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# NUMBER-THEORETIC FORMULAE FOR THE HOMOLOGICAL TORSION OF THE BIANCHI GROUPS

ALEXANDER D. RAHM

ABSTRACT. We give number-theoretic formulae for the homological torsion of the Bianchi groups. The Bianchi groups are the arithmetic groups  $\mathrm{PSL}_2(\mathcal{O})$  over rings  $\mathcal{O}$  of integers in an imaginary quadratic number field.

## CONTENTS

1. Introduction	1
2. The conjugacy classes of finite order elements.	2
3. The Kramer numbers and group homology	6
4. Numerical evaluation of Kramer’s formulae	8
References	16

## 1. INTRODUCTION

Denote by  $\mathbb{Q}(\sqrt{-m})$ , with  $m$  a square-free positive integer, an imaginary quadratic number field, and by  $\mathcal{O}_{-m}$  its ring of integers. The *Bianchi groups* are the groups  $(\mathrm{P})\mathrm{SL}_2(\mathcal{O}_{-m})$ . The Bianchi groups may be considered as a key to the study of a larger class of groups, the *Kleinian* groups, which date back to work of Henri Poincaré [9]. In fact, each non-cocompact arithmetic Kleinian group is commensurable with some Bianchi group [8]. A wealth of information on the Bianchi groups can be found in the monographs [4, 5, 8]. These groups act in a natural way on hyperbolic three-space, which is isomorphic to the symmetric space associated to them. The kernel of this action is the centre  $\{\pm 1\}$  of the groups. Thus it is useful to study the quotient of  $\mathrm{SL}_2(\mathcal{O}_{-m})$  by its centre, namely  $\mathrm{PSL}_2(\mathcal{O}_{-m})$ . In 1892, Luigi Bianchi [3] computed fundamental domains for this action when  $m = 1, 2, 3, 5, 6, 7, 10, 11, 13, 15$  and 19. Such a fundamental domain has the shape of a hyperbolic polyhedron (up to a missing vertex at certain cusps, which represent the ideal classes of  $\mathcal{O}_{-m}$ ), so we will call it the *Bianchi fundamental polyhedron*. The computation of the Bianchi fundamental polyhedron has been implemented for all Bianchi groups [11] in the language Pari/GP [1].

Kramer [7] has determined number-theoretic formulae for the numbers of conjugacy classes of finite subgroups in the Bianchi groups. In the last three decades, this was referenced as unpublished work [14], but the author has obtained Kramer’s permission to upload Grunewald’s copy on a thesis archive.

The purpose of this article is to express the homological torsion of the Bianchi groups with these formulae. We use the techniques developed in [10] to achieve this.

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## 2. THE CONJUGACY CLASSES OF FINITE ORDER ELEMENTS.

Let  $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-m})$  be a Bianchi group. Then any element of  $\Gamma$  fixing a point inside real hyperbolic 3-space  $\mathcal{H}_{\mathbb{R}}^3$  acts as a rotation of finite order.

We make the following group-theoretic construction in order to build a bridge between the Krämer numbers and the  $\ell$ -torsion subcomplexes. For a circle to become a graph, we identify the two endpoints of a single edge.

**Definition 1.** *The  $\ell$ -conjugacy classes graph of an arbitrary group  $\Gamma$  is given by the following construction.*

- We take as vertices the conjugacy classes of finite subgroups  $G$  of  $\Gamma$  containing elements  $\gamma$  of order  $\ell$  such that the normaliser of  $\langle \gamma \rangle$  in  $G$  is not  $\langle \gamma \rangle$  itself.
- We connect two vertices by an edge if and only if they admit representatives sharing a common subgroup of order  $\ell$ .
- For every pair of subgroups of order  $\ell$  in  $G$ , which are conjugate in  $\Gamma$  but not in  $G$ , we draw a circle attached to the vertex labelled by  $G$ .
- For every conjugacy class of subgroups of order  $\ell$  which are not properly contained in any finite subgroup of  $\Gamma$ , we add a disjoint circle.

Recall the following classification of Felix Klein [6].

**Lemma 2** (Klein). *The finite subgroups in  $\mathrm{PSL}_2(\mathcal{O})$  are exclusively of isomorphism types the cyclic groups of orders one, two and three, the Klein four-group  $\mathcal{D}_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ , the symmetric group  $\mathcal{S}_3$  and the alternating group  $\mathcal{A}_4$ .*

Let  $\Gamma := \mathrm{PSL}_2(\mathcal{O})$ , let  $Z$  be the refined cellular complex obtained from the action of  $\Gamma$  on hyperbolic 3-space as described in [10], and let  $\ell$  be a prime number.

**Definition 3.** *The  $\ell$ -torsion subcomplex is the subcomplex of  $\Gamma \backslash Z$  consisting of all the cells, the pre-images of which have stabilisers in  $\Gamma$  containing elements of order  $\ell$ .*

For  $\ell$  being one of the two occurring primes 2 and 3, this subcomplex is a finite graph, because the cells of dimension greater than 1 are trivially stabilised in the refined cellular complex. We reduce this subcomplex with the procedure of [10].

Except for the Gaussian and Eisenstein integers, which have to be treated separately, all the rings of integers of imaginary quadratic number fields admit as only units  $\{\pm 1\}$ . In the latter case, we call  $\mathrm{PSL}_2(\mathcal{O})$  a Bianchi group with units  $\{\pm 1\}$ .

**Theorem 4.** *Let  $\Gamma$  be any Bianchi group with units  $\{\pm 1\}$  and  $\ell$  any prime. Then the  $\ell$ -conjugacy classes graph and the reduced  $\ell$ -torsion subcomplex of  $\Gamma$  are isomorphic graphs.*

The rest of this section will be devoted to the proof of this theorem. The first ingredient is the following lemma, which we recall from [10].

**Lemma 5.** *Let  $v$  be a non-singular vertex in the refined cell complex. Then the number  $\mathbf{n}$  of orbits of edges in the refined cell complex adjacent to  $v$ , with stabiliser in  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  isomorphic to  $\mathbb{Z}/\ell$ ,*

is given as follows for  $\ell = 2$  and  $\ell = 3$ .

Isomorphism type of the vertex stabiliser	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$
$\mathbf{n}$ for $\ell = 2$	0	2	0	3	2	1
$\mathbf{n}$ for $\ell = 3$	0	0	2	0	1	2.

Now we investigate the associated normaliser groups. Straight-forward verification using the multiplication tables of the implied finite groups yields the following.

**Lemma 6.** *Let  $G$  be a finite subgroup of  $\mathrm{PSL}_2(\mathcal{O}_{-m})$ . Then the type of the normaliser of any subgroup of type  $\mathbb{Z}/\ell$  in  $G$  is given as follows for  $\ell = 2$  and  $\ell = 3$ , where we print only cases with existing subgroup of type  $\mathbb{Z}/\ell$ .*

Isomorphism type of $G$	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$
normaliser of $\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathcal{D}_2$	$\mathbb{Z}/2$	$\mathcal{D}_2$
normaliser of $\mathbb{Z}/3$			$\mathbb{Z}/3$		$\mathcal{S}_3$	$\mathbb{Z}/3$ .

The final ingredient in the proof of theorem 4 is the following.

**Theorem 7.** *There is a natural bijection between conjugacy classes of subgroups of  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  of order  $\ell$  and edges of the reduced  $\ell$ -torsion subcomplex. It is given by considering the stabiliser of a representative edge in the refined cell complex.*

In order to prove the latter theorem, we need several lemmata, and we establish them now.

**Lemma 8.** *Consider two adjacent edges  $E, E'$  of the non-reduced torsion subcomplex. Then for any representative  $e$  of  $E$ , there is an adjacent representative  $e'$  of  $E'$  on the same geodesic line as  $e$ .*

*Proof.* Consider the element  $\gamma \in \Gamma$  identifying the end  $v$  of  $e$  with the origin  $\gamma(v)$  of  $e'$ . As  $E$  and  $E'$  are distinct in the orbit space,  $\gamma^{-1}$  cannot send  $e'$  onto  $e$ .

Assume that the stabiliser of  $v$  is of isomorphism type  $\mathbb{Z}/\ell\mathbb{Z}$ . Then as the edge  $\gamma^{-1}(e')$  is point-wise fixed by  $\Gamma$ , its stabiliser must contain the rotation with axis passing through  $e$ . Hence  $\gamma^{-1}(e')$  is adjacent to  $e$  and on the same geodesic line.

If the stabiliser of  $v$  is of isomorphism type  $\mathcal{D}_2$ ,  $\mathcal{S}_3$  or  $\mathcal{A}_4$ , there are at most two orbits of  $\ell$ -torsion stabilised edges adjacent to  $v$ . So there is an element  $\alpha$  in the stabiliser of  $v$  such that  $\alpha\gamma^{-1}(e')$  is adjacent to  $e$  and on the same geodesic line.  $\square$

**Corollary 9.** *Any edge of the reduced torsion subcomplex can be represented by a chain of edges on the intersection of one geodesic line with a strict fundamental domain for  $\Gamma$  in  $\mathcal{H}$ .*

*Proof.* Consider the chain of edges in the torsion subcomplex that are reduced to the given edge of the reduced torsion subcomplex. The representatives in  $\mathcal{H}$  of the edges on this chain lie on pairwise different orbits because the torsion subcomplex is a subcomplex of the orbit space  $\mathcal{H}/\Gamma$ . Now we start with a representative  $e$  for the edge sharing its origin with the reduced edge, and use lemma 8 to obtain an adjacent edge  $e'$  on the same geodesic line, representing the adjacent edge of the torsion subcomplex. We proceed this way, assigning the role of  $e$  to  $e'$  and so on, until we have reached the end of the reduced edge.  $\square$

**Corollary 10.** *Any edge of the reduced torsion subcomplex admits only representatives with stabiliser in the same conjugacy class.*

**Lemma 11.** *Let  $\alpha$  and  $\gamma$  be elements of  $\mathrm{PSL}_2(\mathbb{C})$ . Then the fixed point set in  $\mathcal{H}$  of  $\alpha$  is identified by  $\gamma$  with the fixed point set of  $\gamma\alpha\gamma^{-1}$ .*

*Proof.* One immediately checks that any fixed point  $x \in \mathcal{H}$  of  $\alpha$  induces the fixed point  $\gamma(x)$  of  $\gamma\alpha\gamma^{-1}$ . As  $\mathrm{PSL}_2(\mathbb{C})$  acts by isometries, the whole fixed point sets are identified.  $\square$

**Lemma 12.** *Let  $v \in \mathcal{H}_{\mathbb{R}}^3$  be a vertex with stabiliser in  $\Gamma$  of type  $\mathcal{D}_2$  or  $\mathcal{A}_4$ . Let  $\gamma \in \Gamma$  be a rotation of order 2 around an edge  $e$  adjacent to  $v$ .*


*Then the centraliser  $C_{\Gamma}(\gamma)$  reflects  $\mathcal{H}^{\Gamma}$  — which is the geodesic line through  $e$  — onto itself at  $v$ .*


*Proof.* Denote by  $\Gamma_v$  the stabiliser of the vertex  $v$ . In the case that  $\Gamma_v$  is of type  $\mathcal{D}_2$ , which is abelian, it admits two order-2-elements centralising  $\gamma$  and turning the geodesic line through  $e$  onto itself such that the image of  $e$  touches  $v$  from the opposite site to  $e$ . In the case that  $\Gamma_v$  is of type  $\mathcal{A}_4$ , it contains a normal subgroup of type  $\mathcal{D}_2$  that admits again two such elements.  $\square$

Let  $\alpha$  be any torsion element in  $\Gamma$ . We construct the *chain of edges associated to  $\alpha$*  as follows. Consider the edge of the reduced torsion subcomplex to which the edge stabilised by  $\alpha$  belongs. Use corollary 9 to represent it by a chain of edges on a geodesic line. Now,  $\alpha$  is conjugate to an element  $\gamma\alpha\gamma^{-1}$  of the stabiliser of one of the edges in the chain. By lemma 11, the element  $\gamma^{-1} \in \Gamma$  maps the mentioned geodesic line to the rotation axis of  $\alpha$ . The image under  $\gamma^{-1}$  of the chain of edges under consideration is the desired chain associated to  $\alpha$ .

**Lemma 13.** *Let  $\alpha$  be any 2-torsion element in  $\Gamma$ . Then the chain of edges associated to  $\alpha$  is a fundamental domain for the centraliser of  $\alpha$  on the rotation axis of  $\alpha$ .*

*Proof.* Consider the edge of the reduced torsion subcomplex to which the edge stabilised by  $\alpha$  belongs. We distinguish the following homeomorphism types of connected components, on which this edge can lie.


 Choose a first edge  $e$  in the chain. We have an identification  $\gamma$  between its end and the origin of the next edge  $e'$ , and by lemma 8 we can choose it such that the edge stabilisers are conjugate under  $\gamma$ . We can write  $\Gamma_e = \langle \alpha \rangle$  and  $\Gamma_{e'} = \langle \gamma\alpha\gamma^{-1} \rangle$ . By lemma 11, this gives us the identification  $\gamma^{-1}$  from  $e'$  to an edge on the rotation axis of  $\alpha$ , adjacent to  $e$  because of the first condition on  $\gamma$ . We repeat this step until we have attached an edge  $\delta e$  on the orbit of the first edge  $e$ , with  $\delta \in \Gamma$ . As  $\delta$  is an isometry, the whole chain is translated by  $\delta$  from the start at  $e$  to the start at  $\delta e$ . So the group  $\langle \delta \rangle$  acts on the rotation axis with fundamental domain our chain of edges. And  $\delta\alpha\delta^{-1}$  is again the rotation of order 2 around the axis of  $\alpha$ . So,  $\delta\alpha\delta^{-1} = \alpha$  and therefore  $\langle \delta \rangle < C_{\Gamma}(\alpha)$ .

  $\mathcal{A}_4$ -reflections are centralising by lemma 12 and tessellate the rotation axis of  $\alpha$ .

  $\mathcal{D}_2$ -reflections are centralising by lemma 12 and tessellate the rotation axis of  $\alpha$ .

 On the edge:  $\mathcal{A}_4$ -reflections and  $\mathcal{D}_2$ -reflections.

On the circle:  $\mathcal{D}_2$ -reflections.

So in all cases except , we can observe that the vertices in the reduced torsion subgraph provide centralising reflections. So the same arguments apply when the connected component of the reduced torsion subgraph has more cells: In this case, any edge of it has origin and end provided with  $\mathcal{A}_4$ - and  $\mathcal{D}_2$ -stabilisers, possibly like in the last case, with the same stabiliser at origin and end. So, we can prove the statement for every edge with  $\mathcal{A}_4$ -reflections and  $\mathcal{D}_2$ -reflections by lemma 12.  $\square$

**Lemma 14.** *Let  $\alpha$  be any non-trivial torsion element in a Bianchi group  $\Gamma$ . Then the  $\Gamma$ -image of the chain of edges associated to  $\alpha$  contains the rotation axis of  $\alpha$ .*

*Proof.* For  $\alpha$  a 2-torsion element, this follows directly from lemma 13. So we can assume  $\alpha$  to be a 3-torsion element.

• The non-centralising reflections of  $\mathcal{S}_3$ , which is associated to the two vertices, tessellate a geodesic line. Remark that there are exactly three maps  $\mathbb{Z}/3 \rightarrow \mathbb{Z}/3$  induced on the edge stabilisers, given by the three order-2-elements of  $\mathcal{S}_3$ . So, the quotient space by the centraliser of  $\alpha$  must be bigger.

○ After finitely many translations, all the edges in the fundamental domain conjugate to edges on the geodesic line are obtained as images. Hence, the  $\Gamma$ -images contain the whole geodesic line.  $\square$

*Proof of theorem 7.* Corollary 10 associates a conjugacy class of type  $\mathbb{Z}/\ell$  to each edge of the reduced  $\ell$ -torsion subcomplex. Conversely, let  $\alpha$  and  $\gamma\alpha\gamma^{-1}$  be conjugate torsion elements of  $\Gamma$ . We want to show that they stabilise edges representing the same reduced edge. By Klein, we know that the torsion elements are elliptic and hence fix some geodesic line. Every torsion element acts as the stabiliser of a line conjugate to one passing through the Bianchi fundamental polyhedron. So, it is conjugate to one of the representative edge stabilisers. By lemma 11, we know that the line fixed by  $\alpha$  is sent by  $\gamma$  to the line fixed by  $\gamma\alpha\gamma^{-1}$ .

By lemmata 13 and 14, the union of the  $\Gamma$ -images of the chain associated to  $\alpha$  contains the whole geodesic line fixed by  $\alpha$  (because then, as the  $\Gamma$ -action is cellular, any cell stabilised by  $\gamma\alpha\gamma^{-1}$  admits a cell on its orbit stabilised by  $\alpha$ ). So it follows that all the edges on a geodesic line belong to the same reduced edge.  $\square$

*Proof of theorem 4.* Comparing with lemma 5, we see that the vertex set of the  $\ell$ -conjugacy classes graph gives precisely the bifurcation points and vertices with only one adjacent edge of the  $\ell$ -torsion subcomplex. When passing to the reduced  $\ell$ -torsion subcomplex, we get rid of all vertices with two adjacent edges except in the disjoint circles, see [10]. By theorem 7, the edges of the  $\ell$ -conjugacy classes graph give the edges of the reduced  $\ell$ -torsion subcomplex.  $\square$

The author would like to add the following explanation why in our fundamental domain diagrams, there occurs only one representative per torsion-stabilised edge.

**Remark 15.** Let  $e$  be a non-trivially stabilised edge in the fundamental domain for the refined cell complex. Then the fundamental domain for the 2-dimensional retract can be chosen such that it contains  $e$  as the only edge on its orbit.

*Sketch of proof.* Observe that the inner dihedral angle  $\frac{2\pi}{q}$  of the Bianchi fundamental polyhedron is  $\frac{2\pi}{\ell}$  or  $\frac{\pi}{\ell}$  at its edges admitting a rotation of order  $\ell$  from the Bianchi group. We can verify this in the vertical half-plane where the action of  $\mathrm{PSL}_2(\mathbb{Z})$  is embedded into the action of the Bianchi group, for the generators of orders  $\ell = 2$  and  $\ell = 3$  of  $\mathrm{PSL}_2(\mathbb{Z})$  which fix edges orthogonal to the vertical half-plane. These angles are transported to all edges stabilised by Bianchi group elements conjugate to these two rotations. Poincaré [9] partitions the edges of the Bianchi fundamental polyhedron into cycles, consisting of the edges on the same orbit, of length  $\frac{q}{\ell} = 1$  or  $2$ . In the case of length 2, Poincaré's description implies that each of the two 2-cells separated by the first edge of the cycle, is respectively on the same orbit as one of the 2-cells separated by the second edge of the cycle. As the fundamental domain for the 2-dimensional retract is strict with respect to the 2-cells, it can be chosen such that it contains  $e$  as the only edge on its orbit.  $\square$

## 3. THE KRÄMER NUMBERS AND GROUP HOMOLOGY

Krämer [7] has determined number-theoretic formulae for the numbers of conjugacy classes of the following types of finite subgroups in the Bianchi groups, where the symbols in the first row are Krämer's notations for the number of their conjugacy classes:

$\mu_2$	$\mu_T$	$\mu_3$	$\lambda_{2n}$	$\lambda_4^T = l_4^T$	$\lambda_4^*$	$\lambda_6^*$	$\mu_2^-$
$\mathcal{D}_2$	$\mathcal{A}_4$	$\mathcal{S}_3$	$\mathbb{Z}/n$	$\mathbb{Z}/2 \hookrightarrow \mathcal{A}_4$	$\mathbb{Z}/2 \hookrightarrow \mathcal{D}_2$	$\mathbb{Z}/3 \hookrightarrow \mathcal{S}_3$	$\mathcal{D}_2 \not\hookrightarrow \mathcal{A}_4$

Here, the inclusion arrows mean that we only consider copies of  $\mathbb{Z}/n$  admitting the specified inclusion in the given Bianchi group and  $\mathcal{D}_2 \not\hookrightarrow \mathcal{A}_4$  means that we only consider copies of  $\mathcal{D}_2$  not admitting any inclusion into a subgroup of type  $\mathcal{A}_4$  of the Bianchi group. Note that the number  $\mu_2^-$  is simply the difference  $\mu_2 - \mu_T$ , because every copy of  $\mathcal{A}_4$  admits precisely one normal subgroup of type  $\mathcal{D}_2$ . Also, note that for all Bianchi groups with units  $\{\pm 1\}$ , by lemma 5, we have  $\lambda_4^T \leq \mu_T$  and  $2\lambda_6^* = \mu_3$ .

The values given by Krämer's formulae are matching with the values computed with [11].

**Observation 16.** The Krämer numbers determine the 3-conjugacy classes graph and hence the reduced 3-torsion subcomplex for all Bianchi groups with units  $\{\pm 1\}$ , as we can see immediately from theorem 4 and the description of the reduced 3-torsion subcomplex in [10].

**Theorem 17.** *The homology of the Bianchi groups depends in all degrees above 2 (their virtual cohomological dimension) only on the Krämer numbers.*

In 3-torsion, this follows directly from observation 16 and the main result of [10], which is the following.

**Theorem 18** ([10]). *The  $\ell$ -primary part of the integral homology of  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  depends in degrees greater than 2 (the virtual cohomological dimension) only on the homeomorphism type of the  $\ell$ -torsion subcomplex.*

In 2-torsion, we have to recall some lemmata in order to give the proof of theorem 17.

**Lemma 19** ([10]). *The  $\ell$ -primary part of the terms  $E_{p,q}^2$  of the equivariant spectral sequence converging to  $\mathrm{H}_{p+q}(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$  depends in all rows  $q \geq 1$  only on the homeomorphism type of the  $\ell$ -torsion subcomplex.*

**Sub-Lemma 20** ([10]). *The matrix for the  $\ell$ -primary part of the differential  $d_{1,q}^1$  of the equivariant spectral sequence converging to  $\mathrm{H}_{p+q}(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$  can be decomposed as a direct sum of the blocks associated to the connected components of the  $\ell$ -torsion subgraph.*


**Lemma 21** (Schwermer/Vogtmann). *Let  $M$  be  $\mathbb{Z}$  or  $\mathbb{Z}/2$ . Consider group homology with trivial  $M$ -coefficients. Then the following holds.*


- Any inclusion  $\mathbb{Z}/2 \rightarrow \mathcal{S}_3$  induces an injection on homology.
- An inclusion  $\mathbb{Z}/3 \rightarrow \mathcal{S}_3$  induces an injection on homology in degrees congruent to 3 or 0 mod 4, and is otherwise zero.
- Any inclusion  $\mathbb{Z}/2 \rightarrow \mathcal{D}_2$  induces an injection on homology in all degrees.
- An inclusion  $\mathbb{Z}/3 \rightarrow \mathcal{A}_4$  induces injections on homology in all degrees.
- An inclusion  $\mathbb{Z}/2 \rightarrow \mathcal{A}_4$  induces injections on homology in degrees greater than 1, and is zero on  $\mathrm{H}_1$ .

For the proof in  $\mathbb{Z}$ -coefficients, see [12], for  $\mathbb{Z}/2$ -coefficients see [10].



**Lemma 22** ([10], lemma 32). *Let  $q \geq 3$  be an odd integer number. Let  $v$  be a vertex representative of stabiliser type  $\mathcal{D}_2$  in the refined cellular complex for the Bianchi groups. Then the three images in  $(H_q(\mathcal{D}_2; \mathbb{Z}))_{(2)}$  induced by the inclusions of the stabilisers of the edges adjacent to  $v$ , are linearly independent.*

Finally, we deduce one new ingredient for the proof of theorem 17, which might be of interest in its own right.

**Lemma 23.** *In all rows  $q > 1$  and outside connected components of type , the  $d_{p,q}^1$ -differential of the equivariant spectral sequence converging to  $H_{p+q}(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$  is always injective.*

*Proof.* For matrix blocks of the  $d_{p,q}^1$ -differential associated to vertices with just one adjacent edge, we see from lemma 5 that the vertex stabiliser is of type  $\mathcal{A}_4$  in 2-torsion, respectively of type  $\mathcal{S}_3$  in 3-torsion, so injectivity follows from lemma 21. As we have placed ourselves outside connected components of type , the remaining vertices are bifurcation points of stabiliser type  $\mathcal{D}_2$  and injectivity follows from lemma 22.  $\square$

*Proof of theorem 17.* What we need to determine with the Kramer numbers, is the 2-primary part of the  $E_{p,q}^2$ -term of the equivariant spectral sequence converging to  $H_{p+q}(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$  in all rows  $q > 1$ . From there, we see from the deduction of theorem 18 in [10] that we obtain the claim. By sub-lemma 20 and lemma 19, we only need to check this determination on each homeomorphism type of connected components of the 2-torsion subgraph. We use theorem 4 to identify the reduced 2-torsion subgraph and the 2-conjugacy classes graph. Then we can observe that

- Kramer's number  $\lambda_4^* - \lambda_4$  determines the number of connected components of type .
- Kramer's number  $\lambda_4^*$  determines the number of edges of the 2-torsion subgraph outside connected components of type .
- Kramer's number  $\mu_{\bar{2}}$  determines the number of bifurcation points, and  $\mu_T$  determines the number of vertices with only one adjacent edge of the 2-torsion subgraph.

$\square$



## 4. NUMERICAL EVALUATION OF KRÄMER'S FORMULAE

Denote by  $\delta$  the number of finite ramification places of  $\mathbb{Q}(\sqrt{-m})$  over  $\mathbb{Q}$ . Let  $k_+$  be the totally real number field  $\mathbb{Q}(\sqrt{3m})$  and denote its ideal class number by  $h_{k_+}$ . Krämer introduces the following indicators:

$$z := \begin{cases} 2, & \text{if 3 is the norm of an integer of } k_+, \\ 1, & \text{otherwise.} \end{cases}$$

For  $m \equiv 0 \pmod{3}$  and  $m \neq 3$ , denote by  $\epsilon := \frac{1}{2}(a + b\sqrt{\frac{m}{3}}) > 1$  the fundamental unit of  $k_+$  (where  $a, b \in \mathbb{N}$ ). Now, define

$$x := \begin{cases} 2, & \text{if the norm of } \epsilon \text{ is } 1, \\ 1, & \text{if the norm of } \epsilon \text{ is } -1 \end{cases}$$

and

$$y := \begin{cases} 2, & \text{if } b \equiv 0 \pmod{3}, \\ 1, & \text{otherwise.} \end{cases}$$


Then Krämer's formulae are the following in 3-torsion.


$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$	$\lambda_6^*$	$\lambda_6 - \lambda_6^*$
$m \equiv 2 \pmod{3}$	0	$\frac{z}{2}h_{k_+}$
$m \equiv 1 \pmod{3}$ gives either	$2^{\delta-1}$	$\frac{1}{2}(h_{k_+} - 2^{\delta-1})$
or	0	$\frac{1}{2}h_{k_+}$
$m \equiv 6 \pmod{9}$	0	$xyh_{k_+}$
$m \equiv 3 \pmod{9}$ , $m \neq 3$ gives either	$2^{\delta-2}$	$\frac{1}{2}(3xh_{k_+} - 2^{\delta-2})$
or	0	$\frac{1}{2}3xh_{k_+}$




Krämer obtains the following general statements about his numbers:
















condition	implication
$m \equiv 7 \pmod{8}$	$\mu_T = \mu_2^- = \lambda_4^T = \lambda_4^* = 0.$
$m \equiv 2 \pmod{3}$	$\mu_3 = \lambda_6^* = 0.$
$m \equiv 5 \pmod{8}$	$\mu_T = \lambda_4^T = 0.$
$m \equiv 6 \pmod{9}$	$\mu_3 = \lambda_6^* = 0.$
$m \equiv 21 \pmod{24}$	$\lambda_4^* = 0.$
$m \equiv 0 \pmod{6}$ and $\lambda_4^* > 0$	$\lambda_4^T > 0.$
$m \equiv 9 \pmod{24}$ and $\lambda_4^* > 0$	$\lambda_4^T > 0.$
$m$ prime and $m \equiv 1$ or $3 \pmod{8}$	$\lambda_4^T > 0.$
$m \equiv 5 \pmod{8}$	$\lambda_4^* > 0.$
$m = 2p$ with $p$ prime and $p \equiv 3$ or $5 \pmod{8}$	$\lambda_4^* > 0.$
$m = p'p$ with $p$ and $p'$ prime and $p \equiv p' \equiv 3$ or $5 \pmod{8}$	$\lambda_4^* > 0.$
$m = 3p$ with $p$ prime and $p \equiv 1$ or $3 \pmod{8}$	$\lambda_4^T > 0.$
$m$ prime and $m \equiv 1 \pmod{3}$	$\lambda_6^* > 0.$
$m = 3p$ with $p$ prime and $p \equiv 1 \pmod{3}$	$\lambda_6^* > 0.$
$m \equiv 3 \pmod{8}$ and $m$ admits a prime divisor $p$ with $p \equiv 5$ or $7 \pmod{8}$	$\lambda_4 - \lambda_4^* > 0.$
$m \equiv 1 \pmod{8}$ and $w = 1$ and $h(k_+) = 2^{\delta-3}$	$\mu_2^- = 0.$
$m \equiv 2 \pmod{4}$ and $q = 1$ and $h(k_+) = 2^{\delta-1}$ and $w = 2$	$\lambda_4^* = 0.$
$m \equiv 2 \pmod{4}$ and $h(k_+) = 2^{\delta-2}$ and $m$ admits a prime divisor $p$ with $p \equiv 5$ or $7 \pmod{8}$	$\lambda_4^* = 0.$
$m \equiv 3 \pmod{9}$ and $x' = 1$ and $\frac{m}{3}$ admits only prime divisors $p$ with $p \equiv 1 \pmod{12}$	$\lambda_6^* > 0.$
$m \equiv 3 \pmod{9}$ and $x' = 1$ and $\frac{m}{3}$ admits a prime divisor $p$ with $p \equiv 5 \pmod{12}$	$\lambda_6^* = 0.$
$m \equiv 3 \pmod{9}$ and $h(k'_+) = 2^{\delta-3}$ and $\frac{m}{3}$ admits only prime divisors $p$ with $p \equiv \pm 1 \pmod{12}$ or $p = 2$	$\lambda_6^* = 0.$
$m \equiv 3 \pmod{9}$ and $h(k'_+) = 1$ and $\frac{m}{3} = p'p$ with $p', p$ prime and $p' \equiv p \equiv 7 \pmod{12}$	$\lambda_6^* > 0.$



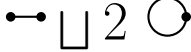
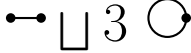

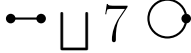
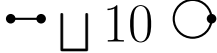
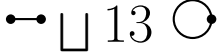
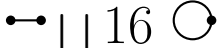
With these statements, we can decide for many Bianchi groups, which of the alternative cases in Kramer's formulae must be used. We do this in the below tables for all such Bianchi groups  $\mathrm{PSL}_2(\mathcal{O}_{-m})$  with  $m$  ranging between 5 and 10.000.

3-conjugacy classes graph	$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 3-conjugacy classes graph
	5, 6, 11, 14, 15, 17, 23, 29, 35, 38, 41, 47, 51, 53, 59, 62, 71, 83, 86, 87, 89, 95, 101, 113, 119, 123, 131, 134, 137, 149, 155, 158, 159, 167, 173, 179, 191, 197, 203, 206, 215, 233, 239, 254, 257, 263, 267, 269, 278, 281, 287, 293, 303, 311, 317, 323, 326, 335, 339, 353, 359, 371, 383, 389, 395, 398, 401, 411, 422, 431, 446, 447, 449, 461, 479, 515, 519, 521, 527, 542, 551, 557, 566, 569, 591, 593, 599, 614, 623, 635, 641, 647, 653, 662, 695, 699, 701, 707, 719, 731, 734, 743, 755, 758, 761, 773, 779, 791, 797, 807, 809, 815, 821, 839, 843, 857, 878, 879, 881, 887, 899, 911, 926, 929, 941, 951, 953, 959, 977, 995, 998, 1013, 1031, 1049, 1055, 1059, 1061, 1094, 1097, 1103, 1109, 1115, 1139, 1142, 1151, 1167, 1181, 1211, 1214, 1217, 1223, 1229, 1238, 1247, 1262, 1271, 1277, 1286, 1289, 1301, 1343, 1347, 1361, 1367, 1373, 1379, 1382, 1383, 1415, 1433, 1439, 1481, 1487, 1493, 1502, 1511, 1527, 1535, 1553, 1559, 1563, 1574, 1601, 1607, 1613, 1622, 1631, 1637, 1643, 1646, 1655, 1671, 1697, 1707, 1709, 1718, 1721, 1733, 1751, 1763, 1766, 1779, 1799, 1814, 1835, 1838, 1851, 1883, 1889, 1901, 1913, 1919, 1923, 1934, 1943, 1949, 1959, 1973, 1982, 2031, 2039, 2069, 2078, 2087, 2102, 2103, 2111, 2126, 2129, 2147, 2153, 2159, 2174, 2195, 2219, 2237, 2246, 2279, 2291, 2319, 2333, 2342, 2357, 2363, 2381, 2391, 2393, 2399, 2417, 2427, 2435, 2441, 2462, 2463, 2471, 2477, 2495, 2543, 2549, 2567, 2571, 2591, 2603, 2606, 2609, 2615, 2621, 2633, 2643, 2654, 2657, 2693, 2711, 2723, 2729, 2735, 2741, 2747, 2753, 2771, 2777, 2787, 2789, 2798, 2807, 2823, 2831, 2837, 2855, 2859, 2861, 2894, 2897, 2909, 2918, 2931, 2942, 2957, 2966, 2969, 2987, 3035, 3039, 3041, 3062, 3086, 3089, 3095, 3119, 3131, 3134, 3143, 3147, 3155, 3183, 3209, 3215, 3221, 3227, 3239, 3254, 3257, 3287, 3291, 3326, 3327, 3329, 3383, 3389, 3398, 3413, 3446, 3449, 3455, 3461, 3494, 3503, 3518, 3533, 3543, 3551, 3557, 3563, 3566, 3579, 3581, 3587, 3593, 3635, 3651, 3662, 3683, 3701, 3734, 3743, 3755, 3758, 3761, 3791, 3797, 3821, 3827, 3831, 3833, 3867, 3881, 3902, 3903, 3911, 3929, 3935, 3974, 3983, 3989, 3998, 4001, 4013, 4022, 4031, 4055, 4073, 4083, 4115, 4127, 4133, 4151, 4157, 4166, 4187, 4217, 4223, 4227, 4241, 4247, 4253, 4286, 4289, 4295, 4299, 4319, 4337, 4343, 4349, 4358, 4373, 4379, 4406, 4421, 4427, 4443, 4479, 4481, 4487, 4493, 4502, 4517, 4535, 4571, 4607, 4619, 4622, 4649, 4659, 4673, 4727, 4733, 4739, 4751, 4766, 4811, 4817, 4839, 4859, 4877, 4883, 4889, 4911, 4919, 4934, 4937, 4955, 5021, 5078, 5081, 5091, 5102, 5111, 5127, 5163, 5189, 5195, 5199, 5231, 5237, 5255, 5261, 5273, 5279, 5309, 5315, 5318, 5327, 5333, 5339, 5342, 5363, 5366, 5381, 5393, 5411, 5414, 5435, 5438, 5441, 5459, 5477, 5501, 5519, 5534, 5567, 5573, 5582, 5615, 5627, 5631, 5663, 5667, 5669, 5693, 5699, 5702, 5717, 5739, 5741, 5747, 5771, 5774, 5801, 5847, 5849, 5855, 5891, 5919, 5942, 5991, 5999, 6023, 6029, 6053, 6089, 6101, 6107, 6113, 6119, 6134, 6155, 6158, 6167, 6173, 6191, 6197, 6207, 6239, 6311, 6317, 6326, 6329, 6353, 6374, 6387, 6389, 6395, 6407, 6423, 6443, 6449, 6455, 6467, 6473, 6503, 6515, 6521, 6542, 6581, 6587, 6614, 6635, 6638, 6653, 6662, 6671, 6683, 6686, 6689, 6707, 6711, 6731, 6737, 6767, 6791, 6819, 6833, 6839, 6857, 6869, 6891, 6917, 6926, 6927, 6977, 6995, 6998, 6999, 7001, 7013, 7022, 7031, 7071, 7091, 7094, 7109, 7115, 7118, 7121, 7142, 7143, 7166, 7179, 7193, 7214, 7223, 7229, 7235, 7251, 7253, 7295, 7323, 7343, 7349, 7355, 7367, 7382, 7391, 7415, 7427, 7431, 7433, 7439, 7454, 7457, 7463, 7478, 7481, 7517, 7541, 7577, 7589, 7619, 7646, 7647, 7649, 7655, 7673, 7679, 7694, 7715, 7757, 7763, 7793, 7814, 7817, 7827, 7829, 7835, 7853, 7859, 7863, 7877, 7886, 7895, 7899, 7901, 7934, 7937, 7949, 7971, 7979, 8003, 8006, 8054, 8069, 8079, 8081, 8093, 8102, 8117, 8135, 8159, 8187, 8222, 8223, 8259, 8273, 8279, 8315, 8318, 8339, 8351, 8367, 8369, 8403, 8438, 8462, 8471, 8495, 8511, 8519, 8531, 8537, 8583, 8597, 8603, 8609, 8615, 8651, 8669, 8678, 8681, 8691, 8693, 8711, 8726, 8727, 8735, 8741, 8753, 8759, 8795, 8846, 8849, 8861, 8871, 8891, 8894, 8903, 8907, 8915, 8927, 8939, 8969, 9023, 9029, 9038, 9041, 9083, 9107, 9123, 9134, 9137, 9143, 9155, 9161, 9167, 9173, 9179, 9182, 9206, 9209, 9221, 9257, 9267, 9278, 9281, 9293, 9299, 9302, 9326, 9335, 9377, 9395, 9413, 9431, 9446, 9461, 9473, 9518, 9527, 9533, 9566, 9599, 9611, 9627, 9629, 9662, 9677, 9689, 9707, 9755, 9771, 9827, 9833, 9857, 9863, 9899, 9929, 9935, 9941, 9974, 9983, 9987, 9998,

3-conjugacy classes graph	$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 3-conjugacy classes graph
	<p>26, 42, 65, 69, 74, 77, 78, 110, 122, 141, 143, 161, 170, 182, 185, 186, 195, 209, 213, 218, 221, 222, 230, 231, 266, 305, 314, 329, 365, 366, 377, 410, 434, 437, 455, 458, 470, 473, 474, 483, 485, 494, 497, 501, 518, 545, 554, 555, 573, 590, 611, 618, 627, 629, 654, 671, 689, 710, 717, 746, 749, 767, 782, 785, 789, 794, 806, 842, 851, 854, 869, 890, 893, 905, 906, 933, 935, 938, 942, 965, 986, 989, 1022, 1037, 1073, 1079, 1086, 1095, 1106, 1118, 1130, 1145, 1149, 1157, 1166, 1169, 1178, 1194, 1199, 1226, 1235, 1241, 1253, 1293, 1295, 1310, 1311, 1313, 1334, 1338, 1349, 1358, 1370, 1374, 1385, 1394, 1397, 1406, 1437, 1442, 1455, 1457, 1463, 1469, 1491, 1509, 1514, 1517, 1526, 1529, 1565, 1577, 1595, 1598, 1599, 1626, 1649, 1661, 1662, 1670, 1673, 1685, 1742, 1745, 1769, 1778, 1781, 1790, 1797, 1802, 1817, 1829, 1841, 1853, 1886, 1887, 1910, 1937, 1941, 1946, 1958, 1961, 1985, 2006, 2015, 2021, 2094, 2105, 2114, 2117, 2135, 2157, 2165, 2177, 2201, 2202, 2229, 2231, 2234, 2238, 2247, 2249, 2255, 2282, 2285, 2294, 2318, 2321, 2330, 2378, 2382, 2387, 2390, 2426, 2429, 2438, 2486, 2489, 2501, 2510, 2513, 2534, 2537, 2561, 2570, 2589, 2607, 2630, 2634, 2639, 2661, 2666, 2669, 2678, 2679, 2681, 2726, 2733, 2778, 2786, 2795, 2810, 2813, 2895, 2906, 2921, 2922, 2949, 2951, 2954, 2967, 2981, 3014, 3017, 3026, 3047, 3053, 3059, 3065, 3077, 3093, 3101, 3110, 3111, 3113, 3122, 3149, 3161, 3173, 3182, 3197, 3219, 3233, 3246, 3281, 3293, 3302, 3305, 3309, 3311, 3317, 3335, 3341, 3353, 3365, 3374, 3377, 3395, 3422, 3431, 3437, 3453, 3471, 3473, 3497, 3515, 3521, 3530, 3569, 3590, 3599, 3629, 3642, 3686, 3713, 3723, 3731, 3737, 3749, 3759, 3782, 3785, 3786, 3794, 3809, 3830, 3839, 3842, 3854, 3869, 3878, 3893, 3914, 3926, 3953, 3957, 3962, 3966, 3977, 3995, 4010, 4011, 4061, 4094, 4097, 4118, 4121, 4142, 4145, 4155, 4190, 4191, 4193, 4199, 4238, 4254, 4265, 4298, 4310, 4317, 4346, 4355, 4362, 4371, 4382, 4385, 4398, 4403, 4461, 4469, 4490, 4506, 4526, 4529, 4533, 4542, 4553, 4577, 4587, 4601, 4613, 4623, 4634, 4646, 4658, 4677, 4685, 4695, 4715, 4731, 4749, 4757, 4769, 4781, 4814, 4821, 4823, 4829, 4838, 4841, 4847, 4853, 4886, 4895, 4898, 4910, 4947, 4958, 4974, 4982, 4983, 4985, 4991, 4997, 5015, 5019, 5030, 5033, 5055, 5057, 5066, 5118, 5129, 5135, 5138, 5159, 5162, 5165, 5174, 5177, 5198, 5201, 5213, 5222, 5246, 5249, 5262, 5307, 5321, 5343, 5345, 5429, 5451, 5465, 5469, 5486, 5487, 5495, 5513, 5514, 5523, 5555, 5558, 5559, 5561, 5585, 5595, 5597, 5609, 5613, 5654, 5678, 5726, 5729, 5735, 5738, 5753, 5765, 5777, 5789, 5798, 5822, 5837, 5846, 5873, 5882, 5918, 5921, 5930, 5933, 5946, 5951, 5955, 5966, 5969, 5982, 5990, 6017, 6035, 6041, 6062, 6083, 6086, 6117, 6126, 6146, 6161, 6170, 6185, 6189, 6206, 6215, 6233, 6234, 6251, 6254, 6261, 6278, 6281, 6333, 6335, 6341, 6351, 6377, 6378, 6386, 6401, 6410, 6414, 6431, 6437, 6470, 6479, 6482, 6485, 6495, 6509, 6522, 6554, 6557, 6558, 6590, 6593, 6603, 6617, 6641, 6677, 6695, 6698, 6702, 6743, 6749, 6755, 6758, 6773, 6794, 6806, 6818, 6821, 6851, 6854, 6855, 6881, 6893, 6905, 6923, 6935, 6963, 6974, 6986, 7037, 7046, 7053, 7055, 7061, 7097, 7107, 7133, 7145, 7163, 7169, 7181, 7190, 7241, 7259, 7269, 7277, 7278, 7298, 7313, 7322, 7334, 7341, 7373, 7386, 7397, 7409, 7421, 7422, 7430, 7467, 7511, 7526, 7535, 7539, 7562, 7574, 7610, 7611, 7613, 7622, 7631, 7658, 7674, 7683, 7709, 7721, 7745, 7751, 7766, 7773, 7818, 7874, 7913, 7922, 7946, 7955, 7958, 7961, 7962, 7982, 7985, 7989, 7994, 8015, 8021, 8033, 8057, 8061, 8090, 8105, 8126, 8129, 8141, 8153, 8174, 8195, 8213, 8249, 8255, 8282, 8285, 8286, 8309, 8333, 8345, 8357, 8366, 8390, 8394, 8399, 8414, 8417, 8435, 8439, 8441, 8465, 8474, 8489, 8498, 8538, 8549, 8555, 8558, 8561, 8570, 8574, 8582, 8618, 8621, 8630, 8637, 8655, 8657, 8666, 8687, 8702, 8705, 8709, 8717, 8718, 8738, 8765, 8774, 8781, 8810, 8822, 8826, 8834, 8858, 8873, 8879, 8885, 8909, 8921, 8945, 8987, 8997, 9002, 9017, 9035, 9051, 9062, 9089, 9101, 9110, 9146, 9158, 9191, 9195, 9197, 9215, 9231, 9233, 9254, 9258, 9263, 9290, 9294, 9303, 9305, 9329, 9339, 9353, 9357, 9362, 9365, 9389, 9398, 9434, 9443, 9447, 9455, 9494, 9501, 9509, 9515, 9530, 9542, 9554, 9557, 9569, 9573, 9582, 9593, 9638, 9641, 9665, 9674, 9686, 9695, 9699, 9701, 9710, 9726, 9734, 9770, 9773, 9779, 9815, 9830, 9854, 9869, 9878, 9879, 9893, 9911, 9917, 9926, 9951, 9953, 9962, 9965, 9978,</p>

3-conjugacy classes graph	$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 3-conjugacy classes graph
3 	30, 66, 107, 138, 174, 255, 282, 302, 318, 354, 419, 498, 503, 509, 606, 617, 642, 759, 771, 786, 795, 863, 894, 974, 1002, 1007, 1046, 1074, 1146, 1182, 1193, 1319, 1355, 1362, 1409, 1434, 1454, 1478, 1479, 1506, 1551, 1578, 1583, 1614, 1619, 1691, 1695, 1758, 1866, 1871, 1895, 1902, 1947, 1967, 1997, 2051, 2055, 2081, 2082, 2141, 2154, 2213, 2273, 2283, 2298, 2309, 2334, 2343, 2351, 2514, 2586, 2595, 2658, 2739, 2759, 2766, 2801, 2846, 2874, 2879, 3018, 3054, 3243, 3299, 3342, 3359, 3495, 3522, 3531, 3567, 3594, 3617, 3647, 3671, 3687, 3695, 3855, 3899, 3917, 3918, 3954, 4049, 4062, 4071, 4098, 4119, 4206, 4229, 4262, 4314, 4323, 4395, 4397, 4415, 4457, 4478, 4574, 4595, 4637, 4694, 4755, 4782, 4793, 4799, 4835, 4899, 4926, 5006, 5009, 5034, 5178, 5219, 5297, 5322, 5417, 5466, 5511, 5579, 5591, 5606, 5682, 5703, 5727, 5763, 5813, 5826, 5861, 5898, 5907, 5981, 6114, 6186, 6221, 6269, 6366, 6518, 6519, 6546, 6551, 6569, 6618, 6639, 6654, 6701, 6761, 6911, 6978, 6983, 7122, 7286, 7338, 7374, 7383, 7491, 7529, 7554, 7698, 7703, 7743, 7841, 7842, 7862, 7871, 7887, 8198, 8202, 8207, 8237, 8238, 8283, 8297, 8319, 8331, 8355, 8501, 8513, 8562, 8654, 8679, 8787, 8807, 8837, 8895, 8922, 8951, 8994, 8999, 9039, 9066, 9138, 9255, 9354, 9426, 9437, 9497, 9642, 9663, 9671, 9678, 9795, 9822, 9831,
4 	33, 105, 114, 146, 177, 249, 258, 285, 290, 299, 321, 330, 341, 357, 374, 393, 402, 413, 429, 465, 506, 510, 530, 537, 546, 581, 582, 609, 638, 663, 665, 674, 681, 737, 753, 770, 830, 834, 861, 866, 870, 902, 915, 917, 962, 987, 1001, 1010, 1023, 1034, 1041, 1070, 1085, 1113, 1121, 1133, 1158, 1185, 1190, 1205, 1239, 1257, 1265, 1266, 1346, 1401, 1430, 1446, 1473, 1482, 1490, 1505, 1545, 1554, 1562, 1581, 1586, 1589, 1590, 1634, 1689, 1698, 1727, 1730, 1743, 1757, 1761, 1770, 1793, 1806, 1842, 1869, 1878, 1898, 1905, 1955, 1970, 1986, 1995, 2013, 2030, 2033, 2045, 2049, 2085, 2090, 2093, 2121, 2139, 2189, 2210, 2211, 2222, 2261, 2265, 2270, 2274, 2301, 2345, 2346, 2405, 2418, 2454, 2481, 2498, 2519, 2546, 2562, 2573, 2585, 2598, 2618, 2690, 2702, 2717, 2742, 2751, 2774, 2814, 2834, 2841, 2849, 2870, 2886, 2913, 2930, 2933, 2958, 2985, 2990, 2993, 2994, 3003, 3005, 3030, 3057, 3066, 3071, 3074, 3102, 3129, 3138, 3165, 3170, 3230, 3237, 3255, 3263, 3269, 3282, 3290, 3345, 3363, 3390, 3399, 3401, 3410, 3426, 3435, 3458, 3489, 3498, 3507, 3534, 3542, 3561, 3597, 3615, 3638, 3641, 3714, 3770, 3777, 3806, 3815, 3845, 3849, 3857, 3858, 3905, 3921, 3930, 3939, 3941, 3965, 4029, 4038, 4047, 4065, 4074, 4082, 4085, 4109, 4110, 4130, 4137, 4146, 4169, 4226, 4277, 4281, 4301, 4334, 4353, 4430, 4433, 4454, 4470, 4497, 4505, 4514, 4551, 4559, 4565, 4578, 4610, 4614, 4670, 4686, 4697, 4706, 4713, 4722, 4730, 4745, 4758, 4767, 4857, 4862, 4865, 4866, 4929, 4965, 4970, 5001, 5090, 5093, 5117, 5126, 5141, 5154, 5183, 5190, 5226, 5235, 5270, 5271, 5282, 5330, 5357, 5369, 5370, 5379, 5397, 5405, 5423, 5442, 5453, 5474, 5489, 5505, 5510, 5570, 5621, 5645, 5649, 5681, 5685, 5694, 5705, 5714, 5757, 5762, 5793, 5795, 5829, 5870, 5885, 5910, 5937, 5945, 5954, 5957, 6005, 6009, 6014, 6054, 6071, 6095, 6099, 6110, 6149, 6153, 6162, 6182, 6194, 6230, 6245, 6266, 6279, 6290, 6305, 6306, 6314, 6315, 6365, 6383, 6398, 6446, 6461, 6497, 6549, 6567, 6585, 6594, 6605, 6657, 6665, 6693, 6710, 6729, 6734, 6738, 6747, 6774, 6783, 6785, 6801, 6815, 6830, 6837, 6846, 6873, 6882, 6890, 6902, 6941, 6945, 6953, 6965, 6990, 7017, 7026, 7049, 7070, 7085, 7106, 7130, 7134, 7215, 7233, 7238, 7242, 7271, 7305, 7310, 7337, 7359, 7361, 7370, 7379, 7395, 7413, 7445, 7469, 7493, 7503, 7553, 7565, 7566, 7593, 7598, 7602, 7667, 7685, 7697, 7710, 7718, 7719, 7730, 7733, 7737, 7739, 7746, 7781, 7782, 7790, 7799, 7826, 7847, 7926, 7970, 7973, 7997, 8018, 8030, 8043, 8045, 8066, 8070, 8097, 8151, 8162, 8165, 8169, 8177, 8201, 8210, 8246, 8261, 8277, 8294, 8393, 8453, 8457, 8502, 8510, 8529, 8565, 8585, 8642, 8690, 8729, 8763, 8789, 8790, 8799, 8817, 8853, 8855, 8862, 8889, 8897, 8898, 8934, 8961, 8990, 9005, 9006, 9015, 9033, 9087, 9095, 9159, 9170, 9222, 9249, 9269, 9285, 9314, 9338, 9393, 9449, 9465, 9470, 9483, 9485, 9503, 9510, 9545, 9546, 9581, 9591, 9609, 9614, 9617, 9618, 9681, 9713, 9717, 9737, 9762, 9797, 9807, 9809, 9821, 9842, 9897, 9905, 9906, 9942, 9969, 9989,
5 	677, 1043, 1203, 1451, 1877, 2297, 2315, 2558, 2582, 3137, 3158, 3803, 4409, 4721, 5039, 5153, 5462, 5657, 5807, 6038, 6203, 6243, 6459, 6782, 7838, 7967, 8009, 8111, 8423, 8483, 8486, 8543, 8639, 8933, 9014, 9341, 9521, 9806, 9902, 9995,

3-conjugacy classes graph	$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 3-conjugacy classes graph
6 	246, 362, 390, 435, 533, 602, 713, 762, 822, 858, 1077, 1110, 1335, 1337, 1419, 1541, 1547, 1794, 1830, 1865, 2054, 2138, 2183, 2186, 2198, 2262, 2355, 2369, 2414, 2442, 2517, 2526, 2555, 2703, 2705, 2822, 2885, 2910, 2915, 3029, 3098, 3126, 3198, 3206, 3315, 3414, 3434, 3545, 3614, 3653, 3665, 3666, 3669, 3774, 4101, 4103, 4134, 4154, 4181, 4541, 4589, 4710, 4790, 4818, 4826, 4854, 4938, 5069, 5105, 5106, 5210, 5306, 5430, 5538, 5541, 5543, 5594, 5690, 5718, 5790, 5811, 5894, 5909, 5993, 6063, 6077, 6209, 6222, 6438, 6474, 6494, 6621, 6623, 6629, 6746, 6870, 6878, 6942, 7157, 7194, 7199, 7265, 7287, 7289, 7358, 7629, 7635, 7661, 7878, 7995, 8078, 8133, 8186, 8310, 8346, 8418, 8459, 8534, 8606, 8682, 8777, 8801, 8870, 8942, 8943, 8981, 9069, 9071, 9113, 9194, 9266, 9374, 9390, 9435, 9626, 9843, 9867,
7 	587, 971, 2699, 3677, 4742, 4803, 4907, 4973, 5207, 5294, 7262, 7571, 8387, 8429, 8573, 9623, 9743,
8 	438, 645, 690, 798, 897, 969, 1005, 1122, 1131, 1221, 1230, 1410, 1610, 1635, 1653, 1833, 1874, 1914, 2022, 2067, 2130, 2258, 2310, 2337, 2373, 2409, 2445, 2453, 2465, 2483, 2490, 2553, 2706, 2765, 2769, 2877, 2945, 3021, 3210, 3245, 3266, 3278, 3318, 3354, 3417, 3462, 3470, 3570, 3605, 3606, 3674, 3689, 3705, 3710, 3741, 3795, 3813, 3885, 3890, 3894, 3938, 4002, 4037, 4173, 4209, 4245, 4290, 4326, 4370, 4389, 4407, 4445, 4466, 4515, 4605, 4661, 4718, 4785, 4794, 4830, 4902, 5010, 5037, 5045, 5073, 5109, 5181, 5217, 5234, 5253, 5258, 5285, 5289, 5402, 5406, 5478, 5546, 5549, 5622, 5630, 5642, 5658, 5730, 5810, 5838, 5858, 5865, 5874, 5883, 5973, 6002, 6018, 6026, 6135, 6198, 6270, 6293, 6342, 6369, 6441, 6486, 6513, 6530, 6545, 6578, 6630, 6666, 6765, 6770, 6797, 6809, 6810, 6954, 6981, 6989, 7035, 7062, 7073, 7089, 7161, 7170, 7205, 7206, 7314, 7385, 7449, 7458, 7490, 7494, 7505, 7521, 7557, 7601, 7637, 7638, 7755, 7805, 7845, 7881, 7890, 7910, 7917, 7953, 8106, 8115, 8142, 8205, 8241, 8270, 8295, 8313, 8342, 8358, 8421, 8430, 8466, 8547, 8601, 8633, 8645, 8646, 8762, 8798, 8813, 8835, 8930, 8970, 9053, 9086, 9149, 9177, 9230, 9321, 9330, 9366, 9401, 9407, 9410, 9429, 9590, 9605, 9635, 9645, 9654, 9735, 9758, 9761, 9782, 9789, 9798, 9834, 9845, 9870, 9881, 9933, 9977,
9 	1038, 2091, 2955, 4638, 5295, 5646, 5897, 7499, 7662, 8823, 8966, 9411, 9749,
10 	1082, 1322, 3002, 3107, 3678, 4106, 4331, 4709, 6065, 7067, 7197, 7217, 7931, 8099, 8189, 9402,
11 	2027, 3467, 6257, 8267,
12 	678, 978, 1326, 1329, 1518, 1977, 2522, 2697, 3273, 3485, 3633, 4070, 4218, 4313, 4434, 4569, 5042, 5291, 5361, 5433, 5721, 6098, 6297, 6531, 6626, 7010, 7178, 7377, 7446, 7485, 7530, 7670, 7769, 7815, 8034, 8426, 8493, 8754, 9074, 9077, 9105, 9141, 9185, 9186, 9347, 9422, 9474, 9519, 9606, 9915,
13 	8747,
14 	5802, 6074, 6302, 6533,
15 	5963, 7086, 7806, 8958,
16 	1302, 1365, 2193, 2670, 2715, 2805, 2882, 3201, 4182, 4641, 4893, 5334, 5901, 6045, 6090, 6405, 6477, 6918, 7665, 7701, 7802, 7809, 7854, 8007, 8322, 8385, 8610, 8745, 9042, 9078, 9690, 9746, 9785, 9890,
18 	2190, 3270, 4566, 5142, 8321,
20 	4562, 5298, 5633, 6081, 6347, 7202, 9753,
21 	6078,
24 	6915, 7998, 8178, 8979, 9213, 9861,

3-conjugacy classes graph	$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 3-conjugacy classes graph
	<p>7, 19, 31, 43, 67, 79, 103, 127, 139, 151, 163, 199, 211, 223, 271, 283, 307, 379, 439, 463, 487, 499, 523, 571, 607, 619, 631, 691, 727, 739, 751, 787, 811, 823, 859, 883, 907, 919, 967, 991, 1039, 1051, 1063, 1123, 1171, 1231, 1279, 1303, 1399, 1423, 1447, 1459, 1471, 1483, 1531, 1543, 1567, 1579, 1627, 1663, 1699, 1723, 1759, 1783, 1831, 1867, 1987, 1999, 2011, 2083, 2131, 2143, 2179, 2203, 2239, 2251, 2287, 2311, 2347, 2371, 2383, 2467, 2503, 2539, 2551, 2647, 2659, 2671, 2719, 2731, 2767, 2803, 2851, 2971, 3019, 3067, 3079, 3163, 3187, 3259, 3307, 3319, 3331, 3343, 3391, 3463, 3499, 3511, 3559, 3583, 3607, 3631, 3691, 3727, 3739, 3823, 3847, 3907, 3931, 3967, 4003, 4051, 4099, 4159, 4231, 4327, 4339, 4483, 4507, 4519, 4591, 4603, 4639, 4651, 4663, 4723, 4759, 4783, 4831, 4903, 4951, 4999, 5023, 5059, 5107, 5179, 5227, 5323, 5347, 5407, 5431, 5443, 5479, 5503, 5527, 5623, 5647, 5659, 5743, 5779, 5791, 5827, 5839, 5851, 5923, 6007, 6043, 6091, 6151, 6163, 6199, 6247, 6271, 6343, 6367, 6379, 6427, 6451, 6547, 6571, 6619, 6679, 6703, 6823, 6907, 6967, 6991, 7027, 7039, 7159, 7219, 7243, 7411, 7507, 7591, 7603, 7639, 7687, 7699, 7723, 7759, 7867, 7879, 7927, 7951, 7963, 8011, 8167, 8191, 8263, 8287, 8311, 8419, 8443, 8467, 8527, 8539, 8563, 8599, 8623, 8647, 8719, 8731, 8779, 8803, 8839, 8863, 8887, 8923, 8971, 9007, 9067, 9091, 9103, 9127, 9151, 9187, 9199, 9283, 9319, 9343, 9391, 9403, 9439, 9511, 9547, 9619, 9631, 9643, 9679, 9739, 9787, 9811, 9859, 9871, 9883, 9907, 9931,</p>
	<p>39, 111, 183, 219, 291, 327, 331, 367, 471, 543, 579, 643, 723, 831, 939, 1011, 1047, 1087, 1119, 1191, 1227, 1263, 1291, 1299, 1327, 1371, 1623, 1803, 1839, 1879, 1951, 1983, 2019, 2127, 2271, 2307, 2487, 2559, 2631, 2791, 2811, 2991, 3063, 3099, 3207, 3271, 3351, 3459, 3547, 3603, 3639, 3643, 3711, 3747, 3919, 3943, 3963, 4027, 4111, 4143, 4359, 4363, 4423, 4647, 4791, 4827, 4863, 4971, 5007, 5011, 5079, 5119, 5167, 5223, 5259, 5331, 5367, 5403, 5563, 5583, 5619, 5799, 5979, 6051, 6067, 6159, 6339, 6411, 6483, 6607, 6663, 6691, 6763, 6807, 6843, 6871, 6879, 6883, 7023, 7131, 7167, 7311, 7351, 7419, 7459, 7563, 7779, 7851, 8059, 8067, 8247, 8391, 8499, 8707, 8859, 9003, 9043, 9111, 9147, 9183, 9327, 9463, 9507, 9651, 9903, 9939, 9967,</p>
	<p>547, 1747, 2683, 2707, 4243, 4447, 4987, 5419, 5683, 6079, 7207,</p>
	<p>2887, 3571, 4567, 8179, 8431,</p>
	<p>687, 2199, 4219, 4467, 6211, 6267, 7671, 8031, 8139, 8571, 8751, 9687,</p>
	<p>3279, 4287, 9363, 9543, 9759,</p>
	<p>1731, 3027, 6087,</p>
	<p>3387,</p>
	<p>3891,</p>

3-conjugacy classes graph	$m$ specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 3-conjugacy classes graph
$2 \dashrightarrow$	13, 37, 61, 109, 157, 181, 229, 277, 349, 373, 421, 541, 613, 661, 709, 733, 757, 829, 853, 877, 997, 1021, 1069, 1093, 1117, 1213, 1237, 1381, 1429, 1549, 1597, 1621, 1669, 1693, 1741, 1789, 1861, 2053, 2221, 2293, 2341, 2389, 2677, 2749, 2797, 2917, 3037, 3061, 3109, 3181, 3229, 3253, 3301, 3373, 3517, 3541, 3613, 3637, 3709, 3853, 4021, 4093, 4261, 4549, 4597, 4621, 4789, 4813, 4861, 4933, 4957, 5077, 5101, 5197, 5413, 5557, 5653, 5701, 5749, 5821, 5869, 6037, 6229, 6277, 6301, 6373, 6397, 6469, 6637, 6661, 6709, 6733, 6781, 6829, 6949, 6997, 7069, 7237, 7309, 7333, 7477, 7549, 7621, 7669, 7717, 7741, 7789, 8053, 8101, 8221, 8317, 8461, 8581, 8629, 8821, 8893, 8941, 9109, 9133, 9157, 9181, 9277, 9349, 9397, 9421, 9613, 9661, 9733, 9781, 9901, 9949, 9973,
$2 \dashrightarrow \sqcup \bigcirc$	73, 97, 193, 241, 313, 337, 409, 457, 577, 601, 769, 1009, 1033, 1129, 1153, 1201, 1249, 1297, 1489, 1609, 1753, 1801, 1993, 2017, 2089, 2113, 2137, 2161, 2281, 2377, 2473, 2617, 2689, 2713, 2833, 2857, 2953, 3001, 3049, 3121, 3217, 3313, 3361, 3433, 3457, 3529, 3697, 3793, 4057, 4129, 4153, 4177, 4201, 4273, 4441, 4729, 4969, 4993, 5113, 5281, 5449, 5521, 5569, 5641, 5689, 5737, 5881, 5953, 6121, 6217, 6361, 6481, 6529, 6577, 6673, 6841, 6961, 7129, 7297, 7393, 7417, 7489, 7537, 7561, 7753, 7873, 8161, 8209, 8233, 8329, 8353, 8377, 8521, 8689, 8713, 8761, 9001, 9049, 9337, 9433, 9649, 9697, 9721, 9769, 9817,
$2 \dashrightarrow \sqcup 2 \bigcirc$	21, 57, 93, 129, 201, 309, 381, 397, 399, 417, 453, 489, 597, 633, 651, 813, 849, 903, 921, 993, 1101, 1137, 1389, 1443, 1461, 1569, 1641, 1659, 1713, 1767, 1821, 1857, 1893, 1929, 1933, 2073, 2163, 2217, 2253, 2269, 2361, 2433, 2437, 2469, 2557, 2577, 2649, 2667, 2721, 2757, 2847, 2901, 2919, 2973, 3117, 3153, 3171, 3189, 3369, 3423, 3469, 3693, 3733, 3783, 3819, 3837, 3873, 3877, 3909, 3999, 4197, 4251, 4269, 4341, 4357, 4377, 4413, 4431, 4449, 4503, 4593, 4629, 4683, 4737, 4909, 4989, 5097, 5169, 5241, 5277, 5349, 5493, 5581, 5601, 5637, 5853, 5943, 5997, 6033, 6231, 6249, 6393, 6421, 6447, 6537, 6609, 6717, 6771, 6861, 6933, 6951, 7041, 7113, 7149, 7213, 7239, 7347, 7509, 7527, 7617, 7653, 7707, 7923, 7941, 7959, 8013, 8049, 8121, 8157, 8193, 8301, 8373, 8389, 8409, 8553, 8607, 8643, 8661, 8677, 8931, 9013, 9057, 9201, 9219, 9237, 9291, 9399, 9561, 9579, 9723, 9777, 9813, 9829, 9921, 9957, 9993,
$2 \dashrightarrow \sqcup 3 \bigcirc$	433, 673, 937, 1321, 1657, 2521, 2593, 3769, 3889, 4297, 4561, 5209, 5233, 6337, 6553, 6793, 7177, 7321, 7369, 7681, 7993, 8017, 8089, 8737, 8929, 9601,
$2 \dashrightarrow \sqcup 4 \bigcirc$	1453, 2029, 6133, 7573, 7933, 8269,
$2 \dashrightarrow \sqcup 5 \bigcirc$	1777, 1873, 3169, 4513, 4657, 5857, 7057, 9241,
$2 \dashrightarrow \sqcup 6 \bigcirc$	8293,
$2 \dashrightarrow \sqcup 7 \bigcirc$	3673, 4801, 6073, 8641,
$2 \dashrightarrow \sqcup 8 \bigcirc$	237, 669, 3513, 4701, 4881, 5437, 5961, 6429, 7977, 8913, 9489,
$2 \dashrightarrow \sqcup 14 \bigcirc$	1317, 1497, 2181, 3981,
$2 \dashrightarrow \sqcup 20 \bigcirc$	3261, 6753, 7401.



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