \sum'_{k,l} \int_{O \cap S} (D_h \underline{c}_{k,l}^s) \partial_{x_k} v^s \partial_{x_l} D_h \bar{v}^s dx \right. \\ \left. + \lambda \int_O (D_h \alpha) v D_h \bar{v} dx + \delta \lambda \int_{O \cap S} (D_h \beta) v^s D_h \bar{v}^s dx \right| \\ \lesssim \|V\|_{\mathcal{H}_\delta^1} \|D_h V\|_{\mathcal{H}_\delta^1}.$$

If  $\rho \in H_0^1(O)$  with its support finitely away from the boundary  $\partial O$  then  $|D_h(\rho)|_{L^2(O)} \leq |\nabla \rho|_{L^2(O)}$  for  $|h|$  sufficiently small [Bre83, Proposition IX.3]. We thus have

$$\left| \int_O \alpha \phi D_{-h}(D_h \bar{v}) dx + \delta \int_{O \cap S} \beta \phi^s D_{-h}(D_h \bar{v}^s) dx \right| \lesssim \|\Phi\|_{\mathcal{H}_\delta^0} \|D_h V\|_{\mathcal{H}_\delta^1}.$$

We thus find

$$|a_\lambda(D_h v, D_h v)| \lesssim (\|V\|_{\mathcal{H}_\delta^1} + \|\Phi\|_{\mathcal{H}_\delta^0}) \|D_h V\|_{\mathcal{H}_\delta^1} \lesssim \|\Phi\|_{\mathcal{H}_\delta^0} \|D_h V\|_{\mathcal{H}_\delta^1}.$$

uniformly in  $\delta$ , using (C.1). The coercivity of  $a_\lambda$  gives

$$\|D_h V\|_{\mathcal{H}_\delta^1} \lesssim \|F\|_{\mathcal{H}_\delta^0}.$$

For  $k \in \{1, \dots, n\}$  and  $\psi \in \mathcal{C}_c^\infty(O)$ , we choose  $h$  in the direction of the  $x_l$  coordinate,  $l \in \{1, \dots, n-1\}$ . we then have

$$|(\partial_{x_k} v, D_{-h}\psi)_{L^2(O)}| = |(D_h \partial_{x_k} v, \psi)_{L^2(O)}| \leq \|D_h V\|_{\mathcal{H}_\delta^1} \|\psi\|_{L^2} \lesssim \|F\|_{\mathcal{H}_\delta^0} \|\psi\|_{L^2}.$$

As the l.h.s. converges to  $|(\partial_{x_k} v, \partial_{x_l} \psi)_{L^2(O)}|$  we obtain that  $\partial_{x_k x_l}^2 v \in L^2$  and

$$\|\partial_{x_k x_l}^2 v\|_{L^2} \lesssim \|F\|_{\mathcal{H}_\delta^0}, \quad (k, l) \neq (n, n)$$

Similarly for  $k \in \{1, \dots, n-1\}$  and  $\psi^s \in \mathcal{C}_c^\infty(O \cap S)$ , we choose  $h$  in the direction of the  $x_l$  coordinate,  $l \in \{1, \dots, n-1\}$ . We have

$$\delta^{\frac{1}{2}} |(\partial_{x_k} v^s, D_{-h}\psi^s)_{L^2(O \cap S)}| = \delta^{\frac{1}{2}} |(D_h \partial_{x_k} v^s, \psi^s)_{L^2(O \cap S)}| \leq \|D_h V\|_{\mathcal{H}_\delta^1} \|\psi^s\|_{L^2} \lesssim \|F\|_{\mathcal{H}_\delta^0} \|\psi^s\|_{L^2}.$$

and we obtain  $\partial_{x_k x_l}^2 v^s \in L^2(O)$  and moreover

$$\delta^{\frac{1}{2}} \|\partial_{x_k x_l}^2 v^s\|_{L^2(O)} \lesssim \|F\|_{\mathcal{H}_\delta^0}, \quad k, l \in \{1, \dots, n-1\}$$

From (C.4) observe now that (in a weak sense) we have in  $\Omega_i \cap O$ ,  $i = 1, 2$ ,

$$\partial_{x_n}^2 v = -\frac{1}{c_{nn}} \left( \sum_{(k,l) \neq (n,n)} \partial_{x_k} c_{kl} \partial_{x_l} v + \alpha \phi + (\partial_{x_n} c_{nn}) \partial_{x_n} v - \lambda \alpha v \right).$$

It follows that  $\partial_{x_n}^2 v|_{\Omega_i} \in L^2(\Omega_i)$  and

$$\|\partial_{x_n}^2 v|_{\Omega_i}\|_{L^2(\Omega_i)} \lesssim \|F\|_{\mathcal{H}_\delta^0}, \quad i = 1, 2,$$

which concludes the proof.  $\square$

### C.3 Proof of Proposition 2.3

An inspection of the proof shows that is sufficient to assume that  $c$  is piecewise  $\mathcal{C}^{m+1}$ ,  $c^s$  is  $\mathcal{C}^{m+1}$  and that  $S$  is of class  $\mathcal{C}^{m+2}$ . We proceed by induction. The case  $m = 0$  is treated in Proposition 2.2. Let  $m_0 > 0$ . Assume the result is true for  $0 \leq m \leq m_0 - 1$  and  $f|_{\Omega_i} \in H^{m_0}(\Omega_i)$ ,  $i = 0, 1$ , and  $f^s \in H^{m_0}(S)$ . We thus have  $Z = (z, z^s) \in H^{m_0+1}(\Omega_1 \cup \Omega_2) \times H^{m_0+1}(S)$  with

$$\sum_{i=1,2} \|z|_{\Omega_i}\|_{H^{m_0+1}(\Omega_i)} + \delta^{\frac{1}{2}} |z^s|_{H^{m_0+1}(S)} \lesssim \sum_{i=1,2} \|f|_{\Omega_i}\|_{H^{m_0}(\Omega_i)} + \delta^{\frac{1}{2}} |f^s|_{H^{m_0}(S)} = N_{m_0}(F).$$

We use the same partition of unity  $\theta_j$ ,  $j = 1, \dots, N$ , as in the proof of Proposition 2.2. Since the result is known away from  $S$  by standard elliptic regularity theory. We thus place ourselves in  $O = O_j$  such that  $O_j \cap S \neq \emptyset$ . With  $\theta = \theta_j$  we set  $v = \theta z$  and  $v^s = \theta z^s$  and  $V = (v, v^s)$ . With the notation of the proof of Proposition 2.2 we obtain after a local change of variables

$$\begin{aligned} & \sum_{k,l} \int_O c_{k,l} \partial_{x_k} v \partial_{x_l} \bar{\psi} dx + \delta \sum'_{k,l} \int_{O \cap S} c_{k,l}^s \partial_{x_k} v^s \partial_{x_l} \bar{\psi}^s dx + \lambda \int_O \alpha v \bar{\psi} dx + \delta \lambda \int_{O \cap S} \beta v^s \bar{\psi}^s dx \\ &= \int_O \alpha \phi \bar{\psi} dx + \delta \int_{O \cap S} \beta \phi^s \bar{\psi}^s dx, \end{aligned} \tag{C.5}$$

for  $\Psi = (\psi, \psi^s) \in \mathcal{H}_\delta^1$ , where  $\sum_{k,l}$  is a sum with  $k, l$  running over  $\{1, \dots, n\}$  and  $\sum'_{k,l}$  is a sum with  $k, l$  running over  $\{1, \dots, n-1\}$ . We have  $v \in H^{m_0+1}(O)$  and  $v^s \in H^{m_0+1}(O \cap S)$  with their supports finitely away from  $\partial O$  and  $\Phi = (\phi, \phi^s)$  is such that

$$\sum_{i=1,2} \|\phi|_{\Omega_i}\|_{H^{m_0}(\Omega_i)} + \delta^{\frac{1}{2}} |\phi^s|_{H^{m_0}(S)} \lesssim N_{m_0}(F).$$

The functions  $c_{k,l}$  are piecewise  $\mathcal{C}^{m+1}$  with a discontinuity across the interface  $S$  and the functions  $c_{k,l}^s$  are  $\mathcal{C}^{m+1}$ .

For  $j = 1, \dots, n-1$ , if  $\Psi$  is chosen such that

$$\psi = \partial_{x_j} \tilde{\psi}, \quad \psi^s = \partial_{x_j} \tilde{\psi}^s, \quad \text{with } (\tilde{\psi}, \tilde{\psi}^s) \in (\mathcal{C}_c^\infty(O) \times \mathcal{C}_c^\infty(S \cap O)) \cap \mathcal{H}_\delta^1,$$

as we have  $\partial_{x_j} v|_{\Omega_i} \in H^1(\Omega_i)$  and  $\partial_{x_j} v^s \in H^1(S)$ , we find

$$\begin{aligned} \sum_{k,l} \int_O \underline{c}_{k,l} \partial_{x_k x_j}^2 v \partial_{x_l} \bar{\psi} dx + \delta \sum'_{k,l} \int_{O \cap S} \underline{c}_{k,l}^s \partial_{x_k x_j}^2 v^s \partial_{x_l} \bar{\psi}^s dx + \lambda \int_O \alpha \partial_{x_j} v \bar{\psi} dx + \delta \lambda \int_{O \cap S} \beta \partial_{x_j} v^s \bar{\psi}^s dx \\ = \int_O \tilde{\phi} \bar{\psi} dx + \delta \int_{O \cap S} \tilde{\phi}^s \bar{\psi}^s dx, \end{aligned}$$

with

$$\sum_{i=1,2} \|\tilde{\phi}|_{\Omega_i}\|_{H^{m_0-1}(\Omega_i)} + \delta^{\frac{1}{2}} |\tilde{\phi}^s|_{H^{m_0-1}(S)} \lesssim N_{m_0}(F).$$

The induction assumption then yields<sup>6</sup>

$$\sum_{i=1,2} \|\partial_{x_j} v|_{\Omega_i}\|_{H^{m_0+1}(\Omega_i \cap O)} + \delta^{\frac{1}{2}} |\partial_{x_j} v^s|_{H^{m_0+1}(S \cap O)} \lesssim N_{m_0}(F). \quad (\text{C.6})$$

From (C.5) we have in  $\Omega_i \cap O$ , in a weak sense,  $i = 1, 2$ ,

$$\partial_{x_n}^2 v = -\frac{1}{\underline{c}_{nn}} \left( \sum_{(k,l) \neq (n,n)} \partial_{x_k} \underline{c}_{kl} \partial_{x_l} v + \alpha \phi + (\partial_{x_n} \underline{c}_{nn}) \partial_{x_n} v - \lambda \alpha v \right).$$

Yet, as  $v|_{\Omega_i} \in H^2(\Omega_i)$ , this also holds in  $L^2(\Omega_i)$ . We thus conclude that  $\partial_{x_n}^2 v|_{\Omega_i} \in H^{m_0}(\Omega_i \cap O)$  and

$$\|\partial_{x_n}^2 v|_{\Omega_i}\|_{H^{m_0}(\Omega_i)} \lesssim N_{m_0}(F),$$

by (C.6). This concludes the proof.  $\square$

## C.4 Proof of Lemma 3.6

The proof we give extends that of Lemma 3 page 480 in [LR97]. We drop the “ $\gamma_i$ ” notation here since the same argument holds for both cases. We have  $p_\varphi = \xi_n^2 + 2i(\partial_{x_n} \varphi) \xi_n + q_2 + 2iq_1$ . We set  $\alpha \in \mathbb{C}$  such that  $\alpha^2 = (\partial_{x_n} \varphi)^2 + q_2 + 2iq_1$ . Then the imaginary parts of the two roots of  $p_\varphi$  are  $-\partial_{x_n} \varphi \pm \text{Re}(\alpha)$  and have opposite signs if and only if  $|\text{Re}(\alpha)| > |\partial_{x_n} \varphi|$ . We note that

$$|\text{Re}(z)| > A \quad \Leftrightarrow \quad \text{Re}(z^2) > A^2 - \frac{(\text{Im}(z^2))^2}{4A^2}, \quad z \in \mathbb{C}, \quad (\text{C.7})$$

with a similar equivalence in the case of equalities on both sides. Substituting  $\alpha$  for  $z$ , and  $|\partial_{x_n} \varphi|$  for  $A$ , we thus obtain that the imaginary part of the roots have opposite signs if and only if  $\mu > 0$ , as  $\mu = q_2 + q_1^2 / (\partial_{x_n} \varphi)^2$ . In the case  $\mu = 0$  only one of the roots is real and the imaginary part of the second one is of the opposite sign of  $\partial_{x_n} \varphi$ . In the case  $\mu < 0$  both imaginary parts of the roots have the same sign equal to the opposite sign of  $\partial_{x_n} \varphi$ .

If we have  $\text{Im}(\rho^+) \geq C_0 > 0$  and  $\text{Im}(\rho^-) \leq -C_0$  then  $|\text{Re}(\alpha)| \geq |\partial_{x_n} \varphi| + C_0$  and by (C.7) we obtain

$$(\partial_{x_n} \varphi)^2 + q_2 = \text{Re}(\alpha^2) \geq (|\partial_{x_n} \varphi| + C_0)^2 - \frac{q_1^2}{(|\partial_{x_n} \varphi| + C_0)^2}.$$

which gives

$$\mu \geq C_0^2 + 2C_0 |\partial_{x_n} \varphi| + q_1^2 \left( \frac{1}{(\partial_{x_n} \varphi)^2} - \frac{1}{(|\partial_{x_n} \varphi| + C_0)^2} \right) \geq C > 0.$$

Conversely, let us assume that  $\mu \geq C_1 > 0$ . Note that for all  $M > 0$ , there exists  $R > 0$  such that  $|\xi_0| + |\eta|_g \geq R \Rightarrow |\text{Re}(\alpha)| \geq M$ . Actually, we have

$$\text{Re}(\alpha^2) - M^2 + \frac{\text{Im}(\alpha^2)^2}{4M^2} > 0$$

for  $|\xi_0| + |\eta|_g \geq R \Rightarrow |\text{Re}(\alpha)|$  sufficiently large, which yields  $|\text{Re}(\alpha)| \geq M$ . Taking now  $M = |\partial_{x_n} \varphi| + C$ , we obtain  $|\text{Im}(\rho^\pm)| \geq C$ .

<sup>6</sup>The induction assumption is applied to the local form of the elliptic problem here, i.e., (C.5).

It suffices to take  $|\xi_0| + |\eta|_g \leq R$ ,  $x_0 \in [0, X_0]$ ,  $x_n \in [-2\varepsilon, 2\varepsilon]$ . The variables  $(x_0, y, x_n, \xi_0, \eta)$  such that  $\mu \geq C_1$  are in a compact set  $K$ . Then,  $\min_K |\operatorname{Im}(\rho^\pm)|$  is reached. Finally,  $\mu \geq C_1$  implies  $|\operatorname{Im}(\rho^\pm)| \geq C > 0$  as  $\operatorname{Im}(\rho^\pm)$  does not vanish if  $\mu > 0$ . This concludes the proof of the first part of the lemma.

We now address the last point of the lemma. Let  $0 < l < L < \inf_{V_\varepsilon^+} |\partial_{x_n} \varphi|$  and let  $H = L^2 - l^2$ . We consider the region  $\{\mu \geq -H\}$ . In this region we have

$$\mu \geq l^2 - (\partial_{x_n} \varphi)^2 \geq (l^2 - (\partial_{x_n} \varphi)^2) \left( 1 + \frac{q_1^2}{l^2 (\partial_{x_n} \varphi)^2} \right) = l^2 - (\partial_{x_n} \varphi)^2 + q_1^2 \left( \frac{1}{(\partial_{x_n} \varphi)^2} - \frac{1}{l^2} \right).$$

Since  $\mu = q_2 + q_1^2 / (\partial_{x_n} \varphi)^2$  we then have  $q_2 + (\partial_{x_n} \varphi)^2 \geq l^2 - \frac{q_1^2}{l^2}$  which by (C.7) yields  $|\operatorname{Re}(\alpha)| \geq l$ . We conclude by observing that  $|\rho^+ - \rho^-| \geq |\operatorname{Im} \rho^+ - \operatorname{Im} \rho^-| = 2|\operatorname{Re}(\alpha)|$ .  $\square$

## C.5 Proof of Lemma 3.8

We follow the notation of the proof of Lemma 3.6 above and we drop the “ $r_l$ ” notation here since the same argument holds for both cases. We choose  $\alpha \in \mathbb{C}$  such that  $\alpha^2 = (\partial_{x_n} \varphi)^2 + q_2 + 2iq_1 = r(x, \xi_0, \eta) - r(x, \partial_{x_0} \varphi, d_y \varphi) + 2i\tilde{r}(x, \xi_0, \eta, \partial_{x_0} \varphi, d_y \varphi)$  which yields the roots to be  $-i\partial_{x_n} \varphi \pm i\alpha$ . We set  $\lambda_{\mathcal{T}} = (1 + \xi_0^2 + |\eta|_g^2)^{\frac{1}{2}} \in S_{\mathcal{T}}^1(\mathcal{M}_+^*)$  and write  $(\alpha/\lambda_{\mathcal{T}})^2 = \nu_1 + \nu_2$  with

$$\nu_1 = \frac{r(x, \xi_0, \eta)}{\lambda_{\mathcal{T}}^2} \quad \text{and} \quad \nu_2 = \frac{1}{\lambda_{\mathcal{T}}^2} \left( -r(x, \partial_{x_0} \varphi, d_y \varphi) + 2i\tilde{r}(x, \xi_0, \eta, \partial_{x_0} \varphi, d_y \varphi) \right).$$

To prove the first result, i.e.,  $\chi\rho^\pm \in S_{\mathcal{T}}^1(\mathcal{M}_+^*)$ , it suffices to consider  $\lambda_{\mathcal{T}}$  large, as we already know that the two roots are smooth in  $\operatorname{supp}(\chi)$ . Note that there exists  $L > 0$  such that  $|\nu_1| \geq 2L$ , and  $|\nu_2| \leq L$  for  $\lambda_{\mathcal{T}}$  large, say  $\lambda_{\mathcal{T}} \geq R_1$ . In this region we have  $\operatorname{Re}(\alpha^2/\lambda_{\mathcal{T}}^2) \geq \nu_1 - |\operatorname{Re}(\nu_2)| \geq L$ . In particular,

$$\operatorname{Re}(\alpha/\lambda_{\mathcal{T}}) \geq C > 0. \tag{C.8}$$

If  $\lambda_{\mathcal{T}} \geq R_1$ , we have thus obtained that  $(\alpha/\lambda_{\mathcal{T}})^2$  remains away from a neighborhood of the branch  $\overline{\mathbb{R}}_-$  for the complex square root and we may thus choose  $\alpha/\lambda_{\mathcal{T}} = F((\alpha/\lambda_{\mathcal{T}})^2)$  with  $F = \mathcal{C}^\infty(\mathbb{C})$ . Since  $(\alpha/\lambda_{\mathcal{T}})^2 \in S_{\mathcal{T}}^0(\mathcal{M}_+^*)$ , it follows from Theorem 18.1.10 in [Hör85a] that  $\alpha/\lambda_{\mathcal{T}} \in S_{\mathcal{T}}^0(\mathcal{M}_+^*)$ , for  $\lambda_{\mathcal{T}} \geq R_1$ , and it yields the first conclusion.

Let  $C_0 > 0$  and let us place ourselves in a region  $\{\mu \geq C_0\}$ . By Lemma 3.6 we have  $\operatorname{Im}(\rho^+) \geq C > 0$  and  $\operatorname{Im}(\rho^-) \leq -C$ . By (C.8), we obtain  $|\operatorname{Im}(\rho^\pm)| \geq C\lambda_{\mathcal{T}}$ . Since  $\operatorname{Im}(\rho^+) - \operatorname{Im}(\rho^-) = 2\operatorname{Re}(\alpha)$ , and since  $|\operatorname{Im}(\rho^+) - \operatorname{Im}(\rho^-)| \geq C$ , we obtain the final result with (C.8).  $\square$

## C.6 Proof of Lemma 4.2

Using (3.24) We have

$$\operatorname{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j})(\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r = (\phi_j^{-1})^* \Xi_{\mathcal{G},j} \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r.$$

We then write

$$\operatorname{Op}_{\mathcal{T}}(\chi_{\mathcal{G},j})(\phi_j^{-1})^* \zeta_j \tilde{\Xi}_{\mathcal{G}\mathcal{F}} v^r = (\phi_j^{-1})^* \Xi_{\mathcal{G},j} v^r - (\phi_j^{-1})^* \Xi_{\mathcal{G},j} (1 - \tilde{\Xi}_{\mathcal{G}\mathcal{F}}) v^r.$$

Note that  $u_{\mathcal{G},j} = (\phi_j^{-1})^* \Xi_{\mathcal{G},j} v^r$  by (4.1). We have  $\Xi_{\mathcal{G},j} (1 - \tilde{\Xi}_{\mathcal{G}\mathcal{F}}) \in h^\infty \Psi_{\mathcal{T}}^{-\infty}(\mathcal{M}_+)$  as their local symbols in every chart have disjoint supports by Proposition B.14, because of the supports of  $\zeta^3$  and  $\tilde{\chi}_{\mathcal{G}\mathcal{F}}$ . This concludes the proof.  $\square$

## C.7 Proof of Lemma 4.5

Here, all functions are evaluated at the interface, i.e.  $x_n = 0^+$ . From (4.67) we have

$$\tilde{\sigma}_\delta = \tilde{\sigma}_\delta^{(0)} + \tilde{r}$$

with

$$\tilde{\sigma}_\delta^{(0)} = 2\partial_{x_n} \varphi_j^r |\sigma_\delta|^2 + 4q_{1,j}^r \operatorname{Re}(\sigma_\delta) - 2(\partial_{x_n} \varphi_j^r) q_{2,j}^r$$

and  $\tilde{r} \in \frac{\delta^2}{h} S_{\mathcal{T}}^3 + \delta S_{\mathcal{T}}^2 + h S_{\mathcal{T}}^1$ , according to the definitions of  $\tilde{\Sigma}_\delta$  in (4.69), of  $\sigma_\delta$  in (4.67), and  $B_1^r, B_1^{r'}, B_2^r$  in (4.59). Observe that  $\hat{\chi} \frac{h^2}{\delta^2 + h^2} \tilde{\sigma}_\delta^{(0)} \in S_{\mathcal{T}}^0$  and that the remainder satisfies

$$\hat{\chi} \frac{h^2}{\delta^2 + h^2} \tilde{r} \in h S_{\mathcal{T}}^0, \quad (\text{C.9})$$

since  $\langle (\xi_0, \xi') \rangle \leq C$  in  $\text{supp}(\hat{\chi})$ .

Now, let us produce a lower bound for the symbol  $\tilde{\sigma}_\delta^{(0)}$ . Recalling the definition of  $\mu^r$  in (3.12), denoting  $\mu_j^r = (\phi_j^{-1})^* \mu^r$  and  $\vartheta = -i \frac{\delta c_j^s}{h c_j^l} p_{\varphi, j}^s$ , we find

$$\begin{aligned} \tilde{\sigma}_\delta^{(0)} &= \frac{2}{\beta^2} (\partial_{x_n} \varphi_j^r) |\vartheta - \rho_j^{l,+} + k|^2 + \frac{4}{\beta} \text{Re}(\vartheta - \rho_j^{l,+} + k) q_{1,j}^r - 2(\partial_{x_n} \varphi_j^r) q_{2,j}^r \\ &= 2\partial_{x_n} \varphi_j^r \left( |\vartheta - \rho_j^{l,+} + k|/\beta + q_{1,j}^r / \partial_{x_n} \varphi_j^r \right)^2 - \mu_j^r \\ &\geq 2\partial_{x_n} \varphi_j^r \left( \left( \text{Im}(-\vartheta + \rho_j^{l,+} - k)/\beta \right)^2 - \mu_j^r \right), \end{aligned}$$

since  $\partial_{x_n} \varphi_j^r \geq C > 0$  and  $q_{1,j}^r$  is real. We hence have

$$\text{Im}(-\vartheta + \rho_j^{l,+} - k) = \frac{\delta c_j^s}{h c_j^l} \text{Re} p_\varphi^s + (\text{Im} \rho_j^{l,+} - \text{Im} \rho_j^{l,-})/2 + \beta \partial_{x_n} \varphi_j^r,$$

as (4.10) gives  $-\text{Im} k = \partial_{x_n} \varphi_j^l + \beta \partial_{x_n} \varphi_j^r$  and the properties of the roots of the polynomial  $p_\varphi^{l,+}$  given in (3.10) yield  $\partial_{x_n} \varphi_j^l = -(\text{Im} \rho_j^{l,+} + \text{Im} \rho_j^{l,-})/2$ . The first point of Lemma 3.6 gives  $\text{Im} \rho_j^{l,+} - \text{Im} \rho_j^{l,-} \geq 0$ , and we thus obtain

$$\text{Im}(-\vartheta + \rho_j^{l,+} - k) \geq K_2 \delta / h + \beta \partial_{x_n} \varphi_j^r,$$

since in the present region,  $\text{Re}(p_{\varphi, j}^s)$  is positive elliptic by Proposition 3.5 and the form of (3.11). Using condition (4.51), i.e.,  $(\partial_{x_n} \varphi_j^r)^2 - \mu_j^r \geq K_1 > 0$  we find

$$\tilde{\sigma}_\delta^{(0)} \geq 2\partial_{x_n} \varphi_j^r \left( \frac{\delta^2 K_2^2}{h^2 \beta^2} + K_1 \right).$$

This, together with (C.9) concludes the proof.  $\square$

## C.8 Proof of Lemma 4.7

Let  $X = (x_0, x', \xi_0, \xi') \in W$  and  $\tilde{X} = (\tilde{x}_0, \tilde{x}', \tilde{\xi}_0, \tilde{\xi}') \in W$ . If

$$g_{W, X}(X - \tilde{X}) = |(x_0, x') - (\tilde{x}_0, \tilde{x}')|^2 + \frac{|(\xi_0, \xi') - (\tilde{\xi}_0, \tilde{\xi}')|^2}{\langle (\xi_0, \xi') \rangle^2} < r^2,$$

then, for  $r$  sufficiently small, we have  $C^{-1} \leq \frac{\langle (\tilde{\xi}_0, \tilde{\xi}') \rangle}{\langle (\xi_0, \xi') \rangle} \leq C$  for some  $C > 0$ . As a consequence, we obtain  $C^{-1} \leq \frac{\Lambda(\tilde{X})}{\Lambda(X)} \leq C$  with  $h, \delta > 0$  arbitrary. Hence  $\Lambda$  is slowly varying.

Next, we have

$$\langle (\tilde{\xi}_0, \tilde{\xi}') \rangle^2 \lesssim \langle (\xi_0, \xi') \rangle^2 (1 + |(\xi_0, \xi') - (\tilde{\xi}_0, \tilde{\xi}')|^2)$$

so that

$$\frac{\Lambda(\tilde{X})^2}{\Lambda(X)^2} \lesssim (1 + |(\xi_0, \xi') - (\tilde{\xi}_0, \tilde{\xi}')|^2) \lesssim (1 + g_{W, X}^\sigma(X - \tilde{X})),$$

for  $h, \delta > 0$  arbitrary. Here  $g_{W, X}^\sigma$  denotes the dual metric on  $W$ ,  $g_{W, X}^\sigma = \langle (\xi_0, \xi') \rangle^2 |d(x_0, x')|^2 + |d(\xi_0, \xi')|^2$ . Hence, the order function  $\Lambda$  is temperate, which concludes the proof.  $\square$



## C.9 Proof of Theorem B.1: change of variables for semi-classical operators

Here we consider operators on the whole space  $\mathbb{R}^n$  of the form

$$a(x, D_x, \tau) = u(x) = \iint e^{i\langle x-y, \xi \rangle} a(x, \xi, \tau) u(y) dy d\xi, \quad d\xi = (2\pi)^{-n} d\xi, \quad (\text{C.10})$$

where  $a(x, \xi, \tau)$  is smooth in  $x$  and  $\xi$  and satisfies for some  $m \in \mathbb{R}$ ,

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, \tau)| \leq C_{\alpha, \beta} \mu^{m-|\beta|}, \quad \mu^2 = \tau^2 + |\xi|^2, \quad \tau \geq 1. \quad (\text{C.11})$$

We say that  $a \in S(\mu^m)$ . We shall prove a change of variables formula for this kind of operators. We choose this form of operator to make use of parts of existing proofs. Operators of the form (C.10) are also called semi-classical.

We recall that the semi-classical operators we consider in the main part of the article, i.e., with a small parameter  $h$ , can be put in the form (C.10). In fact, with  $a(x, \xi, h) \in S^m$ , we write

$$\text{op}(a)u(x) = (2\pi h)^{-n} \iint e^{i\langle x-y, \xi \rangle/h} a(x, \xi, h) u(y) dy d\xi = \iint e^{i\langle x-y, \xi \rangle} a(x, h\xi, h) u(y) dy d\xi,$$

and we have  $|\partial_x^\alpha \partial_\xi^\beta a(x, h\xi, h)| \lesssim h^{|\beta|} \langle h\xi \rangle^{m-|\beta|}$ . With  $\tau = 1/h$  we find

$$|\partial_x^\alpha \partial_\xi^\beta \tau^m a(x, h\xi, h)| \lesssim \tau^{m-|\beta|} (1 + |\xi|/\tau)^{m-|\beta|} \lesssim \mu^{m-|\beta|}.$$

Hence, the symbol  $h^{-m} a(x, h\xi, h)$  satisfies (C.11).

Theorem B.1 is the translation for semi-classical tangential operators with a small parameter  $h$  of the following theorem.

**Theorem C.1.** *Let  $X$  and  $X_\kappa$  be open subsets of  $\mathbb{R}^n$  and let  $\kappa : X \rightarrow X_\kappa$  be a diffeomorphism. If  $a \in S(\mu^m)$  and the kernel of  $a(x, D_x, \tau)$  has compact support in  $X \times X$  then the function*

$$a_\kappa(y, \eta, \tau) = \begin{cases} e^{-i\langle \kappa(x), \eta \rangle} a(x, D_x, \tau) e^{i\langle \kappa(x), \eta \rangle} & \text{if } y = \kappa(x) \in X_\kappa, \\ 0 & \text{if } y \notin X_\kappa, \end{cases} \quad (\text{C.12})$$

is in  $S(\mu^m)$ , the kernel of  $a_\kappa(x, D_x, \tau)$  has compact support in  $X_\kappa \times X_\kappa$ , and

$$(a_\kappa(x, D_x, \tau)u) \circ \kappa = a(x, D_x, \tau)(u \circ \kappa), \quad u \in \mathcal{S}'(\mathbb{R}^n). \quad (\text{C.13})$$

For  $a_\kappa$  we have the following asymptotic expansion

$$a_\kappa(\kappa(x), \eta, \tau) - \sum_{\alpha < N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x, {}^t\kappa(x)'\eta, \tau) \partial_y^\alpha e^{i\langle \rho_x(y), \eta \rangle} \Big|_{y=x} \in S(\mu^{m-N/2}), \quad (\text{C.14})$$

where  $\rho_x(y) = \kappa(y) - \kappa(x) - \kappa'(x)(y-x)$ .

Note that  $\rho_x(y)$  vanishes at second order at  $x$  and that the terms in the series are in  $S(\mu^{m-|\alpha|/2})$ . In fact the order of each term in the asymptotic series (C.14) is explained by the following result that we shall use below.

**Lemma C.2.** *We can write  $\partial_y^\alpha e^{i\langle \rho_x(y), \eta \rangle}$  as a linear combination of terms*

$$\prod_{j \in \mathcal{I}} \langle x-y, \rho_{x,j}(y)\eta \rangle \prod_{j \in \mathcal{J}} \langle \partial_y^{\alpha_j} \rho_x(y), \eta \rangle e^{i\langle \rho_x(y), \eta \rangle},$$

for some matrix-valued function  $\rho_{x,j}$ ,  $j \in \mathcal{I} \cup \mathcal{J}$ , with  $|\alpha_j| \geq 2$  if  $j \in \mathcal{J}$ ,  $k = |\mathcal{I}|$  and  $\ell = |\mathcal{I}| + |\mathcal{J}|$  such that  $k \leq \ell \leq |\alpha|$  and  $\ell \leq \frac{|\alpha|+k}{2}$ . In particular,  $|\partial_y^\alpha e^{i\langle \rho_x(y), \eta \rangle} \Big|_{x=y}| \leq C_\alpha \langle \eta \rangle^{\frac{\alpha}{2}}$ .

*Proof.* We note that  $\partial_y^\alpha (e^{i\langle \rho_x(y), \eta \rangle})$  can be written linear combination of terms of the form

$$e^{i\langle \rho_x(y), \eta \rangle} \prod_{1 \leq j \leq p} \langle \partial_y^{\alpha_j} \rho_x(y), \eta \rangle, \quad \text{with} \quad \sum_{1 \leq j \leq p} |\alpha_j| = |\alpha|, \quad p \leq |\alpha|, \quad |\alpha_j| \geq 1.$$

We set  $\mathcal{I} = \{1 \leq j \leq p; |\alpha_j| = 1\}$  and  $\mathcal{J} = \{1 \leq j \leq p; |\alpha_j| \geq 2\}$ . We have  $|\mathcal{I}| + |\mathcal{J}| = p \leq |\alpha|$  and moreover  $|\alpha| \geq |\mathcal{I}| + 2|\mathcal{J}|$ , which gives  $|\mathcal{I}| + |\mathcal{J}| \leq (|\alpha| + |\mathcal{I}|)/2$ . As  $\rho_x(y)$  vanishes at second order at  $y = x$  we obtain  $\langle \partial_y^{\alpha_j} \rho_x(y), \eta \rangle = \langle x-y, \rho_{x,j}(y)\eta \rangle$  for some function  $\rho_{x,j}$  if  $j \in \mathcal{I}$ .  $\square$

*Proof of Theorem C.1.* Let the kernel of  $a(x, D_x, \tau)$  be supported in  $K \times K$ ,  $K \subset X$ , compact. In particular  $a(x, \xi, \tau) = 0$  if  $x \notin K$ . Let  $\phi \in \mathcal{C}_c^\infty(X)$  be such that  $\phi = 1$  in a neighborhood of  $K$ , and  $\tilde{\phi} \in \mathcal{C}_c^\infty(X)$  be such that  $\tilde{\phi} = 1$  in a neighborhood of  $\text{supp}(\phi)$ . Here, we follow the proof of Theorem 18.1.17 in [Hör85a], and we first obtain that for  $\tau$  fixed formula (C.13) holds for  $a_\kappa$  given by (C.12). Moreover  $a_\kappa$  is smooth w.r.t.  $x$  and  $\xi$  and we have

$$a_\kappa(\kappa(x), \xi) = \phi(x) \iint e^{i\langle x-y, \xi \rangle + i\langle \kappa(y) - \kappa(x), \eta \rangle} a(x, \xi, \tau) \tilde{\phi}(y) dy d\xi, \quad x \in X. \quad (\text{C.15})$$

It thus remains to prove that  $a_\kappa \in S(\mu^m)$  and that the asymptotic representation (C.14) holds.

For the proof we shall distinguish two regimes:  $\tau \lesssim |\eta|$  and  $\tau \gtrsim |\eta|$ . We thus introduce  $w \in \mathcal{C}_c^\infty(\mathbb{R})$  such that  $w = 1$  in a neighborhood of 0 and set

$$\gamma_1(x, \eta, \tau) = w(\tau/\langle \eta \rangle) a_\kappa(\kappa(x), \eta, \tau), \quad \gamma_2(x, \eta, \tau) = (1-w)(\tau/\langle \eta \rangle) a_\kappa(\kappa(x), \eta, \tau).$$

We shall prove the following two propositions below.

**Proposition C.3.** *We have  $\gamma_1(x, \eta, \tau) \in S(\mu^m)$  and*

$$\gamma_1(x, \eta, \tau) - w(\tau/\langle \eta \rangle) \sum_{\alpha < N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x, {}^t\kappa(x)'\eta) \partial_y^\alpha e^{i\langle \rho_x(y), \eta \rangle} \Big|_{y=x} \in S(\mu^{m-N/2}). \quad (\text{C.16})$$

**Proposition C.4.** *We have  $\gamma_2(x, \eta, \tau) \in S(\mu^m)$  and*

$$\gamma_2(x, \eta, \tau) - (1-w)(\tau/\langle \eta \rangle) \sum_{\alpha < N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x, {}^t\kappa(x)'\eta) \partial_y^\alpha e^{i\langle \rho_x(y), \eta \rangle} \Big|_{y=x} \in S(\mu^{m-N/2}). \quad (\text{C.17})$$

With these two results the proof of Theorem C.1 clearly follows as  $\kappa$  is a diffeomorphism.  $\square$

We shall need the following result in the course of the proofs, which is the counterpart of Proposition 18.1.4 in [Hör85a] for semi-classical symbols.

**Lemma C.5.** *Let  $a_j(x, \xi, \tau) \in S(\mu^{m_j})$ ,  $j \in N$ , with  $m_j \rightarrow -\infty$  as  $j \rightarrow \infty$ . Let  $a(x, \xi, \tau)$  be smooth with respect to  $x$  and  $\xi$  such that for all  $\alpha, \beta$  for some  $C > 0$  and  $\nu$  depending on  $\alpha$  and  $\beta$*

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, \tau)| \leq C \mu^\nu, \quad x, \xi \in \mathbb{R}^n, \quad \tau \geq 1. \quad (\text{C.18})$$

Assume there is a sequence  $\nu_k \rightarrow -\infty$  such that

$$|a(x, \xi, \tau) - \sum_{j < k} a_j(x, \xi, \tau)| \leq C_k \mu^{\nu_k}, \quad x, \xi \in \mathbb{R}^n, \quad \tau \geq 1, \quad (\text{C.19})$$

then  $a \in S(\mu^m)$ ,  $m = \sup m_j$ , and  $a(x, \xi, \tau) - \sum_{j < k} a_j(x, \xi, \tau) \in S(\mu^{m'_k})$ , with  $m'_k = \max_{j \geq k} m_j$ .

The proof of lemma C.5 is similar to that of Proposition 18.1.4 in [Hör85a]. It is left to the reader.

**Proof of Proposition C.3.** We have

$$\begin{aligned} \gamma_1(x, \eta, \tau) &= w(\tau/\langle \eta \rangle) a_\kappa(\kappa(x), \xi) \\ &= w(\tau/\langle \eta \rangle) \phi(x) \iint e^{i\langle x-y, \xi \rangle + i\langle \kappa(y) - \kappa(x), \eta \rangle} a(x, \xi, \tau) \tilde{\phi}(y) dy d\xi, \quad x \in X. \end{aligned} \quad (\text{C.20})$$

Let  $C_0$  be such that  $\max(|\kappa'(y)|, |\kappa'(y)^{-1}|) \leq C_0$ . Setting  $\Phi(\xi, \eta) = \int e^{i\langle \kappa(y), \eta \rangle - i\langle y, \xi \rangle} \tilde{\phi}(y) dy$ , one obtains through a non-stationary phase argument [Hör85a, page 82]

$$|\Phi(\xi, \eta)| \leq C_N (1 + |\xi| + |\eta|)^{-N}, \quad \text{if } |\xi| \leq \frac{|\eta|}{2C_0} \text{ or } |\xi| \geq 2C_0 |\eta|. \quad (\text{C.21})$$

Let then  $\chi(\xi) \in \mathcal{C}_c^\infty(\mathbb{R}^n)$  be equal to 1 if  $(2C_0)^{-1} < |\xi| < 2C_0$  and equal to 0 if  $|\xi| < (4C_0)^{-1}$  and let us write  $\gamma_1 = I_1 + I_2$  with

$$\begin{aligned} I_1(x, \eta, \tau) &= w(\tau/\langle \eta \rangle) \phi(x) \iint e^{i\langle x-y, \xi \rangle + i\langle \kappa(y) - \kappa(x), \eta \rangle} a(x, \xi, \tau) \tilde{\phi}(y) (1 - \chi)(\xi/|\eta|) dy d\xi \\ &= w(\tau/\langle \eta \rangle) \phi(x) e^{-i\langle \kappa(x), \eta \rangle} \int e^{i\langle x, \xi \rangle} a(x, \xi, \tau) \Phi(\xi, \eta) (1 - \chi)(\xi/|\eta|) d\xi, \end{aligned}$$

and

$$\begin{aligned} I_2 &= w(\tau/\langle\eta\rangle)\phi(x) \iint e^{i\langle x-y,\xi\rangle+i\langle\kappa(y)-\kappa(x),\eta\rangle} a(x,\xi,\tau)\tilde{\phi}(y)\chi(\xi/|\eta|)dyd\xi \\ &= w(\tau/\langle\eta\rangle)\phi(x)\omega^n e^{-i\langle\kappa(x),\eta\rangle} \iint e^{i\omega\langle x-y,\xi\rangle+i\omega\langle\kappa(y),\eta/\omega\rangle} a(x,\omega\xi,\tau)\tilde{\phi}(y)\chi(\xi)d\eta d\xi, \quad \omega = |\eta|. \end{aligned}$$

With (C.21) and as  $\tau \lesssim |\eta|$  here we find

$$|I_1(x,\eta,\tau)| \leq C_N |w(\tau/\langle\eta\rangle)| \int (\tau + |\xi|)^m (1 + |\xi| + |\eta|)^{-(N+n+1+m)} d\xi, \quad N \in \mathbb{N}.$$

which gives

$$|I_1(x,\eta,\tau)| \leq \begin{cases} C_N |w(\tau/\langle\eta\rangle)| \int (|\eta| + |\xi|)^m (1 + |\xi| + |\eta|)^{-(N+n+1+m)} d\xi & \text{if } m \geq 0, \\ C_N |w(\tau/\langle\eta\rangle)| \int (1 + |\xi|)^m (1 + |\xi| + |\eta|)^{-(N+n+1+m)} d\xi, & \text{if } m < 0. \end{cases}$$

In any case we find

$$|I_1(x,\eta,\tau)| \leq C'_N \frac{|w(\tau/\langle\eta\rangle)|}{(1 + |\eta|)^N} \leq C'_N \frac{1}{(\tau + |\eta|)^N}, \quad N \in \mathbb{N}. \quad (\text{C.22})$$

For the term  $I_2$  we first write

$$I_2 = w(\tau/\langle\eta\rangle)\phi(x)\omega^n e^{-i\langle\kappa(x),\eta\rangle} \iint e^{i\omega\langle\kappa(x+y),\eta/\omega\rangle - \langle y,\xi\rangle} a(x,\omega\xi,\tau)\tilde{\phi}(x+y)\chi(\xi)dyd\xi, \quad \omega = |\eta|.$$

to apply the stationary-phase result of Theorem 7.7.7 in [Hör90], which yields for  $k \geq n$ :

$$\begin{aligned} & \left| I_2(x,\eta,\tau) - w(\tau/\langle\eta\rangle)\phi(x) \sum_{\nu=0}^{k-n} \frac{(-i)^\nu}{\nu!} \langle \partial_y, \partial_\xi / \omega \rangle^\nu \left( e^{i\omega\langle \rho_x(x+y), \eta/\omega \rangle} a(x,\omega\xi,\tau)\tilde{\phi}(x+y)\chi(\xi) \Big|_{y=0, \xi=t\kappa'(x)\eta/\omega} \right) \right| \\ & \leq C\omega^{(n-k)/2} w(\tau/\langle\eta\rangle) \sum_{|\alpha| \leq 2k} \sup_{y,\xi} |D_{y,\xi}^\alpha (\tilde{\phi}(x+y)\chi(\xi)a(x,\omega\xi,\tau))| \end{aligned}$$

As  $\tau \lesssim \omega$ , and  $\xi$  is bounded, we observe that

$$|\omega^\alpha (D_\xi^\alpha a)(x,\omega\xi)| \lesssim \omega^{|\alpha|} (\tau + \omega|\xi|)^{m-|\alpha|} \lesssim (\tau + \omega)^m.$$

We also have  $\chi(\xi) = 1$  in a neighborhood of  ${}^t\kappa'(x)\eta/\omega$ . As  $\phi = 1$  and  $\tilde{\phi} = 1$  in a neighborhood of  $K$  we thus obtain

$$\begin{aligned} & \left| I_2(x,\eta,\tau) - w(\tau/\langle\eta\rangle) \sum_{\nu=0}^{k-n} \frac{(-i)^\nu}{\nu!} \langle \partial_y, \partial_\xi \rangle^\nu \left( e^{i\langle \rho_x(y), \eta \rangle} a(x,\xi,\tau) \Big|_{y=x, \xi=t\kappa'(x)\eta/\omega} \right) \right| \\ & \leq C\omega^{(n-k)/2} (\tau + \omega)^m w(\tau/\langle\eta\rangle) \lesssim (\tau + |\eta|)^{m+\frac{n}{2}-\frac{k}{2}}, \quad x \in K. \end{aligned}$$

We have thus obtained an asymptotic development in the form of (C.19). As each term in the series is also a semi-classical symbol, by (C.22) we find that an estimate of the form (C.18) is achieved when no derivation is applied to  $\gamma_1$ . Applying partial derivatives w.r.t.  $x$  and  $\eta$  to  $\gamma_1(x,\eta,\tau)$  results in a sum of terms with the same form as (C.20) with additional expressions with at most polynomial growth in  $\eta$ . The analysis carried out above also yields an estimate of the form (C.18). With Lemma C.5 this completes the proof.  $\square$

**Proof of Proposition C.4.** We have

$$\begin{aligned} \gamma_2(x,\eta,\tau) &= (1-w)(\tau/\langle\eta\rangle)a_\kappa(\kappa(x),\xi) \\ &= (1-w)(\tau/\langle\eta\rangle)\phi(x) \iint e^{i\langle x-y,\xi\rangle+i\langle\kappa(y)-\kappa(x),\eta\rangle} a(x,\xi,\tau)\tilde{\phi}(y)dyd\xi, \quad x \in X. \end{aligned} \quad (\text{C.23})$$

This representation is to be understood in the sense of oscillatory integrals, which justifies the manipulations we perform below.

In the support of  $(1-w)(\tau/\langle\eta\rangle)$  we have  $\tau \gtrsim |\eta|$ . As  $\rho_x(y) = \kappa(y) - \kappa(x) - \kappa'(x)(y-x)$  we write

$$\begin{aligned} \gamma_2(x,\eta,\tau) &= (1-w)(\tau/\langle\eta\rangle)\phi(x) \iint e^{i\langle x-y,\xi-t\kappa'(x)\eta\rangle+i\langle\rho_x(y),\eta\rangle} a(x,\xi,\tau)\tilde{\phi}(y)dyd\xi \\ &= (1-w)(\tau/\langle\eta\rangle)\phi(x) \iint e^{i\langle x-y,\xi\rangle+i\langle\rho_x(y),\eta\rangle} a(x,\xi+{}^t\kappa'(x)\eta,\tau)\tilde{\phi}(y)dyd\xi, \end{aligned}$$

which by the Taylor formula gives  $\gamma_2 = \gamma_{2,N} + r_N$  with

$$\gamma_{2,N}(x,\eta,\tau) = \sum_{|\alpha| < N} \frac{1}{\alpha!} (1-w)(\tau/\langle\eta\rangle)\phi(x) \iint e^{i\langle x-y,\xi\rangle+i\langle\rho_x(y),\eta\rangle} \xi^\alpha \partial_\xi^\alpha a(x,{}^t\kappa'(x)\eta,\tau)\tilde{\phi}(y) dyd\xi,$$

and

$$r_N = N \sum_{|\alpha|=N} \int_0^1 \frac{(1-\sigma)^{N-1}}{\alpha!} \iint e^{i\langle x-y, \xi \rangle + i\langle \rho_x(y), \eta \rangle} \xi^\alpha \partial_\xi^\alpha a(x, \sigma\xi + {}^t\kappa'(x)\eta, \tau) \tilde{\phi}(y) dy d\xi d\sigma.$$

Observing that  $\xi^\alpha e^{i\langle x-y, \xi \rangle} = i^{|\alpha|} \partial_y^\alpha e^{i\langle x-y, \xi \rangle}$  we find

$$\begin{aligned} \gamma_2(x, \eta, \tau) &= (1-w)(\tau/\langle \eta \rangle) \phi(x) \sum_{|\alpha| < N} \frac{(-i)^{|\alpha|}}{\alpha!} \iint e^{i\langle x-y, \xi \rangle} \partial_y^\alpha (\tilde{\phi}(y) e^{i\langle \rho_x(y), \eta \rangle}) \partial_\xi^\alpha a(x, {}^t\kappa'(x)\eta, \tau) dy d\xi \\ &= (1-w)(\tau/\langle \eta \rangle) \phi(x) \sum_{|\alpha| < N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_y^\alpha (\tilde{\phi}(y) e^{i\langle \rho_x(y), \eta \rangle}) \Big|_{y=x} \partial_\xi^\alpha a(x, {}^t\kappa'(x)\eta, \tau) \\ &= (1-w)(\tau/\langle \eta \rangle) \sum_{|\alpha| < N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_y^\alpha (e^{i\langle \rho_x(y), \eta \rangle}) \Big|_{y=x} \partial_\xi^\alpha a(x, {}^t\kappa'(x)\eta, \tau), \end{aligned}$$

for  $x \in K$ , because of the supports of  $\phi$  and  $\tilde{\phi}$ . From the properties of  $\rho_y(x)$  given in Lemma C.2 each term in the sum is in  $S(\mu^{m-|\alpha|/2})$ . Similarly we have

$$\begin{aligned} r_N(x, \eta, \tau) &= (1-w)(\tau/\langle \eta \rangle) \phi(x) N (-i)^N \sum_{|\alpha|=N} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_0^1 \frac{(1-\sigma)^{N-1}}{\alpha!} \\ &\quad \times \iint e^{i\langle x-y, \xi \rangle} (\partial_y^{\beta-\alpha} \tilde{\phi}(y)) (\partial_y^\beta e^{i\langle \rho_x(y), \eta \rangle}) \partial_\xi^\alpha a(x, \sigma\xi + {}^t\kappa'(x)\eta, \tau) dy d\xi d\sigma. \quad (\text{C.24}) \end{aligned}$$

If we prove that  $r_N \lesssim (\tau + |\eta|)^{m+n+1-N/2}$  if  $N \geq m$ , we then obtain an estimate of the form (C.19). In particular this yield  $|\gamma_2| \lesssim \mu^\nu$  for some  $\nu \in \mathbb{R}$ .

Applying partial derivatives w.r.t.  $x$  and  $\eta$  to  $\gamma_2(x, \eta, \tau)$  results in a sum of terms with the same form as (C.23) with additional expressions with at most polynomial growth in  $\eta$ . Computing  $\partial_x^\alpha \partial_\eta^\beta \gamma_2$  we may apply a similar analysis and find  $|\partial_x^\alpha \partial_\eta^\beta \gamma_2| \lesssim \mu^\nu$  for some  $\nu \in \mathbb{R}$ . We thus have an estimate of the form (C.18). With Lemma C.5 this will complete the proof.

With Lemma C.2 the remainder term  $r_N$  in (C.24) is a linear combination of terms of the form

$$\begin{aligned} r'_N(x, \eta, \tau) &= (1-w)(\tau/\langle \eta \rangle) \phi(x) \int_0^1 (1-\sigma)^{N-1} \iint e^{i\langle x-y, \xi \rangle} \hat{\phi}(y) \prod_{j \in \mathcal{J}} \langle \partial_y^{\alpha_j} \rho_x(y), \eta \rangle e^{i\langle \rho_x(y), \eta \rangle} \\ &\quad \times \prod_{j \in \mathcal{I}} \langle x-y, \rho_{x,j}(y) \eta \rangle \partial_\xi^\alpha a(x, \sigma\xi + {}^t\kappa'(x)\eta, \tau) dy d\xi d\sigma, \end{aligned}$$

with  $|\alpha_j| \geq 2$  if  $j \in \mathcal{J}$  and  $k = |\mathcal{I}|$  and  $\ell = |\mathcal{I}| + |\mathcal{J}|$  such that

$$k \leq \ell \leq |\beta| \leq |\alpha| = N, \quad \ell \leq \frac{|\beta| + k}{2}. \quad (\text{C.25})$$

Here the function  $\hat{\phi}$  has support in  $K$  and is constant on  $\text{supp}(\phi)$ .

As  $\langle x-y, \rho_{x,j}(y) \eta \rangle e^{i\langle x-y, \xi \rangle} = -i \langle \partial_\xi, \rho_{x,j}(y) \eta \rangle e^{i\langle x-y, \xi \rangle}$  we obtain

$$\begin{aligned} r'_N(x, \eta, \tau) &= i^k (1-w)(\tau/\langle \eta \rangle) \phi(x) \int_0^1 (1-\sigma)^{N-1} \iint e^{i\langle x-y, \xi \rangle} \hat{\phi}(y) \prod_{j \in \mathcal{J}} \langle \partial_y^{\alpha_j} \rho_x(y), \eta \rangle e^{i\langle \rho_x(y), \eta \rangle} \\ &\quad \times \prod_{j \in \mathcal{I}} \langle \partial_\xi, \rho_{x,j}(y) \eta \rangle \partial_\xi^\alpha a(x, \sigma\xi + {}^t\kappa'(x)\eta, \tau) dy d\xi d\sigma, \end{aligned}$$

and we may thus write  $r_N$  as a linear combination of terms of the following form

$$\begin{aligned} r''_N(x, \eta, \tau) &= (1-w)(\tau/\langle \eta \rangle) \phi(x) \int_0^1 \sigma^{|\gamma|} (1-\sigma)^{N-1} \iint e^{i\langle x-y, \xi \rangle} \hat{\phi}(y) p(x, y, \eta) e^{i\langle \rho_x(y), \eta \rangle} \\ &\quad \times \partial_\xi^{\alpha+\gamma} a(x, \sigma\xi + {}^t\kappa'(x)\eta, \tau) dy d\xi d\sigma, \end{aligned}$$

where  $|\gamma| = k$  and  $p(x, y, \eta)$  is a polynomial in  $\eta$  of order  $\ell$  with smooth coefficients.

We note that  $\langle \xi \rangle^{-2} (1 + i\langle \xi, \partial_y \rangle) e^{i\langle x-y, \xi \rangle} = e^{i\langle x-y, \xi \rangle}$ . This yields, for  $q \in \mathbb{N}$ ,

$$\begin{aligned} r''_N(x, \eta, \tau) &= (1-w)(\tau/\langle \eta \rangle) \phi(x) \int_0^1 \sigma^{|\gamma|} (1-\sigma)^{N-1} \iint e^{i\langle x-y, \xi \rangle} (1 - i\langle \xi, \partial_y \rangle)^q \left( \hat{\phi}(y) p(x, y, \eta) e^{i\langle \rho_x(y), \eta \rangle} \right) \\ &\quad \times \langle \xi \rangle^{-2q} \partial_\xi^{\alpha+\gamma} a(x, \sigma\xi + {}^t\kappa'(x)\eta, \tau) dy d\xi d\sigma, \end{aligned}$$

We choose  $|\alpha| = N \geq m$  and  $q = n + 1$ . We then have

$$|\partial_\xi^{\alpha+\gamma} a(x, \sigma\xi + {}^t\kappa'(x)\eta, \tau)| \leq (\tau + |\sigma\xi + {}^t\kappa'(x)\eta|)^{m-N-|\gamma|} \lesssim \tau^{m-N-|\gamma|}.$$

We thus obtain

$$|r_N''(x, \eta, \tau)| \leq |(1-w)(\tau/\langle\eta\rangle)| |\eta|^{\ell+n+1} \tau^{m-N-|\gamma|} \lesssim \tau^{m+\ell+n+1-N-|\gamma|},$$

as  $m - N - |\gamma| \leq 0$  and  $|\eta| \lesssim \tau$ . Since  $\ell \leq (N+k)/2 = (N+|\gamma|)/2$  this yields

$$|r_N''(x, \eta, \tau)| \lesssim \tau^{m+n+1-(N+|\gamma|)/2} \lesssim (\tau + |\eta|)^{m+n+1-N/2},$$

as claimed above. This concludes the proof.  $\square$

## C.10 Proof of Proposition B.7

The proof follows some of the lines of that of Proposition 18.1.19 in [Hör85a]. We fix  $j \in J$ . We take  $\tilde{\lambda}_l, l \in L$ , a locally finite partition of unity of  $(0, X_0) \times \tilde{U}_j$ . For all  $k, l \in L$ , we set  $\sigma_{kl} \in S_{\mathcal{T}}^m(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$  as

$$\text{Op}_{\mathcal{T}}(\sigma_{kl}) = (\phi_j^{-1})^* \lambda_k A \phi_j^* \tilde{\lambda}_l, \quad \text{where } \lambda_k = \phi_j^* \tilde{\lambda}_k.$$

Note that  $\text{supp}(\sigma_{kl}) \subset (0, X_0) \times \tilde{U}_j \times \mathbb{R} \times \mathbb{R}^n$ . We define  $a_j := \sum'_{k,l} \sigma_{kl}$  where  $\sum'$  denotes the sum over the pairs  $k, l$  such that  $\text{supp}(\lambda_l) \cap \text{supp}(\lambda_k) \neq \emptyset$ . This sum is locally finite, which gives  $a_j \in S_{\mathcal{T}}^m((0, X_0) \times \tilde{U}_j \times \mathbb{R} \times \mathbb{R}^n)$ .

For  $\lambda \in \mathcal{C}_c^\infty((0, X_0) \times U_j)$ ,  $\tilde{\lambda} \in \mathcal{C}_c^\infty((0, X_0) \times \tilde{U}_j)$  we consider

$$R = (\phi_j^{-1})^* \lambda A \phi_j^* \tilde{\lambda} - \left( (\phi_j^{-1})^* \lambda \right) \text{Op}_{\mathcal{T}}(a_j) \tilde{\lambda} = \sum_{k,l} (\phi_j^{-1})^* \lambda \lambda_k A \phi_j^* \tilde{\lambda}_l \tilde{\lambda} - \left( (\phi_j^{-1})^* \lambda \right) \text{Op}_{\mathcal{T}}(a_j) \tilde{\lambda}.$$

Note that the sum only involves  $k, l$  such that  $\text{supp}(\lambda_k) \cap \text{supp}(\lambda) \neq \emptyset$ ,  $\text{supp}(\tilde{\lambda}_l) \cap \text{supp}(\tilde{\lambda}) \neq \emptyset$ . Hence, the sum is finite. We find

$$R = \sum'_{k,l} (\phi_j^{-1})^* \lambda \lambda_k A \phi_j^* \tilde{\lambda}_l \tilde{\lambda} - \left( (\phi_j^{-1})^* \lambda \right) \text{Op}_{\mathcal{T}}(a_j) \tilde{\lambda} + R_1$$

where  $R_1$  is a finite sum of operators in  $\Psi_{\mathcal{T}}^m(\mathbb{R}^n \times \mathbb{R})$  (and also in  $\Psi_{\mathcal{T}}^m((0, X_0) \times \tilde{U}_j \times \mathbb{R})$ ) with kernels vanishing in a neighborhood of the diagonal. By Lemma B.2, we have  $R_1 \in h^\infty \Psi_{\mathcal{T}}^{-\infty}(\mathbb{R}^n \times \mathbb{R})$ . Moreover, observe that

$$(\phi_j^{-1})^* \lambda \lambda_k A \phi_j^* \tilde{\lambda}_l \tilde{\lambda} = \left( (\phi_j^{-1})^* \lambda \right) \text{Op}_{\mathcal{T}}(\sigma_{kl}) \tilde{\lambda}.$$

We thus have  $R = R_1$  from the definition of  $a_j$ .

We now prove uniqueness. Let  $\tilde{a}_j$  satisfy the same properties as  $a_j$ . Introducing  $b = a_j - \tilde{a}_j$ , for all  $\lambda \in \mathcal{C}_c^\infty((0, X_0) \times U_j)$ ,  $\tilde{\lambda} \in \mathcal{C}_c^\infty((0, X_0) \times \tilde{U}_j)$  we have

$$\left( (\phi_j^{-1})^* \lambda \right) \text{Op}_{\mathcal{T}}(b) \tilde{\lambda} \in h^\infty \Psi_{\mathcal{T}}^{-\infty}(\mathbb{R}^n \times \mathbb{R}).$$

Let  $K$  be a compact set in  $(0, X_0) \times \tilde{U}_j$  and we choose  $\lambda, \tilde{\lambda}$  such that  $(\phi_j^{-1})^* \lambda = 1$  on  $K$  and  $\tilde{\lambda} = 1$  on  $\text{supp}((\phi_j^{-1})^* \lambda)$ . The symbol of  $\left( (\phi_j^{-1})^* \lambda \right) \text{Op}_{\mathcal{T}}(b) \tilde{\lambda}$  is in  $h^\infty S_{\mathcal{T}}^{-\infty}(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$  and is given by

$$\left( (\phi_j^{-1})^* \lambda \right) b \# \tilde{\lambda} \in (\phi_j^{-1})^* \lambda b + h^\infty S_{\mathcal{T}}^{-\infty}(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$$

by the composition formula (B.1). As a consequence, according to Definition B.3, we have

$$b \in h^\infty S_{\mathcal{T}}^{-\infty}((0, X_0) \times \tilde{U}_j \times \mathbb{R} \times \mathbb{R}^n).$$

$\square$

### C.11 Proof of Proposition B.8

Let  $K$  be a compact set in  $(0, X_0) \times (U_j \cap U_k)$ . Let  $\lambda, \hat{\lambda} \in \mathcal{C}_c^\infty((0, X_0) \times (U_j \cap U_k))$  be equal to one on  $K$ . We set  $\tilde{\lambda}_\ell = (\phi_\ell^{-1})^* \hat{\lambda}$ ,  $\ell = j, k$ . We also introduce  $A_{\lambda, \hat{\lambda}, \ell} = (\phi_\ell^{-1})^* \lambda A \phi_\ell^* \tilde{\lambda}_\ell$  and find

$$A_{\lambda, \hat{\lambda}, \ell} = \text{Op}_{\mathcal{T}}(\tilde{a}_\ell) \quad \text{mod } h^\infty \Psi_{\mathcal{T}}^{-\infty}(\mathbb{R}^n \times \mathbb{R}), \quad \text{with } \tilde{a}_\ell = \left( (\phi_\ell^{-1})^* \lambda \right) a_\ell \# \tilde{\lambda}_\ell, \quad \ell = j, k.$$

The kernel of  $A_{\lambda, \hat{\lambda}, j}$  (resp.  $A_{\lambda, \hat{\lambda}, k}$ ) has a compact support in  $((0, X_0) \times \tilde{U}_{j,k})^2$  (resp.  $((0, X_0) \times \tilde{U}_{k,j})^2$ ). Observe that we have

$$A_{\lambda, \hat{\lambda}, k} = (\phi_{jk}^{-1})^* A_{\lambda, \hat{\lambda}, j} \phi_{jk}^*.$$

From Theorem B.1, we have for all  $N \in \mathbb{N}$ ,

$$\tilde{a}_k - T_{\phi_{jk}, N}(\tilde{a}_j) \in h^N S_{\mathcal{T}}^{m-N/2}(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n).$$

Set  $K_k = \phi_k(K)$  and  $\chi \in \mathcal{C}_c^\infty(K_k)$ . Since  $\lambda = \hat{\lambda} = 1$  on  $K$ , we have

$$\chi a_k = \chi \tilde{a}_k \quad \text{mod } h^\infty S_{\mathcal{T}}^{-\infty}(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n).$$

We also have

$$\begin{aligned} \chi \tilde{a}_k &= \chi T_{\phi_{jk}, N}(\tilde{a}_j) \quad \text{mod } h^N S_{\mathcal{T}}^{m-N/2}(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n) \\ &= \chi T_{\phi_{jk}, N}(a_j) \quad \text{mod } h^N S_{\mathcal{T}}^{m-N/2}(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n), \end{aligned}$$

because of the form of  $T_{\phi_{jk}, N}$  in (B.5). We thus obtain

$$\chi(a_k - T_{\phi_{jk}, N}(a_j)) \in h^N S_{\mathcal{T}}^{m-N/2}(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n). \quad (\text{C.26})$$

As  $K$  is arbitrary, (C.26) holds for any  $\chi \in \mathcal{C}_c^\infty(\tilde{U}_{k,j})$ . This gives the conclusion according to Definition B.3.  $\square$

### C.12 Proof of Proposition B.10

Let  $K_{A,h}$  and  $K_{B,h}$  be the kernels of  $A$  and  $B$ . We shall use the notation of Definition B.5.

As the two operators are properly supported the composition makes sense and  $AB : \mathcal{C}_c^\infty(\mathcal{X}) \rightarrow \mathcal{C}_c^\infty(\mathcal{X})$ . We denote its distribution kernel by  $K_{AB,h}$ . (Note that we use the Riemannian structure here to identify function, densities, and half-densities on  $\mathcal{X}$ ). We have

$$\mathcal{K}_{AB,h}(x_0, y; \hat{x}_0, \hat{y}) = \iint_{\mathcal{X}'} \mathcal{K}_{A,h}(x_0, y; \check{x}_0, \check{y}) \mathcal{K}_{B,h}(\check{x}_0, \check{y}; \hat{x}_0, \hat{y}) \, d\check{x}_0 \, d\check{y}$$

in the sense given at the end of Section 8.2 in [Hör90]. We choose  $\chi, \hat{\chi} \in \mathcal{C}_c^\infty(\mathcal{X}')$  such that  $\text{supp}(\chi) \cap \text{supp}(\hat{\chi}) = \emptyset$ . In addition we introduce  $\check{\chi}$  such that  $\text{supp}(\chi) \cap \text{supp}(\check{\chi}) = \emptyset$  and  $\check{\chi} = 1$  on  $\text{supp}(\hat{\chi})$ . We then write

$$\begin{aligned} &\chi(x_0, y) \hat{\chi}(\hat{x}_0, \hat{y}) \mathcal{K}_{AB,h}(x_0, y; \hat{x}_0, \hat{y}) \\ &= \hat{\chi}(\hat{x}_0, \hat{y}) \iint_{\mathcal{X}'} \chi(x_0, y) \check{\chi}(\check{x}_0, \check{y}) \mathcal{K}_{A,h}(x_0, y; \check{x}_0, \check{y}) \mathcal{K}_{B,h}(\check{x}_0, \check{y}; \hat{x}_0, \hat{y}) \, d\check{x}_0 \, d\check{y} \\ &\quad + \chi(x_0, y) \iint_{\mathcal{X}'} \hat{\chi}(\hat{x}_0, \hat{y}) (1 - \check{\chi}(\check{x}_0, \check{y})) \mathcal{K}_{A,h}(x_0, y; \check{x}_0, \check{y}) \mathcal{K}_{B,h}(\check{x}_0, \check{y}; \hat{x}_0, \hat{y}) \, d\check{x}_0 \, d\check{y}. \end{aligned}$$

We note that in the first sum  $\chi(x_0, y) \check{\chi}(\check{x}_0, \check{y}) \mathcal{K}_{A,h}(x_0, y; \check{x}_0, \check{y})$  is smooth and compactly supported because of the disjoint supports of the cut-off functions and the regularity of the kernel  $\mathcal{K}_{A,h}$  off the diagonal. In the second sum  $\hat{\chi}(\hat{x}_0, \hat{y}) (1 - \check{\chi}(\check{x}_0, \check{y})) \mathcal{K}_{B,h}(\check{x}_0, \check{y}; \hat{x}_0, \hat{y})$  is also smooth as  $\text{supp}(\hat{\chi}) \cap \text{supp}(1 - \check{\chi}) = \emptyset$  and compactly supported as  $\mathcal{K}_{B,h}$  is properly supported. Because of (B.8) both terms then yield a smooth function in the variables  $x_0, y, \hat{x}_0, \hat{y}$  and estimating derivatives then yields a proper estimate of the form of (B.7).

We now consider  $j \in J$  and  $\lambda \in \mathcal{C}_c^\infty((0, X_0) \times U_j)$ ,  $\tilde{\lambda} \in \mathcal{C}_c^\infty((0, X_0) \times \tilde{U}_j)$ . We set

$$\alpha = (\phi_j^{-1})^* \lambda A B \phi_j^*(\tilde{\lambda}).$$

We then introduce  $\chi, \hat{\chi} \in \mathcal{C}_c^\infty((0, X_0) \times U_j)$  such that both  $\chi = 1$  and  $\hat{\chi} = 1$  on  $\text{supp}(\phi_j^* \tilde{\lambda})$ . We write  $\alpha = \beta + R$  with

$$\beta = (\phi_j^{-1})^* \lambda A \chi \hat{\chi} B \phi_j^*(\tilde{\lambda}), \quad R = (\phi_j^{-1})^* \lambda A (1 - \chi \hat{\chi}) B \phi_j^*(\tilde{\lambda}).$$

Arguing as above we find that the kernel  $K_R$  of  $R$  is a smooth function and it satisfies an estimate of the form  $\sup_{x_n} q(K_R) \leq Ch^N$ , for any  $N \in \mathbb{N}$  and  $q$  semi-norm on  $\mathcal{C}^\infty((\mathbb{R}^n \times \mathbb{R})^2)$ . Moreover its support is compact. Hence  $R \in h^\infty \Psi_{\mathcal{T}}^{-\infty}(\mathbb{R}^n \times \mathbb{R})$ .

Next, with  $\tilde{\chi} = (\phi_j^{-1})^* \chi$  we write

$$\beta = (\phi_j^{-1})^* \lambda A \phi_j^* \tilde{\chi} (\phi_j^{-1})^* \hat{\chi} B \phi_j^* (\tilde{\lambda}).$$

By Proposition B.7 we obtain a semi-classical tangential operator on  $\mathbb{R}^n \times \mathbb{R}$  with symbol

$$\beta_j = \left( ((\phi_j^{-1})^* \lambda) a_j \right) \# \left( \tilde{\chi} ((\phi_j^{-1})^* \hat{\chi}) b_j \right) \# \tilde{\lambda} \quad \text{mod } h^\infty S_{\mathcal{T}}^{-\infty}(\mathbb{R}^n \times \mathbb{R}).$$

which belongs to  $S_{\mathcal{T}}^{m+m'}(\mathbb{R}^n \times \mathbb{R})$ . The operator  $AB$  is thus in  $\Psi_{\mathcal{T}}^{m+m'}(\mathbb{R}^n \times \mathbb{R})$ .

From the composition formula B.1, because of the supports of  $\tilde{\chi}$  and  $(\phi_j^{-1})^* \hat{\chi}$  we further obtain

$$\beta_j = \left( (\phi_j^{-1})^* \lambda (a_j \# b_j) \right) \# \tilde{\lambda} \quad \text{mod } h^\infty S_{\mathcal{T}}^{-\infty}(\mathbb{R}^n \times \mathbb{R}).$$

Hence by Proposition B.7  $a_j \# b_j$  is a representative of the local symbol of  $AB$  in this chart.  $\square$

### C.13 Proof of Proposition B.12

The existence of  $L$  is only related to the proper support of the kernel of  $A$ . We have

$$|(Au)|_{x_n=0}|_k = \sum_j |(\phi_j^{-1})^* \psi_j (Au)|_{x_n=0}|_k.$$

Let  $j \in J$ . It suffices to prove that

$$|(\phi_j^{-1})^* \psi_j (Au)|_{x_n=0}|_k \leq C_K |u|_{k+\ell}.$$

We choose a partition of unity  $\sum_k \hat{\psi}_k = 1$ , subordinated to the open covering  $(U_k)_{k \in J}$  such that  $\hat{\psi}_j = 1$  in a neighborhood of  $\text{supp}(\psi_j)$ . Then  $\text{supp}(\hat{\psi}_k) \cap \text{supp}(\psi_j) = \emptyset$  for  $k \neq j$ . We then have

$$\psi_j Au = \sum_{k \neq j} \psi_j A \hat{\psi}_k u + \psi_j A \hat{\psi}_j u$$

The terms in the sum are then associated with properly supported operators with smooth kernels for which the operator continuity (after restriction to  $x_n = 0$ ) is clear. To treat the last term we choose  $\lambda \in \mathcal{C}_c^\infty((0, X_0) \times U_j)$  such that  $\lambda = 1$  on  $L \cap ((0, X_0) \times \text{supp}(\psi_j))$ , and  $\hat{\lambda} \in \mathcal{C}_c^\infty((0, X_0) \times \tilde{U}_j)$  such that  $\hat{\lambda} = 1$  on  $K \cap ((0, X_0) \times \text{supp}((\phi_j^{-1})^* \psi_j))$ . We then have

$$(\phi_j^{-1})^* \psi_j A \hat{\psi}_j u = (\phi_j^{-1})^* \lambda \psi_j A \phi_j^* \hat{\lambda} (\phi_j^{-1})^* \hat{\psi}_j u = B (\phi_j^{-1})^* \hat{\psi}_j u \quad \text{at } x_n = 0,$$

with  $B \in \Psi_{\mathcal{T}}^\ell(\mathbb{R}^{n+1})$  by Definition B.5. Hence

$$|(\phi_j^{-1})^* \psi_j (A \hat{\psi}_j u)|_{x_n=0}|_k \lesssim |(\phi_j^{-1})^* \hat{\psi}_j u|_{k+\ell} \lesssim |u|_{k+\ell}$$

by (1.22).  $\square$

**Acknowledgment.** The authors wish to thank E. Zuazua for suggesting the study of the controllability of a model with a diffusive interface. Most of this research was carried out at Institut Henri Poincaré (Paris, France). Moreover, the authors wish to thank Institut Henri Poincaré for providing a very stimulating environment during the ‘‘Control of Partial and Differential Equations and Applications’’ program in the Fall 2010.

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