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# ACCESSING THE FARRELL–TATE COHOMOLOGY OF DISCRETE GROUPS

ALEXANDER D. RAHM

ABSTRACT. We introduce a method to explicitly determine the Farrell–Tate cohomology of discrete groups. We apply this method to the Coxeter triangle groups as well as to the *Bianchi groups*, i.e.  $\mathrm{PSL}_2(\mathcal{O})$  for  $\mathcal{O}$  the ring of integers in an imaginary quadratic number field. We show that the Farrell–Tate cohomology of the Bianchi groups is completely determined by the numbers of conjugacy classes of finite subgroups. In fact, our access to the Farrell–Tate cohomology of the Bianchi groups allows us to detach the information about it from geometric models and to express it only with the group structure. Formulae for the numbers of conjugacy classes of finite subgroups in the Bianchi groups have been determined in a thesis of Krämer, in terms of elementary number-theoretic information on  $\mathcal{O}$ . An evaluation of these formulae for a large number of Bianchi groups is provided numerically in the appendix. Our new insights about the homological torsion allow us to give a conceptual description of the cohomology ring structure of the Bianchi groups.

## 1. INTRODUCTION

Farrell–Tate cohomology  $\widehat{H}^q$  (which we will by now just call Farrell cohomology) is identical to the classical cohomology  $H^q$  of groups in all degrees  $q$  above their virtual cohomological dimension [7]. In section 2, we will introduce a method of how to explicitly determine it: By reducing the conjugacy in the torsion subcomplexes. We apply this to the Coxeter triangle groups in section 3, and as follows to the Bianchi groups.

Denote by  $\mathbb{Q}(\sqrt{-m})$ , with  $m$  a square-free positive integer, an imaginary quadratic number field, and by  $\mathcal{O}_{-m}$  its ring of integers. The *Bianchi groups* are the groups  $\mathrm{PSL}_2(\mathcal{O}_{-m})$ . The Bianchi groups may be considered as a key to the study of a larger class of groups, the *Kleinian groups*, which date back to work of Henri Poincaré [17]. In fact, each non-cocompact arithmetic Kleinian group is commensurable with some Bianchi group [16]. A wealth of information on the Bianchi groups can be found in the monographs [11], [10], [16]. Krämer [15] has determined number-theoretic formulae for the numbers of conjugacy classes of finite subgroups in the Bianchi groups, using numbers of ideal classes in orders of cyclotomic extensions of  $\mathbb{Q}(\sqrt{-m})$ .

In this article, we express the homological torsion of the Bianchi groups as a function of these numbers of conjugacy classes. To achieve this, we build on the geometric techniques of [19], which depend on the explicit knowledge of the quotient space of geometric models for the Bianchi groups — like any technique effectively accessing the (co)homology of the Bianchi groups, either directly [23], [27] or via a group presentation [5]. For the Bianchi groups, we can in this article detach invariants of the group actions from the geometric models, in order to express them only by the group structure itself, in terms of conjugacy classes of finite subgroups,

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normalisers of the latter, and their interactions. This information is gathered, and reduced to its essence, in the conjugacy-reduced torsion subcomplexes we construct for this purpose.

Not only does this provide us with exact formulae for the homological torsion of the Bianchi groups, the power of which we can see in the numerical evaluations of appendices A.1 and A.2, also it allows us to understand the rôle of the centralisers of the finite subgroups, and this is how in [18], some more fruits of the present results are harvested (in terms of the Chen/Ruan orbifold cohomology of the orbifolds given by the action of the Bianchi groups on complexified hyperbolic space).

**Results for the Bianchi groups.** Except for the Gaussian and Eisenstein integers, which can easily be treated separately [23], [19], all the rings of integers of imaginary quadratic number fields admit as only units  $\{\pm 1\}$ . In the latter case, we call  $\mathrm{PSL}_2(\mathcal{O})$  a *Bianchi group with units*  $\{\pm 1\}$ . For the possible types of finite subgroups in the Bianchi groups, see lemma 13 : there are five non-trivial possibilities. Our main result is the following. We make it more explicit in theorem 2 and give its proof in section 5. We deduce the structure of the cohomology ring in section 6.

**Corollary 1.** *The Farrell cohomology of the Bianchi groups with units  $\{\pm 1\}$  depends only on the numbers of conjugacy classes of non-trivial finite subgroups of the occurring five types.*

The main step in order to prove this, is to read off the Farrell cohomology from the conjugacy-reduced torsion subcomplexes.

Krämer's formulae express the numbers of conjugacy classes of the five types of non-trivial finite subgroups in the Bianchi groups, where the symbols in the first row are Krämer's notations for the number of their conjugacy classes:

$\lambda_4$	$\lambda_6$	$\mu_2$	$\mu_3$	$\mu_T$
$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$

Recall that we can express the homology in degrees above the virtual cohomological dimension of the Bianchi groups by the two Poincaré series — for  $\ell = 2$  and  $\ell = 3$  — in the dimensions over the field with  $\ell$  elements, of the homology with  $\mathbb{Z}/\ell$ -coefficients of  $\mathrm{PSL}_2(\mathcal{O}_{-m})$ ,

$$P_m^\ell(t) := \sum_{q>2}^{\infty} \dim_{\mathbb{F}_\ell} H_q(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z}/\ell) t^q,$$

which have been suggested by Grunewald. Further let  $P_{\mathcal{O}}(t) := \frac{-2t^3}{t-1}$ , which equals the series  $P_m^2(t)$  of the groups  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  the conjugacy-reduced 2-torsion subcomplex of which is a circle. Denote by

- $P_{\mathcal{D}_2}^*(t) := \frac{-t^3(3t-5)}{2(t-1)^2}$ , the Poincaré series over  $\dim_{\mathbb{F}_2} H_q(\mathcal{D}_2; \mathbb{Z}/2) - \frac{3}{2} \dim_{\mathbb{F}_2} H_q(\mathbb{Z}/2; \mathbb{Z}/2)$
- and by  $P_{\mathcal{A}_4}^*(t) := \frac{-t^3(t^3-2t^2+2t-3)}{2(t-1)^2(t^2+t+1)}$ , the Poincaré series over

$$\dim_{\mathbb{F}_2} H_q(\mathcal{A}_4; \mathbb{Z}/2) - \frac{1}{2} \dim_{\mathbb{F}_2} H_q(\mathbb{Z}/2; \mathbb{Z}/2).$$

In 3-torsion, let  $P_{\bullet\bullet\bullet}(t) := \frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}$ , which equals the series  $P_m^3(t)$  for the Bianchi groups the conjugacy-reduced 3-torsion subcomplex of which is a single edge without identifications.

**Theorem 2.** *For all Bianchi groups with units  $\{\pm 1\}$ , the homology in degrees above their virtual cohomological dimension is given by the Poincaré series*

$$P_m^2(t) = \left( \lambda_4 - \frac{3\mu_2 - 2\mu_T}{2} \right) P_{\circlearrowleft}(t) + (\mu_2 - \mu_T) P_{\mathcal{D}_2}^*(t) + \mu_T P_{\mathcal{A}_4}^*(t)$$

and

$$P_m^3(t) = \left( \lambda_6 - \frac{\mu_3}{2} \right) P_{\circlearrowleft}(t) + \frac{\mu_3}{2} P_{\bullet\text{---}\bullet}(t).$$

**Organisation of the paper.** In section 2, we introduce our method to explicitly determine Farrell cohomology: By reducing the conjugacy in the torsion subcomplexes. We apply our method to the Coxeter triangle groups in section 3. In section 4, we show how to read off the Farrell cohomology of the Bianchi groups from the conjugacy-reduced torsion subcomplexes. We achieve this by showing that for the Bianchi groups, the conjugacy-reduced torsion subcomplexes are homeomorphic to conjugacy classes graphs that we can define without reference to any geometric model. This enables us in section 5 to prove the formulae for the homological torsion of the Bianchi groups in terms of numbers of conjugacy classes of finite subgroups. We use this to establish the structure of the classical cohomology rings of the Bianchi groups in section 6. Krämer has given number-theoretic formulae for these numbers of conjugacy classes, and we evaluate them numerically in appendices A.1 and A.2. Finally, we present some numerical asymptotics on the numbers of conjugacy classes in appendix A.3.

**Acknowledgements.** The author is indebted to the late great mathematician Fritz Grunewald, for telling him about the existence and providing him a copy of Krämer’s Diplom thesis. Warmest thanks go to Ruben Sanchez-Garcia for providing his implementation of the Davis complex, to Oliver Braunling for a correspondence on the occurrence of given norms on rings of integers, to Nicolas Bergeron for discussions on asymptotics, to Matthias Wendt for a very careful lecture of the manuscript and helpful suggestions, and to Graham Ellis and Stephen S. Gelbart for support and encouragement.

## 2. CONJUGACY REDUCTION OF TORSION SUBCOMPLEXES

Let  $X$  be a finite-dimensional cell complex with a cellular action of a discrete group  $\Gamma$ , such that each cell stabiliser fixes its cell pointwise. Let  $p$  be a prime such that the fixed point set  $X^G$  is acyclic for every non-trivial finite  $p$ -subgroup  $G$  of  $\Gamma$ . Then, the  $\Gamma$ -equivariant Farrell cohomology of  $X$  gives us the  $p$ -primary part of the Farrell cohomology of  $\Gamma$ , as follows.

**Proposition 3** (Brown [7]). *Under the above assumptions, the canonical map*

$$\widehat{H}^*(\Gamma; M)_{(p)} \rightarrow \widehat{H}_\Gamma^*(X; M)_{(p)}$$

*is an isomorphism for any  $\Gamma$ -module  $M$  of coefficients.*

We are going to deduce from  $X$  a cell complex  $Z(p)$  for every prime  $p$ , which still yields the  $p$ -primary part of the Farrell cohomology of  $\Gamma$ , but can have considerably less orbits of cells, be easier to handle in practice, and, for certain classes of groups, lead us to an explicit structural description of the Farrell cohomology of  $\Gamma$ . We will call the complex  $Z(p)$  the conjugacy-reduced  $p$ -torsion subcomplex of  $\Gamma$ . Its pivotal property will be given in theorem 9.

The classical choice [7] is to take for  $X$  the geometric realisation of the partially ordered set of non-trivial finite subgroups (respectively, non-trivial elementary Abelian  $p$ -subgroups) of

$\Gamma$ , the latter acting by conjugation. The stabilisers are then the normalisers, which in many discrete groups are infinite. And it can impose great computational challenges to determine a group presentation for them. When we want to compute the module  $\widehat{H}_\Gamma^*(X; M)_{(p)}$  subject to proposition 3, at least we must get to know the ( $p$ -primary part of the) Farrell cohomology of these normalisers. The Bianchi groups are an instance that different isomorphism types can occur herefor at different conjugacy classes of elementary Abelian  $p$ -subgroups, both for  $p = 2$  and  $p = 3$ . As the only non-trivial elementary Abelian 3-subgroups in the Bianchi groups are of rank 1, the orbit space  $\Gamma \backslash X$  consists only of one point for each conjugacy class of type  $\mathbb{Z}/3$  and a corollary [7] from proposition 3 decomposes the 3-primary part of the Farrell cohomology of the Bianchi groups into the direct product over their normalisers. However, due to the different possible homological types of the normalisers (in fact, two of them occur), the final result remains unclear and subject to tedious case-by-case computations of the normalisers.

In contrast, in the cell complex we are going to develop, the connected components of the orbit space are for the 3-torsion in the Bianchi groups not simple points, but have either the shape  $\bullet\bullet$  or  $\bigcirc$ . This dichotomy already contains the information about the occurring normaliser.

**Definition 4.** *The  $p$ -torsion subcomplex  $X_p$  of a  $\Gamma$ -cell complex  $X$  consists of all the cells of  $X$  the stabilisers in  $\Gamma$  of which contain elements of order  $p$ .*

In  $X_p$ , let  $\sigma$  be a cell of dimension  $n - 1$ , lying in the boundary of precisely two  $n$ -cells  $\tau_1$  and  $\tau_2$ , the latter cells representing two different orbits. Then in the  $p$ -torsion subcomplex, we merge the cells  $\tau_1$  and  $\tau_2$  along  $\sigma$  and do so for their entire orbits, if and only if they meet the following condition.

**Condition 5.** The cell stabilisers admit isomorphisms  $\Gamma_{\tau_1} \cong \Gamma_{\tau_2}$  and either  $\Gamma_{\tau_1} \cong \Gamma_\sigma$  or  $\Gamma_{\tau_1} \cong N_{\Gamma_\sigma}(\text{center}(\text{Sylow}_p(\Gamma_\sigma)))$ .

Here, we write  $N_{\Gamma_\sigma}$  for taking the normaliser in  $\Gamma_\sigma$  and  $\text{Sylow}_p$  for picking an arbitrary Sylow  $p$ -subgroup. This is well defined because all Sylow  $p$ -subgroups are conjugate.

**Lemma 6.** *Let  $\widetilde{X}_p$  be the  $\Gamma$ -complex obtained by orbit-wise merging two  $n$ -cells of  $X_p$  satisfying condition 5. Then,*

$$\widehat{H}_\Gamma^*(\widetilde{X}_p; M)_{(p)} \cong \widehat{H}_\Gamma^*(X_p; M)_{(p)}.$$

For the proof, we will make use of Swan's extension [26, final corollary] to Farrell cohomology of the Second Theorem of Grün [12, Satz 5].

**Theorem 7 (Swan).** *Let  $G$  be a  $p$ -normal finite group, and let  $N$  be the normaliser of the center of a Sylow  $p$ -subgroup of  $G$ . Let  $M$  be any  $G$ -module. Then the inclusion and transfer maps both are isomorphisms between the  $p$ -primary components of  $\widehat{H}^*(G; M)$  and  $\widehat{H}^*(N; M)$ .*

*Proof of lemma 6.* Consider the equivariant spectral sequence in Farrell cohomology [7]. On  $X_p$ , it includes a map

$$\widehat{H}^*(\Gamma_\sigma; M)_{(p)} \xrightarrow{d_1^{(n-1),*} |_{\widehat{H}^*(\Gamma_\sigma; M)_{(p)}}} \widehat{H}^*(\Gamma_{\tau_1}; M)_{(p)} \oplus \widehat{H}^*(\Gamma_{\tau_2}; M)_{(p)},$$

which is the diagonal map with blocks the isomorphisms

$$\widehat{H}^*(\Gamma_\sigma; M)_{(p)} \xrightarrow{\cong} \widehat{H}^*(\Gamma_{\tau_i}; M)_{(p)},$$

induced by the inclusions  $\Gamma_{\tau_i} \hookrightarrow \Gamma_\sigma$ . The latter inclusions induce isomorphisms because of theorem 7. If for the orbit of  $\tau_1$  or  $\tau_2$  we have chosen a representative which is not adjacent to  $\sigma$ , than this isomorphism is composed with the isomorphism induced by conjugation with the element of  $\Gamma$  carrying the cell to one adjacent to  $\sigma$ . Hence, the map  $d_1^{(n-1),*} |_{\widehat{H}^*(\Gamma_\sigma; M)_{(p)}}$  has vanishing kernel, and dividing its image out of  $\widehat{H}^*(\Gamma_{\tau_1}; M)_{(p)} \oplus \widehat{H}^*(\Gamma_{\tau_2}; M)_{(p)}$  gives the  $p$ -primary part  $\widehat{H}^*(\Gamma_{\tau_1 \cup \tau_2}; M)_{(p)}$  of the Farrell cohomology of the union  $\tau_1 \cup \tau_2$  of the two  $n$ -cells.  $\square$

**Definition 8.** *The conjugacy-reduced  $p$ -torsion subcomplex  $Z(p)$  associated to a  $\Gamma$ -cell complex  $X$  is the cell complex obtained by recursively merging orbit-wise all the pairs of cell satisfying condition 5.*

**Theorem 9.** *There is an isomorphism between the  $p$ -primary parts of the Farrell cohomology of  $\Gamma$  and the  $\Gamma$ -equivariant Farrell cohomology of the conjugacy-reduced  $p$ -torsion subcomplex  $Z(p)$ .*

*Proof.* We apply proposition 3 to the cell complex  $X$ , and then we apply lemma 6 each time that we orbit-wise merge a pair of cells of  $X_p$ .  $\square$

### 3. RESULTS FOR THE COXETER TRIANGLE GROUPS

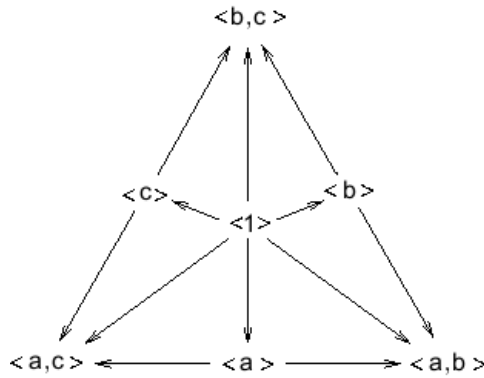


FIGURE 1. Quotient of the Davis complex for a triangle group (diagram by Sanchez-Garcia [22]).

The *Coxeter triangle groups* are given by the presentation

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1 \rangle,$$

where  $2 \leq p, q, r \in \mathbb{N}$  and  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$ .

**Theorem 10.** *For any prime  $\ell > 2$ , the  $\ell$ -primary part of the Farrell cohomology of a Coxeter triangle group is given as the direct sum over the  $\ell$ -primary part of the Farrell cohomology of the dihedral groups  $\mathcal{D}_p$ ,  $\mathcal{D}_q$  and  $\mathcal{D}_r$ .*

*Proof.* The Coxeter triangle groups act cellularly on their Davis complex (a finite-dimensional cell complex), such that each cell stabiliser fixes its cell pointwise. The quotient space of their Davis complex can be realized as the barycentric subdivision of an euclidean or hyperbolic triangle with interior angles  $\frac{\pi}{p}$ ,  $\frac{\pi}{q}$  and  $\frac{\pi}{r}$ , and  $a$ ,  $b$  and  $c$  acting as reflections through the corresponding

sides. We obtain this triangle by realizing the partially ordered set (where arrows stand for inclusions) of Figure 1. The whole Davis complex of the Coxeter triangle groups is then given as a tessellation of the euclidean or hyperbolic plane by these triangles. The quotient space of the  $\ell$ -torsion subcomplex then consists of one vertex for each of the dihedral groups  $\mathcal{D}_p$ ,  $\mathcal{D}_q$  and  $\mathcal{D}_r$  which contain an element of order  $\ell$ . Theorem 9 now yields the result.  $\square$

As Coxeter groups have trivial rational cohomology [9], the integral cohomology ring of the Coxeter triangle groups is now obtained from a computation of their 2-torsion homology.

#### 4. THE CONJUGACY CLASSES OF FINITE ORDER ELEMENTS IN THE BIANCHI GROUPS

The groups  $\mathrm{SL}_2(\mathcal{O}_{-m})$  act in a natural way on hyperbolic three-space, which is isomorphic to the symmetric space  $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$  associated to them. The kernel of this action is the centre  $\{\pm 1\}$  of the groups. Thus it is useful to study the quotient of  $\mathrm{SL}_2(\mathcal{O}_{-m})$  by its centre, namely  $\mathrm{PSL}_2(\mathcal{O}_{-m})$ .

Let  $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-m})$  be a Bianchi group. Then any element of  $\Gamma$  fixing a point inside real hyperbolic 3-space  $\mathcal{H}_{\mathbb{R}}^3$  acts as a rotation of finite order. Let  $X$  be the refined cellular complex obtained from the action of  $\Gamma$  on hyperbolic 3-space as described in [19], namely we subdivide  $\mathcal{H}_{\mathbb{R}}^3$  until the stabiliser in  $\Gamma$  of any cell  $\sigma$  fixes  $\sigma$  pointwise. We achieve this by computing Bianchi's fundamental polyhedron for the action of  $\Gamma$ , taking as preliminary set of 2-cells its facets lying on the Euclidean hemispheres and vertical planes of the upper-half space model for  $\mathcal{H}_{\mathbb{R}}^3$ , and then subdividing along the rotation axes of the elements of  $\Gamma$ . Let  $\ell$  be a prime number.

It is well-known that if  $\gamma$  is an element of Bianchi group of finite order  $n$ , then  $n$  must be 1, 2, 3, 4 or 6, because  $\gamma$  has eigenvalues  $\rho$  and  $\bar{\rho}$  with  $\rho$  a primitive  $n$ -th root of unity and the trace of  $\gamma$  is  $\rho + \bar{\rho} \in \mathcal{O}_{-m} \cap \mathbb{R} = \mathbb{Z}$ . For  $\ell$  being one of the two occurring primes 2 and 3, this subcomplex is a finite graph, because the cells of dimension greater than 1 are trivially stabilised in the refined cellular complex. We reduce this subcomplex with the procedure of [19], which consists in taking the pairs of edges with a common endpoint such that no further edge is adjacent to this endpoint, and replacing them together with this endpoint by a single edge.

We construct the following graph purely group-theoretically in order to locate conjugacy classes of finite groups on the conjugacy-reduced torsion subcomplex. Let  $\ell$  be a prime number. For a circle to become a graph, we identify the two endpoints of a single edge.

**Definition 11.** *The  $\ell$ -conjugacy classes graph of an arbitrary group  $\Gamma$  is given by the following construction.*

- We take as vertices the conjugacy classes of finite subgroups  $G$  of  $\Gamma$  containing elements  $\gamma$  of order  $\ell$  such that the normaliser of  $\langle \gamma \rangle$  in  $G$  is not  $\langle \gamma \rangle$  itself.
- We connect two vertices by an edge if and only if they admit representatives sharing a common subgroup of order  $\ell$ .
- For every pair of subgroups of order  $\ell$  in  $G$ , which are conjugate in  $\Gamma$  but not in  $G$ , we draw a circle attached to the vertex labelled by  $G$ .
- For every conjugacy class of subgroups of order  $\ell$  which are not properly contained in any finite subgroup of  $\Gamma$ , we add a disjoint circle.

**Theorem 12.** *Let  $\Gamma$  be any Bianchi group with units  $\{\pm 1\}$  and  $\ell$  any prime number. Then the  $\ell$ -conjugacy classes graph and the reduced  $\ell$ -torsion subcomplex of  $\Gamma$  are isomorphic graphs.*

The rest of this section will be devoted to the proof of this theorem. The first ingredient is the following classification of Felix Klein [13].

**Lemma 13** (Klein). *The finite subgroups in  $\mathrm{PSL}_2(\mathcal{O})$  are exclusively of isomorphism types the cyclic groups of orders one, two and three, the Klein four-group  $\mathcal{D}_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ , the symmetric group  $\mathcal{S}_3$  and the alternating group  $\mathcal{A}_4$ .*

Recall the following lemma from [19].

**Lemma 14.** *Let  $v$  be a non-singular vertex in the refined cell complex. Then the number  $\mathbf{n}$  of orbits of edges in the refined cell complex adjacent to  $v$ , with stabiliser in  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  isomorphic to  $\mathbb{Z}/\ell$ , is given as follows for  $\ell = 2$  and  $\ell = 3$ .*

Isomorphism type of the vertex stabiliser	{1}	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$
$\mathbf{n}$ for $\ell = 2$	0	2	0	3	2	1
$\mathbf{n}$ for $\ell = 3$	0	0	2	0	1	2.

Now we investigate the associated normaliser groups. Straight-forward verification using the multiplication tables of the concerned finite groups yields the following.

**Lemma 15.** *Let  $G$  be a finite subgroup of  $\mathrm{PSL}_2(\mathcal{O}_{-m})$ . Then the type of the normaliser of any subgroup of type  $\mathbb{Z}/\ell$  in  $G$  is given as follows for  $\ell = 2$  and  $\ell = 3$ , where we print only cases with existing subgroup of type  $\mathbb{Z}/\ell$ .*

Isomorphism type of $G$	{1}	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$
normaliser of $\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathcal{D}_2$	$\mathbb{Z}/2$	$\mathcal{D}_2$
normaliser of $\mathbb{Z}/3$			$\mathbb{Z}/3$		$\mathcal{S}_3$	$\mathbb{Z}/3$ .

The final ingredient in the proof of theorem 12 is the following.

**Theorem 16.** *There is a natural bijection between conjugacy classes of subgroups of  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  of order  $\ell$  and edges of the reduced  $\ell$ -torsion subcomplex. It is given by considering the stabiliser of a representative edge in the refined cell complex.*

In order to prove the latter theorem, we need several lemmata, and we establish them now.

**Lemma 17.** *Consider two adjacent edges  $E, E'$  of the non-reduced torsion subcomplex. Then for any representative  $e$  of  $E$ , there is an adjacent representative  $e'$  of  $E'$  on the same geodesic line as  $e$ .*

*Proof.* Consider the element  $\gamma \in \Gamma$  identifying the end  $v$  of  $e$  with the origin  $\gamma(v)$  of  $e'$ . As  $E$  and  $E'$  are distinct in the orbit space,  $\gamma^{-1}$  cannot send  $e'$  onto  $e$ .

Assume that the stabiliser of  $v$  is of isomorphism type  $\mathbb{Z}/\ell\mathbb{Z}$ . Then as the edge  $\gamma^{-1}(e')$  is point-wise fixed by  $\Gamma$ , its stabiliser must contain the rotation with axis passing through  $e$ . Hence  $\gamma^{-1}(e')$  is adjacent to  $e$  and on the same geodesic line.

If the stabiliser of  $v$  is of isomorphism type  $\mathcal{D}_2, \mathcal{S}_3$  or  $\mathcal{A}_4$ , there are at most two orbits of  $\ell$ -torsion stabilised edges adjacent to  $v$ . So there is an element  $\alpha$  in the stabiliser of  $v$  such that  $\alpha\gamma^{-1}(e')$  is adjacent to  $e$  and on the same geodesic line.  $\square$

**Corollary 18.** *Any edge of the reduced torsion subcomplex can be represented by a chain of edges on the intersection of one geodesic line with a strict fundamental domain for  $\Gamma$  in  $\mathcal{H}$ .*

*Proof.* Consider the chain of edges in the torsion subcomplex that are reduced to the given edge of the reduced torsion subcomplex. The representatives in  $\mathcal{H}$  of the edges on this chain lie on pairwise different orbits because the torsion subcomplex is a subcomplex of the orbit space  $\Gamma \backslash \mathcal{H}$ . Now we start with a representative  $e$  for the edge sharing its origin with the reduced edge, and use lemma 17 to obtain an adjacent edge  $e'$  on the same geodesic line, representing the adjacent edge of the torsion subcomplex. We proceed this way, assigning the role of  $e$  to  $e'$  and so on, until we have reached the end of the reduced edge.  $\square$

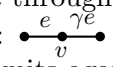
**Corollary 19.** *Any edge of the reduced torsion subcomplex admits only representatives with stabiliser in the same conjugacy class.*

**Lemma 20.** *Let  $\alpha$  and  $\gamma$  be elements of  $\mathrm{PSL}_2(\mathbb{C})$ . Then the fixed point set in  $\mathcal{H}$  of  $\alpha$  is identified by  $\gamma$  with the fixed point set of  $\gamma\alpha\gamma^{-1}$ .*

*Proof.* One immediately checks that any fixed point  $x \in \mathcal{H}$  of  $\alpha$  induces the fixed point  $\gamma(x)$  of  $\gamma\alpha\gamma^{-1}$ . As  $\mathrm{PSL}_2(\mathbb{C})$  acts by isometries, the whole fixed point sets are identified.  $\square$

**Lemma 21.** *Let  $v \in \mathcal{H}_{\mathbb{R}}^3$  be a vertex with stabiliser in  $\Gamma$  of type  $\mathcal{D}_2$  or  $\mathcal{A}_4$ . Let  $\gamma \in \Gamma$  be a rotation of order 2 around an edge  $e$  adjacent to  $v$ .*

*Then the centraliser  $C_{\Gamma}(\gamma)$  reflects  $\mathcal{H}^{\gamma}$  — which is the geodesic line through  $e$  — onto itself at  $v$ .*

*Proof.* Denote by  $\Gamma_v$  the stabiliser of the vertex  $v$ . In the case that  $\Gamma_v$  is of type  $\mathcal{D}_2$ , which is Abelian, it admits two order-2-elements centralising  $\gamma$  and turning the geodesic line through  $e$  onto itself such that the image of  $e$  touches  $v$  from the side opposite to  $e$  (illustration: ). In the case that  $\Gamma_v$  is of type  $\mathcal{A}_4$ , it contains a normal subgroup of type  $\mathcal{D}_2$  that admits again two such elements.  $\square$

Let  $\alpha$  be any torsion element in  $\Gamma$ . We construct the *chain of edges associated to  $\alpha$*  as follows. Consider the edge of the reduced torsion subcomplex to which the edge stabilised by  $\alpha$  belongs. Use corollary 18 to represent it by a chain of edges on a geodesic line. Now,  $\alpha$  is conjugate to an element  $\gamma\alpha\gamma^{-1}$  of the stabiliser of one of the edges in the chain. By lemma 20, the element  $\gamma^{-1} \in \Gamma$  maps the mentioned geodesic line to the rotation axis of  $\alpha$ . The image under  $\gamma^{-1}$  of the chain of edges under consideration is the desired chain associated to  $\alpha$ . It is well defined up to translation along the rotation axis of  $\alpha$ .

**Lemma 22.** *Let  $\alpha$  be any 2-torsion element in  $\Gamma$ . Then the chain of edges associated to  $\alpha$  is a fundamental domain for the action of the centraliser of  $\alpha$  on the rotation axis of  $\alpha$ .*

*Proof.* We distinguish the following two cases of how  $\langle \alpha \rangle \cong \mathbb{Z}/2$  is included into  $\Gamma$ .

• Suppose that there is no subgroup of type  $\mathcal{D}_2$  in  $\Gamma$  which contains  $\langle \alpha \rangle$ . Then the connected component of the 2-torsion subgraph to which any edge stabilised by  $\langle \alpha \rangle$  belongs, is homeomorphic to a circle. Choose a first edge  $e$  in the chain associated to  $\alpha$ . We have an identification  $\gamma$  between its end and the origin of the next edge  $e'$ , and by lemma 17 we can choose it such that the edge stabilisers are conjugate under  $\gamma$ . We can write  $\Gamma_e = \langle \alpha \rangle$  and  $\Gamma_{e'} = \langle \gamma\alpha\gamma^{-1} \rangle$ . By lemma 20, this gives us the identification  $\gamma^{-1}$  from  $e'$  to an edge on the rotation axis of  $\alpha$ , adjacent to  $e$  because of the first condition on  $\gamma$ . We repeat this step until we have attached an edge  $\delta e$  on the orbit of the first edge  $e$ , with  $\delta \in \Gamma$ . As  $\delta$  is an isometry, the whole chain is translated by  $\delta$  from the start at  $e$  to the start at  $\delta e$ . So the group  $\langle \delta \rangle$  acts on the rotation axis with fundamental domain our chain of edges. And  $\delta\alpha\delta^{-1}$  is again the rotation of order 2 around the axis of  $\alpha$ . So,  $\delta\alpha\delta^{-1} = \alpha$  and therefore  $\langle \delta \rangle < C_{\Gamma}(\alpha)$ .

$\bullet\text{---}\bullet$ ,  $\bigcirc$ ,  $\bigcirc\text{---}\bullet$ ,  $\dots$ . Suppose that there is a subgroup  $G$  of  $\Gamma$  of type  $G \cong \mathcal{D}_2$  containing  $\langle \alpha \rangle$ . If there is no further inclusion  $G < G' < \Gamma$  with  $G' \cong \mathcal{A}_4$ , let  $G' := G$ . Then the chain associated to  $\alpha$  can be chosen such that one of its endpoints is stabilised by  $G'$ . The other endpoint of this chain must then lie on a different  $\Gamma$ -orbit, and admit as stabiliser a group  $H'$  containing  $\langle \alpha \rangle$ , of type  $\mathcal{D}_2$  or  $\mathcal{A}_4$ . By lemma 21, each  $G'$  and  $H'$  contain a reflection of the rotation axis of  $\alpha$ , centralising  $\alpha$ . These two reflections must differ from one another because they do not fix the chain of edges. So their free product tessellates the rotation axis of  $\alpha$  with images of the chain of edges associated to  $\alpha$ .  $\square$

**Lemma 23.** *Let  $\alpha$  be any non-trivial torsion element in a Bianchi group  $\Gamma$ . Then the  $\Gamma$ -image of the chain of edges associated to  $\alpha$  contains the rotation axis of  $\alpha$ .*

*Proof.* For  $\alpha$  a 2-torsion element, this follows directly from lemma 22. So we can assume  $\alpha$  to be a 3-torsion element.

$\bullet\text{---}\bullet$  The non-centralising reflections of  $\mathcal{S}_3$ , which is associated to the two vertices, tessellate a geodesic line. Remark that there are exactly three maps  $\mathbb{Z}/3 \rightarrow \mathbb{Z}/3$  induced on the edge stabilisers, given by the three order-2-elements of  $\mathcal{S}_3$ . So, the quotient space by the centraliser of  $\alpha$  must be bigger.

$\bigcirc$  After finitely many translations, all the edges in the fundamental domain conjugate to edges on the geodesic line are obtained as images. Hence, the  $\Gamma$ -image contains the whole geodesic line.  $\square$

*Proof of theorem 16.* Corollary 19 associates a conjugacy class of type  $\mathbb{Z}/\ell$  to each edge of the reduced  $\ell$ -torsion subcomplex. Conversely, let  $\alpha$  and  $\gamma\alpha\gamma^{-1}$  be conjugate torsion elements of  $\Gamma$ . We want to show that they stabilise edges representing the same reduced edge. By Klein, we know that the torsion elements are elliptic and hence fix some geodesic line. Every torsion element acts as the stabiliser of a line conjugate to one passing through the Bianchi fundamental polyhedron. So, it is conjugate to one of the representative edge stabilisers. By lemma 20, we know that the line fixed by  $\alpha$  is sent by  $\gamma$  to the line fixed by  $\gamma\alpha\gamma^{-1}$ .

By lemmata 22 and 23, the union of the  $\Gamma$ -images of the chain associated to  $\alpha$  contains the whole geodesic line fixed by  $\alpha$  (because then, as the  $\Gamma$ -action is cellular, any cell stabilised by  $\gamma\alpha\gamma^{-1}$  admits a cell on its orbit stabilised by  $\alpha$ ). So it follows that all the edges on a geodesic line belong to the same reduced edge.  $\square$

*Proof of theorem 12.* Comparing with lemma 14, we see that the vertex set of the  $\ell$ -conjugacy classes graph gives precisely the bifurcation points and vertices with only one adjacent edge of the  $\ell$ -torsion subcomplex. When passing to the reduced  $\ell$ -torsion subcomplex, we get rid of all vertices with two adjacent edges except in the disjoint circles, see [19]. By theorem 16, the edges of the  $\ell$ -conjugacy classes graph give the edges of the reduced  $\ell$ -torsion subcomplex.  $\square$

## 5. THE NUMBERS OF CONJUGACY CLASSES OF FINITE SUBGROUPS AND GROUP HOMOLOGY

In this section, we are going to prove theorem 2. In order to compare with Kramer's formulae that we evaluate in the appendix, we make use of his notations for the numbers of conjugacy classes of the five types of non-trivial finite subgroups in the Bianchi groups. His symbols for these numbers are printed in the first row of the below table, and the second row gives the

symbol for the type of counted subgroup.

$\mu_2$	$\mu_T$	$\mu_3$	$\lambda_{2n}$	$\lambda_4^T$	$\lambda_4^*$	$\lambda_6^*$	$\mu_2^-$
$\mathcal{D}_2$	$\mathcal{A}_4$	$\mathcal{S}_3$	$\mathbb{Z}/n$	$\mathbb{Z}/2 \subset \mathcal{A}_4$	$\mathbb{Z}/2 \subset \mathcal{D}_2$	$\mathbb{Z}/3 \subset \mathcal{S}_3$	$\mathcal{D}_2 \not\subset \mathcal{A}_4$

Here, the inclusion signs “ $\subset$ ” mean that we only consider copies of  $\mathbb{Z}/n$  admitting the specified inclusion in the given Bianchi group and  $\mathcal{D}_2 \not\subset \mathcal{A}_4$  means that we only consider copies of  $\mathcal{D}_2$  not admitting any inclusion into a subgroup of type  $\mathcal{A}_4$  of the Bianchi group.

Note that the number  $\mu_2^-$  is simply the difference  $\mu_2 - \mu_T$ , because every copy of  $\mathcal{A}_4$  admits precisely one normal subgroup of type  $\mathcal{D}_2$ . Also, note the following graph-theoretical properties of the reduced torsion subgraphs, the latter of which we obtain by restricting our attention to the connected components not homeomorphic to  $\mathcal{O}$ .

**Corollary 24** (Corollary to lemma 14). *For all Bianchi groups with units  $\{\pm 1\}$ , the numbers of conjugacy classes of finite subgroups satisfy  $\lambda_4^T \leq \mu_T$  and  $2\lambda_6^* = \mu_3$ , and even*

$$2\lambda_4^* = \mu_T + 3\mu_2^-.$$

The values given by Krämer’s formulae are matching with the values computed with [20].

**Observation 25.** The numbers of conjugacy classes of finite subgroups determine the 3-conjugacy classes graph and hence the reduced 3-torsion subcomplex for all Bianchi groups with units  $\{\pm 1\}$ , as we can see immediately from theorem 12 and the description of the reduced 3-torsion subcomplex in [19].

For the proof of theorem 2, we need the following ingredients.

**Remark 26.** In the equivariant spectral sequence converging to the Farrell cohomology of  $\mathrm{PSL}_2(\mathcal{O}_{-m})$ , the restriction of the differential to maps between cohomology groups of cells that are not adjacent in the orbit space, are zero. So, the  $\ell$ -primary part of the degree-1-differentials of this sequence can be decomposed as a direct sum of the blocks associated to the connected components of the  $\ell$ -torsion subcomplex (Compare with sub-lemma 45 of [19]).

**Lemma 27** (Schwermer/Vogtmann). *Let  $M$  be  $\mathbb{Z}$  or  $\mathbb{Z}/2$ . Consider group homology with trivial  $M$ -coefficients. Then the following holds.*

- Any inclusion  $\mathbb{Z}/2 \rightarrow \mathcal{S}_3$  induces an injection on homology.
- An inclusion  $\mathbb{Z}/3 \rightarrow \mathcal{S}_3$  induces an injection on homology in degrees congruent to 3 or 0 mod 4, and is otherwise zero.
- Any inclusion  $\mathbb{Z}/2 \rightarrow \mathcal{D}_2$  induces an injection on homology in all degrees.
- An inclusion  $\mathbb{Z}/3 \rightarrow \mathcal{A}_4$  induces injections on homology in all degrees.
- An inclusion  $\mathbb{Z}/2 \rightarrow \mathcal{A}_4$  induces injections on homology in degrees greater than 1, and is zero on  $H_1$ .

For the proof in  $\mathbb{Z}$ -coefficients, see [23], for  $\mathbb{Z}/2$ -coefficients see [19].

**Lemma 28** ([19], lemma 32). *Let  $q \geq 3$  be an odd integer number. Let  $v$  be a vertex representative of stabiliser type  $\mathcal{D}_2$  in the refined cellular complex for the Bianchi groups. Then the three images in  $(H_q(\mathcal{D}_2; \mathbb{Z}))_{(2)}$  induced by the inclusions of the stabilisers of the edges adjacent to  $v$ , are linearly independent.*

Finally, we establish the following last ingredient for the proof of theorem 2, which might be of interest in its own right.

**Lemma 29.** *In all rows  $q > 1$  and outside connected components of type  $\mathcal{O}$ , the  $d_{p,q}^1$ -differential of the equivariant spectral sequence converging to  $H_{p+q}(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$  is always injective.*

*Proof.* For matrix blocks of the  $d_{p,q}^1$ -differential associated to vertices with just one adjacent edge, we see from lemma 14 that the vertex stabiliser is of type  $\mathcal{A}_4$  in 2-torsion, respectively of type  $\mathcal{S}_3$  in 3-torsion, so injectivity follows from lemma 27. As we have placed ourselves outside connected components of type  $\mathcal{O}$ , the remaining vertices are bifurcation points of stabiliser type  $\mathcal{D}_2$  and injectivity follows from lemma 28.  $\square$

*Proof of theorem 2.* In 3-torsion, theorem 2 follows directly from observation 25, corollary 24 and theorem 9. In 2-torsion, what we need to determine with the numbers of conjugacy classes of finite subgroups, is the 2-primary part of the  $E_{p,q}^2$ -term of the equivariant spectral sequence converging to  $H_{p+q}(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$  in all rows  $q > 1$ . From there, we see from theorem 9 that we obtain the claim. By remark 26, we only need to check this determination on each homeomorphism type of connected components of the 2-torsion subgraph. We use theorem 12 to identify the reduced 2-torsion subgraph and the 2-conjugacy classes graph. Then we can observe that

- Kramer’s number  $\lambda_4^* - \lambda_4$  determines the number of connected components of type  $\mathcal{O}$ .
- Kramer’s number  $\lambda_4^*$  determines the number of edges of the 2-torsion subgraph outside connected components of type  $\mathcal{O}$ . Lemma 29 tells us that the block of the  $d_{p,q}^1$ -differential of the equivariant spectral sequence associated to such edges is always injective.
- Kramer’s number  $\mu_2$  determines the number of bifurcation points, and  $\mu_T$  determines the number of vertices with only one adjacent edge of the 2-torsion subgraph.

Using corollary 24, we obtain the explicit formulae in theorem 2.  $\square$

## 6. THE COHOMOLOGY RING STRUCTURE OF THE BIANCHI GROUPS

In [5], Berkove has found a compatibility of the cup product of the cohomology ring of a Bianchi group with the cup product of the cohomology rings of its finite subgroups. This compatibility within the equivariant spectral sequence implies that all products that come from different connected components of the conjugacy-reduced torsion subcomplex (which we turn into the conjugacy classes graph in section 4) are zero. It follows that the cohomology ring of any Bianchi group splits into a restricted sum over subrings, which depend in degrees above the virtual cohomological dimension only on the homeomorphism type of the associated connected component of the conjugacy-reduced torsion subcomplex. The analogue in cohomology of theorem 2 and Berkove’s computations of sample cohomology rings [4] yield the following corollary in 3-torsion.

We use Berkove’s notation, in which the degree  $j$  of a cohomology generator  $x_j$  is appended as a subscript. Furthermore, writing cohomology classes inside square brackets means that they are polynomial (of infinite multiplicative order), and writing them inside parentheses means that they are exterior (their powers vanish). The restricted sum  $\oplus$  identifies all the degree zero classes into a single copy of  $\mathbb{Z}$ ; when we write it with a power, we specify the number of summands. Recall that  $\lambda_6$  (respectively  $\mu_3$ ) counts the number of conjugacy classes of subgroups of type  $\mathbb{Z}/3$  (respectively  $\mathcal{S}_3$ ) in the Bianchi group.





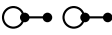

$T$	subring associated to connected components of type $T$ in 2 – conjugacy classes graph
	$\mathbb{F}_2[n_1](m_1)$
	$\mathbb{F}_2[m_3, u_2, v_3, w_3]/\langle m_3 v_3 = 0, \quad u_2^3 + w_3^2 + v_3^2 + m_3^2 + w_3(v_3 + m_3) = 0 \rangle$
	$\mathbb{F}_2[n_1, m_2, n_3, m_3]/\langle n_1 n_3 = 0, \quad m_2^3 + m_3^2 + n_3^2 + m_3 n_3 + n_1 m_2 m_3 = 0 \rangle$
	$\mathbb{F}_2[n_1, m_1, m_3]/\langle m_3(m_3 + n_1^2 m_1 + n_1 m_1^2) = 0 \rangle$

TABLE 1. Restricted summands of the mod-2 cohomology ring  $H^*(\Gamma; \mathbb{F}_2)$  of a Bianchi group  $\Gamma$  above its virtual cohomological dimension.

**Corollary 30.** *In degrees above the virtual cohomological dimension, the 3–primary part of the cohomology ring of any Bianchi group  $\Gamma$  with units  $\{\pm 1\}$  is given by*

$$H^*(\Gamma; \mathbb{Z})_{(3)} \cong \tilde{\oplus}^{(\lambda_6 - \frac{\mu_3}{2})} \mathbb{Z}[x_2](\sigma_1) \tilde{\oplus}^{\frac{\mu_3}{2}} \mathbb{Z}[x_4](x_3),$$

where the generators  $x_j$  are of additive order 3.

In 2–torsion, it does in general not suffice to know only the numbers of conjugacy classes of finite subgroups to obtain the cohomology ring structure, because for the two conjugacy-reduced 2-torsion subcomplexes  and , we obtain the same numbers of conjugacy classes and homological 2–torsion, but different multiplicative structures of the mod-2 cohomology rings, as we can see from table 1, which we compile from the results of [5] (and [19]).

**Observation 31.** In the cases of class numbers 1 and 2, only the homeomorphism types  $T$  listed in table 1 occur as connected components in the conjugacy-reduced 2-torsion subcomplex. So for all such Bianchi groups  $\Gamma$  with units  $\{\pm 1\}$ , the mod-2 cohomology ring  $H^*(\Gamma; \mathbb{F}_2)$  splits, above the virtual cohomological dimension, as a restricted sum over the subrings specified in table 1, with powers according to the multiplicities of the occurrences of the types  $T$ .

## APPENDIX A. NUMERICAL EVALUATION OF KRÄMER’S FORMULAE

**A.1. Numbers of conjugacy classes in 3–torsion.** Denote by  $\delta$  the number of finite ramification places of  $\mathbb{Q}(\sqrt{-m})$  over  $\mathbb{Q}$ . Let  $k_+$  be the totally real number field  $\mathbb{Q}(\sqrt{3m})$  and denote its ideal class number by  $h_{k_+}$ . Krämer introduces the following indicators:

$$z := \begin{cases} 2, & \text{if 3 is the norm of an integer of } k_+, \\ 1, & \text{otherwise.} \end{cases}$$

For  $m \equiv 0 \pmod{3}$  and  $m \neq 3$ , denote by  $\epsilon := \frac{1}{2}(a + b\sqrt{\frac{m}{3}}) > 1$  the fundamental unit of  $k_+$  (where  $a, b \in \mathbb{N}$ ). Now, define

$$x' := \begin{cases} 2, & \text{if the norm of } \epsilon \text{ is } 1, \\ 1, & \text{if the norm of } \epsilon \text{ is } -1 \end{cases}$$

and

$$y := \begin{cases} 2, & \text{if } b \equiv 0 \pmod{3}, \\ 1, & \text{otherwise.} \end{cases}$$

Then [15, 20.39 and 20.41] yield the following formulae in 3–torsion.

$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$	$\lambda_6^*$	$\lambda_6 - \lambda_6^*$
$m \equiv 2 \pmod{3}$	0	$\frac{z}{2}h_{k_+}$
$m \equiv 1 \pmod{3}$ gives either	$2^{\delta-1}$	$\frac{1}{2}(h_{k_+} - 2^{\delta-1})$
or	0	$\frac{1}{2}h_{k_+}$
$m \equiv 6 \pmod{9}$	0	$x'yh_{k_+}$
$m \equiv 3 \pmod{9}$ , $m \neq 3$ gives either	$2^{\delta-2}$	$\frac{1}{2}(3x'h_{k_+} - 2^{\delta-2})$
or	0	$\frac{1}{2}3x'h_{k_+}$

The above case distinctions come from the fact that Krämer’s theorem 20.39 ranges over all types of maximal orders in quaternion algebras over  $\mathbb{Q}(\sqrt{-m})$ , in which Krämer determines the numbers of conjugacy classes in the norm-1-group. The remaining task in order to decide which of the cases applies, is to find out of which type considered in the mentioned theorem is the maximal order  $M_2(\mathcal{O}_{-m})$ . Some methods to cope with this task are introduced in [15, §27].

Kr amer's resulting criteria can be summarised as follows for 3-torsion.

condition	implication
$m \equiv 2 \pmod{3}$	$\mu_3 = \lambda_6^* = 0.$
$m \equiv 6 \pmod{9}$	$\mu_3 = \lambda_6^* = 0.$
$m$ prime and $m \equiv 1 \pmod{3}$	$\lambda_6^* > 0.$
$m = 3p$ with $p$ prime and $p \equiv 1 \pmod{3}$	$\lambda_6^* > 0.$
$m \equiv 1 \pmod{3}$ and $-3$ occurs as norm on $\mathcal{O}_{k_+}$	$\lambda_6^* > 0.$
$m \equiv 1 \pmod{3}$ and $-3$ does not occur as norm on $\mathcal{O}_{k_+}$	$\lambda_6 - \lambda_6^* > 0.$
$m \equiv 1 \pmod{3}$ and $m$ admits a prime divisor $p$ with $p \equiv 2 \pmod{3}$	$\lambda_6 - \lambda_6^* > 0.$
$m \equiv 3 \pmod{9}$ and $x' = 1$ and $\frac{m}{3}$ admits only prime divisors $p$ with $p \equiv 1 \pmod{12}$	$\lambda_6^* > 0.$
$m \equiv 3 \pmod{9}$ and $x' = 1$ and $\frac{m}{3}$ admits a prime divisor $p$ with $p \equiv 5 \pmod{12}$	$\lambda_6^* = 0.$
$m \equiv 3 \pmod{9}$ and $h(k'_+) = 2^{\delta-3}$ and $\frac{m}{3}$ admits only prime divisors $p$ with $p \equiv \pm 1 \pmod{12}$ or $p = 2$	$\lambda_6^* = 0.$
$m \equiv 3 \pmod{9}$ and $h(k'_+) = 1$ and $\frac{m}{3} = p'p$ with $p', p$ prime and $p' \equiv p \equiv 7 \pmod{12}$	$\lambda_6^* > 0.$

In order to determine Kr amer's indicator  $z$ , we need to determine if a given value occurs as the norm on the ring of integers of an imaginary quadratic number field. This is implemented in Pari/GP [1] (the first step is computing the answer under the Generalised Riemann hypothesis, and the second step is a check computation which confirms that we arrive at that answer without this hypothesis). Additionally, we compare with the below criterion [15, (20.13)].

**Lemma 32** (Kr amer). *Let  $m$  be not divisible by 3.*








- *If the number  $-3$  is the norm of an integer in the totally real number field  $k_+$ , then all prime divisors  $p \in \mathbb{N}$  of  $m$  satisfy the congruence  $p \equiv 1 \pmod{3}$ .  
Especially, the congruence  $m \equiv 1 \pmod{3}$  is implied.*
- *If the number 3 is the norm of an integer in the totally real number field  $k_+$ , then all prime divisors  $p \in \mathbb{N}$  of  $m$  satisfy either  $p = 2$  or the congruence  $p \equiv \pm 1 \pmod{12}$ .  
Additionally, the congruence  $m \equiv 2 \pmod{3}$  is implied.*























With Kramer’s criteria at hand, we can decide for many Bianchi groups, which of the alternative cases in Kramer’s formulae must be used. We do this in the below tables for all such Bianchi groups  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  with absolute value of the discriminant  $\Delta$  ranging between 7 and

2003, where we recall that the discriminant is  $\Delta = \begin{cases} -m, & m \equiv 3 \pmod{4}, \\ -4m, & \text{else.} \end{cases}$

In the cases  $m \in \{102, 133, 165, 259, 559, 595, 763, 835, 1435\}$ , where these statements are not sufficient to eliminate the wrong alternatives, we insert the results of [20]. This way, the below tables treat all Bianchi groups with units  $\{\pm 1\}$  and discriminant of absolute value less than 615. The cases where an ambiguity remains (so to exclude them from our tables) are  $m \in \{210, 262, 273, 298, 345, 426, 430, 462, 481, 615, 1155, 1159, 1195, 1339, 1351, 1407, 1515, 1807\}$ . For tables of the cases without ambiguity, with  $m$  ranging up to 10000, see the preprint version 2 of this paper on HAL.

In [14], a theorem is established which solves all these ambiguities by giving for each type of finite subgroups in  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  criteria equivalent to its occurrence, in terms of congruence conditions on the prime divisors of  $m$ .

3–conjugacy classes graph	$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 3–conjugacy classes graph
	2, 5, 6, 10, 11, 14, 15, 17, 22, 23, 29, 34, 35, 38, 41, 46, 47, 51, 53, 55, 58, 59, 62, 71, 82, 83, 86, 87, 89, 94, 95, 101, 106, 113, 115, 118, 119, 123, 131, 134, 137, 142, 149, 155, 158, 159, 166, 167, 173, 178, 179, 187, 191, 197, 202, 203, 206, 214, 215, 226, 227, 233, 235, 239, 251, 254, 257, 263, 267, 269, 274, 278, 281, 287, 293, 295, 303, 311, 317, 319, 323, 326, 334, 335, 339, 346, 347, 353, 355, 358, 359, 371, 382, 383, 389, 391, 394, 395, 398, 401, 411, 415, 422, 431, 443, 446, 447, 449, 451, 454, 461, 466, 467, 478, 479, 491, 515, 519, 527, 535, 551, 563, 583, 591, 599, 623, 635, 647, 655, 659, 667, 683, 695, 699, 707, 719, 731, 743, 755, 779, 791, 799, 807, 815, 827, 839, 843, 879, 887, 895, 899, 911, 943, 947, 951, 955, 959, 979, 983, 995, 1003, 1019, 1031, 1055, 1059, 1091, 1103, 1111, 1115, 1135, 1139, 1151, 1163, 1167, 1187, 1207, 1211, 1219, 1223, 1243, 1247, 1255, 1259, 1271, 1283, 1307, 1315, 1343, 1347, 1363, 1367, 1379, 1383, 1411, 1415, 1439, 1487, 1499, 1507, 1511, 1523, 1527, 1535, 1555, 1559, 1563, 1571, 1607, 1631, 1639, 1643, 1655, 1667, 1671, 1707, 1711, 1735, 1751, 1763, 1779, 1787, 1795, 1799, 1811, 1819, 1823, 1835, 1847, 1851, 1883, 1903, 1907, 1915, 1919, 1923, 1927, 1931, 1943, 1959, 1979, 2003,
2 	26, 42, 65, 69, 70, 74, 77, 78, 85, 110, 122, 130, 141, 143, 145, 154, 161, 170, 182, 185, 186, 190, 194, 195, 205, 209, 213, 218, 221, 222, 230, 231, 238, 253, 265, 266, 286, 305, 310, 314, 322, 329, 365, 366, 370, 377, 386, 406, 407, 410, 418, 434, 437, 442, 445, 455, 458, 470, 473, 474, 483, 485, 493, 494, 497, 555, 611, 627, 671, 715, 767, 803, 851, 923, 935, 1015, 1079, 1095, 1199, 1235, 1295, 1311, 1391, 1403, 1455, 1463, 1491, 1495, 1595, 1599, 1615, 1679, 1703, 1739, 1771, 1855, 1887, 1991,
3 	30, 66, 107, 138, 174, 255, 282, 302, 318, 354, 419, 498, 503, 759, 771, 795, 835, 863, 1007, 1319, 1355, 1427, 1479, 1551, 1583, 1619, 1691, 1695, 1871, 1895, 1947, 1967,
4 	33, 105, 114, 146, 177, 249, 258, 285, 290, 299, 321, 330, 341, 357, 374, 385, 393, 402, 413, 429, 465, 482, 595, 663, 915, 987, 1023, 1067, 1239, 1435, 1727, 1743, 1955, 1995,
5 	1043, 1203, 1451,
6 	102, 165, 246, 362, 390, 435, 1335, 1419, 1547,
7 	587, 971,

3-conjugacy classes graph	$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 3-conjugacy classes graph
8 	438, 1131, 1635.
	7, 19, 31, 43, 67, 79, 103, 127, 139, 151, 163, 199, 211, 223, 271, 283, 307, 379, 439, 463, 487, 499, 523, 571, 607, 619, 631, 691, 727, 739, 751, 787, 811, 823, 859, 883, 907, 919, 967, 991, 1039, 1051, 1063, 1123, 1171, 1231, 1279, 1303, 1399, 1423, 1447, 1459, 1471, 1483, 1531, 1543, 1567, 1579, 1627, 1663, 1699, 1723, 1759, 1783, 1831, 1867, 1987, 1999,
 $\amalg$ 	39, 111, 183, 219, 291, 327, 331, 367, 471, 543, 579, 643, 723, 831, 939, 1011, 1047, 1087, 1119, 1191, 1227, 1263, 1291, 1299, 1327, 1371, 1623, 1803, 1839, 1879, 1951, 1983,
 $\amalg$ 2 	547, 1747,
 $\amalg$ 4 	687,
 $\amalg$ 10 	1731,
2 	13, 37, 61, 91, 109, 157, 181, 229, 247, 277, 349, 373, 403, 421, 427, 511, 679, 703, 871, 1099, 1147, 1267, 1591, 1603, 1687, 1891, 1963,
2  $\amalg$ 	73, 97, 193, 241, 259, 313, 337, 409, 457, 559, 763, 1651, 1939,
2  $\amalg$ 2 	21, 57, 93, 129, 201, 309, 381, 397, 399, 417, 453, 489, 651, 903, 1443, 1659, 1767, 1843,
2  $\amalg$ 3 	433, 1027, 1387,
2  $\amalg$ 8 	237,
4 	217, 301, 469,
4  $\amalg$ 2 	133,

A.2. **Numbers of conjugacy classes in 2-torsion.** Denote by  $\delta$  the number of finite ramification places of  $\mathbb{Q}(\sqrt{-m})$  over  $\mathbb{Q}$ . Let  $k_+$  be the totally real number field  $\mathbb{Q}(\sqrt{m})$  and denote its ideal class number by  $h_{k_+}$ . For  $m \neq 1$ , Krämer introduces the following indicators:

$$z := \begin{cases} 2, & \text{if } 2 \text{ is the norm of an integer of } k_+, \\ 1, & \text{otherwise,} \end{cases} \quad q := \begin{cases} 2, & \text{if } \pm 2 \text{ is the norm of an integer of } k_+, \\ 1, & \text{otherwise,} \end{cases}$$

$$w := \begin{cases} 2, & \text{if } \forall \text{ prime divisors } p \text{ of } m \text{ with } p \neq 2 \text{ we have } p \equiv \pm 1 \pmod{8}, \\ 1, & \text{if } m \text{ admits prime divisors } p \equiv \pm 3 \pmod{8}. \end{cases}$$

Furthermore, denote by  $\epsilon := \frac{1}{2}(a + b\sqrt{m}) > 1$  the fundamental unit of  $k_+$  (where  $a, b \in \mathbb{N}$ ). Now, define

$$x := \begin{cases} 2, & \text{if the norm of } \epsilon \text{ is } 1, \\ 1, & \text{if the norm of } \epsilon \text{ is } -1 \end{cases} \quad \text{and} \quad y := \begin{cases} 3, & \text{if } b \equiv 0 \pmod{2}, \\ 1, & \text{if } b \equiv 1 \pmod{2}. \end{cases}$$

Then [15, 26.12 and 26.14] yield the following formulae in 2-torsion.

$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$	$\mu_T$	$\mu_2^-$	$\lambda_4^T$	$\lambda_4^*$	$\lambda_4 - \lambda_4^*$
$m \equiv 7 \pmod{8}$	0	0	0	0	$\frac{z}{2}h_{k_+}$
$m \equiv 3 \pmod{8}$ gives either or (provided that $2^{\delta-1} > 1$ )	$2^\delta$ 0	0 0	$2^{\delta-1}$ 0	$2^{\delta-1}$ 0	$\frac{1}{2}(h_{k_+} - 2^{\delta-1})$ $\frac{1}{2}h_{k_+}$
$m \equiv 2 \pmod{4}$ and $w = 2$ gives either or (provided that $2^{\delta-1} > 1$ )	$2^{\delta-1}$ 0	$2^{\delta-1}$ 0	$2^{\delta-2}z$ 0	$2^\delta$ 0	$\frac{1}{4}x(z+2)h_{k_+} - 2^{\delta-1}$ $\frac{1}{4}x(z+2)h_{k_+}$
$m \equiv 2 \pmod{4}$ and $w = 1$ gives either or or (provided that $2^{\delta-1} > 2$ )	$2^{\delta-1}$ 0 0	0 $2^{\delta-1}$ 0	$2^{\delta-2}$ 0 0	$2^{\delta-2}$ $2^{\delta-2}3$ 0	$\frac{1}{2}(\frac{3}{2}xh_{k_+} - 2^{\delta-2})$ $\frac{3}{2}(\frac{1}{2}xh_{k_+} - 2^{\delta-2})$ $\frac{3}{4}xh_{k_+}$
$m \equiv 1 \pmod{8}$ and $m \neq 1$ and $w = 2$ gives either or (provided that $2^{\delta-2} > 1$ )	$2^{\delta-1}$ 0	$2^{\delta-1}$ 0	$2^{\delta-2}$ 0	$2^\delta$ 0	$2xh_{k_+} - 2^{\delta-1}$ $2xh_{k_+}$
$m \equiv 1 \pmod{8}$ and $w = 1$ gives either or or (provided that $2^{\delta-2} > 2$ )	$2^{\delta-1}$ 0 0	0 $2^{\delta-1}$ 0	$2^{\delta-2}$ 0 0	$2^{\delta-2}$ $2^{\delta-2}3$ 0	$2xh_{k_+} - 2^{\delta-3}$ $2xh_{k_+} - 2^{\delta-3}3$ $2xh_{k_+}$
$m \equiv 5 \pmod{8}$ or (provided that $2^{\delta-2} > 1$ )	0 0	$2^{\delta-1}$ 0	0 0	$2^{\delta-2}3$ 0	$\frac{1}{2}(x(2y+1)h_{k_+} - 2^{\delta-2}3)$ $\frac{1}{2}x(2y+1)h_{k_+}$

The above case distinctions come from the fact that Kramer's theorem 26.12 ranges over all types of maximal orders in quaternion algebras over  $\mathbb{Q}(\sqrt{-m})$ , in which Kramer determines the numbers of conjugacy classes in the norm-1-group. The remaining task in order to decide which of the cases applies, is to find out of which type considered in the mentioned theorem is the maximal order  $M_2(\mathcal{O}_{-m})$ . Some methods to cope with this task are introduced in [15, §27], where Kramer obtains the following criteria for the 2-torsion numbers:

condition	implication
$m \equiv 7 \pmod{8}$	$\mu_T = \mu_2^- = \lambda_4^T = \lambda_4^* = 0.$
$m \equiv 5 \pmod{8}$	$\mu_T = \lambda_4^T = 0.$
$m \equiv 21 \pmod{24}$	$\lambda_4^* = 0.$
$m \equiv 0 \pmod{6}$ and $\lambda_4^* > 0$	$\lambda_4^T > 0.$
$m \equiv 9 \pmod{24}$ and $\lambda_4^* > 0$	$\lambda_4^T > 0.$
$m$ prime and $m \equiv 1$ or $3 \pmod{8}$	$\lambda_4^T > 0.$
$m \equiv 5 \pmod{8}$ and $m$ prime	$\lambda_4^* > 0.$
$m = 2p$ with $p$ prime and $p \equiv 3$ or $5 \pmod{8}$	$\lambda_4^* > 0.$
$m = p'p$ with $p$ and $p'$ prime and $p \equiv p' \equiv 3$ or $5 \pmod{8}$	$\lambda_4^* > 0.$
$m = 3p$ with $p$ prime and $p \equiv 1$ or $3 \pmod{8}$	$\lambda_4^T > 0.$
$m \equiv 1$ or $2 \pmod{4}$ and $m \neq 1$ and $x = 1$	$\lambda_4^* > 0$ and $\mu_2^- > 0.$
$m \equiv 1$ or $2 \pmod{4}$ and $m \neq 1$ and $x = 2$	$\lambda_4 - \lambda_4^* > 0.$
$m \equiv 3 \pmod{8}$ and $-2$ occurs as norm on $\mathcal{O}_{k_+}$	$\lambda_4^* > 0$ and $\lambda_4^T > 0.$
$m \equiv 3 \pmod{8}$ and $-2$ does not occur as norm on $\mathcal{O}_{k_+}$	$\lambda_4 - \lambda_4^* > 0.$
$m \equiv 3 \pmod{8}$ and $m$ admits a prime divisor $p$ with $p \equiv 5$ or $7 \pmod{8}$	$\lambda_4 - \lambda_4^* > 0.$
$m \equiv 1 \pmod{8}$ and $w = 1$ and $h(k_+) = 2^{\delta-3}$	$\mu_2^- = 0.$
$m \equiv 2 \pmod{4}$ and $-2$ occurs as norm on $\mathcal{O}_{k_+}$	$\lambda_4^T > 0.$
$m \equiv 2 \pmod{4}$ and $-2$ does not occur as norm on $\mathcal{O}_{k_+}$ and $h(k_+) = 2^{\delta-2}$	$\lambda_4^* = 0.$
$m \equiv 2 \pmod{4}$ and $q = 1$ and $h(k_+) = 2^{\delta-1}$ and $w = 2$	$\lambda_4^* = 0.$
$m \equiv 2 \pmod{4}$ and $h(k_+) = 2^{\delta-2}$ and $m$ admits a prime divisor $p$ with $p \equiv 5$ or $7 \pmod{8}$	$\lambda_4^* = 0.$

With the above criteria at hand, we can decide for many Bianchi groups, which of the alternative cases in Kramer’s formulae must be used. We do this in the below tables for all such Bianchi groups  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  with absolute value of the discriminant  $\Delta$  ranging between 7 and 2003. In the cases  $m \in \{34, 105, 141, 142, 194, 235, 323, 427, 899, 979, 1243, 1507\}$ , where these statements are not sufficient to eliminate the wrong alternatives, we insert the results of [20]. This way, the below tables treat all Bianchi groups with units  $\{\pm 1\}$  and discriminant of absolute value less than 820. The cases where an ambiguity remains (so to exclude them from our tables) are the following values of  $m$ : 205, 221, 254, 273, 305, 321, 322, 326, 345, 377, 381, 385, 386, 410, 438, 465, 469, 473, 482, 1067, 1139, 1211, 1339, 1443, 1763, 1771, 1947.

The above mentioned theorem on subgroup occurrences [14] solves all these ambiguities.

2–torsion homology	$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 2–torsion homology
$P \circlearrowleft$	7, 15, 23, 31, 35, 39, 47, 55, 71, 87, 91, 95, 103, 111, 115, 127, 143, 151, 155, 159, 167, 183, 191, 199, 203, 215, 239, 247, 259, 263, 271, 295, 299, 303, 311, 319, 327, 335, 355, 367, 371, 383, 395, 403, 407, 415, 431, 447, 463, 471, 479, 487, 503, 515, 519, 535, 543, 551, 559, 583, 591, 599, 607, 611, 631, 635, 647, 655, 667, 671, 687, 695, 703, 707, 719, 743, 751, 755, 763, 767, 807, 815, 823, 831, 835, 851, 863, 871, 879, 887, 911, 919, 923, 951, 955, 967, 983, 991, 995, 1007, 1027, 1031, 1039, 1043, 1047, 1055, 1063, 1079, 1099, 1103, 1115, 1119, 1135, 1147, 1151, 1159, 1167, 1195, 1199, 1219, 1231, 1247, 1255, 1263, 1267, 1279, 1303, 1315, 1319, 1355, 1363, 1379, 1383, 1391, 1399, 1403, 1415, 1423, 1439, 1447, 1471, 1487, 1511, 1535, 1543, 1555, 1559, 1583, 1591, 1603, 1607, 1623, 1643, 1651, 1655, 1663, 1671, 1703, 1711, 1727, 1739, 1759, 1783, 1795, 1807, 1823, 1831, 1835, 1839, 1871, 1879, 1883, 1891, 1895, 1903, 1915, 1919, 1939, 1943, 1951, 1959, 1963, 1983, 1991, 1999,
$2P \circlearrowleft$	14, 46, 62, 94, 119, 158, 195, 206, 231, 255, 287, 302, 334, 382, 391, 398, 435, 446, 455, 478, 483, 511, 527, 555, 595, 615, 623, 651, 663, 679, 715, 759, 791, 795, 903, 915, 935, 943, 987, 1015, 1095, 1131, 1207, 1235, 1271, 1295, 1311, 1335, 1343, 1407, 1435, 1455, 1463, 1479, 1491, 1515, 1547, 1551, 1595, 1615, 1631, 1635, 1659, 1687, 1695, 1751, 1767, 1799, 1855, 1887, 1927, 1955, 1967,
$3P \circlearrowleft$	21, 30, 42, 69, 70, 77, 78, 79, 93, 110, 133, 138, 154, 174, 182, 186, 190, 213, 222, 223, 230, 235, 237, 253, 266, 282, 286, 301, 309, 310, 318, 341, 359, 366, 406, 413, 426, 427, 430, 437, 453, 470, 474, 494, 839, 895, 899, 1191, 1223, 1367, 1527, 1567, 1639, 1735, 1847,
$4P \circlearrowleft$	161, 217, 238, 329, 399, 497, 799, 959, 1023, 1155, 1239, 1351, 1679, 1743, 1995,
$5P \circlearrowleft$	439, 727, 1111, 1327,
$6P \circlearrowleft$	142, 165, 210, 285, 330, 357, 390, 429, 434, 462, 1495, 1599,
$7P \circlearrowleft$	141, 1087,
$8P \circlearrowleft$	105,
$2P_{D_2}^*$	5, 10, 13, 26, 29, 53, 58, 61, 74, 106, 109, 122, 149, 157, 173, 181, 202, 218, 277, 293, 298, 314, 317, 362, 394, 397, 421, 458, 461,

2-torsion homology	$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 2-torsion homology
$2P_{\mathcal{D}_2}^* + 2P_{\circlearrowleft}$	37, 101, 197, 269, 349, 373, 389,
$2P_{\mathcal{D}_2}^* + 3P_{\circlearrowleft}$	229, 346,
$4P_{\mathcal{D}_2}^*$	85, 130, 170, 290, 365, 370, 493,
$4P_{\mathcal{D}_2}^* + P_{\circlearrowleft}$	65, 185, 265, 481,
$4P_{\mathcal{D}_2}^* + 3P_{\circlearrowleft}$	442, 445,
$4P_{\mathcal{D}_2}^* + 4P_{\circlearrowleft}$	485,
$4P_{\mathcal{D}_2}^* + 5P_{\circlearrowleft}$	145,
$P_{\mathcal{A}_4}^* + P_{\mathcal{D}_2}^*$	2,
$2P_{\mathcal{A}_4}^*$	11, 19, 43, 59, 67, 83, 107, 131, 139, 163, 179, 211, 227, 251, 283, 307, 331, 347, 379, 419, 467, 491, 523, 547, 563, 571, 587, 619, 643, 683, 691, 739, 787, 811, 827, 859, 883, 907, 947, 971, 1019, 1051, 1123, 1163, 1187, 1259, 1283, 1291, 1307, 1427, 1451, 1459, 1483, 1499, 1531, 1571, 1579, 1619, 1667, 1699, 1723, 1747, 1867, 1931, 1979, 2003,
$2P_{\mathcal{A}_4}^* + P_{\circlearrowleft}$	6, 22, 38, 86, 118, 134, 166, 214, 262, 278, 358, 422, 443, 454, 659, 1091, 1171, 1523, 1627, 1787, 1811, 1907, 1987,
$2P_{\mathcal{A}_4}^* + 2P_{\circlearrowleft}$	499,
$2P_{\mathcal{A}_4}^* + 2P_{\mathcal{D}_2}^*$	17, 41, 73, 89, 97, 113, 137, 193, 233, 241, 281, 313, 337, 353, 409, 433, 449, 457,
$2P_{\mathcal{A}_4}^* + 2P_{\mathcal{D}_2}^* + P_{\circlearrowleft}$	82, 146, 178, 274, 466,
$2P_{\mathcal{A}_4}^* + 2P_{\mathcal{D}_2}^* + 2P_{\circlearrowleft}$	34, 194,
$2P_{\mathcal{A}_4}^* + 2P_{\mathcal{D}_2}^* + 4P_{\circlearrowleft}$	226, 257,
$2P_{\mathcal{A}_4}^* + 2P_{\mathcal{D}_2}^* + 8P_{\circlearrowleft}$	401,
$4P_{\mathcal{A}_4}^*$	51, 123, 187, 267, 339, 411, 451, 699, 771, 779, 803, 843, 1059, 1203, 1347, 1563, 1691, 1707, 1779, 1819, 1843, 1923,
$4P_{\mathcal{A}_4}^* + P_{\circlearrowleft}$	219, 291, 323, 579, 723, 731, 939, 979, 1003, 1011, 1227, 1243, 1371, 1387, 1411, 1507, 1731, 1803,
$4P_{\mathcal{A}_4}^* + 2P_{\circlearrowleft}$	66, 102, 114, 246, 258, 354, 374, 402, 418, 498, 1851,
$4P_{\mathcal{A}_4}^* + 3P_{\circlearrowleft}$	33, 57, 129, 177, 201, 209, 249, 393, 417, 489, 1299,
$8P_{\mathcal{A}_4}^*$	627, 1419.

**A.3. Asymptotic behaviour of the number of conjugacy classes.** From Kramer’s above formulae, we see that both in 2–torsion and in 3–torsion, the number of conjugacy classes of finite subgroups, and hence also the cardinality of the homology of the Bianchi groups in degrees above their virtual cohomological dimension, admits only two factors which are not strictly limited:  $h_{k_+}$  and  $2^\delta$ . As for the ideal class number  $h_{k_+}$ , it is subject to the predictions of the Cohen–Lenstra heuristic [8]. As for the factor  $2^\delta$ , the number  $\delta$  of finite ramification places of  $\mathbb{Q}(\sqrt{-m})$  over  $\mathbb{Q}$  is well-known to equal the number of prime divisors of the discriminant of  $\mathbb{Q}(\sqrt{-m})$ .

The numerical evaluation of Kramer’s formulae provides us with databases which are over a thousand times larger than what is reasonable to print in sections A.1 and A.2. We now give an instance of how these databases can be exploited. Denote the discriminant of  $\mathbb{Q}(\sqrt{-m})$  by  $\Delta$ . In the cases  $m \equiv 3 \pmod{4}$ , we have  $\Delta = -m$ . Denote the number  $\lambda_6 - \lambda_6^*$  of connected components of type  $\mathcal{O}$  in the 3-conjugacy classes graph by  $\lambda'_6(\Delta)$ . Then clearly, the subgroup in  $H_q(\mathrm{PSL}_2(\mathcal{O}_{-m}))$ ,  $q > 2$ , generated by the order-3-elements coming from the connected components of this type, is of order  $3^{\lambda'_6(\Delta)}$ . Denote by  $\mathrm{covolume}(\Delta)$  the volume of the quotient space  $\mathrm{PSL}_2(\mathcal{O}_{-m}) \backslash \mathcal{H}$ . The study of the ratio  $\frac{3^{\lambda'_6(\Delta)}}{\mathrm{covolume}(\Delta)}$  is motivated by the formulae in [3]. In figure 2, we print the logarithm of the average of this ratio over the cases  $|\Delta| \equiv 3 \pmod{4}$ , scaled by a factor  $m^{-\frac{2}{3}}$ , so to say

$$m^{-\frac{2}{3}} \log \left( \frac{1}{\#\{\Delta : |\Delta| \leq m\}} \sum_{|\Delta| \leq m} \frac{3^{\lambda'_6(\Delta)}}{\mathrm{covolume}(\Delta)} \right),$$

where we consider  $m$  and  $\Delta$  as independent variables,  $m$  running through the square-free positive rational integers. In order to cope with the fact that in some cases, Kramer’s formulae leave an ambiguity, we print a function assuming the lowest possible values of  $\lambda'_6(\Delta)$  and one assuming the highest possible values of  $\lambda'_6(\Delta)$  in the same diagram.

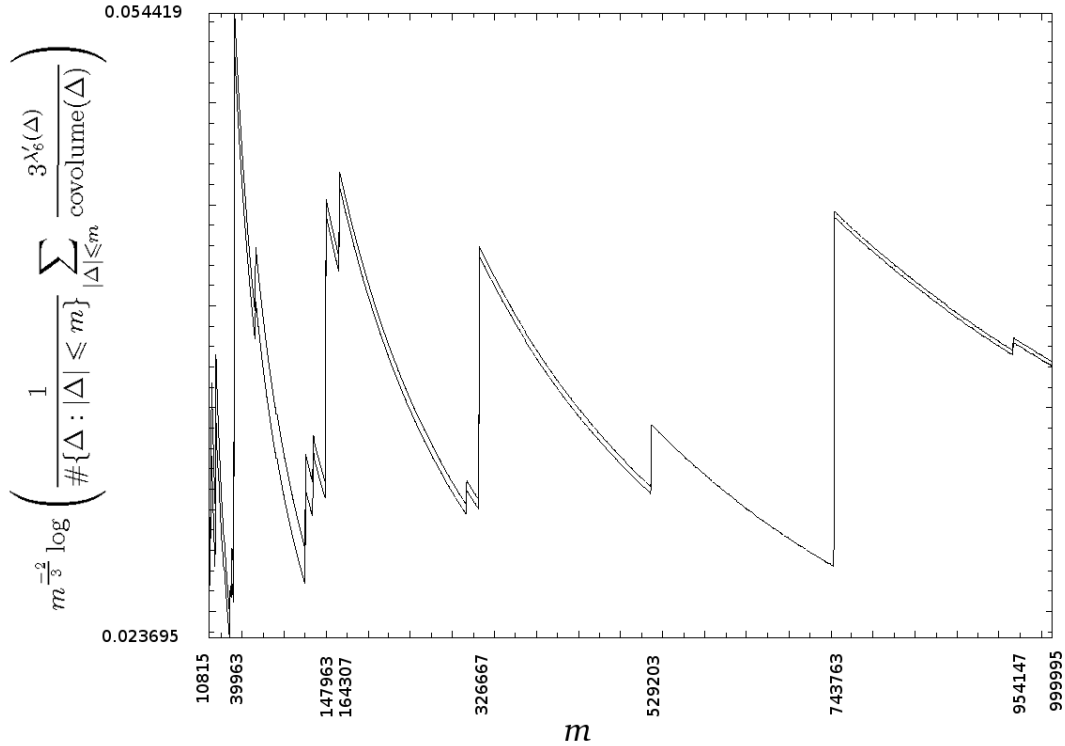
So for  $m$  greater than 10815 and less than one million, we can observe that the average of the above ratio oscillates between  $\exp(m^{\frac{2}{3}}0.023695)$  and  $\exp(m^{\frac{2}{3}}0.054419)$ . For  $m$  less than 10815, this oscillation is much stronger, and the diagram might be seen as suggesting that possibly the oscillation could remain between these two bounds for  $m$  greater than one million.

For related asymptotics, see the recent works of Bergeron/Venkatesh [3] and Sengun [24]. For an alternative computer program treating the Bianchi groups, see the SAGE package of Cremona’s student Aranes [2], and for  $\mathrm{GL}_2(\mathcal{O})$  see Yasaki’s program [28].

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FIGURE 2. Average homological 3-torsion outside subgroups of type  $\mathcal{S}_3$ , scaled as indicated.



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