



HAL
open science

NUMBER-THEORETIC FORMULAE FOR THE COHOMOLOGY RING STRUCTURE OF THE BIANCHI GROUPS

Alexander D. Rahm

► **To cite this version:**

Alexander D. Rahm. NUMBER-THEORETIC FORMULAE FOR THE COHOMOLOGY RING STRUCTURE OF THE BIANCHI GROUPS. 2011. hal-00618167v3

HAL Id: hal-00618167

<https://hal.science/hal-00618167v3>

Preprint submitted on 18 Dec 2011 (v3), last revised 26 Sep 2013 (v8)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

NUMBER-THEORETIC FORMULAE FOR THE COHOMOLOGY RING STRUCTURE OF THE BIANCHI GROUPS

ALEXANDER D. RAHM

ABSTRACT. We describe the torsion in the homology of *Bianchi groups*, i.e. $\mathrm{PSL}_2(\mathcal{O})$ for \mathcal{O} the ring of integers in an imaginary quadratic number field. We show that the homological torsion is completely determined by the numbers of conjugacy classes of finite subgroups of the Bianchi groups, by detaching information from geometric models and expressing it only with the group structure. Formulae for the numbers of conjugacy classes of finite subgroups have been determined in a thesis of Krämer, in terms of elementary number-theoretic information on \mathcal{O} . An evaluation of these formulae for a large number of Bianchi groups is provided numerically. Our new insights about the homological torsion allow us to give a conceptual description of the cohomology ring structure of the Bianchi groups.

1. INTRODUCTION

Denote by $\mathbb{Q}(\sqrt{-m})$, with m a square-free positive integer, an imaginary quadratic number field, and by \mathcal{O}_{-m} its ring of integers. The *Bianchi groups* are the groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$. The Bianchi groups may be considered as a key to the study of a larger class of groups, the *Kleinian groups*, which date back to work of Henri Poincaré [15]. In fact, each non-cocompact arithmetic Kleinian group is commensurable with some Bianchi group [14]. A wealth of information on the Bianchi groups can be found in the monographs [10], [9], [14].

Norbert Krämer [13] has determined number-theoretic formulae for the numbers of conjugacy classes of finite subgroups in the Bianchi groups, using numbers of ideal classes in orders of cyclotomic extensions of $\mathbb{Q}(\sqrt{-m})$.

The purpose of this article is to express the homological torsion of the Bianchi groups as a function of these numbers of conjugacy classes. To achieve this, we build on the geometric techniques of [17], which depend on the explicit knowledge of the quotient space of geometric models for the Bianchi groups — like any technique effectively accessing the (co)homology of the Bianchi groups, either directly [20], [23] or via a group presentation [5]. The main achievement of this article is to detach invariants of the group actions from the geometric models, in order to express them only by the group structure itself, in terms of conjugacy classes of finite subgroups, normalisers of the latter, and their interactions. This information is gathered, and reduced to its essence, in the conjugacy classes graphs we construct for this purpose.

Not only does this provide us with exact formulae for the homological torsion of the Bianchi groups, the power of which we can see in the numerical evaluations of sections 5 and 6, also it allows us to understand the rôle of the centralisers of the finite subgroups, and this is how in [16], some more fruits of the present results are harvested.

Date: December 18, 2011.

2000 *Mathematics Subject Classification.* 11F75, Cohomology of arithmetic groups.

Funded by the Irish Research Council for Science, Engineering and Technology.

Statement of the results. Except for the Gaussian and Eisenstein integers, which can easily be treated separately [20], [17], all the rings of integers of imaginary quadratic number fields admit as only units $\{\pm 1\}$. In the latter case, we call $\mathrm{PSL}_2(\mathcal{O})$ a *Bianchi group with units $\{\pm 1\}$* . For the possible types of finite subgroups in the Bianchi groups, see lemma 8 : there are five non-trivial possibilities. Our main result is the following. We make it more explicit in theorem 3 and give its proof in section 4. We deduce the structure of the cohomology ring in section 2.

Corollary 1. *The homology of the Bianchi groups with units $\{\pm 1\}$ depends in all degrees above 2 (their virtual cohomological dimension) only on the numbers of conjugacy classes of non-trivial finite subgroups of the occurring five types.*

The main step in order to prove this, is to show how the homological torsion is determined by the following graph, which we construct purely group-theoretically. Let ℓ be a prime number. For a circle to become a graph, we identify the two endpoints of a single edge.

Definition 2. *The ℓ -conjugacy classes graph of an arbitrary group Γ is given by the following construction.*

- We take as vertices the conjugacy classes of finite subgroups G of Γ containing elements γ of order ℓ such that the normaliser of $\langle \gamma \rangle$ in G is not $\langle \gamma \rangle$ itself.
- We connect two vertices by an edge if and only if they admit representatives sharing a common subgroup of order ℓ .
- For every pair of subgroups of order ℓ in G , which are conjugate in Γ but not in G , we draw a circle attached to the vertex labelled by G .
- For every conjugacy class of subgroups of order ℓ which are not properly contained in any finite subgroup of Γ , we add a disjoint circle.

Kr amer's formulae express the numbers of conjugacy classes of the five types of non-trivial finite subgroups in the Bianchi groups, where the symbols in the first row are Kr amer's notations for the number of their conjugacy classes:

λ_4	λ_6	μ_2	μ_3	μ_T
$\mathbb{Z}/2$	$\mathbb{Z}/3$	\mathcal{D}_2	\mathcal{S}_3	\mathcal{A}_4

Recall that we can express the homology in degrees above the virtual cohomological dimension of the Bianchi groups by the two Poincar e series — for $\ell = 2$ and $\ell = 3$ — in the dimensions over the field with ℓ elements, of the homology with \mathbb{Z}/ℓ -coefficients of $\mathrm{PSL}_2(\mathcal{O}_{-m})$,

$$P_m^\ell(t) := \sum_{q>2}^{\infty} \dim_{\mathbb{F}_\ell} H_q(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z}/\ell) t^q,$$

which have been suggested by Grunewald. Further let $P_{\mathcal{O}}(t) := \frac{-2t^3}{t-1}$, which equals the series $P_m^2(t)$ of the groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ the 2-conjugacy classes graph of which is a circle. Denote by

- $P_{\mathcal{D}_2}^*(t) := \frac{-t^3(3t-5)}{2(t-1)^2}$, the Poincar e series over $\dim_{\mathbb{F}_2} H_q(\mathcal{D}_2; \mathbb{Z}/2) - \frac{3}{2} \dim_{\mathbb{F}_2} H_q(\mathbb{Z}/2; \mathbb{Z}/2)$
- and by $P_{\mathcal{A}_4}^*(t) := \frac{-t^3(t^3-2t^2+2t-3)}{2(t-1)^2(t^2+t+1)}$, the Poincar e series over

$$\dim_{\mathbb{F}_2} H_q(\mathcal{A}_4; \mathbb{Z}/2) - \frac{1}{2} \dim_{\mathbb{F}_2} H_q(\mathbb{Z}/2; \mathbb{Z}/2).$$

In 3-torsion, let $P_{\bullet\bullet}(t) := \frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}$, which equals the series $P_m^3(t)$ for the Bianchi groups the 3-conjugacy classes graph of which is a single edge without identifications.

Theorem 3. *For all Bianchi groups with units $\{\pm 1\}$, the homology in degrees above their virtual cohomological dimension is given by the Poincaré series*

$$P_m^2(t) = \left(\lambda_4 - \frac{3\mu_2 - 2\mu_T}{2} \right) P_{\circlearrowleft}(t) + (\mu_2 - \mu_T) P_{\mathcal{D}_2}^*(t) + \mu_T P_{\mathcal{A}_4}^*(t)$$

and

$$P_m^3(t) = \left(\lambda_6 - \frac{\mu_3}{2} \right) P_{\circlearrowleft}(t) + \frac{\mu_3}{2} P_{\bullet\bullet}(t).$$

Organisation of the paper. We establish the structure of the cohomology rings in section 2. In section 3, we show that the conjugacy classes graphs determine the homology of the Bianchi groups above their virtual cohomological dimension. We achieve this by showing that for the Bianchi groups, the conjugacy classes graphs are homeomorphic to the torsion subcomplexes extracted from geometric models. This enables us in section 4 to prove the formulae for the homological torsion of the Bianchi groups in terms of numbers of conjugacy classes of finite subgroups. Krämer has given number-theoretic formulae for these numbers of conjugacy classes, and we evaluate them numerically in sections 5 and 6. Finally, we present some numerical asymptotics on the numbers of conjugacy classes in section 7.

Acknowledgements. The author is indebted to the late great mathematician Fritz Grunewald, for telling him about the existence and providing him a copy of Krämer's Diplom thesis. Warmest thanks go to Oliver Braunling for a correspondence on the occurrence of given norms on rings of integers, to Nicolas Bergeron for discussions on asymptotics, to Matthias Wendt for a very careful lecture of the manuscript and helpful suggestions, and to Stephen Gelbart for support and encouragement.

2. THE COHOMOLOGY RING STRUCTURE OF THE BIANCHI GROUPS

In [5], Berkove has found a compatibility of the cup product of the cohomology ring of a Bianchi group with the cup product of the cohomology rings of its finite subgroups. This compatibility within the equivariant spectral sequence implies that all products that come from different connected components of the torsion subcomplex, which we identify with the conjugacy classes graph in section 3, are zero. It follows that the cohomology ring of any Bianchi group splits into a restricted sum over subrings, which depend in degrees above the virtual cohomological dimension only on the homeomorphism type of the associated connected component of the conjugacy classes graph. The analogue in cohomology of theorem 3 and Berkove's computations of sample cohomology rings [4] yield the following corollary in 3-torsion.

We use Berkove's notation, in which the degree j of a cohomology generator x_j is appended as a subscript. Furthermore, writing cohomology classes inside square brackets means that they are polynomial (of infinite multiplicative order), and writing them inside parentheses means that they are exterior (their powers vanish). The restricted sum \bigoplus identifies all the degree zero classes into a single copy of \mathbb{Z} ; when we write it with a power, we specify the number of summands. Recall that λ_6 (respectively μ_3) counts the number of conjugacy classes subgroups of type $\mathbb{Z}/3$ (respectively \mathcal{S}_3) in the Bianchi group.

T	subring associated to connected components of type T in 2 – conjugacy classes graph
	$\mathbb{F}_2[n_1](m_1)$
	$\mathbb{F}_2[m_3, u_2, v_3, w_3]/\langle m_3 v_3 = 0, \quad u_2^3 + w_3^2 + v_3^2 + m_3^2 + w_3(v_3 + m_3) = 0 \rangle$
	$\mathbb{F}_2[n_1, m_2, n_3, m_3]/\langle n_1 n_3 = 0, \quad m_2^3 + m_3^2 + n_3^2 + m_3 n_3 + n_1 m_2 m_3 = 0 \rangle$
	$\mathbb{F}_2[n_1, m_1, m_3]/\langle m_3(m_3 + n_1^2 m_1 + n_1 m_1^2) = 0 \rangle$

TABLE 1. Restricted summands of the mod-2 cohomology ring $H^*(\Gamma; \mathbb{F}_2)$ of a Bianchi group Γ above its virtual cohomological dimension.

Corollary 4. *In degrees above the virtual cohomological dimension, the 3–primary part of the cohomology ring of any Bianchi group Γ with units $\{\pm 1\}$ is given by*

$$H^*(\Gamma; \mathbb{Z})_{(3)} \cong \tilde{\oplus}^{(\lambda_6 - \frac{\mu_3}{2})} \mathbb{Z}[x_2](\sigma_1) \tilde{\oplus}^{\frac{\mu_3}{2}} \mathbb{Z}[x_4](x_3),$$

where the generators x_j are of additive order 3.

In 2–torsion, it does in general not suffice to know only the Kramer numbers to obtain the cohomology ring structure, because for the two 2–conjugacy classes graphs  and , we obtain the same Kramer numbers and homological 2–torsion, but different multiplicative structures of the mod-2 cohomology rings, as we can see from table 1, which we compile from the results of [5] and [17].

Observation 5. In the cases of class numbers 1 and 2, and in the cases where the absolute value of the discriminant is inferior to 200, only the homeomorphism types T listed in table 1 occur as connected components in the 2–conjugacy classes graph. So for all these Bianchi groups Γ with units $\{\pm 1\}$, the mod-2 cohomology ring $H^*(\Gamma; \mathbb{F}_2)$ splits, above the virtual cohomological dimension, as a restricted sum over the subrings specified in table 1, with powers according to the multiplicities of the occurrences of the types T .

3. THE CONJUGACY CLASSES OF FINITE ORDER ELEMENTS

The groups $SL_2(\mathcal{O}_{-m})$ act in a natural way on hyperbolic three-space, which is isomorphic to the symmetric space $SL_2(\mathbb{C})/SU(2)$ associated to them. The kernel of this action is the centre $\{\pm 1\}$ of the groups. Thus it is useful to study the quotient of $SL_2(\mathcal{O}_{-m})$ by its centre, namely $PSL_2(\mathcal{O}_{-m})$.

Let $\Gamma = PSL_2(\mathcal{O}_{-m})$ be a Bianchi group. Then any element of Γ fixing a point inside real hyperbolic 3-space $\mathcal{H}_{\mathbb{R}}^3$ acts as a rotation of finite order. Let Z be the refined cellular complex obtained from the action of Γ on hyperbolic 3-space as described in [17], namely we subdivide $\mathcal{H}_{\mathbb{R}}^3$ until the stabiliser in Γ of any cell σ fixes σ pointwise. We achieve this by computing Bianchi’s fundamental polyhedron for the action of Γ , taking as preliminary set of 2-cells its facets lying on the Euclidean hemispheres and vertical planes of the upper-half space model for $\mathcal{H}_{\mathbb{R}}^3$, and then subdividing along the rotation axes of the elements of Γ . Let ℓ be a prime number.

Definition 6. *The ℓ -torsion subcomplex is the subcomplex of $\Gamma \backslash Z$ consisting of all the cells, the pre-images of which have stabilisers in Γ containing elements of order ℓ .*

It is well-known that if γ is an element of Bianchi group of finite order n , then n must be 1, 2, 3, 4 or 6, because γ has eigenvalues ρ and $\bar{\rho}$ with ρ a primitive n -th root of unity and the trace of γ is $\rho + \bar{\rho} \in \mathcal{O}_{-m} \cap \mathbb{R} = \mathbb{Z}$. For ℓ being one of the two occurring primes 2 and 3, this subcomplex is a finite graph, because the cells of dimension greater than 1 are trivially stabilised in the refined cellular complex. We reduce this subcomplex with the procedure of [17], which consists in taking the pairs of edges with a common endpoint such that no further edge is adjacent to this endpoint, and replacing them together with this endpoint by a single edge.

Theorem 7. *Let Γ be any Bianchi group with units $\{\pm 1\}$ and ℓ any prime number. Then the ℓ -conjugacy classes graph and the reduced ℓ -torsion subcomplex of Γ are isomorphic graphs.*

The rest of this section will be devoted to the proof of this theorem. The first ingredient is the following classification of Felix Klein [11].

Lemma 8 (Klein). *The finite subgroups in $\mathrm{PSL}_2(\mathcal{O})$ are exclusively of isomorphism types the cyclic groups of orders one, two and three, the Klein four-group $\mathcal{D}_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, the symmetric group \mathcal{S}_3 and the alternating group \mathcal{A}_4 .*

Recall the following lemma from [17].

Lemma 9. *Let v be a non-singular vertex in the refined cell complex. Then the number \mathbf{n} of orbits of edges in the refined cell complex adjacent to v , with stabiliser in $\mathrm{PSL}_2(\mathcal{O}_{-m})$ isomorphic to \mathbb{Z}/ℓ , is given as follows for $\ell = 2$ and $\ell = 3$.*

Isomorphism type of the vertex stabiliser	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	\mathcal{D}_2	\mathcal{S}_3	\mathcal{A}_4
\mathbf{n} for $\ell = 2$	0	2	0	3	2	1
\mathbf{n} for $\ell = 3$	0	0	2	0	1	2.

Now we investigate the associated normaliser groups. Straight-forward verification using the multiplication tables of the concerned finite groups yields the following.

Lemma 10. *Let G be a finite subgroup of $\mathrm{PSL}_2(\mathcal{O}_{-m})$. Then the type of the normaliser of any subgroup of type \mathbb{Z}/ℓ in G is given as follows for $\ell = 2$ and $\ell = 3$, where we print only cases with existing subgroup of type \mathbb{Z}/ℓ .*

Isomorphism type of G	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	\mathcal{D}_2	\mathcal{S}_3	\mathcal{A}_4
normaliser of $\mathbb{Z}/2$		$\mathbb{Z}/2$		\mathcal{D}_2	$\mathbb{Z}/2$	\mathcal{D}_2
normaliser of $\mathbb{Z}/3$			$\mathbb{Z}/3$		\mathcal{S}_3	$\mathbb{Z}/3$.

The final ingredient in the proof of theorem 7 is the following.

Theorem 11. *There is a natural bijection between conjugacy classes of subgroups of $\mathrm{PSL}_2(\mathcal{O}_{-m})$ of order ℓ and edges of the reduced ℓ -torsion subcomplex. It is given by considering the stabiliser of a representative edge in the refined cell complex.*

In order to prove the latter theorem, we need several lemmata, and we establish them now.

Lemma 12. *Consider two adjacent edges E, E' of the non-reduced torsion subcomplex. Then for any representative e of E , there is an adjacent representative e' of E' on the same geodesic line as e .*

Proof. Consider the element $\gamma \in \Gamma$ identifying the end v of e with the origin $\gamma(v)$ of e' . As E and E' are distinct in the orbit space, γ^{-1} cannot send e' onto e .

Assume that the stabiliser of v is of isomorphism type $\mathbb{Z}/\ell\mathbb{Z}$. Then as the edge $\gamma^{-1}(e')$ is pointwise fixed by Γ , its stabiliser must contain the rotation with axis passing through e . Hence $\gamma^{-1}(e')$ is adjacent to e and on the same geodesic line.

If the stabiliser of v is of isomorphism type $\mathcal{D}_2, \mathcal{S}_3$ or \mathcal{A}_4 , there are at most two orbits of ℓ -torsion stabilised edges adjacent to v . So there is an element α in the stabiliser of v such that $\alpha\gamma^{-1}(e')$ is adjacent to e and on the same geodesic line. \square

Corollary 13. *Any edge of the reduced torsion subcomplex can be represented by a chain of edges on the intersection of one geodesic line with a strict fundamental domain for Γ in \mathcal{H} .*

Proof. Consider the chain of edges in the torsion subcomplex that are reduced to the given edge of the reduced torsion subcomplex. The representatives in \mathcal{H} of the edges on this chain lie on pairwise different orbits because the torsion subcomplex is a subcomplex of the orbit space $\Gamma \backslash \mathcal{H}$. Now we start with a representative e for the edge sharing its origin with the reduced edge, and use lemma 12 to obtain an adjacent edge e' on the same geodesic line, representing the adjacent edge of the torsion subcomplex. We proceed this way, assigning the role of e to e' and so on, until we have reached the end of the reduced edge. \square

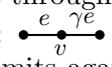
Corollary 14. *Any edge of the reduced torsion subcomplex admits only representatives with stabiliser in the same conjugacy class.*

Lemma 15. *Let α and γ be elements of $\mathrm{PSL}_2(\mathbb{C})$. Then the fixed point set in \mathcal{H} of α is identified by γ with the fixed point set of $\gamma\alpha\gamma^{-1}$.*

Proof. One immediately checks that any fixed point $x \in \mathcal{H}$ of α induces the fixed point $\gamma(x)$ of $\gamma\alpha\gamma^{-1}$. As $\mathrm{PSL}_2(\mathbb{C})$ acts by isometries, the whole fixed point sets are identified. \square

Lemma 16. *Let $v \in \mathcal{H}_{\mathbb{R}}^3$ be a vertex with stabiliser in Γ of type \mathcal{D}_2 or \mathcal{A}_4 . Let $\gamma \in \Gamma$ be a rotation of order 2 around an edge e adjacent to v .*

Then the centraliser $C_{\Gamma}(\gamma)$ reflects \mathcal{H}^{γ} — which is the geodesic line through e — onto itself at v .

Proof. Denote by Γ_v the stabiliser of the vertex v . In the case that Γ_v is of type \mathcal{D}_2 , which is Abelian, it admits two order-2-elements centralising γ and turning the geodesic line through e onto itself such that the image of e touches v from the side opposite to e (illustration: ). In the case that Γ_v is of type \mathcal{A}_4 , it contains a normal subgroup of type \mathcal{D}_2 that admits again two such elements. \square

Let α be any torsion element in Γ . We construct the *chain of edges associated to α* as follows. Consider the edge of the reduced torsion subcomplex to which the edge stabilised by α belongs. Use corollary 13 to represent it by a chain of edges on a geodesic line. Now, α is conjugate to an element $\gamma\alpha\gamma^{-1}$ of the stabiliser of one of the edges in the chain. By lemma 15, the element $\gamma^{-1} \in \Gamma$ maps the mentioned geodesic line to the rotation axis of α . The image under γ^{-1} of the chain of edges under consideration is the desired chain associated to α . It is well defined up to translation along the rotation axis of α .

Proof of theorem 7. Comparing with lemma 9, we see that the vertex set of the ℓ -conjugacy classes graph gives precisely the bifurcation points and vertices with only one adjacent edge of the ℓ -torsion subcomplex. When passing to the reduced ℓ -torsion subcomplex, we get rid of all vertices with two adjacent edges except in the disjoint circles, see [17]. By theorem 11, the edges of the ℓ -conjugacy classes graph give the edges of the reduced ℓ -torsion subcomplex. \square

4. THE KRÄMER NUMBERS AND GROUP HOMOLOGY

Krämer [13] has determined number-theoretic formulae for the numbers of conjugacy classes of the five types of non-trivial finite subgroups in the Bianchi groups, where the symbols in the first row are Krämer's notations for the number of their conjugacy classes:

μ_2	μ_T	μ_3	λ_{2n}	λ_4^T	λ_4^*	λ_6^*	μ_2^-
\mathcal{D}_2	\mathcal{A}_4	\mathcal{S}_3	\mathbb{Z}/n	$\mathbb{Z}/2 \subset \mathcal{A}_4$	$\mathbb{Z}/2 \subset \mathcal{D}_2$	$\mathbb{Z}/3 \subset \mathcal{S}_3$	$\mathcal{D}_2 \not\subset \mathcal{A}_4$

Here, the inclusion signs “ \subset ” mean that we only consider copies of \mathbb{Z}/n admitting the specified inclusion in the given Bianchi group and $\mathcal{D}_2 \not\subset \mathcal{A}_4$ means that we only consider copies of \mathcal{D}_2 not admitting any inclusion into a subgroup of type \mathcal{A}_4 of the Bianchi group.

Note that the number μ_2^- is simply the difference $\mu_2 - \mu_T$, because every copy of \mathcal{A}_4 admits precisely one normal subgroup of type \mathcal{D}_2 . Also, note the following graph-theoretical properties of the reduced torsion subgraphs, the latter of which we obtain by restricting our attention to the connected components not homeomorphic to \mathcal{O} .

Corollary 19 (Corollary to lemma 9). *For all Bianchi groups with units $\{\pm 1\}$, the Krämer numbers satisfy $\lambda_4^T \leq \mu_T$ and $2\lambda_6^* = \mu_3$, and even*

$$2\lambda_4^* = \mu_T + 3\mu_2^-.$$

The values given by Krämer's formulae are matching with the values computed with [18].

Observation 20. The Krämer numbers determine the 3-conjugacy classes graph and hence the reduced 3-torsion subcomplex for all Bianchi groups with units $\{\pm 1\}$, as we can see immediately from theorem 7 and the description of the reduced 3-torsion subcomplex in [17].

From now on, we will use group homology, the virtual cohomological dimension and the equivariant spectral sequence, none of which we do define: they can be looked up in [7]. Except in degrees 1 and 2 (below and at the virtual cohomological dimension of the Bianchi groups), where we still have to resort to the methods of [20] in the principal ideal domain cases and [19] in the other cases, we establish the characterisation of theorem 3.

In 3-torsion, theorem 3 follows directly from observation 20, corollary 19 and the main result of [17], which is the following.

Theorem 21 ([17]). *The ℓ -primary part of the integral homology of $\mathrm{PSL}_2(\mathcal{O}_{-m})$ depends in degrees greater than 2 (the virtual cohomological dimension) only on the homeomorphism type of the ℓ -torsion subcomplex.*

In 2-torsion, we have to recall some lemmata in order to give the proof of theorem 3.

Lemma 22 ([17]). *The ℓ -primary part of the terms $E_{p,q}^2$ of the equivariant spectral sequence converging to $H_{p+q}(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$ depends in all rows $q \geq 1$ only on the homeomorphism type of the ℓ -torsion subcomplex.*

Sub-Lemma 23 ([17]). *The matrix for the ℓ -primary part of the differential $d_{1,q}^1$ of the equivariant spectral sequence converging to $H_{p+q}(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$ can be decomposed as a direct sum of the blocks associated to the connected components of the ℓ -torsion subgraph.*

Lemma 24 (Schwermer/Vogtmann). *Let M be \mathbb{Z} or $\mathbb{Z}/2$. Consider group homology with trivial M -coefficients. Then the following holds.*

- Any inclusion $\mathbb{Z}/2 \rightarrow \mathcal{S}_3$ induces an injection on homology.
- An inclusion $\mathbb{Z}/3 \rightarrow \mathcal{S}_3$ induces an injection on homology in degrees congruent to 3 or 0 mod 4, and is otherwise zero.
- Any inclusion $\mathbb{Z}/2 \rightarrow \mathcal{D}_2$ induces an injection on homology in all degrees.
- An inclusion $\mathbb{Z}/3 \rightarrow \mathcal{A}_4$ induces injections on homology in all degrees.
- An inclusion $\mathbb{Z}/2 \rightarrow \mathcal{A}_4$ induces injections on homology in degrees greater than 1, and is zero on H_1 .

For the proof in \mathbb{Z} -coefficients, see [20], for $\mathbb{Z}/2$ -coefficients see [17].

Lemma 25 ([17], lemma 32). *Let $q \geq 3$ be an odd integer number. Let v be a vertex representative of stabiliser type \mathcal{D}_2 in the refined cellular complex for the Bianchi groups. Then the three images in $(H_q(\mathcal{D}_2; \mathbb{Z}))_{(2)}$ induced by the inclusions of the stabilisers of the edges adjacent to v , are linearly independent.*

Finally, we deduce one new ingredient for the proof of theorem 3, which might be of interest in its own right.

Lemma 26. *In all rows $q > 1$ and outside connected components of type \mathcal{O} , the $d_{p,q}^1$ -differential of the equivariant spectral sequence converging to $H_{p+q}(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$ is always injective.*

Proof. For matrix blocks of the $d_{p,q}^1$ -differential associated to vertices with just one adjacent edge, we see from lemma 9 that the vertex stabiliser is of type \mathcal{A}_4 in 2-torsion, respectively of type \mathcal{S}_3 in 3-torsion, so injectivity follows from lemma 24. As we have placed ourselves outside connected components of type \mathcal{O} , the remaining vertices are bifurcation points of stabiliser type \mathcal{D}_2 and injectivity follows from lemma 25. \square

Proof of theorem 3. What we need to determine with the Kramer numbers, is the 2-primary part of the $E_{p,q}^2$ -term of the equivariant spectral sequence converging to $H_{p+q}(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$ in all rows $q > 1$. From there, we see from the deduction of theorem 21 in [17] that we obtain the claim. By sub-lemma 23 and lemma 22, we only need to check this determination on each homeomorphism type of connected components of the 2-torsion subgraph. We use theorem 7 to identify the reduced 2-torsion subgraph and the 2-conjugacy classes graph. Then we can observe that

- Kramer's number $\lambda_4^* - \lambda_4$ determines the number of connected components of type \mathcal{O} .
- Kramer's number λ_4^* determines the number of edges of the 2-torsion subgraph outside connected components of type \mathcal{O} . Lemma 26 tells us that the block of the $d_{p,q}^1$ -differential of the equivariant spectral sequence associated to such edges is always injective.
- Kramer's number μ_2^- determines the number of bifurcation points, and μ_T determines the number of vertices with only one adjacent edge of the 2-torsion subgraph.

Using corollary 19, we obtain the explicit formulae in theorem 3. \square

5. NUMERICAL EVALUATION OF KRÄMER'S FORMULAE IN 3-TORSION

Denote by δ the number of finite ramification places of $\mathbb{Q}(\sqrt{-m})$ over \mathbb{Q} . Let k_+ be the totally real number field $\mathbb{Q}(\sqrt{3m})$ and denote its ideal class number by h_{k_+} . Krämer introduces the following indicators:

$$z := \begin{cases} 2, & \text{if 3 is the norm of an integer of } k_+, \\ 1, & \text{otherwise.} \end{cases}$$

For $m \equiv 0 \pmod{3}$ and $m \neq 3$, denote by $\epsilon := \frac{1}{2}(a + b\sqrt{\frac{m}{3}}) > 1$ the fundamental unit of k_+ (where $a, b \in \mathbb{N}$). Now, define

$$x' := \begin{cases} 2, & \text{if the norm of } \epsilon \text{ is } 1, \\ 1, & \text{if the norm of } \epsilon \text{ is } -1 \end{cases}$$

and

$$y := \begin{cases} 2, & \text{if } b \equiv 0 \pmod{3}, \\ 1, & \text{otherwise.} \end{cases}$$

Then [13, 20.39 and 20.41] yield the following formulae in 3-torsion.

m specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$	λ_6^*	$\lambda_6 - \lambda_6^*$
$m \equiv 2 \pmod{3}$	0	$\frac{z}{2}h_{k_+}$
$m \equiv 1 \pmod{3}$ gives either	$2^{\delta-1}$	$\frac{1}{2}(h_{k_+} - 2^{\delta-1})$
or	0	$\frac{1}{2}h_{k_+}$
$m \equiv 6 \pmod{9}$	0	$x'yh_{k_+}$
$m \equiv 3 \pmod{9}$, $m \neq 3$ gives either	$2^{\delta-2}$	$\frac{1}{2}(3x'h_{k_+} - 2^{\delta-2})$
or	0	$\frac{1}{2}3x'h_{k_+}$

The above case distinctions come from the fact that Krämer's theorem 20.39 ranges over all types of maximal orders in quaternion algebras over $\mathbb{Q}(\sqrt{-m})$, in which Krämer determines the numbers of conjugacy classes in the norm-1-group. The remaining task in order to decide which of the cases applies, is to find out of which type considered in the mentioned theorem is the maximal order $M_2(\mathcal{O}_{-m})$. Some methods to cope with this task are introduced in [13, §27].

Krämer's resulting criteria can be summarised as follows for 3-torsion.

condition	implication
$m \equiv 2 \pmod{3}$	$\mu_3 = \lambda_6^* = 0.$
$m \equiv 6 \pmod{9}$	$\mu_3 = \lambda_6^* = 0.$
m prime and $m \equiv 1 \pmod{3}$	$\lambda_6^* > 0.$
$m = 3p$ with p prime and $p \equiv 1 \pmod{3}$	$\lambda_6^* > 0.$
$m \equiv 1 \pmod{3}$ and -3 occurs as norm on \mathcal{O}_{k_+}	$\lambda_6^* > 0.$
$m \equiv 1 \pmod{3}$ and -3 does not occur as norm on \mathcal{O}_{k_+}	$\lambda_6 - \lambda_6^* > 0.$
$m \equiv 1 \pmod{3}$ and m admits a prime divisor p with $p \equiv 2 \pmod{3}$	$\lambda_6 - \lambda_6^* > 0.$
$m \equiv 3 \pmod{9}$ and $x' = 1$ and $\frac{m}{3}$ admits only prime divisors p with $p \equiv 1 \pmod{12}$	$\lambda_6^* > 0.$
$m \equiv 3 \pmod{9}$ and $x' = 1$ and $\frac{m}{3}$ admits a prime divisor p with $p \equiv 5 \pmod{12}$	$\lambda_6^* = 0.$
$m \equiv 3 \pmod{9}$ and $h(k'_+) = 2^{\delta-3}$ and $\frac{m}{3}$ admits only prime divisors p with $p \equiv \pm 1 \pmod{12}$ or $p = 2$	$\lambda_6^* = 0.$
$m \equiv 3 \pmod{9}$ and $h(k'_+) = 1$ and $\frac{m}{3} = p'p$ with p', p prime and $p' \equiv p \equiv 7 \pmod{12}$	$\lambda_6^* > 0.$

In order to determine Krämer's indicator z , we need to determine if a given value occurs as the norm on the ring of integers of an imaginary quadratic number field. This is implemented in Pari/GP [1] (the first step is computing the answer under the Generalised Riemann hypothesis, and the second step is a check computation which confirms that we arrive at that answer without this hypothesis). Additionally, we compare with the below criterion [13, (20.13)].

Lemma 27 (Krämer). *Let m be not divisible by 3.*

- *If the number -3 is the norm of an integer in the totally real number field k_+ , then all prime divisors $p \in \mathbb{N}$ of m satisfy the congruence $p \equiv 1 \pmod{3}$. Especially, the congruence $m \equiv 1 \pmod{3}$ is implied.*
- *If the number 3 is the norm of an integer in the totally real number field k_+ , then all prime divisors $p \in \mathbb{N}$ of m satisfy either $p = 2$ or the congruence $p \equiv \pm 1 \pmod{12}$. Additionally, the congruence $m \equiv 2 \pmod{3}$ is implied.*

With Kramer's criteria at hand, we can decide for many Bianchi groups, which of the alternative cases in Kramer's formulae must be used. We do this in the below tables for all such Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with absolute value of the discriminant Δ ranging between 7 and

2003, where we recall that the discriminant is $\Delta = \begin{cases} -m, & m \equiv 3 \pmod{4}, \\ -4m, & \text{else.} \end{cases}$

In the cases $m \in \{102, 133, 165, 259, 559, 595, 763, 835, 1435\}$, where these statements are not sufficient to eliminate the wrong alternatives, we insert the results of [18]. This way, the below tables treat all Bianchi groups with units $\{\pm 1\}$ and discriminant of absolute value less than 615. The cases where an ambiguity remains (so to exclude them from our tables) are $m \in \{210, 262, 273, 298, 345, 426, 430, 462, 481, 615, 1155, 1159, 1195, 1339, 1351, 1407, 1515, 1807\}$. For tables of the cases without ambiguity, with m ranging up to 10000, see the preprint version 2 of this paper on HAL.

In [12], a theorem is established which solves all these ambiguities by giving for each type of finite subgroups in $\mathrm{PSL}_2(\mathcal{O}_{-m})$ criteria equivalent to its occurrence, in terms of congruence conditions on the prime divisors of m .

3-conjugacy classes graph	m specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 3-conjugacy classes graph
	2, 5, 6, 10, 11, 14, 15, 17, 22, 23, 29, 34, 35, 38, 41, 46, 47, 51, 53, 55, 58, 59, 62, 71, 82, 83, 86, 87, 89, 94, 95, 101, 106, 113, 115, 118, 119, 123, 131, 134, 137, 142, 149, 155, 158, 159, 166, 167, 173, 178, 179, 187, 191, 197, 202, 203, 206, 214, 215, 226, 227, 233, 235, 239, 251, 254, 257, 263, 267, 269, 274, 278, 281, 287, 293, 295, 303, 311, 317, 319, 323, 326, 334, 335, 339, 346, 347, 353, 355, 358, 359, 371, 382, 383, 389, 391, 394, 395, 398, 401, 411, 415, 422, 431, 443, 446, 447, 449, 451, 454, 461, 466, 467, 478, 479, 491, 515, 519, 527, 535, 551, 563, 583, 591, 599, 623, 635, 647, 655, 659, 667, 683, 695, 699, 707, 719, 731, 743, 755, 779, 791, 799, 807, 815, 827, 839, 843, 879, 887, 895, 899, 911, 943, 947, 951, 955, 959, 979, 983, 995, 1003, 1019, 1031, 1055, 1059, 1091, 1103, 1111, 1115, 1135, 1139, 1151, 1163, 1167, 1187, 1207, 1211, 1219, 1223, 1243, 1247, 1255, 1259, 1271, 1283, 1307, 1315, 1343, 1347, 1363, 1367, 1379, 1383, 1411, 1415, 1439, 1487, 1499, 1507, 1511, 1523, 1527, 1535, 1555, 1559, 1563, 1571, 1607, 1631, 1639, 1643, 1655, 1667, 1671, 1707, 1711, 1735, 1751, 1763, 1779, 1787, 1795, 1799, 1811, 1819, 1823, 1835, 1847, 1851, 1883, 1903, 1907, 1915, 1919, 1923, 1927, 1931, 1943, 1959, 1979, 2003,
2 	26, 42, 65, 69, 70, 74, 77, 78, 85, 110, 122, 130, 141, 143, 145, 154, 161, 170, 182, 185, 186, 190, 194, 195, 205, 209, 213, 218, 221, 222, 230, 231, 238, 253, 265, 266, 286, 305, 310, 314, 322, 329, 365, 366, 370, 377, 386, 406, 407, 410, 418, 434, 437, 442, 445, 455, 458, 470, 473, 474, 483, 485, 493, 494, 497, 555, 611, 627, 671, 715, 767, 803, 851, 923, 935, 1015, 1079, 1095, 1199, 1235, 1295, 1311, 1391, 1403, 1455, 1463, 1491, 1495, 1595, 1599, 1615, 1679, 1703, 1739, 1771, 1855, 1887, 1991,
3 	30, 66, 107, 138, 174, 255, 282, 302, 318, 354, 419, 498, 503, 759, 771, 795, 835, 863, 1007, 1319, 1355, 1427, 1479, 1551, 1583, 1619, 1691, 1695, 1871, 1895, 1947, 1967,
4 	33, 105, 114, 146, 177, 249, 258, 285, 290, 299, 321, 330, 341, 357, 374, 385, 393, 402, 413, 429, 465, 482, 595, 663, 915, 987, 1023, 1067, 1239, 1435, 1727, 1743, 1955, 1995,
5 	1043, 1203, 1451,
6 	102, 165, 246, 362, 390, 435, 1335, 1419, 1547,
7 	587, 971,

3-conjugacy classes graph	m specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 3-conjugacy classes graph
8 	438, 1131, 1635.
	7, 19, 31, 43, 67, 79, 103, 127, 139, 151, 163, 199, 211, 223, 271, 283, 307, 379, 439, 463, 487, 499, 523, 571, 607, 619, 631, 691, 727, 739, 751, 787, 811, 823, 859, 883, 907, 919, 967, 991, 1039, 1051, 1063, 1123, 1171, 1231, 1279, 1303, 1399, 1423, 1447, 1459, 1471, 1483, 1531, 1543, 1567, 1579, 1627, 1663, 1699, 1723, 1759, 1783, 1831, 1867, 1987, 1999,
 \amalg  	39, 111, 183, 219, 291, 327, 331, 367, 471, 543, 579, 643, 723, 831, 939, 1011, 1047, 1087, 1119, 1191, 1227, 1263, 1291, 1299, 1327, 1371, 1623, 1803, 1839, 1879, 1951, 1983,
 \amalg 2 	547, 1747,
 \amalg 4 	687,
 \amalg 10 	1731,
2 	13, 37, 61, 91, 109, 157, 181, 229, 247, 277, 349, 373, 403, 421, 427, 511, 679, 703, 871, 1099, 1147, 1267, 1591, 1603, 1687, 1891, 1963,
2  \amalg 	73, 97, 193, 241, 259, 313, 337, 409, 457, 559, 763, 1651, 1939,
2  \amalg 2 	21, 57, 93, 129, 201, 309, 381, 397, 399, 417, 453, 489, 651, 903, 1443, 1659, 1767, 1843,
2  \amalg 3 	433, 1027, 1387,
2  \amalg 8 	237,
4 	217, 301, 469,
4  \amalg 2 	133,

6. NUMERICAL EVALUATION OF KRÄMER'S FORMULAE IN 2-TORSION

Denote by δ the number of finite ramification places of $\mathbb{Q}(\sqrt{-m})$ over \mathbb{Q} . Let k_+ be the totally real number field $\mathbb{Q}(\sqrt{m})$ and denote its ideal class number by h_{k_+} . For $m \neq 1$, Krämer introduces the following indicators:

$$z := \begin{cases} 2, & \text{if } 2 \text{ is the norm of an integer of } k_+, \\ 1, & \text{otherwise,} \end{cases} \quad q := \begin{cases} 2, & \text{if } \pm 2 \text{ is the norm of an integer of } k_+, \\ 1, & \text{otherwise,} \end{cases}$$

$$w := \begin{cases} 2, & \text{if } \forall \text{ prime divisors } p \text{ of } m \text{ with } p \neq 2 \text{ we have } p \equiv \pm 1 \pmod{8}, \\ 1, & \text{if } m \text{ admits prime divisors } p \equiv \pm 3 \pmod{8}. \end{cases}$$

Furthermore, denote by $\epsilon := \frac{1}{2}(a + b\sqrt{m}) > 1$ the fundamental unit of k_+ (where $a, b \in \mathbb{N}$). Now, define

$$x := \begin{cases} 2, & \text{if the norm of } \epsilon \text{ is } 1, \\ 1, & \text{if the norm of } \epsilon \text{ is } -1 \end{cases} \quad \text{and} \quad y := \begin{cases} 3, & \text{if } b \equiv 0 \pmod{2}, \\ 1, & \text{if } b \equiv 1 \pmod{2}. \end{cases}$$

Then [13, 26.12 and 26.14] yield the following formulae in 2-torsion.

m specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$	μ_T	μ_2^-	λ_4^T	λ_4^*	$\lambda_4 - \lambda_4^*$
$m \equiv 7 \pmod{8}$	0	0	0	0	$\frac{z}{2}h_{k_+}$
$m \equiv 3 \pmod{8}$ gives either	2^δ	0	$2^{\delta-1}$	$2^{\delta-1}$	$\frac{1}{2}(h_{k_+} - 2^{\delta-1})$
or (provided that $2^{\delta-1} > 1$)	0	0	0	0	$\frac{1}{2}h_{k_+}$
$m \equiv 2 \pmod{4}$ and $w = 2$ gives either	$2^{\delta-1}$	$2^{\delta-1}$	$2^{\delta-2}z$	2^δ	$\frac{1}{4}x(z+2)h_{k_+} - 2^{\delta-1}$
or (provided that $2^{\delta-1} > 1$)	0	0	0	0	$\frac{1}{4}x(z+2)h_{k_+}$
$m \equiv 2 \pmod{4}$ and $w = 1$ gives either	$2^{\delta-1}$	0	$2^{\delta-2}$	$2^{\delta-2}$	$\frac{1}{2}(\frac{3}{2}xh_{k_+} - 2^{\delta-2})$
or	0	$2^{\delta-1}$	0	$2^{\delta-2}3$	$\frac{3}{2}(\frac{1}{2}xh_{k_+} - 2^{\delta-2})$
or (provided that $2^{\delta-1} > 2$)	0	0	0	0	$\frac{3}{4}xh_{k_+}$
$m \equiv 1 \pmod{8}$ and $m \neq 1$ and $w = 2$ gives either	$2^{\delta-1}$	$2^{\delta-1}$	$2^{\delta-2}$	2^δ	$2xh_{k_+} - 2^{\delta-1}$
or (provided that $2^{\delta-2} > 1$)	0	0	0	0	$2xh_{k_+}$
$m \equiv 1 \pmod{8}$ and $w = 1$ gives either	$2^{\delta-1}$	0	$2^{\delta-2}$	$2^{\delta-2}$	$2xh_{k_+} - 2^{\delta-3}$
or	0	$2^{\delta-1}$	0	$2^{\delta-2}3$	$2xh_{k_+} - 2^{\delta-3}3$
or (provided that $2^{\delta-2} > 2$)	0	0	0	0	$2xh_{k_+}$
$m \equiv 5 \pmod{8}$	0	$2^{\delta-1}$	0	$2^{\delta-2}3$	$\frac{1}{2}(x(2y+1)h_{k_+} - 2^{\delta-2}3)$
or (provided that $2^{\delta-2} > 1$)	0	0	0	0	$\frac{1}{2}x(2y+1)h_{k_+}$

The above case distinctions come from the fact that Kramer's theorem 26.12 ranges over all types of maximal orders in quaternion algebras over $\mathbb{Q}(\sqrt{-m})$, in which Kramer determines the numbers of conjugacy classes in the norm-1-group. The remaining task in order to decide which of the cases applies, is to find out of which type considered in the mentioned theorem is the maximal order $M_2(\mathcal{O}_{-m})$. Some methods to cope with this task are introduced in [13, §27], where Kramer obtains the following criteria for the 2-torsion numbers:

condition	implication
$m \equiv 7 \pmod{8}$	$\mu_T = \mu_2^- = \lambda_4^T = \lambda_4^* = 0.$
$m \equiv 5 \pmod{8}$	$\mu_T = \lambda_4^T = 0.$
$m \equiv 21 \pmod{24}$	$\lambda_4^* = 0.$
$m \equiv 0 \pmod{6}$ and $\lambda_4^* > 0$	$\lambda_4^T > 0.$
$m \equiv 9 \pmod{24}$ and $\lambda_4^* > 0$	$\lambda_4^T > 0.$
m prime and $m \equiv 1$ or $3 \pmod{8}$	$\lambda_4^T > 0.$
$m \equiv 5 \pmod{8}$ and m prime	$\lambda_4^* > 0.$
$m = 2p$ with p prime and $p \equiv 3$ or $5 \pmod{8}$	$\lambda_4^* > 0.$
$m = p'p$ with p and p' prime and $p \equiv p' \equiv 3$ or $5 \pmod{8}$	$\lambda_4^* > 0.$
$m = 3p$ with p prime and $p \equiv 1$ or $3 \pmod{8}$	$\lambda_4^T > 0.$
$m \equiv 1$ or $2 \pmod{4}$ and $m \neq 1$ and $x = 1$	$\lambda_4^* > 0$ and $\mu_2^- > 0.$
$m \equiv 1$ or $2 \pmod{4}$ and $m \neq 1$ and $x = 2$	$\lambda_4 - \lambda_4^* > 0.$
$m \equiv 3 \pmod{8}$ and -2 occurs as norm on \mathcal{O}_{k_+}	$\lambda_4^* > 0$ and $\lambda_4^T > 0.$
$m \equiv 3 \pmod{8}$ and -2 does not occur as norm on \mathcal{O}_{k_+}	$\lambda_4 - \lambda_4^* > 0.$
$m \equiv 3 \pmod{8}$ and m admits a prime divisor p with $p \equiv 5$ or $7 \pmod{8}$	$\lambda_4 - \lambda_4^* > 0.$
$m \equiv 1 \pmod{8}$ and $w = 1$ and $h(k_+) = 2^{\delta-3}$	$\mu_2^- = 0.$
$m \equiv 2 \pmod{4}$ and -2 occurs as norm on \mathcal{O}_{k_+}	$\lambda_4^T > 0.$
$m \equiv 2 \pmod{4}$ and -2 does not occur as norm on \mathcal{O}_{k_+} and $h(k_+) = 2^{\delta-2}$	$\lambda_4^* = 0.$
$m \equiv 2 \pmod{4}$ and $q = 1$ and $h(k_+) = 2^{\delta-1}$ and $w = 2$	$\lambda_4^* = 0.$
$m \equiv 2 \pmod{4}$ and $h(k_+) = 2^{\delta-2}$ and m admits a prime divisor p with $p \equiv 5$ or $7 \pmod{8}$	$\lambda_4^* = 0.$

With the above criteria at hand, we can decide for many Bianchi groups, which of the alternative cases in Kramer's formulae must be used. We do this in the below tables for all such Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with absolute value of the discriminant Δ ranging between 7 and 2003. In the cases $m \in \{34, 105, 141, 142, 194, 235, 323, 427, 899, 979, 1243, 1507\}$, where these statements are not sufficient to eliminate the wrong alternatives, we insert the results of [18]. This way, the below tables treat all Bianchi groups with units $\{\pm 1\}$ and discriminant of absolute value less than 820. The cases where an ambiguity remains (so to exclude them from our tables) are the following values of m : 205, 221, 254, 273, 305, 321, 322, 326, 345, 377, 381, 385, 386, 410, 438, 465, 469, 473, 482, 1067, 1139, 1211, 1339, 1443, 1763, 1771, 1947.

The above mentioned theorem on subgroup occurrences [12] solves all these ambiguities.

2-torsion homology	m specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 2-torsion homology
$P \circlearrowleft$	7, 15, 23, 31, 35, 39, 47, 55, 71, 87, 91, 95, 103, 111, 115, 127, 143, 151, 155, 159, 167, 183, 191, 199, 203, 215, 239, 247, 259, 263, 271, 295, 299, 303, 311, 319, 327, 335, 355, 367, 371, 383, 395, 403, 407, 415, 431, 447, 463, 471, 479, 487, 503, 515, 519, 535, 543, 551, 559, 583, 591, 599, 607, 611, 631, 635, 647, 655, 667, 671, 687, 695, 703, 707, 719, 743, 751, 755, 763, 767, 807, 815, 823, 831, 835, 851, 863, 871, 879, 887, 911, 919, 923, 951, 955, 967, 983, 991, 995, 1007, 1027, 1031, 1039, 1043, 1047, 1055, 1063, 1079, 1099, 1103, 1115, 1119, 1135, 1147, 1151, 1159, 1167, 1195, 1199, 1219, 1231, 1247, 1255, 1263, 1267, 1279, 1303, 1315, 1319, 1355, 1363, 1379, 1383, 1391, 1399, 1403, 1415, 1423, 1439, 1447, 1471, 1487, 1511, 1535, 1543, 1555, 1559, 1583, 1591, 1603, 1607, 1623, 1643, 1651, 1655, 1663, 1671, 1703, 1711, 1727, 1739, 1759, 1783, 1795, 1807, 1823, 1831, 1835, 1839, 1871, 1879, 1883, 1891, 1895, 1903, 1915, 1919, 1939, 1943, 1951, 1959, 1963, 1983, 1991, 1999,
$2P \circlearrowleft$	14, 46, 62, 94, 119, 158, 195, 206, 231, 255, 287, 302, 334, 382, 391, 398, 435, 446, 455, 478, 483, 511, 527, 555, 595, 615, 623, 651, 663, 679, 715, 759, 791, 795, 903, 915, 935, 943, 987, 1015, 1095, 1131, 1207, 1235, 1271, 1295, 1311, 1335, 1343, 1407, 1435, 1455, 1463, 1479, 1491, 1515, 1547, 1551, 1595, 1615, 1631, 1635, 1659, 1687, 1695, 1751, 1767, 1799, 1855, 1887, 1927, 1955, 1967,
$3P \circlearrowleft$	21, 30, 42, 69, 70, 77, 78, 79, 93, 110, 133, 138, 154, 174, 182, 186, 190, 213, 222, 223, 230, 235, 237, 253, 266, 282, 286, 301, 309, 310, 318, 341, 359, 366, 406, 413, 426, 427, 430, 437, 453, 470, 474, 494, 839, 895, 899, 1191, 1223, 1367, 1527, 1567, 1639, 1735, 1847,
$4P \circlearrowleft$	161, 217, 238, 329, 399, 497, 799, 959, 1023, 1155, 1239, 1351, 1679, 1743, 1995,
$5P \circlearrowleft$	439, 727, 1111, 1327,
$6P \circlearrowleft$	142, 165, 210, 285, 330, 357, 390, 429, 434, 462, 1495, 1599,
$7P \circlearrowleft$	141, 1087,
$8P \circlearrowleft$	105,
$2P_{D_2}^*$	5, 10, 13, 26, 29, 53, 58, 61, 74, 106, 109, 122, 149, 157, 173, 181, 202, 218, 277, 293, 298, 314, 317, 362, 394, 397, 421, 458, 461,

2-torsion homology	m specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 2-torsion homology
$2P_{\mathcal{D}_2}^* + 2P_{\circlearrowleft}$	37, 101, 197, 269, 349, 373, 389,
$2P_{\mathcal{D}_2}^* + 3P_{\circlearrowleft}$	229, 346,
$4P_{\mathcal{D}_2}^*$	85, 130, 170, 290, 365, 370, 493,
$4P_{\mathcal{D}_2}^* + P_{\circlearrowleft}$	65, 185, 265, 481,
$4P_{\mathcal{D}_2}^* + 3P_{\circlearrowleft}$	442, 445,
$4P_{\mathcal{D}_2}^* + 4P_{\circlearrowleft}$	485,
$4P_{\mathcal{D}_2}^* + 5P_{\circlearrowleft}$	145,
$P_{\mathcal{A}_4}^* + P_{\mathcal{D}_2}^*$	2,
$2P_{\mathcal{A}_4}^*$	11, 19, 43, 59, 67, 83, 107, 131, 139, 163, 179, 211, 227, 251, 283, 307, 331, 347, 379, 419, 467, 491, 523, 547, 563, 571, 587, 619, 643, 683, 691, 739, 787, 811, 827, 859, 883, 907, 947, 971, 1019, 1051, 1123, 1163, 1187, 1259, 1283, 1291, 1307, 1427, 1451, 1459, 1483, 1499, 1531, 1571, 1579, 1619, 1667, 1699, 1723, 1747, 1867, 1931, 1979, 2003,
$2P_{\mathcal{A}_4}^* + P_{\circlearrowleft}$	6, 22, 38, 86, 118, 134, 166, 214, 262, 278, 358, 422, 443, 454, 659, 1091, 1171, 1523, 1627, 1787, 1811, 1907, 1987,
$2P_{\mathcal{A}_4}^* + 2P_{\circlearrowleft}$	499,
$2P_{\mathcal{A}_4}^* + 2P_{\mathcal{D}_2}^*$	17, 41, 73, 89, 97, 113, 137, 193, 233, 241, 281, 313, 337, 353, 409, 433, 449, 457,
$2P_{\mathcal{A}_4}^* + 2P_{\mathcal{D}_2}^* + P_{\circlearrowleft}$	82, 146, 178, 274, 466,
$2P_{\mathcal{A}_4}^* + 2P_{\mathcal{D}_2}^* + 2P_{\circlearrowleft}$	34, 194,
$2P_{\mathcal{A}_4}^* + 2P_{\mathcal{D}_2}^* + 4P_{\circlearrowleft}$	226, 257,
$2P_{\mathcal{A}_4}^* + 2P_{\mathcal{D}_2}^* + 8P_{\circlearrowleft}$	401,
$4P_{\mathcal{A}_4}^*$	51, 123, 187, 267, 339, 411, 451, 699, 771, 779, 803, 843, 1059, 1203, 1347, 1563, 1691, 1707, 1779, 1819, 1843, 1923,
$4P_{\mathcal{A}_4}^* + P_{\circlearrowleft}$	219, 291, 323, 579, 723, 731, 939, 979, 1003, 1011, 1227, 1243, 1371, 1387, 1411, 1507, 1731, 1803,
$4P_{\mathcal{A}_4}^* + 2P_{\circlearrowleft}$	66, 102, 114, 246, 258, 354, 374, 402, 418, 498, 1851,
$4P_{\mathcal{A}_4}^* + 3P_{\circlearrowleft}$	33, 57, 129, 177, 201, 209, 249, 393, 417, 489, 1299,
$8P_{\mathcal{A}_4}^*$	627, 1419.

7. ASYMPTOTIC BEHAVIOUR OF THE NUMBER OF CONJUGACY CLASSES

From Kramer's above formulae, we see that both in 2-torsion and in 3-torsion, the number of conjugacy classes of finite subgroups, and hence also the cardinality of the homology of the Bianchi groups in degrees above their virtual cohomological dimension, admits only two factors which are not strictly limited: h_{k_+} and 2^δ . As for the ideal class number h_{k_+} , we recall that the Cohen-Lenstra heuristic [8] predicts the following. Define the constant A as the product $\prod_{k \geq 2} \zeta(k) \approx 2.29486$, where ζ is the classical Riemann zeta-function. Furthermore, set $B := \prod_{k \in \mathbb{N}} (1 - 2^{-k}) \approx 0.288788$. Then, the probability that the ideal class group of k_+ is isomorphic to a given finite Abelian group G of odd order $|G|$, is equal to

$$\frac{1}{2 \cdot |G| \cdot B \cdot A \cdot |\text{Aut}(G)|}.$$

This prediction implies that more than 75% of all totally real quadratic number fields admit the trivial ideal class group ($h_{k_+} = 1$) and over 12.5% of them admit $h_{k_+} = 3$.

As for the factor 2^δ , the number δ of finite ramification places of $\mathbb{Q}(\sqrt{-m})$ over \mathbb{Q} is well-known to equal the number of prime divisors of the discriminant of $\mathbb{Q}(\sqrt{-m})$.

The numerical evaluation of Kramer's formulae provides us with databases which are over a thousand times larger than what is reasonable to print in sections 5 and 6. We now give an instance of how these databases can be exploited. Denote the discriminant of $\mathbb{Q}(\sqrt{-m})$ by Δ . In the cases $m \equiv 3 \pmod{4}$, we have $\Delta = -m$. Denote the number $\lambda_6 - \lambda_6^*$ of connected components of type \mathcal{O} in the 3-conjugacy classes graph by $\lambda'_6(\Delta)$. Then clearly, the subgroup in $H_q(\text{PSL}_2(\mathcal{O}_{-m}))$, $q > 2$, generated by the order-3-elements coming from the connected components of this type, is of order $3^{\lambda'_6(\Delta)}$. Denote by $\text{covolume}(\Delta)$ the volume of the quotient space $\text{PSL}_2(\mathcal{O}_{-m}) \backslash \mathcal{H}$. The study of the ratio $\frac{3^{\lambda'_6(\Delta)}}{\text{covolume}(\Delta)}$ is motivated by the formulae in [3]. In figure 1, we print the logarithm of the average of this ratio over the cases $|\Delta| \equiv 3 \pmod{4}$, scaled by a factor $m^{-\frac{2}{3}}$, so to say

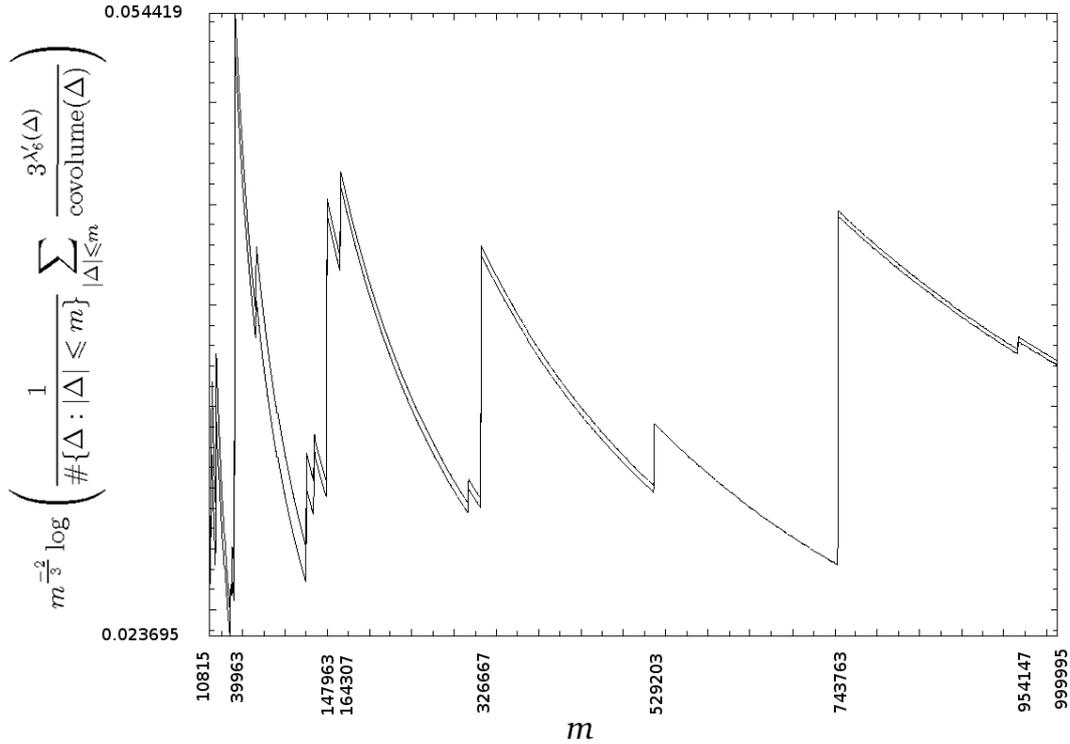
$$m^{-\frac{2}{3}} \log \left(\frac{1}{\#\{\Delta : |\Delta| \leq m\}} \sum_{|\Delta| \leq m} \frac{3^{\lambda'_6(\Delta)}}{\text{covolume}(\Delta)} \right),$$

where we consider m and Δ as independent variables, m running through the square-free positive rational integers. In order to cope with the fact that in some cases, Kramer's formulae leave an ambiguity, we print a function assuming the lowest possible values of $\lambda'_6(\Delta)$ and one assuming the highest possible values of $\lambda'_6(\Delta)$ in the same diagram.

So for m greater than 10815 and less than one million, we can observe that the average of the above ratio oscillates between $\exp(m^{\frac{2}{3}} 0.023695)$ and $\exp(m^{\frac{2}{3}} 0.054419)$. For m less than 10815, this oscillation is much stronger, and the diagram might be seen as suggesting that possibly the oscillation could remain between these two bounds for m greater than one million.

For related asymptotics, see the recent works of Bergeron/Venkatesh [3] and Sengun [21]. For an alternative computer program treating the Bianchi groups, see the SAGE package of Cremona's student Aranes [2], and for $\text{GL}_2(\mathcal{O})$ see Yasaki's program [24].

FIGURE 1. Average homological 3-torsion outside subgroups of type \mathcal{S}_3 , scaled as indicated.



REFERENCES

- [1] Bill Allombert, Christian Batut, Karim Belabas, Dominique Bernardi, Henri Cohen, Francisco Diaz y Diaz, Yves Eichenlaub, Xavier Gourdon, Louis Granboulan, Bruno Haible, Guillaume Hanrot, Pascal Letard, Gerhard Niklasch, Michel Olivier, Thomas Papanikolaou, Xavier Roblot, Denis Simon, Emmanuel Tollis, Ilya Zakharevitch, and the PARI group, *PARI/GP, version 2.4.3*, specialized computer algebra system, Bordeaux, 2010, <http://pari.math.u-bordeaux.fr/>.
- [2] Maria T. Aranés, *Modular symbols over number fields*, Ph.D. Thesis, University of Warwick, www.warwick.ac.uk/staff/J.E.Cremona/theses/maite_thesis.pdf, 2010.
- [3] Nicolas Bergeron and Akshay Venkatesh, *The asymptotic growth of torsion homology for arithmetic groups*, ArXiv e-prints (2010), available at 1004.1083.
- [4] Ethan Berkove, *The integral cohomology of the Bianchi groups*, *Trans. Amer. Math. Soc.* **358** (2006), no. 3, 1033–1049 (electronic). MR2187644 (2006h:20073), Zbl pre02237880
- [5] ———, *The mod-2 cohomology of the Bianchi groups*, *Trans. Amer. Math. Soc.* **352** (2000), no. 10, 4585–4602, DOI 10.1090/S0002-9947-00-02505-8. MR1675241 (2001b:11043)
- [6] Luigi Bianchi, *Sui gruppi di sostituzioni lineari con coefficienti appartenenti a corpi quadratici immaginari*, *Math. Ann.* **40** (1892), no. 3, 332–412 (Italian). MR1510727, JFM 24.0188.02
- [7] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. **87**, Springer-Verlag, 1982. MR672956 (83k:20002), Zbl 0584.20036
- [8] Henri Cohen, *A course in computational algebraic number theory*, Graduate Texts in Mathematics, vol. 138, Springer-Verlag, Berlin, 1993. MR1228206 (94i:11105)
- [9] Jürgen Elstrodt, Fritz Grunewald, and Jens Mennicke, *Groups acting on hyperbolic space*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. MR1483315 (98g:11058), Zbl 0888.11001

- [10] Benjamin Fine, *Algebraic theory of the Bianchi groups*, Monographs and Textbooks in Pure and Applied Mathematics, vol. **129**, Marcel Dekker Inc., New York, 1989. MR1010229 (90h:20002), Zbl 0760.20014
- [11] Felix Klein, *Ueber binäre Formen mit linearen Transformationen in sich selbst*, Math. Ann. **9** (1875), no. 2, 183–208. MR1509857
- [12] Norbert Krämer, handwritten manuscript of around 1984. Presently again in preparation for publication.
- [13] ———, *Die Konjugationsklassenzahlen der endlichen Untergruppen in der Norm-Eins-Gruppe von Maximalordnungen in Quaternionenalgebren*, Bonn, 1980. <http://tel.archives-ouvertes.fr/tel-00628809/> (German).
- [14] Colin Maclachlan and Alan W. Reid, *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Mathematics, vol. **219**, Springer-Verlag, New York, 2003. MR1937957 (2004i:57021), Zbl 1025.57001
- [15] Henri Poincaré, *Mémoire*, Acta Math. **3** (1966), no. 1, 49–92 (French). Les groupes kleinéens. MR1554613
- [16] Alexander D. Rahm, *Chen/Ruan orbifold cohomology of the Bianchi groups* (preprint, arXiv : 1109.5923, <http://hal.archives-ouvertes.fr/hal-00627034/>, 2011).
- [17] ———, *The homological torsion of PSL_2 of the imaginary quadratic integers*, to appear in Transactions of the AMS (<http://hal.archives-ouvertes.fr/hal-00578383/en/>, 2011).
- [18] ———, *Bianchi.gp*, Open source program (GNU general public license), 2010. Based on Pari/GP [1]. Available on the author’s homepage, this program computes a fundamental domain for the Bianchi groups in hyperbolic 3-space, the associated quotient space and essential information about the group homology of the Bianchi groups.
- [19] Alexander D. Rahm and Mathias Fuchs, *The integral homology of PSL_2 of imaginary quadratic integers with nontrivial class group*, J. Pure Appl. Algebra **215** (2011), no. 6, 1443–1472. MR2769243
- [20] Joachim Schwermer and Karen Vogtmann, *The integral homology of SL_2 and PSL_2 of Euclidean imaginary quadratic integers*, Comment. Math. Helv. **58** (1983), no. 4, 573–598. MR728453 (86d:11046), Zbl 0545.20031
- [21] Mehmet H. Sengün, *On the (co)homology of Bianchi groups*, to appear in Exp. Math. (2010).
- [22] Richard G. Swan, *Generators and relations for certain special linear groups*, Advances in Math. **6** (1971), 1–77. MR0284516 (44 #1741), Zbl 0221.20060
- [23] Karen Vogtmann, *Rational homology of Bianchi groups*, Math. Ann. **272** (1985), no. 3, 399–419. MR799670 (87a:22025), Zbl 0545.20031
- [24] Dan Yasaki, *Hyperbolic tessellations associated to Bianchi groups*, Algorithmic number theory, Lecture Notes in Comput. Sci. 6197, 2010, pp. 385–396. MR2721434

NATIONAL UNIVERSITY OF IRELAND AT GALWAY, DEPARTMENT OF MATHEMATICS
E-mail address: Alexander.Rahm@nuigalway.ie
URL: www-fourier.ujf-grenoble.fr/~rahm