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# ACCESSING THE FARRELL–TATE COHOMOLOGY OF DISCRETE GROUPS

ALEXANDER D. RAHM

ABSTRACT. We introduce a method to explicitly determine the Farrell–Tate cohomology of discrete groups. We apply this method to the Coxeter triangle and tetrahedral groups as well as to the Bianchi groups, i.e.  $\mathrm{PSL}_2(\mathcal{O})$  for  $\mathcal{O}$  the ring of integers in an imaginary quadratic number field. We show that the Farrell–Tate cohomology of the Bianchi groups is completely determined by the numbers of conjugacy classes of finite subgroups. In fact, our access to Farrell–Tate cohomology allows us to detach the information about it from geometric models for the Bianchi groups and to express it only with the group structure. Formulae for the numbers of conjugacy classes of finite subgroups have been determined in a thesis of Krämer, in terms of elementary number-theoretic information on  $\mathcal{O}$ . An evaluation of these formulae for a large number of Bianchi groups is provided numerically in the appendix. Our new insights about the homological torsion allow us to give a conceptual description of the cohomology ring structure of the Bianchi groups.

## 1. INTRODUCTION

Farrell–Tate cohomology  $\widehat{H}^q$  (which we will by now just call Farrell cohomology) is identical to the classical cohomology  $H^q$  of groups in all degrees  $q$  above their virtual cohomological dimension [7]. So for instance for the Coxeter groups, the Farrell cohomology equals all of the classical cohomology. In section 2, we will introduce a method of how to explicitly determine the Farrell cohomology : By reducing the torsion subcomplexes. This method has also been implemented on the computer [10, HAP version of June 22nd, 2012, respectively more recent], which allows us to check the results that we obtain by our arguments. We apply our method to the Coxeter triangle and tetrahedral groups in section 3, and to the Bianchi groups in sections 4 through 6.

We require any discrete group  $\Gamma$  under our study to be provided with a cell complex on which it acts cellularly.

**Definition 1.** Let  $\ell$  be a prime number. The  $\ell$ -torsion subcomplex of a  $\Gamma$ -cell complex  $X$  consists of all the cells of  $X$  the stabilisers in  $\Gamma$  of which contain elements of order  $\ell$ .

For the Coxeter tetrahedral groups, generated by the reflections on the sides of a tetrahedron in hyperbolic 3-space, we obtain the following. Denote by  $\mathcal{D}_\ell$  the dihedral group of order  $2\ell$ .

**Corollary 2** (Corollary to theorem 14.). *Let  $\Gamma$  be a Coxeter tetrahedral group, and  $\ell > 2$  be a prime number. Then there is an isomorphism  $H_q(\Gamma; \mathbb{Z}/\ell) \cong (H_q(\mathcal{D}_\ell; \mathbb{Z}/\ell))^m$ , with  $m$  the number of connected components of the orbit space of the  $\ell$ -torsion subcomplex of the Davis complex of  $\Gamma$ .*

We specify the exponent  $m$  in the tables in figures 2 through 5.

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**Results for the Bianchi groups.** Denote by  $\mathbb{Q}(\sqrt{-m})$ , with  $m$  a square-free positive integer, an imaginary quadratic number field, and by  $\mathcal{O}_{-m}$  its ring of integers. The *Bianchi groups* are the groups  $\mathrm{PSL}_2(\mathcal{O}_{-m})$ . The Bianchi groups may be considered as a key to the study of a larger class of groups, the *Kleinian groups*, which date back to work of Henri Poincaré [18]. In fact, each non-cocompact arithmetic Kleinian group is commensurable with some Bianchi group [17]. A wealth of information on the Bianchi groups can be found in the monographs [12], [11], [17]. Krämer [16] has determined number-theoretic formulae for the numbers of conjugacy classes of finite subgroups in the Bianchi groups, using numbers of ideal classes in orders of cyclotomic extensions of  $\mathbb{Q}(\sqrt{-m})$ .

In section 5, we express the homological torsion of the Bianchi groups as a function of these numbers of conjugacy classes. To achieve this, we build on the geometric techniques of [20], which depend on the explicit knowledge of the quotient space of geometric models for the Bianchi groups — like any technique effectively accessing the (co)homology of the Bianchi groups, either directly [24], [28] or via a group presentation [5]. For the Bianchi groups, we can in this article detach invariants of the group actions from the geometric models, in order to express them only by the group structure itself, in terms of conjugacy classes of finite subgroups, normalisers of the latter, and their interactions. This information is gathered, and reduced to its essence, in the reduced torsion subcomplexes we construct for this purpose.

Not only does this provide us with exact formulae for the homological torsion of the Bianchi groups, the power of which we can see in the numerical evaluations of appendices A.1 and A.2, also it allows us to understand the rôle of the centralisers of the finite subgroups, and this is how in [19], some more fruits of the present results are harvested (in terms of the Chen/Ruan orbifold cohomology of the orbifolds given by the action of the Bianchi groups on complexified hyperbolic space).

Except for the Gaussian and Eisenstein integers, which can easily be treated separately [24], [20], all the rings of integers of imaginary quadratic number fields admit as only units  $\{\pm 1\}$ . In the latter case, we call  $\mathrm{PSL}_2(\mathcal{O})$  a *Bianchi group with units  $\{\pm 1\}$* . For the possible types of finite subgroups in the Bianchi groups, see lemma 20 : there are five non-trivial possibilities. Our main result is the following. We make it more explicit in theorem 4 and give its proof in section 5. We deduce the structure of the cohomology ring in section 6.

**Corollary 3.** *The Farrell cohomology of the Bianchi groups with units  $\{\pm 1\}$  depends only on the numbers of conjugacy classes of non-trivial finite subgroups of the occurring five types.*

The main step in order to prove this, is to read off the Farrell cohomology from the reduced torsion subcomplexes.

Krämer's formulae express the numbers of conjugacy classes of the five types of non-trivial finite subgroups in the Bianchi groups, where the symbols in the first row are Krämer's notations for the number of their conjugacy classes:

$\lambda_4$	$\lambda_6$	$\mu_2$	$\mu_3$	$\mu_T$
$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$

Recall that we can express the homology in degrees above the virtual cohomological dimension of the Bianchi groups by the two Poincaré series — for  $\ell = 2$  and  $\ell = 3$  — in the dimensions

over the field with  $\ell$  elements, of the homology with  $\mathbb{Z}/\ell$ -coefficients of  $\mathrm{PSL}_2(\mathcal{O}_{-m})$ ,

$$P_m^\ell(t) := \sum_{q>2}^{\infty} \dim_{\mathbb{F}_\ell} H_q(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z}/\ell) t^q,$$

which have been suggested by Grunewald. Further let  $P_{\bullet\circ}(t) := \frac{-2t^3}{t-1}$ , which equals the series  $P_m^2(t)$  of the groups  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  the reduced 2-torsion subcomplex of which is a circle. Denote by

- $P_{\mathcal{D}_2}^*(t) := \frac{-t^3(3t-5)}{2(t-1)^2}$ , the Poincaré series over  $\dim_{\mathbb{F}_2} H_q(\mathcal{D}_2; \mathbb{Z}/2) - \frac{3}{2} \dim_{\mathbb{F}_2} H_q(\mathbb{Z}/2; \mathbb{Z}/2)$
- and by  $P_{\mathcal{A}_4}^*(t) := \frac{-t^3(t^3-2t^2+2t-3)}{2(t-1)^2(t^2+t+1)}$ , the Poincaré series over

$$\dim_{\mathbb{F}_2} H_q(\mathcal{A}_4; \mathbb{Z}/2) - \frac{1}{2} \dim_{\mathbb{F}_2} H_q(\mathbb{Z}/2; \mathbb{Z}/2).$$

In 3-torsion, let  $P_{\bullet\bullet}(t) := \frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}$ , which equals the series  $P_m^3(t)$  for the Bianchi groups the reduced 3-torsion subcomplex of which is a single edge without identifications.

**Theorem 4.** *For all Bianchi groups with units  $\{\pm 1\}$ , the homology in degrees above their virtual cohomological dimension is given by the Poincaré series*

$$P_m^2(t) = \left( \lambda_4 - \frac{3\mu_2 - 2\mu_T}{2} \right) P_{\bullet\circ}(t) + (\mu_2 - \mu_T) P_{\mathcal{D}_2}^*(t) + \mu_T P_{\mathcal{A}_4}^*(t)$$

and

$$P_m^3(t) = \left( \lambda_6 - \frac{\mu_3}{2} \right) P_{\bullet\circ}(t) + \frac{\mu_3}{2} P_{\bullet\bullet}(t).$$

**Organisation of the paper.** In section 2, we introduce our method to explicitly determine Farrell cohomology: By reducing the torsion subcomplexes. We apply our method to the Coxeter triangle and tetrahedral groups in section 3. In section 4, we show how to read off the Farrell cohomology of the Bianchi groups from the reduced torsion subcomplexes. We achieve this by showing that for the Bianchi groups, the reduced torsion subcomplexes are homeomorphic to conjugacy classes graphs that we can define without reference to any geometric model. This enables us in section 5 to prove the formulae for the homological torsion of the Bianchi groups in terms of numbers of conjugacy classes of finite subgroups. We use this to establish the structure of the classical cohomology rings of the Bianchi groups in section 6. Krämer has given number-theoretic formulae for these numbers of conjugacy classes, and we evaluate them numerically in appendices A.1 and A.2. Finally, we present some numerical asymptotics on the numbers of conjugacy classes in appendix A.3.

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## 2. REDUCTION OF TORSION SUBCOMPLEXES

Let  $X$  be a finite-dimensional cell complex with a cellular action of a discrete group  $\Gamma$ , such that each cell stabiliser fixes its cell pointwise. Let  $\ell$  be a prime such that the fixed point set  $X^G$  is acyclic for every non-trivial finite  $\ell$ -subgroup  $G$  of  $\Gamma$ . Then, the  $\Gamma$ -equivariant Farrell cohomology of  $X$  gives us the  $\ell$ -primary part of the Farrell cohomology of  $\Gamma$ , as follows.

**Proposition 5** (Brown [7]). *Under the above assumptions, the canonical map*

$$\widehat{H}^*(\Gamma; M)_{(\ell)} \rightarrow \widehat{H}_\Gamma^*(X; M)_{(\ell)}$$

*is an isomorphism for any  $\Gamma$ -module  $M$  of coefficients.*

We are going to deduce from  $X$  a cell complex  $Z(\ell)$  for every prime  $\ell$ , which still yields the  $\ell$ -primary part of the Farrell cohomology of  $\Gamma$ , but can have considerably less orbits of cells, be easier to handle in practice, and, for certain classes of groups, lead us to an explicit structural description of the Farrell cohomology of  $\Gamma$ . We will call the complex  $Z(\ell)$  the reduced  $\ell$ -torsion subcomplex of  $\Gamma$ . Its pivotal property will be given in theorem 13.

The classical choice [7] is to take for  $X$  the geometric realisation of the partially ordered set of non-trivial finite subgroups (respectively, non-trivial elementary Abelian  $\ell$ -subgroups) of  $\Gamma$ , the latter acting by conjugation. The stabilisers are then the normalisers, which in many discrete groups are infinite. And it can impose great computational challenges to determine a group presentation for them. When we want to compute the module  $\widehat{H}_\Gamma^*(X; M)_{(\ell)}$  subject to proposition 5, at least we must get to know the ( $\ell$ -primary part of the) Farrell cohomology of these normalisers. The Bianchi groups are an instance that different isomorphism types can occur herefore at different conjugacy classes of elementary Abelian  $\ell$ -subgroups, both for  $\ell = 2$  and  $\ell = 3$ . As the only non-trivial elementary Abelian 3-subgroups in the Bianchi groups are of rank 1, the orbit space  $\Gamma \backslash X$  consists only of one point for each conjugacy class of type  $\mathbb{Z}/3$  and a corollary [7] from proposition 5 decomposes the 3-primary part of the Farrell cohomology of the Bianchi groups into the direct product over their normalisers. However, due to the different possible homological types of the normalisers (in fact, two of them occur), the final result remains unclear and subject to tedious case-by-case computations of the normalisers.

In contrast, in the cell complex we are going to develop, the connected components of the orbit space are for the 3-torsion in the Bianchi groups not simple points, but have either the shape  $\bullet \rightarrow$  or  $\bigcirc$ . This dichotomy already contains the information about the occurring normaliser.

**Condition 6.** In the  $\ell$ -torsion subcomplex, let  $\sigma$  be a cell of dimension  $n - 1$ , lying in the boundary of precisely two  $n$ -cells  $\tau_1$  and  $\tau_2$ , the latter cells representing two different orbits. Assume further that no higher-dimensional cells of the  $\ell$ -torsion subcomplex touch  $\sigma$ ; and that the  $n$ -cell stabilisers admit an isomorphism  $\Gamma_{\tau_1} \cong \Gamma_{\tau_2}$ .

Where this condition is fulfilled in the  $\ell$ -torsion subcomplex, we merge the cells  $\tau_1$  and  $\tau_2$  along  $\sigma$  and do so for their entire orbits, if and only if they meet the following additional condition. Here, we use Zassenhaus's notion for a finite group to be  $\ell$ -normal, if the center of one of its Sylow  $\ell$ -subgroups is the center of every Sylow  $\ell$ -subgroup in which it is contained.

**Condition 7.** Either  $\Gamma_{\tau_1} \cong \Gamma_\sigma$ , or  $\Gamma_\sigma$  is  $\ell$ -normal and  $\Gamma_{\tau_1} \cong N_{\Gamma_\sigma}(\text{center}(\text{Sylow}_p(\Gamma_\sigma)))$ .

Here, we write  $N_{\Gamma_\sigma}$  for taking the normaliser in  $\Gamma_\sigma$  and  $\text{Sylow}_p$  for picking an arbitrary Sylow  $\ell$ -subgroup. This is well defined because all Sylow  $\ell$ -subgroups are conjugate.

**Lemma 8.** *Let  $\widetilde{X}_\ell$  be the  $\Gamma$ -complex obtained by orbit-wise merging two  $n$ -cells of the  $\ell$ -torsion subcomplex  $X_\ell$  satisfying conditions 6 and 7. Then,*

$$\widehat{H}_\Gamma^*(\widetilde{X}_\ell; M)_{(\ell)} \cong \widehat{H}_\Gamma^*(X_\ell; M)_{(\ell)}.$$

For the proof, we will make use of Swan’s extension [27, final corollary] to Farrell cohomology of the Second Theorem of Grün [13, Satz 5].

**Theorem 9** (Swan). *Let  $G$  be a  $\ell$ -normal finite group, and let  $N$  be the normaliser of the center of a Sylow  $\ell$ -subgroup of  $G$ . Let  $M$  be any  $G$ -module. Then the inclusion and transfer maps both are isomorphisms between the  $\ell$ -primary components of  $\widehat{H}^*(G; M)$  and  $\widehat{H}^*(N; M)$ .*

*Proof of lemma 8.* Consider the equivariant spectral sequence in Farrell cohomology [7]. On the  $\ell$ -torsion subcomplex, it includes a map

$$\widehat{H}^*(\Gamma_\sigma; M)_{(\ell)} \xrightarrow{d_1^{(n-1),*} |_{\widehat{H}^*(\Gamma_\sigma; M)_{(\ell)}}} \widehat{H}^*(\Gamma_{\tau_1}; M)_{(\ell)} \oplus \widehat{H}^*(\Gamma_{\tau_2}; M)_{(\ell)},$$

which is the diagonal map with blocks the isomorphisms

$$\widehat{H}^*(\Gamma_\sigma; M)_{(\ell)} \xrightarrow{\cong} \widehat{H}^*(\Gamma_{\tau_i}; M)_{(\ell)},$$

induced by the inclusions  $\Gamma_{\tau_i} \hookrightarrow \Gamma_\sigma$ . The latter inclusions induce isomorphisms because of theorem 9. If for the orbit of  $\tau_1$  or  $\tau_2$  we have chosen a representative which is not adjacent to  $\sigma$ , than this isomorphism is composed with the isomorphism induced by conjugation with the element of  $\Gamma$  carrying the cell to one adjacent to  $\sigma$ . Hence, the map  $d_1^{(n-1),*} |_{\widehat{H}^*(\Gamma_\sigma; M)_{(\ell)}}$  has vanishing kernel, and dividing its image out of  $\widehat{H}^*(\Gamma_{\tau_1}; M)_{(\ell)} \oplus \widehat{H}^*(\Gamma_{\tau_2}; M)_{(\ell)}$  gives the  $\ell$ -primary part  $\widehat{H}^*(\Gamma_{\tau_1 \cup \tau_2}; M)_{(\ell)}$  of the Farrell cohomology of the union  $\tau_1 \cup \tau_2$  of the two  $n$ -cells. As by condition 6 no higher-dimensional cells are touching  $\sigma$ , there are no higher degree differentials interfering.  $\square$

The following consequence of the Lyndon–Hochschild–Serre spectral sequence allows us to weaken condition 7 in a way that lemma 8 still holds. Let  $\ell$  be a prime number, and denote homology with non-twisted  $\mathbb{Z}/\ell$ -coefficients by “mod  $\ell$  homology”.

**Lemma 10.** *Let  $T$  be a group with trivial mod  $\ell$  homology, and consider any group extension*

$$1 \rightarrow T \rightarrow E(Q, T) \rightarrow Q \rightarrow 1.$$

*Then the map  $E(Q, T) \rightarrow Q$  induces an isomorphism on mod  $\ell$  homology.*

This statement may look like a triviality, but it becomes wrong as soon as we exchange the rôles of  $T$  and  $Q$  in the group extension. In degrees 1 and 2, our claim follows from [7, VII.(6.4)]. In arbitrary degree, we can proceed as follows.

*Proof.* Consider the Lyndon–Hochschild–Serre spectral sequence associated to the group extension, namely

$$E_{p,q}^2 = H_p(Q; H_q(T; \mathbb{Z}/\ell)) \text{ converges to } H_{p+q}(E(Q, T); \mathbb{Z}/\ell).$$

By our assumption,  $H_q(T; \mathbb{Z}/\ell)$  is trivial, so this spectral sequence concentrates in the row  $q = 0$ , degenerates on the second page and yields isomorphisms

$$(1) \quad H_p(Q; H_0(T; \mathbb{Z}/\ell)) \cong H_p(E(Q, T); \mathbb{Z}/\ell).$$

As  $H_0(T; \mathbb{Z}/\ell) \cong (\mathbb{Z}/\ell)/\langle tm - m \mid t \in T, m \in \mathbb{Z}/\ell \rangle$  with trivial  $\mathbb{Z}/\ell$ -coefficients, we observe that for all  $t \in T$  and  $m \in \mathbb{Z}/\ell$ , we have  $gm = m$  and hence for all  $q \in Q$ , we have  $qgm = m$ . Therefore, the ideal divided out in the above expression is trivial and we only need to determine the action of  $Q$  on the coefficients  $\mathbb{Z}/\ell$  in order to obtain its action on  $H_0(T; \mathbb{Z}/\ell)$ .

If  $qm \neq m$  for some  $q \in Q$ ,  $m \in \mathbb{Z}/\ell$ , then  $\ell$  being a prime yields  $\langle qm - m \rangle = \mathbb{Z}/\ell$  and further  $H_0(Q; H_0(T; \mathbb{Z}/\ell)) \cong H_0(T; \mathbb{Z}/\ell)/\langle qm - m \mid q \in Q, m \in H_0(T; \mathbb{Z}/\ell) \rangle = 0$ . This contradicts the existence of the isomorphism (1) to  $H_0(E(Q, T); \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$ . Hence the action of  $Q$  on the coefficients of  $H_0(T; \mathbb{Z}/\ell)$  must be trivial. Thus, the isomorphism (1) becomes

$$H_p(Q; \mathbb{Z}/\ell) \cong H_p(E(Q, T); \mathbb{Z}/\ell).$$

□

The above lemma directly implies that any extension of two groups both having trivial mod  $\ell$  homology, again has trivial mod  $\ell$  homology.

Our condition 7 can now be weakened as follows.

**Condition 11.** The group  $\Gamma_\sigma$  admits a (possibly trivial) normal subgroup  $T_\sigma$  with trivial mod  $\ell$  homology and with quotient group  $G_\sigma$ ; and the group  $\Gamma_\tau$  admits a (possibly trivial) normal subgroup  $T_\tau$  with trivial mod  $\ell$  homology and with quotient group  $G_\tau$  making the sequences

$$1 \rightarrow T_\sigma \rightarrow \Gamma_\sigma \rightarrow G_\sigma \rightarrow 1 \text{ and } 1 \rightarrow T_\tau \rightarrow \Gamma_\tau \rightarrow G_\tau \rightarrow 1$$

exact and satisfying one of the following.

- Either  $G_\tau \cong G_\sigma$ ,
- or  $G_\sigma$  is  $\ell$ -normal and  $G_\tau \cong N_{G_\sigma}(\text{center}(\text{SyLOW}_p(G_\sigma)))$ ,
- or both  $G_\sigma$  and  $G_\tau$  are  $\ell$ -normal and there is a (possibly trivial) group  $T$  with trivial mod  $\ell$  homology making the sequence

$$1 \rightarrow T \rightarrow N_{G_\sigma}(\text{center}(\text{SyLOW}_p(G_\sigma))) \rightarrow N_{G_\tau}(\text{center}(\text{SyLOW}_p(G_\tau))) \rightarrow 1$$

exact.

Note that this weaker condition is not yet implemented on the machine, so the above quoted program carries out only the reductions that are subject to condition 7.

By a “terminal vertex”, we will denote a vertex with no adjacent higher-dimensional cells and precisely one adjacent edge in the quotient space, and by “cutting off” the latter edge, we will mean that we remove the edge together with the terminal vertex from our cell complex.

**Definition 12.** The *reduced  $\ell$ -torsion subcomplex* associated to a  $\Gamma$ -cell complex  $X$  is the cell complex obtained by recursively merging orbit-wise all the pairs of cells satisfying conditions 6 and 11; and cutting off edges that admit a terminal vertex together with which they satisfy condition 11.

**Theorem 13.** *There is an isomorphism between the  $\ell$ -primary parts of the Farrell cohomology of  $\Gamma$  and the  $\Gamma$ -equivariant Farrell cohomology of the reduced  $\ell$ -torsion subcomplex.*

*Proof.* We apply proposition 5 to the cell complex  $X$ , and then we apply lemma 8 each time that we orbit-wise merge a pair of cells of the  $\ell$ -torsion subcomplex, or that we cut off an edge. Lemma 10 allows us to replace condition 7 by condition 11 as hypothesis for lemma 8.  $\square$

### 3. FARRELL COHOMOLOGY OF THE COXETER TETRAHEDRAL GROUPS

As Coxeter groups have trivial rational cohomology [9], their Farrell cohomology is identical with their classical cohomology. So in this section, we make use of this fact to determine the latter. For facts about Coxeter groups, and especially for the Davis complex, we refer to [9]. Recall that the simplest example of a Coxeter group, the dihedral group  $\mathcal{D}_n$ , is an extension

$$1 \rightarrow \mathbb{Z}/n \rightarrow \mathcal{D}_n \rightarrow \mathbb{Z}/2 \rightarrow 1,$$

so we can make use of the original application [29] of Wall's lemma to obtain its mod  $\ell$  homology for prime numbers  $\ell > 2$ ,

$$H_q(\mathcal{D}_n; \mathbb{Z}/\ell) \cong \begin{cases} \mathbb{Z}/\ell, & q = 0, \\ \mathbb{Z}/\gcd(n, \ell), & q \equiv 3 \text{ or } 4 \pmod{4}, \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem 14.** *Let  $\ell > 2$  be a prime number. Let  $\Gamma$  be a Coxeter group admitting a Coxeter system with at most four generators, and relator orders not divisible by  $\ell^2$ . Let  $Z(\ell)$  be the  $\ell$ -torsion subcomplex of the Davis complex of  $\Gamma$ . If  $Z(\ell)$  is at most one-dimensional and contains no loop nor bifurcation, then the mod  $\ell$  homology of  $\Gamma$  is isomorphic to  $(H_q(\mathcal{D}_\ell; \mathbb{Z}/\ell))^m$ , with  $m$  the number of connected components of the orbit space of  $Z(\ell)$ .*

The conditions of this theorem are for instance fulfilled by the Coxeter tetrahedral groups; we specify the exponent  $m$  for them in the tables in figures 2 through 5. In order to prove theorem 14, we lean on the following intermediary statement.

**Lemma 15.** *Let  $\ell > 2$  be a prime number; and let  $\Gamma_\sigma$  be a finite Coxeter group with  $n \leq 4$  generators. If  $\Gamma_\sigma$  is not a direct product of two dihedral groups and not associated to the Coxeter diagram  $\mathbf{F}_4$  or  $\mathbf{H}_4$ , then condition 11 is fulfilled for the triple consisting of  $\ell$ , the group  $\Gamma_\sigma$  and any of its Coxeter subgroups  $\Gamma_{\tau_1}$  with  $(n - 1)$  generators that contains  $\ell$ -torsion elements.*

*Proof.* The dihedral groups admit only Coxeter subgroups with two elements, so without  $\ell$ -torsion. There are only finitely many other isomorphism types of irreducible finite Coxeter groups with at most four generators, specified by the Coxeter diagrams

$\mathbf{A}_1$	$\mathbf{A}_3$	$\mathbf{A}_4$	$\mathbf{B}_3$	$\mathbf{B}_4$	$\mathbf{D}_4$	$\mathbf{H}_3$
•	•—•	•—•—•	• <sup>4</sup> —•—•	• <sup>4</sup> —•—•—•	• • •	• <sup>5</sup> —•—•

on which we can check the condition case by case.

$\mathbf{A}_1$ . The symmetric group  $\mathcal{S}_2$  admits no Coxeter subgroups.

$\mathbf{A}_3$ . The symmetric group  $\mathcal{S}_4$  is 3-normal; and its Sylow-3-subgroups are of type  $\mathbb{Z}/3$ , so they are identical to their center. Their normalisers in  $\mathcal{S}_4$  match the Coxeter subgroups of type  $\mathcal{D}_3$  that one obtains by omitting one of the generators of  $\mathcal{S}_4$  at an end of its Coxeter

diagram. The other possible Coxeter subgroup type is  $(\mathbb{Z}/2)^2$ , obtained by omitting the middle generator in this diagram, and contains no 3-torsion.

- A<sub>4</sub>.** The Coxeter subgroups with three generators in the symmetric group  $\mathcal{S}_5$  are  $\mathcal{D}_3 \times \mathbb{Z}/2$  and  $\mathcal{S}_4$ , so we only need to consider 3-torsion. The group  $\mathcal{S}_5$  is 3-normal; the normaliser of the center of any of its Sylow-3-subgroups is of type  $\mathcal{D}_3 \times \mathbb{Z}/2$ . So for the Coxeter subgroup  $\mathcal{S}_4$ , we use the normaliser  $\mathcal{D}_3$  of its Sylow-3-subgroup  $\mathbb{Z}/3$ ; and apply the last option of condition 11.
- B<sub>3</sub>.** We apply lemma 10 to the Coxeter group  $(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$ , and retain only  $\mathcal{S}_3$ , which is isomorphic to the only Coxeter subgroup admitting 3-torsion.
- B<sub>4</sub>.** The Coxeter subgroups with three generators are of type  $\mathcal{S}_4$ ,  $\mathbb{Z}/2 \times \mathcal{D}_3$ ,  $\mathcal{D}_4 \times \mathbb{Z}/2$  or  $(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$ , thus for the three of them containing 3-torsion, we use the above methods to relate them to  $\mathcal{D}_3$ . The Coxeter group  $(\mathbb{Z}/2)^4 \rtimes \mathcal{S}_4$  is 3-normal; its Sylow-3-subgroup is of type  $\mathbb{Z}/3$  and admits a normaliser  $N$  fitting into the exact sequence

$$1 \rightarrow (\mathbb{Z}/2)^2 \rightarrow N \rightarrow \mathcal{D}_3 \rightarrow 1.$$

- D<sub>4</sub>.** From the Coxeter diagram, we see that the Coxeter subgroups with three generators are  $(\mathbb{Z}/2)^3$  and  $\mathcal{S}_4$ . So we only need to compare with the 3-torsion of  $\mathcal{S}_4$ . For this purpose, we apply lemma 10 to the Coxeter group  $(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_4$ .
- H<sub>3</sub>.** The symmetry group  $\text{Icos}_{120}$  of the icosahedron splits as a direct product  $\mathbb{Z}/2 \times \mathcal{A}_5$ , so by lemma 10, we can for all primes  $\ell > 2$  make use of the alternating group  $\mathcal{A}_5$  as the quotient group in condition 11. The primes other than 2, at which the homology of  $\mathcal{A}_5$  admits torsion, are 3 and 5. So now let  $\ell$  be 3 or 5. Then the group  $\mathcal{A}_5$  is  $\ell$ -normal; and its Sylow- $\ell$ -subgroups are of type  $\mathbb{Z}/\ell$ , so they are identical to their center. Their normalisers in  $\mathcal{A}_4$  are of type  $\mathcal{D}_\ell$ . From the Coxeter diagram, we see that this is the only Coxeter subgroup type with two generators that contains  $\ell$ -torsion.

The case where we have a direct product of the one-generator Coxeter group  $\mathbb{Z}/2$  with one of the above groups, is already absorbed by condition 11.  $\square$

*Proof of theorem 14.* The Davis complex is a finite-dimensional cell complex with a cellular action of the Coxeter group  $\Gamma$  with respect to which it is constructed, such that each cell stabiliser fixes its cell pointwise. Also, it admits the property that the fixed point sets of the finite subgroups of  $\Gamma$  are acyclic [9]. Thus by proposition 5, the  $\Gamma$ -equivariant Farrell cohomology of the Davis complex gives us the  $\ell$ -primary part of the Farrell cohomology of  $\Gamma$ . As the 3-torsion subcomplex for the group generated by the Coxeter diagram  $\mathbf{F}_4$  (the symmetry group of the 24-cell) and the 3- and 5-torsion subcomplexes for the group generated by the Coxeter diagram  $\mathbf{H}_4$  (the symmetry group of the 600-cell) as well as the  $\ell$ -torsion subcomplex of a direct product of two dihedral groups with  $\ell$ -torsion all contain 2-cells, we are either in the case where the  $\ell$ -torsion subcomplex is trivial or in the case in which we suppose to be from now on, namely where  $\Gamma$  is not one of the groups just mentioned. Then all the finite Coxeter subgroups of  $\Gamma$  fulfill the hypothesis of lemma 15, and hence all pairs of a vertex stabiliser and the stabiliser of an adjacent edge satisfy condition 11. By the assumptions on  $Z(\ell)$ , also condition 6 is fulfilled for any pair of adjacent edges in  $Z(\ell)$ . Hence, every connected component of the reduced  $\ell$ -torsion subcomplex is a single vertex. From recursive use of lemma 15 and the assumption that the relator orders are not divisible by  $\ell^2$ , we see that the stabiliser of the latter vertex has the mod  $\ell$  homology of  $\mathcal{D}_\ell$ . Theorem 13 now yields our claim.  $\square$

Let us determine the exponent  $m$  of theorem 14 for some classes of examples. The Coxeter triangle groups are given by the presentation

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1 \rangle,$$

where  $2 \leq p, q, r \in \mathbb{N}$  and  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$ .

**Proposition 16.** *For any prime  $\ell > 2$ , the mod  $\ell$  homology of a Coxeter triangle group is given as the direct sum over the mod  $\ell$  homology of the dihedral groups  $\mathcal{D}_p$ ,  $\mathcal{D}_q$  and  $\mathcal{D}_r$ .*

*Proof.* The quotient space of the Davis complex of a Coxeter triangle group can be realized as the barycentric subdivision of an Euclidean or hyperbolic triangle with interior angles  $\frac{\pi}{p}$ ,  $\frac{\pi}{q}$  and  $\frac{\pi}{r}$ , and  $a$ ,  $b$  and  $c$  acting as reflections through the corresponding sides. We obtain this triangle

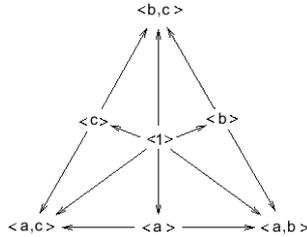


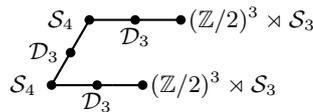
FIGURE 1. Quotient of the Davis complex for a triangle group (diagram reprinted with the kind permission of Sanchez-Garcia [23]).

by realizing the partially ordered set (where arrows stand for inclusions) of Figure 1. The whole Davis complex of the Coxeter triangle groups is then given as a tessellation of the Euclidean or hyperbolic plane by these triangles. The quotient space of the  $\ell$ -torsion subcomplex then consists of one vertex for each of the dihedral groups  $\mathcal{D}_p$ ,  $\mathcal{D}_q$  and  $\mathcal{D}_r$  which contain an element of order  $\ell$ . Theorem 13 now yields the result.  $\square$

**3.1. Results for the Coxeter tetrahedral groups.** Consider the groups that are generated by the reflections on the four sides of a tetrahedron in hyperbolic 3-space, such that the images of the tetrahedron tessellate the latter. Up to isomorphism, there are only thirty-two such groups [11]; and we call them the Coxeter tetrahedral groups  $CT(n)$ , with  $n$  running from 1 through 32.

**Proposition 17.** *For all prime numbers  $\ell > 2$ , the mod  $\ell$  homology of all the Coxeter tetrahedral groups is specified in the tables in figures 2 through 5 in all the cases where it is non-trivial.*

*Proof.* Consider the Coxeter tetrahedral group  $CT(25)$ , generated by the Coxeter diagram . Then the Davis complex of  $CT(25)$  has a strict fundamental domain isomorphic to the barycentric subdivision of the hyperbolic tetrahedron the reflections on the sides of which generate  $CT(25)$  geometrically. A strict fundamental domain for the action on the 3-torsion subcomplex is then the graph



Name	Coxeter graph	3-torsion subcomplex quotient	reduced 3-torsion subcomplex quotient	$H_q(CT(m); \mathbb{Z}/3)$
$CT(1)$		$(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3 \cdot \mathcal{D}_3 \cdot \mathcal{S}_3 \times \mathbb{Z}/2$	$\bullet \mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(2)$		$(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3 \cdot \mathcal{D}_3 \cdot (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$	$\bullet \mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(3)$		$\mathcal{D}_3 \cdot (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3 \cdot \mathcal{S}_4 \cdot \mathcal{D}_3 \cdot (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$	$\bullet \mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(7)$		$\mathcal{D}_6 \cdot \mathcal{D}_6 \times \mathbb{Z}/2$ $\mathcal{D}_3 \cdot (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$	$\bullet \mathcal{D}_6 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(8)$		$\mathcal{D}_3 \cdot \mathcal{D}_3 \times \mathbb{Z}/2$ $\mathcal{D}_3 \cdot \mathcal{S}_4 \cdot \mathcal{D}_3 \cdot \mathcal{S}_4 \cdot \mathcal{D}_3$	$\bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(9)$		$\mathcal{D}_3 \cdot \mathcal{D}_3 \times \mathbb{Z}/2$ $\mathcal{D}_3 \cdot (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$ $\mathcal{D}_3 \cdot (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$	$\bullet \mathcal{D}_3 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^3$
$CT(10)$		$\mathcal{D}_3 \cdot \mathcal{D}_6 \cdot \mathcal{D}_6 \times \mathbb{Z}/2$ $\mathcal{S}_4 \cdot \mathcal{D}_3 \cdot \mathcal{D}_3 \times \mathbb{Z}/2$	$\bullet \mathcal{D}_6 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(11)$		$\mathcal{D}_3 \cdot \mathcal{D}_3 \times \mathbb{Z}/2 \cdot \mathcal{D}_6$	$\bullet \mathcal{D}_6 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(12)$		six copies of $\bullet \mathcal{D}_3$	six copies of $\bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^6$
$CT(13)$		$\bullet \mathcal{D}_6 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3$ $\mathcal{D}_3 \cdot \mathcal{D}_3 \times \mathbb{Z}/2$	$\bullet \mathcal{D}_6 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^4$

FIGURE 2. 3-torsion subcomplexes of the Coxeter tetrahedral groups  $CT(1)$  through  $CT(13)$ , in the cases where they are non-trivial.

where the labels specify the isomorphism types of the stabilisers, namely the dihedral group  $\mathcal{D}_3$ , which also stabilises the edges, the symmetric group  $\mathcal{S}_4$  and the semi-direct product  $(\mathbb{Z}/2)^2 \rtimes \mathcal{S}_3$ . The  $\ell$ -torsion subcomplexes for all greater primes  $\ell$  are empty. By theorem 14, we can reduce the 3-torsion subcomplex to a single vertex and obtain  $H_*(CT(25); \mathbb{Z}/3) \cong H_*(\mathcal{D}_3; \mathbb{Z}/3)$ . For the other Coxeter tetrahedral groups, we proceed analogously.  $\square$

The entries in the tables in figures 2 through 5 have additionally been checked on the machine [10].

Name	Coxeter graph	3-torsion subcomplex quotient	reduced 3-torsion subcomplex quotient	$H_q(CT(m); \mathbb{Z}/3)$
$CT(14)$		$\mathcal{D}_6 \bullet \mathcal{D}_6 \times \mathbb{Z}/2$ $\mathcal{D}_6 \bullet \mathcal{D}_6 \times \mathbb{Z}/2$ $\bullet \mathcal{D}_3$	$\bullet \mathcal{D}_6 \bullet \mathcal{D}_6 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^3$
$CT(15)$		$\mathcal{D}_3 \bullet \mathcal{D}_3 \times \mathbb{Z}/2$ $\mathcal{D}_3 \bullet \mathcal{D}_3 \times \mathbb{Z}/2$ $\bullet \mathcal{D}_6$	$\bullet \mathcal{D}_6 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^3$
$CT(16)$		$\mathcal{D}_3 \bullet \mathcal{S}_4 \bullet \mathcal{D}_3$ $\mathcal{D}_3 \bullet \mathcal{S}_4 \bullet \mathcal{D}_3$ $\bullet \mathcal{D}_3$	$\bullet \mathcal{D}_3 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^3$
$CT(17)$		$\bullet \mathcal{D}_6 \bullet \mathcal{D}_6 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$\bullet \mathcal{D}_6 \bullet \mathcal{D}_6 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^4$
$CT(18)$		$(\mathbb{Z}/2)^3 \times \mathcal{S}_3 \bullet \mathcal{D}_3 \bullet (\mathbb{Z}/2)^3 \times \mathcal{S}_3$ $(\mathbb{Z}/2)^3 \times \mathcal{S}_3 \bullet \mathcal{D}_3 \bullet (\mathbb{Z}/2)^3 \times \mathcal{S}_3$	$\bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(19)$		$(\mathbb{Z}/2)^3 \times \mathcal{S}_3 \bullet \mathcal{D}_3 \bullet \text{Icos}_{120}$	$\bullet \mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(20)$		$\mathcal{D}_3 \bullet \text{Icos}_{120}$ $\mathcal{S}_4 \bullet \mathcal{D}_3 \bullet \text{Icos}_{120}$	$\bullet \mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(21)$		$\text{Icos}_{120} \bullet \mathcal{D}_3 \bullet \text{Icos}_{120}$ $\text{Icos}_{120} \bullet \mathcal{D}_3 \bullet \text{Icos}_{120}$	$\bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(22)$		$\text{Icos}_{120} \bullet \mathcal{D}_3 \bullet \mathcal{D}_3 \times \mathbb{Z}/2$ $\text{Icos}_{120} \bullet \mathcal{D}_3 \bullet \mathcal{D}_3 \times \mathbb{Z}/2$	$\bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(23)$		$\mathcal{S}_4 \bullet \mathcal{D}_3 \bullet \text{Icos}_{120}$ $\mathcal{S}_4 \bullet \mathcal{D}_3 \bullet \text{Icos}_{120}$	$\bullet \mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(24)$		$\text{Icos}_{120} \bullet \mathcal{D}_3 \bullet \text{Icos}_{120}$	$\bullet \mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(25)$		$\mathcal{S}_4 \bullet \mathcal{D}_3 \bullet (\mathbb{Z}/2)^3 \times \mathcal{S}_3$ $\mathcal{S}_4 \bullet \mathcal{D}_3 \bullet (\mathbb{Z}/2)^3 \times \mathcal{S}_3$	$\bullet \mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(26)$		$\text{Icos}_{120} \bullet \mathcal{D}_3 \bullet (\mathbb{Z}/2)^3 \times \mathcal{S}_3$ $\text{Icos}_{120} \bullet \mathcal{D}_3 \bullet (\mathbb{Z}/2)^3 \times \mathcal{S}_3$	$\bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(27)$		$\mathcal{D}_3 \bullet \mathcal{D}_3 \times \mathbb{Z}/2$ $\mathcal{D}_3 \bullet \text{Icos}_{120}$ $\mathcal{D}_3 \bullet \text{Icos}_{120}$	$\bullet \mathcal{D}_3 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^3$

FIGURE 3. 3-torsion subcomplexes of the Coxeter tetrahedral groups  $CT(14)$  through  $CT(27)$ .

Name	Coxeter graph	3-torsion subcomplex quotient	reduced 3-torsion subcomplex quotient	$H_q(CT(m); \mathbb{Z}/3)$
$CT(28)$		$\mathcal{D}_6 \bullet \mathcal{D}_6 \times \mathbb{Z}/2$ $\mathcal{D}_3 \bullet \text{Icos}_{120}$	$\bullet \mathcal{D}_6 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(29)$		$\mathcal{D}_3 \bullet \text{Icos}_{120}$ $\mathcal{D}_3 \bullet \text{Icos}_{120}$	$\bullet \mathcal{D}_6$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^3$
$CT(30)$		$\mathcal{D}_3 \bullet (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$ $\mathcal{D}_3 \bullet (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$	$\bullet \mathcal{D}_6$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^3$
$CT(31)$		$(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$	$\bullet \mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(32)$		$\bullet \mathcal{D}_6$ $\mathcal{D}_3 \bullet \mathcal{S}_4 \bullet \mathcal{D}_3 \bullet \mathcal{S}_4$	$\bullet \mathcal{D}_6 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$

FIGURE 4. 3-torsion subcomplexes of the Coxeter tetrahedral groups  $CT(28)$  through  $CT(32)$ .

Name and Coxeter graph	5-torsion subcomplex quotient	reduced 5-torsion subcomplex quotient	$H_q(CT(m); \mathbb{Z}/5)$
$CT(19), CT(28)$ 	$\text{Icos}_{120} \bullet \mathcal{D}_5 \bullet \mathcal{D}_5 \times \mathbb{Z}/2$	$\bullet \mathcal{D}_5$	$H_q(\mathcal{D}_5; \mathbb{Z}/5)$
$CT(20), CT(22), CT(23),$ 	$\text{Icos}_{120} \bullet \mathcal{D}_5 \bullet \text{Icos}_{120}$	$\bullet \mathcal{D}_5$	$H_q(\mathcal{D}_5; \mathbb{Z}/5)$
$CT(21)$ 	$\text{Icos}_{120} \bullet \mathcal{D}_5 \bullet \text{Icos}_{120}$ $\text{Icos}_{120} \bullet \mathcal{D}_5 \bullet \text{Icos}_{120}$	$\bullet \mathcal{D}_5 \bullet \mathcal{D}_5$	$(H_q(\mathcal{D}_5; \mathbb{Z}/5))^2$
$CT(24)$ 	$\text{Icos}_{120} \bullet \mathcal{D}_5 \bullet \mathcal{D}_5 \times \mathbb{Z}/2$ $\text{Icos}_{120} \bullet \mathcal{D}_5 \bullet \mathcal{D}_5 \times \mathbb{Z}/2$	$\bullet \mathcal{D}_5 \bullet \mathcal{D}_5$	$(H_q(\mathcal{D}_5; \mathbb{Z}/5))^2$

FIGURE 5. 5-torsion subcomplexes of the Coxeter tetrahedral groups, in the cases where they are non-trivial.

## 4. THE CONJUGACY CLASSES OF FINITE ORDER ELEMENTS IN THE BIANCHI GROUPS

The groups  $\mathrm{SL}_2(\mathcal{O}_{-m})$  act in a natural way on hyperbolic three-space, which is isomorphic to the symmetric space  $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$  associated to them. The kernel of this action is the centre  $\{\pm 1\}$  of the groups. Thus it is useful to study the quotient of  $\mathrm{SL}_2(\mathcal{O}_{-m})$  by its centre, namely  $\mathrm{PSL}_2(\mathcal{O}_{-m})$ .

Let  $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-m})$  be a Bianchi group. Then any element of  $\Gamma$  fixing a point inside real hyperbolic 3-space  $\mathcal{H}_{\mathbb{R}}^3$  acts as a rotation of finite order. Let  $X$  be the refined cellular complex obtained from the action of  $\Gamma$  on hyperbolic 3-space as described in [20], namely we subdivide  $\mathcal{H}_{\mathbb{R}}^3$  until the stabiliser in  $\Gamma$  of any cell  $\sigma$  fixes  $\sigma$  pointwise. We achieve this by computing Bianchi's fundamental polyhedron for the action of  $\Gamma$ , taking as preliminary set of 2-cells its facets lying on the Euclidean hemispheres and vertical planes of the upper-half space model for  $\mathcal{H}_{\mathbb{R}}^3$ , and then subdividing along the rotation axes of the elements of  $\Gamma$ . Let  $\ell$  be a prime number.

It is well-known that if  $\gamma$  is an element of Bianchi group of finite order  $n$ , then  $n$  must be 1, 2, 3, 4 or 6, because  $\gamma$  has eigenvalues  $\rho$  and  $\bar{\rho}$  with  $\rho$  a primitive  $n$ -th root of unity and the trace of  $\gamma$  is  $\rho + \bar{\rho} \in \mathcal{O}_{-m} \cap \mathbb{R} = \mathbb{Z}$ . For  $\ell$  being one of the two occurring primes 2 and 3, this subcomplex is a finite graph, because the cells of dimension greater than 1 are trivially stabilised in the refined cellular complex. We reduce this subcomplex with the procedure of [20], which consists in taking the pairs of edges with a common endpoint such that no further edge is adjacent to this endpoint, and replacing them together with this endpoint by a single edge.

We construct the following graph purely group-theoretically in order to locate conjugacy classes of finite groups on the reduced torsion subcomplex. Let  $\ell$  be a prime number. For a circle to become a graph, we identify the two endpoints of a single edge.

**Definition 18.** *The  $\ell$ -conjugacy classes graph of an arbitrary group  $\Gamma$  is given by the following construction.*

- We take as vertices the conjugacy classes of finite subgroups  $G$  of  $\Gamma$  containing elements  $\gamma$  of order  $\ell$  such that the normaliser of  $\langle \gamma \rangle$  in  $G$  is not  $\langle \gamma \rangle$  itself.
- We connect two vertices by an edge if and only if they admit representatives sharing a common subgroup of order  $\ell$ .
- For every pair of subgroups of order  $\ell$  in  $G$ , which are conjugate in  $\Gamma$  but not in  $G$ , we draw a circle attached to the vertex labelled by  $G$ .
- For every conjugacy class of subgroups of order  $\ell$  which are not properly contained in any finite subgroup of  $\Gamma$ , we add a disjoint circle.

**Theorem 19.** *Let  $\Gamma$  be any Bianchi group with units  $\{\pm 1\}$  and  $\ell$  any prime number. Then the  $\ell$ -conjugacy classes graph and the reduced  $\ell$ -torsion subcomplex of  $\Gamma$  are isomorphic graphs.*

The rest of this section will be devoted to the proof of this theorem. The first ingredient is the following classification of Felix Klein [14].

**Lemma 20** (Klein). *The finite subgroups in  $\mathrm{PSL}_2(\mathcal{O})$  are exclusively of isomorphism types the cyclic groups of orders one, two and three, the Klein four-group  $\mathcal{D}_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ , the symmetric group  $\mathcal{S}_3$  and the alternating group  $\mathcal{A}_4$ .*

Recall the following lemma from [20].

**Lemma 21.** *Let  $v$  be a non-singular vertex in the refined cell complex. Then the number  $\mathbf{n}$  of orbits of edges in the refined cell complex adjacent to  $v$ , with stabiliser in  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  isomorphic*

to  $\mathbb{Z}/\ell$ , is given as follows for  $\ell = 2$  and  $\ell = 3$ .

Isomorphism type of the vertex stabiliser	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$
$\mathbf{n}$ for $\ell = 2$	0	2	0	3	2	1
$\mathbf{n}$ for $\ell = 3$	0	0	2	0	1	2.

Now we investigate the associated normaliser groups. Straight-forward verification using the multiplication tables of the concerned finite groups yields the following.

**Lemma 22.** *Let  $G$  be a finite subgroup of  $\mathrm{PSL}_2(\mathcal{O}_{-m})$ . Then the type of the normaliser of any subgroup of type  $\mathbb{Z}/\ell$  in  $G$  is given as follows for  $\ell = 2$  and  $\ell = 3$ , where we print only cases with existing subgroup of type  $\mathbb{Z}/\ell$ .*

Isomorphism type of $G$	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$
normaliser of $\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathcal{D}_2$	$\mathbb{Z}/2$	$\mathcal{D}_2$
normaliser of $\mathbb{Z}/3$			$\mathbb{Z}/3$		$\mathcal{S}_3$	$\mathbb{Z}/3$ .

The final ingredient in the proof of theorem 19 is the following.

**Theorem 23.** *There is a natural bijection between conjugacy classes of subgroups of  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  of order  $\ell$  and edges of the reduced  $\ell$ -torsion subcomplex. It is given by considering the stabiliser of a representative edge in the refined cell complex.*

In order to prove the latter theorem, we need several lemmata, and we establish them now.

**Lemma 24.** *Consider two adjacent edges  $E, E'$  of the non-reduced torsion subcomplex. Then for any representative  $e$  of  $E$ , there is an adjacent representative  $e'$  of  $E'$  on the same geodesic line as  $e$ .*

*Proof.* Consider the element  $\gamma \in \Gamma$  identifying the end  $v$  of  $e$  with the origin  $\gamma(v)$  of  $e'$ . As  $E$  and  $E'$  are distinct in the orbit space,  $\gamma^{-1}$  cannot send  $e'$  onto  $e$ .

Assume that the stabiliser of  $v$  is of isomorphism type  $\mathbb{Z}/\ell\mathbb{Z}$ . Then as the edge  $\gamma^{-1}(e')$  is pointwise fixed by  $\Gamma$ , its stabiliser must contain the rotation with axis passing through  $e$ . Hence  $\gamma^{-1}(e')$  is adjacent to  $e$  and on the same geodesic line.

If the stabiliser of  $v$  is of isomorphism type  $\mathcal{D}_2, \mathcal{S}_3$  or  $\mathcal{A}_4$ , there are at most two orbits of  $\ell$ -torsion stabilised edges adjacent to  $v$ . So there is an element  $\alpha$  in the stabiliser of  $v$  such that  $\alpha\gamma^{-1}(e')$  is adjacent to  $e$  and on the same geodesic line.  $\square$

**Corollary 25.** *Any edge of the reduced torsion subcomplex can be represented by a chain of edges on the intersection of one geodesic line with a strict fundamental domain for  $\Gamma$  in  $\mathcal{H}$ .*

*Proof.* Consider the chain of edges in the torsion subcomplex that are reduced to the given edge of the reduced torsion subcomplex. The representatives in  $\mathcal{H}$  of the edges on this chain lie on pairwise different orbits because the torsion subcomplex is a subcomplex of the orbit space  $\Gamma \backslash \mathcal{H}$ . Now we start with a representative  $e$  for the edge sharing its origin with the reduced edge, and use lemma 24 to obtain an adjacent edge  $e'$  on the same geodesic line, representing the adjacent edge of the torsion subcomplex. We proceed this way, assigning the role of  $e$  to  $e'$  and so on, until we have reached the end of the reduced edge.  $\square$

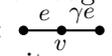
**Corollary 26.** *Any edge of the reduced torsion subcomplex admits only representatives with stabiliser in the same conjugacy class.*

**Lemma 27.** *Let  $\alpha$  and  $\gamma$  be elements of  $\mathrm{PSL}_2(\mathbb{C})$ . Then the fixed point set in  $\mathcal{H}$  of  $\alpha$  is identified by  $\gamma$  with the fixed point set of  $\gamma\alpha\gamma^{-1}$ .*

*Proof.* One immediately checks that any fixed point  $x \in \mathcal{H}$  of  $\alpha$  induces the fixed point  $\gamma(x)$  of  $\gamma\alpha\gamma^{-1}$ . As  $\mathrm{PSL}_2(\mathbb{C})$  acts by isometries, the whole fixed point sets are identified.  $\square$

**Lemma 28.** *Let  $v \in \mathcal{H}_{\mathbb{R}}^3$  be a vertex with stabiliser in  $\Gamma$  of type  $\mathcal{D}_2$  or  $\mathcal{A}_4$ . Let  $\gamma \in \Gamma$  be a rotation of order 2 around an edge  $e$  adjacent to  $v$ .*

*Then the centraliser  $C_{\Gamma}(\gamma)$  reflects  $\mathcal{H}^{\gamma}$  — which is the geodesic line through  $e$  — onto itself at  $v$ .*

*Proof.* Denote by  $\Gamma_v$  the stabiliser of the vertex  $v$ . In the case that  $\Gamma_v$  is of type  $\mathcal{D}_2$ , which is Abelian, it admits two order-2-elements centralising  $\gamma$  and turning the geodesic line through  $e$  onto itself such that the image of  $e$  touches  $v$  from the side opposite to  $e$  (illustration: ). In the case that  $\Gamma_v$  is of type  $\mathcal{A}_4$ , it contains a normal subgroup of type  $\mathcal{D}_2$  that admits again two such elements.  $\square$

Let  $\alpha$  be any torsion element in  $\Gamma$ . We construct the *chain of edges associated to  $\alpha$*  as follows. Consider the edge of the reduced torsion subcomplex to which the edge stabilised by  $\alpha$  belongs. Use corollary 25 to represent it by a chain of edges on a geodesic line. Now,  $\alpha$  is conjugate to an element  $\gamma\alpha\gamma^{-1}$  of the stabiliser of one of the edges in the chain. By lemma 27, the element  $\gamma^{-1} \in \Gamma$  maps the mentioned geodesic line to the rotation axis of  $\alpha$ . The image under  $\gamma^{-1}$  of the chain of edges under consideration is the desired chain associated to  $\alpha$ . It is well defined up to translation along the rotation axis of  $\alpha$ .

**Lemma 29.** *Let  $\alpha$  be any 2-torsion element in  $\Gamma$ . Then the chain of edges associated to  $\alpha$  is a fundamental domain for the action of the centraliser of  $\alpha$  on the rotation axis of  $\alpha$ .*

*Proof.* We distinguish the following two cases of how  $\langle \alpha \rangle \cong \mathbb{Z}/2$  is included into  $\Gamma$ .

$\circlearrowright$  Suppose that there is no subgroup of type  $\mathcal{D}_2$  in  $\Gamma$  which contains  $\langle \alpha \rangle$ . Then the connected component of the 2-torsion subgraph to which any edge stabilised by  $\langle \alpha \rangle$  belongs, is homeomorphic to a circle. Choose a first edge  $e$  in the chain associated to  $\alpha$ . We have an identification  $\gamma$  between its end and the origin of the next edge  $e'$ , and by lemma 24 we can choose it such that the edge stabilisers are conjugate under  $\gamma$ . We can write  $\Gamma_e = \langle \alpha \rangle$  and  $\Gamma_{e'} = \langle \gamma\alpha\gamma^{-1} \rangle$ . By lemma 27, this gives us the identification  $\gamma^{-1}$  from  $e'$  to an edge on the rotation axis of  $\alpha$ , adjacent to  $e$  because of the first condition on  $\gamma$ . We repeat this step until we have attached an edge  $\delta e$  on the orbit of the first edge  $e$ , with  $\delta \in \Gamma$ . As  $\delta$  is an isometry, the whole chain is translated by  $\delta$  from the start at  $e$  to the start at  $\delta e$ . So the group  $\langle \delta \rangle$  acts on the rotation axis with fundamental domain our chain of edges. And  $\delta\alpha\delta^{-1}$  is again the rotation of order 2 around the axis of  $\alpha$ . So,  $\delta\alpha\delta^{-1} = \alpha$  and therefore  $\langle \delta \rangle < C_{\Gamma}(\alpha)$ .

$\bullet\text{---}\bullet$ ,  $\bigoplus$ ,  $\bigcirc\text{---}\bullet$ ,  $\dots$  Suppose that there is a subgroup  $G$  of  $\Gamma$  of type  $G \cong \mathcal{D}_2$  containing  $\langle \alpha \rangle$ . If there is no further inclusion  $G < G' < \Gamma$  with  $G' \cong \mathcal{A}_4$ , let  $G' := G$ . Then the chain associated to  $\alpha$  can be chosen such that one of its endpoints is stabilised by  $G'$ . The other endpoint of this chain must then lie on a different  $\Gamma$ -orbit, and admit as stabiliser a group  $H'$  containing  $\langle \alpha \rangle$ , of type  $\mathcal{D}_2$  or  $\mathcal{A}_4$ . By lemma 28, each  $G'$  and  $H'$  contain a reflection of the rotation axis of  $\alpha$ , centralising  $\alpha$ . These two reflections must differ from one another because they do not fix the chain of edges. So their free product tessellates the rotation axis of  $\alpha$  with images of the chain of edges associated to  $\alpha$ .  $\square$

**Lemma 30.** *Let  $\alpha$  be any non-trivial torsion element in a Bianchi group  $\Gamma$ . Then the  $\Gamma$ -image of the chain of edges associated to  $\alpha$  contains the rotation axis of  $\alpha$ .*

*Proof.* For  $\alpha$  a 2-torsion element, this follows directly from lemma 29. So we can assume  $\alpha$  to be a 3-torsion element.

• The non-centralising reflections of  $\mathcal{S}_3$ , which is associated to the two vertices, tessellate a geodesic line. Remark that there are exactly three maps  $\mathbb{Z}/3 \rightarrow \mathbb{Z}/3$  induced on the edge stabilisers, given by the three order-2-elements of  $\mathcal{S}_3$ . So, the quotient space by the centraliser of  $\alpha$  must be bigger.

◉ After finitely many translations, all the edges in the fundamental domain conjugate to edges on the geodesic line are obtained as images. Hence, the  $\Gamma$ -image contains the whole geodesic line.  $\square$

*Proof of theorem 23.* Corollary 26 associates a conjugacy class of type  $\mathbb{Z}/\ell$  to each edge of the reduced  $\ell$ -torsion subcomplex. Conversely, let  $\alpha$  and  $\gamma\alpha\gamma^{-1}$  be conjugate torsion elements of  $\Gamma$ . We want to show that they stabilise edges representing the same reduced edge. By Klein, we know that the torsion elements are elliptic and hence fix some geodesic line. Every torsion element acts as the stabiliser of a line conjugate to one passing through the Bianchi fundamental polyhedron. So, it is conjugate to one of the representative edge stabilisers. By lemma 27, we know that the line fixed by  $\alpha$  is sent by  $\gamma$  to the line fixed by  $\gamma\alpha\gamma^{-1}$ .

By lemmata 29 and 30, the union of the  $\Gamma$ -images of the chain associated to  $\alpha$  contains the whole geodesic line fixed by  $\alpha$  (because then, as the  $\Gamma$ -action is cellular, any cell stabilised by  $\gamma\alpha\gamma^{-1}$  admits a cell on its orbit stabilised by  $\alpha$ ). So it follows that all the edges on a geodesic line belong to the same reduced edge.  $\square$

*Proof of theorem 19.* Comparing with lemma 21, we see that the vertex set of the  $\ell$ -conjugacy classes graph gives precisely the bifurcation points and vertices with only one adjacent edge of the  $\ell$ -torsion subcomplex. When passing to the reduced  $\ell$ -torsion subcomplex, we get rid of all vertices with two adjacent edges except in the disjoint circles, see [20]. By theorem 23, the edges of the  $\ell$ -conjugacy classes graph give the edges of the reduced  $\ell$ -torsion subcomplex.  $\square$

## 5. THE FARRELL COHOMOLOGY OF THE BIANCHI GROUPS

In this section, we are going to prove theorem 4. In order to compare with Kramer's formulae that we evaluate in the appendix, we make use of his notations for the numbers of conjugacy classes of the five types of non-trivial finite subgroups in the Bianchi groups. His symbols for these numbers are printed in the first row of the below table, and the second row gives the symbol for the type of counted subgroup.

$\mu_2$	$\mu_T$	$\mu_3$	$\lambda_{2n}$	$\lambda_4^T$	$\lambda_4^*$	$\lambda_6^*$	$\mu_2^-$
$\mathcal{D}_2$	$\mathcal{A}_4$	$\mathcal{S}_3$	$\mathbb{Z}/n$	$\mathbb{Z}/2 \subset \mathcal{A}_4$	$\mathbb{Z}/2 \subset \mathcal{D}_2$	$\mathbb{Z}/3 \subset \mathcal{S}_3$	$\mathcal{D}_2 \not\subset \mathcal{A}_4$

Here, the inclusion signs “ $\subset$ ” mean that we only consider copies of  $\mathbb{Z}/n$  admitting the specified inclusion in the given Bianchi group and  $\mathcal{D}_2 \not\subset \mathcal{A}_4$  means that we only consider copies of  $\mathcal{D}_2$  not admitting any inclusion into a subgroup of type  $\mathcal{A}_4$  of the Bianchi group.

Note that the number  $\mu_2^-$  is simply the difference  $\mu_2 - \mu_T$ , because every copy of  $\mathcal{A}_4$  admits precisely one normal subgroup of type  $\mathcal{D}_2$ . Also, note the following graph-theoretical properties

of the reduced torsion subgraphs, the latter of which we obtain by restricting our attention to the connected components not homeomorphic to  $\mathcal{O}$ .

**Corollary 31** (Corollary to lemma 21). *For all Bianchi groups with units  $\{\pm 1\}$ , the numbers of conjugacy classes of finite subgroups satisfy  $\lambda_4^T \leq \mu_T$  and  $2\lambda_6^* = \mu_3$ , and even*

$$2\lambda_4^* = \mu_T + 3\mu_2^-.$$

The values given by Krämer’s formulae are matching with the values computed with [21].

**Observation 32.** The numbers of conjugacy classes of finite subgroups determine the 3-conjugacy classes graph and hence the reduced 3-torsion subcomplex for all Bianchi groups with units  $\{\pm 1\}$ , as we can see immediately from theorem 19 and the description of the reduced 3-torsion subcomplex in [20].

For the proof of theorem 4, we need the following ingredients.

**Remark 33.** In the equivariant spectral sequence converging to the Farrell cohomology of  $\mathrm{PSL}_2(\mathcal{O}_{-m})$ , the restriction of the differential to maps between cohomology groups of cells that are not adjacent in the orbit space, are zero. So, the  $\ell$ -primary part of the degree-1-differentials of this sequence can be decomposed as a direct sum of the blocks associated to the connected components of the  $\ell$ -torsion subcomplex (Compare with sub-lemma 45 of [20]).

**Lemma 34** (Schwermer/Vogtmann). *Let  $M$  be  $\mathbb{Z}$  or  $\mathbb{Z}/2$ . Consider group homology with trivial  $M$ -coefficients. Then the following holds.*

- Any inclusion  $\mathbb{Z}/2 \rightarrow \mathcal{S}_3$  induces an injection on homology.
- An inclusion  $\mathbb{Z}/3 \rightarrow \mathcal{S}_3$  induces an injection on homology in degrees congruent to 3 or 0 mod 4, and is otherwise zero.
- Any inclusion  $\mathbb{Z}/2 \rightarrow \mathcal{D}_2$  induces an injection on homology in all degrees.
- An inclusion  $\mathbb{Z}/3 \rightarrow \mathcal{A}_4$  induces injections on homology in all degrees.
- An inclusion  $\mathbb{Z}/2 \rightarrow \mathcal{A}_4$  induces injections on homology in degrees greater than 1, and is zero on  $H_1$ .

For the proof in  $\mathbb{Z}$ -coefficients, see [24], for  $\mathbb{Z}/2$ -coefficients see [20].

**Lemma 35** ([20], lemma 32). *Let  $q \geq 3$  be an odd integer number. Let  $v$  be a vertex representative of stabiliser type  $\mathcal{D}_2$  in the refined cellular complex for the Bianchi groups. Then the three images in  $(H_q(\mathcal{D}_2; \mathbb{Z}))_{(2)}$  induced by the inclusions of the stabilisers of the edges adjacent to  $v$ , are linearly independent.*

Finally, we establish the following last ingredient for the proof of theorem 4, which might be of interest in its own right.

**Lemma 36.** *In all rows  $q > 1$  and outside connected components of type  $\mathcal{O}$ , the  $d_{p,q}^1$ -differential of the equivariant spectral sequence converging to  $H_{p+q}(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$  is always injective.*

*Proof.* For matrix blocks of the  $d_{p,q}^1$ -differential associated to vertices with just one adjacent edge, we see from lemma 21 that the vertex stabiliser is of type  $\mathcal{A}_4$  in 2-torsion, respectively of type  $\mathcal{S}_3$  in 3-torsion, so injectivity follows from lemma 34. As we have placed ourselves outside connected components of type  $\mathcal{O}$ , the remaining vertices are bifurcation points of stabiliser type  $\mathcal{D}_2$  and injectivity follows from lemma 35.  $\square$

*Proof of theorem 4.* In 3-torsion, theorem 4 follows directly from observation 32, corollary 31 and theorem 13. In 2-torsion, what we need to determine with the numbers of conjugacy classes of finite subgroups, is the 2-primary part of the  $E_{p,q}^2$ -term of the equivariant spectral sequence converging to  $H_{p+q}(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$  in all rows  $q > 1$ . From there, we see from theorem 13 that we obtain the claim. By remark 33, we only need to check this determination on each homeomorphism type of connected components of the 2-torsion subgraph. We use theorem 19 to identify the reduced 2-torsion subgraph and the 2-conjugacy classes graph. Then we can observe that

- Kramer's number  $\lambda_4^* - \lambda_4$  determines the number of connected components of type  $\odot$ .
- Kramer's number  $\lambda_4^*$  determines the number of edges of the 2-torsion subgraph outside connected components of type  $\odot$ . Lemma 36 tells us that the block of the  $d_{p,q}^1$ -differential of the equivariant spectral sequence associated to such edges is always injective.
- Kramer's number  $\mu_2^-$  determines the number of bifurcation points, and  $\mu_T$  determines the number of vertices with only one adjacent edge of the 2-torsion subgraph.

Using corollary 31, we obtain the explicit formulae in theorem 4.  $\square$

## 6. THE COHOMOLOGY RING STRUCTURE OF THE BIANCHI GROUPS

In [5], Berkove has found a compatibility of the cup product of the cohomology ring of a Bianchi group with the cup product of the cohomology rings of its finite subgroups. This compatibility within the equivariant spectral sequence implies that all products that come from different connected components of the reduced torsion subcomplex (which we turn into the conjugacy classes graph in section 4) are zero. It follows that the cohomology ring of any Bianchi group splits into a restricted sum over subrings, which depend in degrees above the virtual cohomological dimension only on the homeomorphism type of the associated connected component of the reduced torsion subcomplex. The analogue in cohomology of theorem 4 and Berkove's computations of sample cohomology rings [4] yield the following corollary in 3-torsion.

We use Berkove's notation, in which the degree  $j$  of a cohomology generator  $x_j$  is appended as a subscript. Furthermore, writing cohomology classes inside square brackets means that they are polynomial (of infinite multiplicative order), and writing them inside parentheses means that they are exterior (their powers vanish). The restricted sum  $\oplus$  identifies all the degree zero classes into a single copy of  $\mathbb{Z}$ ; when we write it with a power, we specify the number of summands. Recall that  $\lambda_6$  (respectively  $\mu_3$ ) counts the number of conjugacy classes of subgroups of type  $\mathbb{Z}/3$  (respectively  $\mathcal{S}_3$ ) in the Bianchi group.

**Corollary 37.** *In degrees above the virtual cohomological dimension, the 3-primary part of the cohomology ring of any Bianchi group  $\Gamma$  with units  $\{\pm 1\}$  is given by*

$$H^*(\Gamma; \mathbb{Z})_{(3)} \cong \tilde{\oplus}^{(\lambda_6 - \frac{\mu_3}{2})} \mathbb{Z}[x_2](\sigma_1) \tilde{\oplus}^{\frac{\mu_3}{2}} \mathbb{Z}[x_4](x_3),$$

where the generators  $x_j$  are of additive order 3.

In 2-torsion, it does in general not suffice to know only the numbers of conjugacy classes of finite subgroups to obtain the cohomology ring structure, because for the two reduced 2-torsion subcomplexes  $\odot \bullet \odot \bullet$  and  $\odot \bullet \bullet$ , we obtain the same numbers of conjugacy classes and homological 2-torsion, but different multiplicative structures of the mod-2 cohomology rings, as we can see from table 1, which we compile from the results of [5] (and [20]).

$T$	subring associated to connected components of type $T$ in 2 – conjugacy classes graph
	$\mathbb{F}_2[n_1](m_1)$
	$\mathbb{F}_2[m_3, u_2, v_3, w_3]/\langle m_3 v_3 = 0, \quad u_2^3 + w_3^2 + v_3^2 + m_3^2 + w_3(v_3 + m_3) = 0 \rangle$
	$\mathbb{F}_2[n_1, m_2, n_3, m_3]/\langle n_1 n_3 = 0, \quad m_2^3 + m_3^2 + n_3^2 + m_3 n_3 + n_1 m_2 m_3 = 0 \rangle$
	$\mathbb{F}_2[n_1, m_1, m_3]/\langle m_3(m_3 + n_1^2 m_1 + n_1 m_1^2) = 0 \rangle$

TABLE 1. Restricted summands of the mod-2 cohomology ring  $H^*(\Gamma; \mathbb{F}_2)$  of a Bianchi group  $\Gamma$  above its virtual cohomological dimension.

**Observation 38.** In the cases of class numbers 1 and 2, only the homeomorphism types  $T$  listed in table 1 occur as connected components in the reduced 2-torsion subcomplex. So for all such Bianchi groups  $\Gamma$  with units  $\{\pm 1\}$ , the mod-2 cohomology ring  $H^*(\Gamma; \mathbb{F}_2)$  splits, above the virtual cohomological dimension, as a restricted sum over the subrings specified in table 1, with powers according to the multiplicities of the occurrences of the types  $T$ .

## APPENDIX A. NUMERICAL EVALUATION OF KRÄMER'S FORMULAE

**A.1. Numbers of conjugacy classes in 3-torsion.** Denote by  $\delta$  the number of finite ramification places of  $\mathbb{Q}(\sqrt{-m})$  over  $\mathbb{Q}$ . Let  $k_+$  be the totally real number field  $\mathbb{Q}(\sqrt{3m})$  and denote its ideal class number by  $h_{k_+}$ . Krämer introduces the following indicators:

$$z := \begin{cases} 2, & \text{if 3 is the norm of an integer of } k_+, \\ 1, & \text{otherwise.} \end{cases}$$

For  $m \equiv 0 \pmod{3}$  and  $m \neq 3$ , denote by  $\epsilon := \frac{1}{2}(a + b\sqrt{\frac{m}{3}}) > 1$  the fundamental unit of  $k_+$  (where  $a, b \in \mathbb{N}$ ). Now, define

$$x' := \begin{cases} 2, & \text{if the norm of } \epsilon \text{ is } 1, \\ 1, & \text{if the norm of } \epsilon \text{ is } -1 \end{cases}$$

and

$$y := \begin{cases} 2, & \text{if } b \equiv 0 \pmod{3}, \\ 1, & \text{otherwise.} \end{cases}$$

Then [16, 20.39 and 20.41] yield the following formulae in 3-torsion.

$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$	$\lambda_6^*$	$\lambda_6 - \lambda_6^*$
$m \equiv 2 \pmod{3}$	0	$\frac{z}{2}h_{k_+}$
$m \equiv 1 \pmod{3}$ gives either	$2^{\delta-1}$	$\frac{1}{2}(h_{k_+} - 2^{\delta-1})$
or	0	$\frac{1}{2}h_{k_+}$
$m \equiv 6 \pmod{9}$	0	$x'yh_{k_+}$
$m \equiv 3 \pmod{9}$ , $m \neq 3$ gives either	$2^{\delta-2}$	$\frac{1}{2}(3x'h_{k_+} - 2^{\delta-2})$
or	0	$\frac{1}{2}3x'h_{k_+}$

The above case distinctions come from the fact that Krämer's theorem 20.39 ranges over all types of maximal orders in quaternion algebras over  $\mathbb{Q}(\sqrt{-m})$ , in which Krämer determines the numbers of conjugacy classes in the norm-1-group. The remaining task in order to decide which of the cases applies, is to find out of which type considered in the mentioned theorem is the maximal order  $M_2(\mathcal{O}_{-m})$ . Some methods to cope with this task are introduced in [16, §27].

Krämer’s resulting criteria can be summarised as follows for 3-torsion.

condition	implication
$m \equiv 2 \pmod{3}$	$\mu_3 = \lambda_6^* = 0.$
$m \equiv 6 \pmod{9}$	$\mu_3 = \lambda_6^* = 0.$
$m$ prime and $m \equiv 1 \pmod{3}$	$\lambda_6^* > 0.$
$m = 3p$ with $p$ prime and $p \equiv 1 \pmod{3}$	$\lambda_6^* > 0.$
$m \equiv 1 \pmod{3}$ and $-3$ occurs as norm on $\mathcal{O}_{k_+}$	$\lambda_6^* > 0.$
$m \equiv 1 \pmod{3}$ and $-3$ does not occur as norm on $\mathcal{O}_{k_+}$	$\lambda_6 - \lambda_6^* > 0.$
$m \equiv 1 \pmod{3}$ and $m$ admits a prime divisor $p$ with $p \equiv 2 \pmod{3}$	$\lambda_6 - \lambda_6^* > 0.$
$m \equiv 3 \pmod{9}$ and $x' = 1$ and $\frac{m}{3}$ admits only prime divisors $p$ with $p \equiv 1 \pmod{12}$	$\lambda_6^* > 0.$
$m \equiv 3 \pmod{9}$ and $x' = 1$ and $\frac{m}{3}$ admits a prime divisor $p$ with $p \equiv 5 \pmod{12}$	$\lambda_6^* = 0.$
$m \equiv 3 \pmod{9}$ and $h(k'_+) = 2^{\delta-3}$ and $\frac{m}{3}$ admits only prime divisors $p$ with $p \equiv \pm 1 \pmod{12}$ or $p = 2$	$\lambda_6^* = 0.$
$m \equiv 3 \pmod{9}$ and $h(k'_+) = 1$ and $\frac{m}{3} = p'p$ with $p', p$ prime and $p' \equiv p \equiv 7 \pmod{12}$	$\lambda_6^* > 0.$

In order to determine Krämer’s indicator  $z$ , we need to determine if a given value occurs as the norm on the ring of integers of an imaginary quadratic number field. This is implemented in Pari/GP [1] (the first step is computing the answer under the Generalised Riemann hypothesis, and the second step is a check computation which confirms that we arrive at that answer without this hypothesis). Additionally, we compare with the below criterion [16, (20.13)].

**Lemma 39** (Krämer). *Let  $m$  be not divisible by 3.*

- *If the number  $-3$  is the norm of an integer in the totally real number field  $k_+$ , then all prime divisors  $p \in \mathbb{N}$  of  $m$  satisfy the congruence  $p \equiv 1 \pmod{3}$ .  
Especially, the congruence  $m \equiv 1 \pmod{3}$  is implied.*
- *If the number  $3$  is the norm of an integer in the totally real number field  $k_+$ , then all prime divisors  $p \in \mathbb{N}$  of  $m$  satisfy either  $p = 2$  or the congruence  $p \equiv \pm 1 \pmod{12}$ .  
Additionally, the congruence  $m \equiv 2 \pmod{3}$  is implied.*

With Kramer's criteria at hand, we can decide for many Bianchi groups, which of the alternative cases in Kramer's formulae must be used. We do this in the below tables for all such Bianchi groups  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  with absolute value of the discriminant  $\Delta$  ranging between 7 and

2003, where we recall that the discriminant is  $\Delta = \begin{cases} -m, & m \equiv 3 \pmod{4}, \\ -4m, & \text{else.} \end{cases}$

In the cases  $m \in \{102, 133, 165, 259, 559, 595, 763, 835, 1435\}$ , where these statements are not sufficient to eliminate the wrong alternatives, we insert the results of [21]. This way, the below tables treat all Bianchi groups with units  $\{\pm 1\}$  and discriminant of absolute value less than 615. The cases where an ambiguity remains (so to exclude them from our tables) are  $m \in \{210, 262, 273, 298, 345, 426, 430, 462, 481, 615, 1155, 1159, 1195, 1339, 1351, 1407, 1515, 1807\}$ . For tables of the cases without ambiguity, with  $m$  ranging up to 10000, see the preprint version 2 of this paper on HAL.

In [15], a theorem is established which solves all these ambiguities by giving for each type of finite subgroups in  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  criteria equivalent to its occurrence, in terms of congruence conditions on the prime divisors of  $m$ .

3-conjugacy classes graph	$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 3-conjugacy classes graph
	2, 5, 6, 10, 11, 14, 15, 17, 22, 23, 29, 34, 35, 38, 41, 46, 47, 51, 53, 55, 58, 59, 62, 71, 82, 83, 86, 87, 89, 94, 95, 101, 106, 113, 115, 118, 119, 123, 131, 134, 137, 142, 149, 155, 158, 159, 166, 167, 173, 178, 179, 187, 191, 197, 202, 203, 206, 214, 215, 226, 227, 233, 235, 239, 251, 254, 257, 263, 267, 269, 274, 278, 281, 287, 293, 295, 303, 311, 317, 319, 323, 326, 334, 335, 339, 346, 347, 353, 355, 358, 359, 371, 382, 383, 389, 391, 394, 395, 398, 401, 411, 415, 422, 431, 443, 446, 447, 449, 451, 454, 461, 466, 467, 478, 479, 491, 515, 519, 527, 535, 551, 563, 583, 591, 599, 623, 635, 647, 655, 659, 667, 683, 695, 699, 707, 719, 731, 743, 755, 779, 791, 799, 807, 815, 827, 839, 843, 879, 887, 895, 899, 911, 943, 947, 951, 955, 959, 979, 983, 995, 1003, 1019, 1031, 1055, 1059, 1091, 1103, 1111, 1115, 1135, 1139, 1151, 1163, 1167, 1187, 1207, 1211, 1219, 1223, 1243, 1247, 1255, 1259, 1271, 1283, 1307, 1315, 1343, 1347, 1363, 1367, 1379, 1383, 1411, 1415, 1439, 1487, 1499, 1507, 1511, 1523, 1527, 1535, 1555, 1559, 1563, 1571, 1607, 1631, 1639, 1643, 1655, 1667, 1671, 1707, 1711, 1735, 1751, 1763, 1779, 1787, 1795, 1799, 1811, 1819, 1823, 1835, 1847, 1851, 1883, 1903, 1907, 1915, 1919, 1923, 1927, 1931, 1943, 1959, 1979, 2003,
2 	26, 42, 65, 69, 70, 74, 77, 78, 85, 110, 122, 130, 141, 143, 145, 154, 161, 170, 182, 185, 186, 190, 194, 195, 205, 209, 213, 218, 221, 222, 230, 231, 238, 253, 265, 266, 286, 305, 310, 314, 322, 329, 365, 366, 370, 377, 386, 406, 407, 410, 418, 434, 437, 442, 445, 455, 458, 470, 473, 474, 483, 485, 493, 494, 497, 555, 611, 627, 671, 715, 767, 803, 851, 923, 935, 1015, 1079, 1095, 1199, 1235, 1295, 1311, 1391, 1403, 1455, 1463, 1491, 1495, 1595, 1599, 1615, 1679, 1703, 1739, 1771, 1855, 1887, 1991,
3 	30, 66, 107, 138, 174, 255, 282, 302, 318, 354, 419, 498, 503, 759, 771, 795, 835, 863, 1007, 1319, 1355, 1427, 1479, 1551, 1583, 1619, 1691, 1695, 1871, 1895, 1947, 1967,
4 	33, 105, 114, 146, 177, 249, 258, 285, 290, 299, 321, 330, 341, 357, 374, 385, 393, 402, 413, 429, 465, 482, 595, 663, 915, 987, 1023, 1067, 1239, 1435, 1727, 1743, 1955, 1995,
5 	1043, 1203, 1451,
6 	102, 165, 246, 362, 390, 435, 1335, 1419, 1547,
7 	587, 971,

3-conjugacy classes graph	$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 3-conjugacy classes graph
8 	438, 1131, 1635,
	7, 19, 31, 43, 67, 79, 103, 127, 139, 151, 163, 199, 211, 223, 271, 283, 307, 379, 439, 463, 487, 499, 523, 571, 607, 619, 631, 691, 727, 739, 751, 787, 811, 823, 859, 883, 907, 919, 967, 991, 1039, 1051, 1063, 1123, 1171, 1231, 1279, 1303, 1399, 1423, 1447, 1459, 1471, 1483, 1531, 1543, 1567, 1579, 1627, 1663, 1699, 1723, 1759, 1783, 1831, 1867, 1987, 1999,
	39, 111, 183, 219, 291, 327, 331, 367, 471, 543, 579, 643, 723, 831, 939, 1011, 1047, 1087, 1119, 1191, 1227, 1263, 1291, 1299, 1327, 1371, 1623, 1803, 1839, 1879, 1951, 1983,
	547, 1747,
	687,
	1731,
2 	13, 37, 61, 91, 109, 157, 181, 229, 247, 277, 349, 373, 403, 421, 427, 511, 679, 703, 871, 1099, 1147, 1267, 1591, 1603, 1687, 1891, 1963,
2 	73, 97, 193, 241, 259, 313, 337, 409, 457, 559, 763, 1651, 1939,
2 	21, 57, 93, 129, 201, 309, 381, 397, 399, 417, 453, 489, 651, 903, 1443, 1659, 1767, 1843,
2 	433, 1027, 1387,
2 	237,
4 	217, 301, 469,
4 	133.

A.2. **Numbers of conjugacy classes in 2-torsion.** Denote by  $\delta$  the number of finite ramification places of  $\mathbb{Q}(\sqrt{-m})$  over  $\mathbb{Q}$ . Let  $k_+$  be the totally real number field  $\mathbb{Q}(\sqrt{m})$  and denote its ideal class number by  $h_{k_+}$ . For  $m \neq 1$ , Kramer introduces the following indicators:

$$z := \begin{cases} 2, & \text{if } 2 \text{ is the norm of an integer of } k_+, \\ 1, & \text{otherwise,} \end{cases} \quad q := \begin{cases} 2, & \text{if } \pm 2 \text{ is the norm of an integer of } k_+, \\ 1, & \text{otherwise,} \end{cases}$$

$$w := \begin{cases} 2, & \text{if } \forall \text{ prime divisors } p \text{ of } m \text{ with } p \neq 2 \text{ we have } p \equiv \pm 1 \pmod{8}, \\ 1, & \text{if } m \text{ admits prime divisors } p \equiv \pm 3 \pmod{8}. \end{cases}$$

Furthermore, denote by  $\epsilon := \frac{1}{2}(a + b\sqrt{m}) > 1$  the fundamental unit of  $k_+$  (where  $a, b \in \mathbb{N}$ ). Now, define

$$x := \begin{cases} 2, & \text{if the norm of } \epsilon \text{ is } 1, \\ 1, & \text{if the norm of } \epsilon \text{ is } -1 \end{cases} \quad \text{and} \quad y := \begin{cases} 3, & \text{if } b \equiv 0 \pmod{2}, \\ 1, & \text{if } b \equiv 1 \pmod{2}. \end{cases}$$

Then [16, 26.12 and 26.14] yield the following formulae in 2-torsion.

$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$	$\mu_T$	$\mu_2^-$	$\lambda_4^T$	$\lambda_4^*$	$\lambda_4 - \lambda_4^*$
$m \equiv 7 \pmod{8}$	0	0	0	0	$\frac{z}{2}h_{k_+}$
$m \equiv 3 \pmod{8}$ gives either or (provided that $2^{\delta-1} > 1$ )	$2^\delta$ 0	0 0	$2^{\delta-1}$ 0	$2^{\delta-1}$ 0	$\frac{1}{2}(h_{k_+} - 2^{\delta-1})$ $\frac{1}{2}h_{k_+}$
$m \equiv 2 \pmod{4}$ and $w = 2$ gives either or (provided that $2^{\delta-1} > 1$ )	$2^{\delta-1}$ 0	$2^{\delta-1}$ 0	$2^{\delta-2}z$ 0	$2^\delta$ 0	$\frac{1}{4}x(z+2)h_{k_+} - 2^{\delta-1}$ $\frac{1}{4}x(z+2)h_{k_+}$
$m \equiv 2 \pmod{4}$ and $w = 1$ gives either or or (provided that $2^{\delta-1} > 2$ )	$2^{\delta-1}$ 0 0	0 $2^{\delta-1}$ 0	$2^{\delta-2}$ 0 0	$2^{\delta-2}$ $2^{\delta-2}3$ 0	$\frac{1}{2}(\frac{3}{2}xh_{k_+} - 2^{\delta-2})$ $\frac{3}{2}(\frac{1}{2}xh_{k_+} - 2^{\delta-2})$ $\frac{3}{4}xh_{k_+}$
$m \equiv 1 \pmod{8}$ and $m \neq 1$ and $w = 2$ gives either or (provided that $2^{\delta-2} > 1$ )	$2^{\delta-1}$ 0	$2^{\delta-1}$ 0	$2^{\delta-2}$ 0	$2^\delta$ 0	$2xh_{k_+} - 2^{\delta-1}$ $2xh_{k_+}$
$m \equiv 1 \pmod{8}$ and $w = 1$ gives either or or (provided that $2^{\delta-2} > 2$ )	$2^{\delta-1}$ 0 0	0 $2^{\delta-1}$ 0	$2^{\delta-2}$ 0 0	$2^{\delta-2}$ $2^{\delta-2}3$ 0	$2xh_{k_+} - 2^{\delta-3}$ $2xh_{k_+} - 2^{\delta-3}3$ $2xh_{k_+}$
$m \equiv 5 \pmod{8}$ or (provided that $2^{\delta-2} > 1$ )	0 0	$2^{\delta-1}$ 0	0 0	$2^{\delta-2}3$ 0	$\frac{1}{2}(x(2y+1)h_{k_+} - 2^{\delta-2}3)$ $\frac{1}{2}x(2y+1)h_{k_+}$

The above case distinctions come from the fact that Kramer's theorem 26.12 ranges over all types of maximal orders in quaternion algebras over  $\mathbb{Q}(\sqrt{-m})$ , in which Kramer determines the numbers of conjugacy classes in the norm-1-group. The remaining task in order to decide which of the cases applies, is to find out of which type considered in the mentioned theorem is the maximal order  $M_2(\mathcal{O}_{-m})$ . Some methods to cope with this task are introduced in [16, §27], where Kramer obtains the following criteria for the 2-torsion numbers:

condition	implication
$m \equiv 7 \pmod{8}$	$\mu_T = \mu_2^- = \lambda_4^T = \lambda_4^* = 0.$
$m \equiv 5 \pmod{8}$	$\mu_T = \lambda_4^T = 0.$
$m \equiv 21 \pmod{24}$	$\lambda_4^* = 0.$
$m \equiv 0 \pmod{6}$ and $\lambda_4^* > 0$	$\lambda_4^T > 0.$
$m \equiv 9 \pmod{24}$ and $\lambda_4^* > 0$	$\lambda_4^T > 0.$
$m$ prime and $m \equiv 1$ or $3 \pmod{8}$	$\lambda_4^T > 0.$
$m \equiv 5 \pmod{8}$ and $m$ prime	$\lambda_4^* > 0.$
$m = 2p$ with $p$ prime and $p \equiv 3$ or $5 \pmod{8}$	$\lambda_4^* > 0.$
$m = p'p$ with $p$ and $p'$ prime and $p \equiv p' \equiv 3$ or $5 \pmod{8}$	$\lambda_4^* > 0.$
$m = 3p$ with $p$ prime and $p \equiv 1$ or $3 \pmod{8}$	$\lambda_4^T > 0.$
$m \equiv 1$ or $2 \pmod{4}$ and $m \neq 1$ and $x = 1$	$\lambda_4^* > 0$ and $\mu_2^- > 0.$
$m \equiv 1$ or $2 \pmod{4}$ and $m \neq 1$ and $x = 2$	$\lambda_4 - \lambda_4^* > 0.$
$m \equiv 3 \pmod{8}$ and $-2$ occurs as norm on $\mathcal{O}_{k_+}$	$\lambda_4^* > 0$ and $\lambda_4^T > 0.$
$m \equiv 3 \pmod{8}$ and $-2$ does not occur as norm on $\mathcal{O}_{k_+}$	$\lambda_4 - \lambda_4^* > 0.$
$m \equiv 3 \pmod{8}$ and $m$ admits a prime divisor $p$ with $p \equiv 5$ or $7 \pmod{8}$	$\lambda_4 - \lambda_4^* > 0.$
$m \equiv 1 \pmod{8}$ and $w = 1$ and $h(k_+) = 2^{\delta-3}$	$\mu_2^- = 0.$
$m \equiv 2 \pmod{4}$ and $-2$ occurs as norm on $\mathcal{O}_{k_+}$	$\lambda_4^T > 0.$
$m \equiv 2 \pmod{4}$ and $-2$ does not occur as norm on $\mathcal{O}_{k_+}$ and $h(k_+) = 2^{\delta-2}$	$\lambda_4^* = 0.$
$m \equiv 2 \pmod{4}$ and $q = 1$ and $h(k_+) = 2^{\delta-1}$ and $w = 2$	$\lambda_4^* = 0.$
$m \equiv 2 \pmod{4}$ and $h(k_+) = 2^{\delta-2}$ and $m$ admits a prime divisor $p$ with $p \equiv 5$ or $7 \pmod{8}$	$\lambda_4^* = 0.$

With the above criteria at hand, we can decide for many Bianchi groups, which of the alternative cases in Kramer's formulae must be used. We do this in the below tables for all such Bianchi groups  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  with absolute value of the discriminant  $\Delta$  ranging between 7 and 2003. In the cases  $m \in \{34, 105, 141, 142, 194, 235, 323, 427, 899, 979, 1243, 1507\}$ , where these statements are not sufficient to eliminate the wrong alternatives, we insert the results of [21]. This way, the below tables treat all Bianchi groups with units  $\{\pm 1\}$  and discriminant of absolute value less than 820. The cases where an ambiguity remains (so to exclude them from our tables) are the following values of  $m$ : 205, 221, 254, 273, 305, 321, 322, 326, 345, 377, 381, 385, 386, 410, 438, 465, 469, 473, 482, 1067, 1139, 1211, 1339, 1443, 1763, 1771, 1947.

The above mentioned theorem on subgroup occurrences [15] solves all these ambiguities.

2-torsion homology	$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 2-torsion homology
$P \circlearrowleft$	7, 15, 23, 31, 35, 39, 47, 55, 71, 87, 91, 95, 103, 111, 115, 127, 143, 151, 155, 159, 167, 183, 191, 199, 203, 215, 239, 247, 259, 263, 271, 295, 299, 303, 311, 319, 327, 335, 355, 367, 371, 383, 395, 403, 407, 415, 431, 447, 463, 471, 479, 487, 503, 515, 519, 535, 543, 551, 559, 583, 591, 599, 607, 611, 631, 635, 647, 655, 667, 671, 687, 695, 703, 707, 719, 743, 751, 755, 763, 767, 807, 815, 823, 831, 835, 851, 863, 871, 879, 887, 911, 919, 923, 951, 955, 967, 983, 991, 995, 1007, 1027, 1031, 1039, 1043, 1047, 1055, 1063, 1079, 1099, 1103, 1115, 1119, 1135, 1147, 1151, 1159, 1167, 1195, 1199, 1219, 1231, 1247, 1255, 1263, 1267, 1279, 1303, 1315, 1319, 1355, 1363, 1379, 1383, 1391, 1399, 1403, 1415, 1423, 1439, 1447, 1471, 1487, 1511, 1535, 1543, 1555, 1559, 1583, 1591, 1603, 1607, 1623, 1643, 1651, 1655, 1663, 1671, 1703, 1711, 1727, 1739, 1759, 1783, 1795, 1807, 1823, 1831, 1835, 1839, 1871, 1879, 1883, 1891, 1895, 1903, 1915, 1919, 1939, 1943, 1951, 1959, 1963, 1983, 1991, 1999,
$2P \circlearrowleft$	14, 46, 62, 94, 119, 158, 195, 206, 231, 255, 287, 302, 334, 382, 391, 398, 435, 446, 455, 478, 483, 511, 527, 555, 595, 615, 623, 651, 663, 679, 715, 759, 791, 795, 903, 915, 935, 943, 987, 1015, 1095, 1131, 1207, 1235, 1271, 1295, 1311, 1335, 1343, 1407, 1435, 1455, 1463, 1479, 1491, 1515, 1547, 1551, 1595, 1615, 1631, 1635, 1659, 1687, 1695, 1751, 1767, 1799, 1855, 1887, 1927, 1955, 1967,
$3P \circlearrowleft$	21, 30, 42, 69, 70, 77, 78, 79, 93, 110, 133, 138, 154, 174, 182, 186, 190, 213, 222, 223, 230, 235, 237, 253, 266, 282, 286, 301, 309, 310, 318, 341, 359, 366, 406, 413, 426, 427, 430, 437, 453, 470, 474, 494, 839, 895, 899, 1191, 1223, 1367, 1527, 1567, 1639, 1735, 1847,
$4P \circlearrowleft$	161, 217, 238, 329, 399, 497, 799, 959, 1023, 1155, 1239, 1351, 1679, 1743, 1995,
$5P \circlearrowleft$	439, 727, 1111, 1327,
$6P \circlearrowleft$	142, 165, 210, 285, 330, 357, 390, 429, 434, 462, 1495, 1599,
$7P \circlearrowleft$	141, 1087,
$8P \circlearrowleft$	105,
$2P_{D_2}^*$	5, 10, 13, 26, 29, 53, 58, 61, 74, 106, 109, 122, 149, 157, 173, 181, 202, 218, 277, 293, 298, 314, 317, 362, 394, 397, 421, 458, 461,

2–torsion homology	$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 2–torsion homology
$2P_{\mathcal{D}_2}^* + 2P_{\circlearrowleft}$	37, 101, 197, 269, 349, 373, 389,
$2P_{\mathcal{D}_2}^* + 3P_{\circlearrowleft}$	229, 346,
$4P_{\mathcal{D}_2}^*$	85, 130, 170, 290, 365, 370, 493,
$4P_{\mathcal{D}_2}^* + P_{\circlearrowleft}$	65, 185, 265, 481,
$4P_{\mathcal{D}_2}^* + 3P_{\circlearrowleft}$	442, 445,
$4P_{\mathcal{D}_2}^* + 4P_{\circlearrowleft}$	485,
$4P_{\mathcal{D}_2}^* + 5P_{\circlearrowleft}$	145,
$P_{\mathcal{A}_4}^* + P_{\mathcal{D}_2}^*$	2,
$2P_{\mathcal{A}_4}^*$	11, 19, 43, 59, 67, 83, 107, 131, 139, 163, 179, 211, 227, 251, 283, 307, 331, 347, 379, 419, 467, 491, 523, 547, 563, 571, 587, 619, 643, 683, 691, 739, 787, 811, 827, 859, 883, 907, 947, 971, 1019, 1051, 1123, 1163, 1187, 1259, 1283, 1291, 1307, 1427, 1451, 1459, 1483, 1499, 1531, 1571, 1579, 1619, 1667, 1699, 1723, 1747, 1867, 1931, 1979, 2003,
$2P_{\mathcal{A}_4}^* + P_{\circlearrowleft}$	6, 22, 38, 86, 118, 134, 166, 214, 262, 278, 358, 422, 443, 454, 659, 1091, 1171, 1523, 1627, 1787, 1811, 1907, 1987,
$2P_{\mathcal{A}_4}^* + 2P_{\circlearrowleft}$	499,
$2P_{\mathcal{A}_4}^* + 2P_{\mathcal{D}_2}^*$	17, 41, 73, 89, 97, 113, 137, 193, 233, 241, 281, 313, 337, 353, 409, 433, 449, 457,
$2P_{\mathcal{A}_4}^* + 2P_{\mathcal{D}_2}^* + P_{\circlearrowleft}$	82, 146, 178, 274, 466,
$2P_{\mathcal{A}_4}^* + 2P_{\mathcal{D}_2}^* + 2P_{\circlearrowleft}$	34, 194,
$2P_{\mathcal{A}_4}^* + 2P_{\mathcal{D}_2}^* + 4P_{\circlearrowleft}$	226, 257,
$2P_{\mathcal{A}_4}^* + 2P_{\mathcal{D}_2}^* + 8P_{\circlearrowleft}$	401,
$4P_{\mathcal{A}_4}^*$	51, 123, 187, 267, 339, 411, 451, 699, 771, 779, 803, 843, 1059, 1203, 1347, 1563, 1691, 1707, 1779, 1819, 1843, 1923,
$4P_{\mathcal{A}_4}^* + P_{\circlearrowleft}$	219, 291, 323, 579, 723, 731, 939, 979, 1003, 1011, 1227, 1243, 1371, 1387, 1411, 1507, 1731, 1803,
$4P_{\mathcal{A}_4}^* + 2P_{\circlearrowleft}$	66, 102, 114, 246, 258, 354, 374, 402, 418, 498, 1851,
$4P_{\mathcal{A}_4}^* + 3P_{\circlearrowleft}$	33, 57, 129, 177, 201, 209, 249, 393, 417, 489, 1299,
$8P_{\mathcal{A}_4}^*$	627, 1419.

**A.3. Asymptotic behaviour of the number of conjugacy classes.** From Kramer’s above formulae, we see that both in 2-torsion and in 3-torsion, the number of conjugacy classes of finite subgroups, and hence also the cardinality of the homology of the Bianchi groups in degrees above their virtual cohomological dimension, admits only two factors which are not strictly limited:  $h_{k_+}$  and  $2^\delta$ . As for the ideal class number  $h_{k_+}$ , it is subject to the predictions of the Cohen-Lenstra heuristic [8]. As for the factor  $2^\delta$ , the number  $\delta$  of finite ramification places of  $\mathbb{Q}(\sqrt{-m})$  over  $\mathbb{Q}$  is well-known to equal the number of prime divisors of the discriminant of  $\mathbb{Q}(\sqrt{-m})$ .

The numerical evaluation of Kramer’s formulae provides us with databases which are over a thousand times larger than what is reasonable to print in sections A.1 and A.2. We now give an instance of how these databases can be exploited. Denote the discriminant of  $\mathbb{Q}(\sqrt{-m})$  by  $\Delta$ . In the cases  $m \equiv 3 \pmod{4}$ , we have  $\Delta = -m$ . Denote the number  $\lambda_6 - \lambda_6^*$  of connected components of type  $\mathcal{O}$  in the 3-conjugacy classes graph by  $\lambda'_6(\Delta)$ . Then clearly, the subgroup in  $H_q(\mathrm{PSL}_2(\mathcal{O}_{-m}))$ ,  $q > 2$ , generated by the order-3-elements coming from the connected components of this type, is of order  $3^{\lambda'_6(\Delta)}$ . Denote by  $\mathrm{covolume}(\Delta)$  the volume of the quotient space  $\mathrm{PSL}_2(\mathcal{O}_{-m}) \backslash \mathcal{H}$ . The study of the ratio  $\frac{3^{\lambda'_6(\Delta)}}{\mathrm{covolume}(\Delta)}$  is motivated by the formulae in [3]. In figure 6, we print the logarithm of the average of this ratio over the cases  $|\Delta| \equiv 3 \pmod{4}$ , scaled by a factor  $m^{-\frac{2}{3}}$ , so to say

$$m^{-\frac{2}{3}} \log \left( \frac{1}{\#\{\Delta : |\Delta| \leq m\}} \sum_{|\Delta| \leq m} \frac{3^{\lambda'_6(\Delta)}}{\mathrm{covolume}(\Delta)} \right),$$

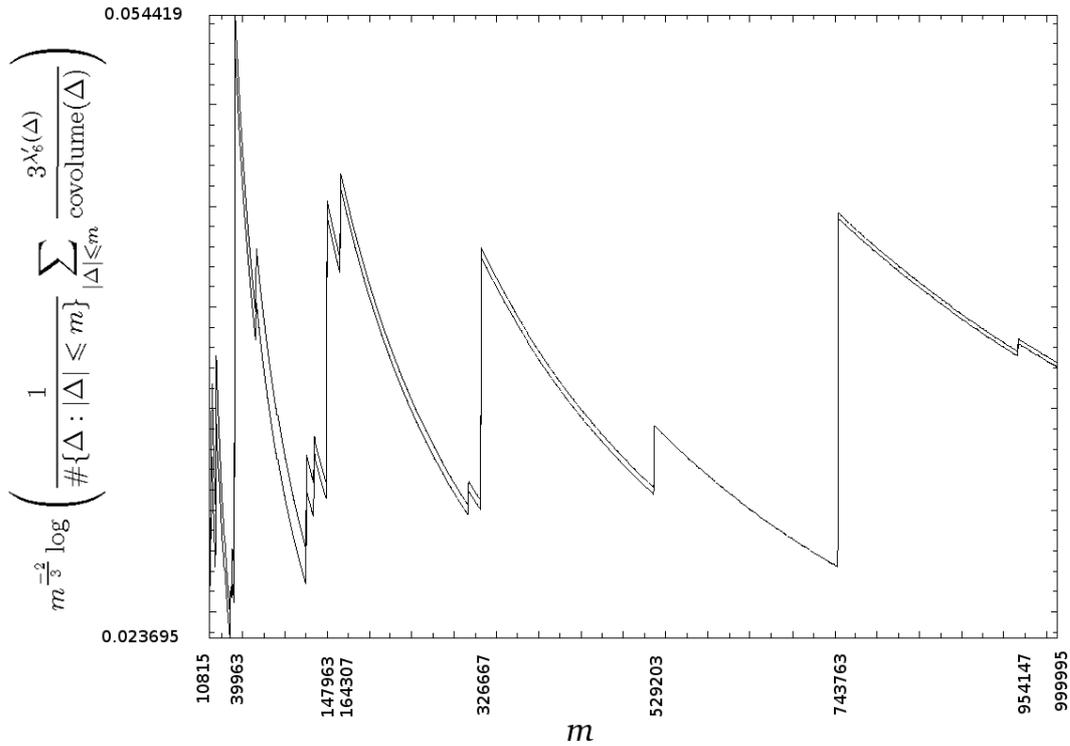
where we consider  $m$  and  $\Delta$  as independent variables,  $m$  running through the square-free positive rational integers. In order to cope with the fact that in some cases, Kramer’s formulae leave an ambiguity, we print a function assuming the lowest possible values of  $\lambda'_6(\Delta)$  and one assuming the highest possible values of  $\lambda'_6(\Delta)$  in the same diagram.

So for  $m$  greater than 10815 and less than one million, we can observe that the average of the above ratio oscillates between  $\exp(m^{\frac{2}{3}}0.023695)$  and  $\exp(m^{\frac{2}{3}}0.054419)$ . For  $m$  less than 10815, this oscillation is much stronger, and the diagram might be seen as suggesting that possibly the oscillation could remain between these two bounds for  $m$  greater than one million.

For related asymptotics, see the recent works of Bergeron/Venkatesh [3] and Sengun [25]. For an alternative computer program treating the Bianchi groups, see the SAGE package of Cremona’s student Aranes [2], and for  $\mathrm{GL}_2(\mathcal{O})$  see Yasaki’s program [30].

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FIGURE 6. Average homological 3-torsion outside subgroups of type  $\mathcal{S}_3$ , scaled as indicated.

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