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A conjecture by Leon Ehrenpreis about zeroes of exponential polynomials

by
Alain Yger
yger@math.u-bordeaux.fr
Institute of Mathematics,
University of Bordeaux, 33405 Talence, France.

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Abstract

Leon Ehrenpreis proposed in his 1970 monograph *Fourier Analysis in several complex variables* the following conjecture : the zeroes of an exponential polynomial $\sum_0^M b_k(z)e^{i\alpha_k z}$, $b_k \in \overline{\mathbb{Q}}[X]$, $\alpha_k \in \overline{\mathbb{Q}} \cap \mathbb{R}$ are well separated with respect to the Paley-Wiener weight. Such a conjecture remains essentially open (besides some very peculiar situations). But it motivated various analytic developments carried by C.A. Berenstein and the author, in relation with the problem of deciding whether an ideal generated by Fourier transforms of differential delayed operators in n variables with algebraic constant coefficients, as well as algebraic delays, is closed or not in the Paley-Wiener algebra $\widehat{\mathcal{E}}(\mathbb{R}^n)$. In this survey, I present various analytic approaches to such a question, involving either the Schanuel-Ax formal conjecture, or $\mathcal{D}$-modules technics based on the use of Bernstein-Sato relations for several functions. Nevertheless, such methods fail to take into account the intrinsic rigidity which arises from arithmetic hypothesis : this is the reason why I also focus on the fact that Gevrey arithmetic methods that where introduced by Y. André to revisit the Lindemann-Weierstrass theorem, could also be understood as an indication for rigidity constraints for example in Ritt's factorisation theorem of exponential sums in one variable. The objective of this survey is to present the state of the art with respect to L. Ehrenpreis's conjecture, as well as to suggest how methods from transcendental number theory could be combined with analytic ideas, in order precisely to take into account such rigidity constraints inherent to arithmetics.

1 The conjecture, various formulations

In [36], page 322, Leon Ehrenpreis formulated the following conjecture.

Conjecture 1.1 (original form, incorrect) *If $F_1, ..., F_N$ are N exponential polynomials in n variables with purely imaginary algebraic frequencies, namely*

$$F_j(z_1, ..., z_n) = \sum_{k=0}^{M_j} b_{jk}(z)\, e^{i\langle \alpha_{jk}, z\rangle}\,,$$
$$b_{jk} \in \mathbb{C}[X_1, ..., X_n]\,,\ \alpha_{jk} \in \overline{\mathbb{Q}}^n \cap \mathbb{R}^n\,,\ j = 1, ..., N,$$

then the ideal $(F_1, ..., F_N)$ they generate in the Paley-Wiener algebra $\widehat{\mathcal{E}'(\mathbb{R}^n)}$ is slowly decreasing respect to the Paley-Wiener weight $p(z) = \log|z| + |\mathrm{Im}\, z|$. As a consequence[1], this ideal is closed in $\widehat{\mathcal{E}'(\mathbb{R}^n)}$. It coincides with the ideal $[I(F_1, ..., F_N)]_{\mathrm{loc}}$, which consists of elements in $\widehat{\mathcal{E}'(\mathbb{R}^n)}$ that belong locally to the ideal generated by $F_1, ..., F_N$ in the algebra of entire functions in n variables.

This conjecture, in a slightly modified form (see Conjecture 1.2), has been the inspiration for the joint work of C.A. Berenstein and the author since 1985. It is a challenging and fascinating question, one that is closely connected with other open questions in number theory and analytic geometry. In this note, I will point out many of these connections, detail some of the progress that has been made on the problem, and hopefully inspire others to continue the work.

As it stands, Conjecture 1.1 would imply, in the one variable setting, the following : *if*

$$f(z) = \sum_{k=0}^{M} b_k(z) e^{i\alpha_k z}\,, \quad b_k \in \mathbb{C}[X]\,, \quad \alpha_k \in \overline{\mathbb{Q}} \cap \mathbb{R} \tag{1.1}$$

is an exponential polynomial in one variable with algebraic frequencies and all simple zeroes, then the ideal (f, f') is a non proper ideal in $\widehat{\mathcal{E}'(\mathbb{R})}$ which would imply

$$|f(z)| + |f'(z)| \geq c\, \frac{e^{-A|\mathrm{Im}\, z|}}{(1 + |z|)^p} \tag{1.2}$$

for some $c, A > 0$ and $p \in \mathbb{N}$. Unfortunately, such an assertion is false if one does not set any condition of arithmetic nature on the polynomial coefficients b_k. Take for example

$$f(z) = f_\gamma(z) = \sin(z - \gamma) - \sin(\sqrt{2}(z - \gamma)),$$

[1] This follows from Theorem 11.2 in [36].

where $2\gamma/\pi$ has excellent approximations belonging to $(2\mathbb{Z}+1)\oplus\sqrt{2}(2\mathbb{Z}+1)$; then some zeroes of f_γ of the form

$$\frac{2l\pi}{1-\sqrt{2}}\,,\ l\in\mathbb{Z}\,,$$

will approach extremely well other zeroes of f_γ of the form

$$\frac{2\alpha+(2l'+1)\pi}{1+\sqrt{2}}\,,\ l'\in\mathbb{Z},$$

and thus the ideal (f_γ, f'_γ) fails to be closed in $\widehat{\mathcal{E}'(\mathbb{R})}$. So Conjecture 1.1 needs to be reformulated as follows.

Conjecture 1.2 (revised form) *If $F_1,...,F_N$ are exponential polynomials in n variables with both algebraic coefficients and purely imaginary algebraic frequencies, namely*

$$\begin{aligned} &F_j(z_1,...,z_n)=\sum_{k=0}^{M_j} b_{jk}(z)\,e^{i\langle\alpha_{jk},z\rangle}\,,\\ &b_{jk}\in\overline{\mathbb{Q}}[X_1,...,X_n]\,,\ \alpha_{jk}\in\overline{\mathbb{Q}}^n\cap\mathbb{R}^n\,,\ j=1,...,N, \end{aligned} \tag{1.3}$$

then the ideal $(F_1,...,F_N)$ they generate in the Paley-Wiener algebra $\widehat{\mathcal{E}'(\mathbb{R}^n)}$ is slowly decreasing respect to the Paley-Wiener weight $p(z)=\log|z|+|\mathrm{Im}\,z|$. As a consequence, this ideal is closed in $\widehat{\mathcal{E}'(\mathbb{R}^n)}$, and thus coincides with the set of elements in $\widehat{\mathcal{E}'(\mathbb{R}^n)}$ which belong locally to the ideal generated by $F_1,...,F_N$ in the algebra of entire functions in n variables.

Such a conjecture appears to be stronger than the following one.

Conjecture 1.3 (weaker revised form) *If $F_1,...,F_N$ are exponential polynomials in n variables as in Conjecture 1.2, namely*

$$\begin{aligned} &F_j(z_1,...,z_n)=\sum_{k=0}^{M_j} b_{jk}(z)\,e^{i\langle\alpha_{jk},z\rangle}\,,\\ &b_{jk}\in\overline{\mathbb{Q}}[X_1,...,X_n]\,,\ \alpha_{jk}\in\overline{\mathbb{Q}}^n\cap\mathbb{R}^n\,,\ j=1,...,N, \end{aligned}$$

then the the closure of the ideal $(F_1,...,F_N)$ they generate in the Paley-Wiener algebra $\widehat{\mathcal{E}'(\mathbb{R}^n)}$ coincides with the set of elements in $\widehat{\mathcal{E}'(\mathbb{R}^n)}$ which belong locally to the ideal generated by $F_1,...,F_N$ in the algebra of entire functions in n variables

The conjecture is equivalent to the assertion that the underlying system of difference-differential equations $\mu_1 * f = \cdots = \mu_N * f = 0$ satisfies the spectral synthesis property.

With C.A. Berenstein, we have been developing since [16] a long-term joint research program originally devoted to various attempts to tackle Conjecture 1.2. Such attempts lead to an approach based on multidimensional analytic residue theory that relies on techniques of analytic continuation in one or several complex variables. Conjecture 1.3 seems harder to deal with since it fits so well with the search for explicit division formulas in $\widehat{\mathcal{E}'(\mathbb{R}^n)}$ that resolve Ehrenpreis's fundamental principle as studied in [36]. (See also [18] or, more recently, [2]). What is known as Ehrenpreis-Montgomery conjecture is the particular case of Conjecture 1.2, when $n = 1$. Thanks to Ritt's theorem [52], Conjecture 1.2 in the case $n = 1$ reduces to the following.

Conjecture 1.4 (Ehrenpreis-Montgomery conjecture) *Let*

$$f(z) = \sum_{k=0}^{M} b_k(z)\, e^{i\alpha_k z}\,,\ \ b_k \in \overline{\mathbb{Q}}[X],\ \ \alpha_k \in \overline{\mathbb{Q}} \cap \mathbb{R} \tag{1.4}$$

be an exponential polynomial with both algebraic coefficients and frequencies. Then, there are constant $c, A > 0$, $p \in \mathbb{N}$ (depending on f) such that

$$\Big(f(z) = f(z') = 0 \text{ and } z \neq z'\Big) \Longrightarrow |z - z'| \geq c\,\frac{e^{-A|\mathrm{Im}\, z|}}{(1+|z|)^p}. \tag{1.5}$$

A possible reason for the terminology is the relation between Conjecture 1.4 and the following conjecture by H. Shapiro (1958) mentioned by H.L. Montgomery in a colloquium in Number Theory (Bolyai Janos ed.), see [57, 58].

Conjecture 1.5 (Montgomery-Shapiro conjecture) *Let f, g be two exponential polynomials that have an infinite number of common zeroes. Then, there is an exponential polynomial h that divides both f and g and has also an infinite number of zeroes.*

Unfortunately, I failed to find a precise reference in H. L. Montgomery's work. There seems to be an oral contribution by H. L. Montgomery linking Conjecture 1.4 and Conjecture 1.5. In 1973, Carlo Moreno, under the supervision of L. Ehrenpreis, quoted in the introduction of [48] an unpublished manuscript [47], which should be devoted to his work towards such a conjecture. His thesis (New York, 1971) was centered around it. The idea there was to prove Conjecture 1.4 for sums of exponentials (that is $b_k \in \overline{\mathbb{Q}}$ for any k), involving only a small number of exponential monomials. This is fundamentally different from the methods that arose later (see e.g. [16]), which depend on the rank of the subgroup $\Gamma(f)$ of the real line generated by the frequencies α_k.

2 What is known in connection with results in transcendental number theory

As mentioned in section 1, besides the approach by C. Moreno in his thesis, most of the attempts towards Conjecture 1.4 rely on an additional hypothesis on the rank of the additive subgroup $\Gamma(f)$ of $\overline{\mathbb{Q}}\cap\mathbb{R}$ generated by the frequencies $\alpha_0, ..., \alpha_M$, not on the number of monomials $e^{i\alpha_k z}$ involved.

An easy case when Conjecture 1.4 holds is the case where the rank of $\Gamma(f)$ equals 2, and the b_k are constant [39]. The result means in that case that the analytic transcendental curve

$$t \in \mathbb{C} \mapsto (e^{it}, e^{i\gamma_1 t}), \ \gamma_1 \in (\overline{\mathbb{Q}} \cap \mathbb{R}) \setminus \mathbb{Q},$$

cannot approach a finite subset in $\overline{\mathbb{Q}}^2$. Explicitly, any finite linear combination of logarithms of r algebraic numbers ($r = 3$ here) α_ι with degrees at most D, logarithmic heights at most h, and with integer coefficients ν_ι having absolute values less than B is either 0 or bounded from below in absolute value,

$$\Big|\sum_{\iota=1}^{r} \nu_\iota \log \alpha_\iota\Big| \geq \frac{1}{B^{c(r)\times D^{r+2}\log D\times h^r}}. \tag{2.6}$$

This is a well known fact originally due to A. Baker, see e.g. [9, 10] or ([59], section 4), for up-to-date results, references or conjectures. When the coefficients ν_ι are algebraic, with heights less than B, the following less explicit estimate continues to hold.

$$\Big|\sum_{\iota=1}^{r} \nu_\iota \log \alpha_\iota\Big| \geq \frac{1}{B^{c(r,D)\times h^{\kappa(r)}}} \tag{2.7}$$

for some constants $c(r, D)$ and $\kappa(r)$, D being the maximum of the degrees of the α_ι and ν_ι.

The next natural step would be to show that, if γ_1, γ_2 are two real algebraic numbers such that $(1, \gamma_1, \gamma_2)$ are $\mathbb{Q}$-linearly independent, the transcendental curve

$$t \in \mathbb{C} \mapsto (e^{it}, e^{i\gamma_1 t}, e^{i\gamma_2 t})$$

cannot approach an algebraic curve in $\mathbb{C}^3$ which is defined over $\overline{\mathbb{Q}}$; That is, the set of common zeroes of polynomials belonging to $\overline{\mathbb{Q}}[X_1, X_2, X_3]$. Here we are close to a quantified version of the so-called Schanuel's conjecture (see [59], section 4, for conjectures respect to its quantitative versions).

Conjecture 2.1 (Schanuel's conjecture, "numerical" version) *Given s complex numbers $y_1, ..., y_s$ which are $\overline{\mathbb{Q}}$-linearly independent, the transcendence degree of the algebraic extension $\overline{\mathbb{Q}}[y_1, ..., y_s, e^{y_1}, ..., e^{y_s}]$ over $\overline{\mathbb{Q}}$ is at least equal to s.*

For $s = 1$, this is Gel'fond-Schneider's theorem. The $s = 2$ case would imply for example the algebraic independence over $\mathbb{Q}$ of the pair of numbers (e, π) or $(\log 2, \log 3)$, and is of course still open. When γ is an algebraic number with degree $D \geq 2$ and ζ a complex number such that $e^{i\zeta} \neq 1$, a result by G. Diaz [34] asserts that, among the exponentials $e^{i\gamma\zeta}, ..., e^{i\gamma^{D-1}\zeta}$, at least $[(d+1)/2]$ are algebraically independent over $\mathbb{Q}$. This result covers Gel'fond's well known result ($D = 3$) and even leads to a quantitative version of it. In fact, the quantitative formulation obtained by D. Brownawell in [14] for $D = 3$ (using Gel'fond-Schneider's method) implies the following (rather weak) result respect to Conjecture 1.4, when the rank of $\Gamma(f)$ equals 3.

Proposition 2.1 ([15]) *If f is an exponential sum in one variable with $b_k \in \overline{\mathbb{Q}}$ and $\Gamma(f) = \mathbb{Z} \oplus \gamma\mathbb{Z} \oplus \gamma^2\mathbb{Z}$, γ being an irrational cubic, then, for any $\epsilon > 0$, there is $c_\epsilon > 0$ depending on f such that*

$$\Big(f(z) = f(z') = 0 \text{ and } z \neq z'\Big) \Longrightarrow |z - z'| \geq c_\epsilon e^{-|z|^{4+\epsilon}} \tag{2.8}$$

The methods introduced by Guy Diaz in [34] in fact allow one to replace $4+\epsilon$ by $1 + \epsilon$ in (2.8). In any case, we are indeed very far from what would be the formulation of Conjecture 1.4 in the particular case where b_k are constant and the algebraic frequencies belong to the group $\mathbb{Z} \oplus \gamma\mathbb{Z} \oplus \gamma^2\mathbb{Z}$, γ being an irrational cubic. This is inherent to the approach of the problem *via* classical methods in diophantine approximation.

Besides these cases and the results of C. Moreno in his unpublished 1971 thesis when the number of monomial terms is small, to my knowledge nothing is really known about Conjecture 1.4, at least in connexion with an approach based on transcendental number theory methods. For some up-to-date survey about Schanuel's conjecture and its quantitative versions, we refer the reader to ([59], sections 3.1 and 4.3).

3 Using the formal counterpart of Schanuel's numerical conjecture

The point of view I developed with C. A. Berenstein in [16] and section 2 of [15] relies on the fact that the formal analog of Schanuel's conjecture holds, despite the fact that very is known about the numerical Schanuel conjecture. This is a result by J. Ax and B. Coleman [6, 31], following the ideas developped by C. Chabauty [28] and E. Kolchin [40], see also [22] for a modern up-to-date presentation. Here is a formulation.

Theorem 3.1 (Schanuel's conjecture, formal version) *Let $y_1, ..., y_s$ be s formal power series in $\mathbb{C}[[t_1, ..., t_k]]$ ($k \geq 1$), and I an ideal in the polynomial*

ring $\mathbb{C}[X_1, ..., X_s, Y_1, ..., Y_s]$*, defining in* $\mathbb{C}^{2s}$ *an algebraic subvariety* $\mathcal{V}(I)$ *with dimension less or equal to* s*, such that*

$$\forall\, P \in I,\ \ P(y_1(t), ..., y_s(t), e^{y_1(t)}, ..., e^{y_s(t)}) \equiv 0.$$

Then, there are rational numbers $r_1, ..., r_s$ *and a complex number*[2] $\gamma \in \mathbb{C}$ *such that*

$$\sum_{j=1}^{s} r_j y_j(t) \equiv \gamma. \tag{3.9}$$

Here is a corollary of the last Theorem that shows the crucial role it plays when studying the *slowly decreasing conditions* introduced by Ehrenpreis (e.g. [36]) for ideals generated by exponential polynomials with frequencies in $(i\mathbb{Z})^n$. We ignore for the moment any condition of arithmetic type on the coefficients.

Corollary 3.1 ([16], Proposition 6.4 and Corollary 6.7) *Let* $P_1, ..., P_N$ *be* N *polynomials in the* $2n$ *variables* $(X_1, ..., X_n, Y_1, ..., Y_n)$*, defining an algebraic variety* $\mathcal{V}(P)$ *in* $\mathbb{C}^{2n}_{z,w}$*. Let* $\pi_z : (z, w) \in \mathbb{C}^{2n} \mapsto z$ *be the projection on the factor* $\mathbb{C}^n_z$*. Let* $W \subset \mathbb{C}^n_z$ *be the subset defined by*

$$(z_1, ..., z_n) \notin W \Longrightarrow \dim(\mathcal{V}(P) \cap \pi^{-1}(z)) = 0 \text{ or } -\infty.$$

Then, any irreducible component with strictly positive dimension of the analytic (transcendental) subset

$$V(F) = \{z \in \mathbb{C}^n\,;\, F_j(z) = P_j(z_1, ..., z_n, e^{iz_1}, ..., e^{iz_n}) = 0,\ j = 1, ..., N\}$$

lies in $\overline{W}$*. In particular, when* $N \geq n$*, any irreducible component with strictly positive dimension of* $V(F)$ *lies in the closure in* $\mathbb{C}^n$ *of the set* $W' \subset \mathbb{C}^n_z$ *defined as*

$$z \notin W' \Longrightarrow \operatorname{rank}\left[\Big(\frac{\partial P_j(z,w)}{\partial w_k}\Big)_{\substack{1\leq j\leq N\\ k\leq 1\leq n}}\right] = n \quad \forall\, w \in \mathbb{C}^n.$$

The formal analog of Schanuel's conjecture also allows one to give refined versions of Ritt's theorem in several variables such as those formulated in [8]. Here is an example.

[2]Unfortunately, even when one specifies arithmetic conditions on the ideal I, such as the generating polynomials have algebraic coefficients, nothing more precise can be asserted about the constant γ. Indeed, this is the main stumbling block to such a result being an efficient tool in proving Conjecture 1.2 or even Conjecture 1.4.

Corollary 3.2 ([16], see also [53]) *Let*

$$F(z_1, ..., z_n) = \sum_{k=0}^{M} b_k(z) e^{i\langle \alpha_k, z\rangle}$$

be an exponential polynomial in n complex variables which is identically zero on an algebraic irreducible curve $\mathcal{C}$. Then either all polynomial factors b_k vanish identically on $\mathcal{C}$ or else $\mathcal{C}$ is contained in some affine subspace $\langle \alpha_{k_1} - \alpha_{k_2}, z\rangle = \gamma$, where γ is a complex constant[3] *and $\alpha_{k_1} \neq \alpha_{k_2}$. If an irreducible polynomial $P \in \mathbb{C}[X_1, ..., X_n]$ divides F (as an entire function) without dividing all the b_k, then P is necessarily of the form*

$$P(X) = \langle \alpha_{k_1} - \alpha_{k_2}, X\rangle - \gamma.$$

The main reason such analytic techniques arising from the formal analog of Schanuel's conjecture fail to imply Conjecture 1.2 (or more specifically Conjecture 1.4), is because they do not allow one to keep track of the arithmetic constraints. Though such a goal can be (partially) achieved when adapting Nœther Normalization's lemma to the frame of exponential polynomials $P(X_1, ..., X_n, e^{Y_1}, ..., e^{Y_n})$ (as in Proposition 6.3 in [16]), it still seems far from providing enough information to make significant advances towards Conjectures 1.2 or 1.4.

4 Arithmetic rigidity and the $\mathcal{D}$-module approach

4.1 Lindemann-Weierstrass theorem *versus* Ritt's factorization

The ubiquity that was pointed out in [4, 5] with respect to the well known Lindemann-Weierstrass theorem suggests how arithmetic rigidity is reflected in Ritt's factorization of exponential sums in the one variable setting. Let us recall the classical "numerical" formulation of Lindemann-Weierstrass theorem.

Theorem 4.1 (Lindemann-Weierstrass, "numerical" formulation) *Let $\alpha_1, ..., \alpha_s$ be s algebraic numbers which are $\mathbb{Q}$-linearly independent. Then their exponentials $e^{\alpha_1}, ..., e^{\alpha_s}$ are algebraically independent over $\overline{\mathbb{Q}}$.*

Here is its equivalent "functional" formulation, which appears to be an arithmetic version of Ritt's factorization theorem. In this situation, arithmetic conditions indeed impose drastic rigidity constraints.

[3] Here again, additional arithmetic information on F does not impose any arithmetic constraint on γ.

Theorem 4.2 (Lindemann-Weierstrass, "functional formulation") *Let $\mathfrak{f}$ be a formal power series in $\mathbb{Q}[[X]]$, which corresponds to the Taylor development about the origin of an exponential polynomial f with constant coefficients*[4]*, such that $f(1) = 0$, that is f can be divided by $z - 1$ as an entire function. Then the quotient*

$$z \mapsto \frac{\mathfrak{f}(X)}{X-1}$$

is also the formal power series at the origin of an exponential polynomial with constant coefficients[5]*.*

4.2 A first ingredient for the proof of Theorem 4.2 : the notion of "size" for a Xd/dX-module over $\mathbb{K}(X)$

One of the major ingredients in the "modern" proof ([4, 5]) of Theorem 4.2 is the notion of "*being of finite size*" for a Xd/dX module over $\mathbb{K}(X)$, where $\mathbb{K}$ is a number field. We keep for the moment to the one variable setting.

Let $\mathbb{K}$ be such a number field, and $\mathcal{M}$ be a Xd/dX-module over $\mathbb{K}(X)$. Assume $\mathcal{M}$ is such that the $\mathbb{K}(X)$ induced module is free with finite rank[6]. Thus, $\mathcal{M}$ can be represented in terms of a basis $\Upsilon = (\upsilon_0, ..., \upsilon_{\mu-1})$ with the action of the differential operator Xd/dX being represented as

$$(Xd/dX)[\upsilon_j] = \sum_{k=0}^{\mu-1} G_{jk}(X)[\upsilon_k].$$

Taking into account the fact that $\mathbb{K}$ is a number field (and thus the arithmetic rigidity), one can introduce a notion of *size* $\sigma(\mathcal{M})$ as

$$\sigma(\mathcal{M}) = \limsup_{N\to\infty} \frac{1}{N} \sum_{v\in\Sigma_{\text{finies}}(K)} \log^+ \max_{0\le p\le N} \left\| \frac{G^{(p)}(X)}{p!} \right\|_v , \tag{4.10}$$

where Σ_{finies} denotes the set of non archimedian (conveniently normalized) absolute values on the number field $\mathbb{K}$, and G^p is the (μ, μ) matrix with entries in $\mathbb{K}(X)$, corresponding to the action of $X^p(d/dX)^p$, expressed within the basis Υ (see for example [3, 33]). The size is in fact independent of the choice of the basis Υ. The module $\mathcal{M}$ is said to satisfy the *Galochkin condition* when its size $\sigma(\mathcal{M})$ is finite.

[4]Certainly, the coefficients and frequencies of such an exponential polynomial f are in $\overline{\mathbb{Q}}$.

[5]That is, of course, is identically zero. Nevertheless, it seems better to keep this formulation to view the statement as the effect of arithmetic rigidity constraints in Ritt's factorization theorem.

[6]More generally, one may replace $\mathbb{K}(X)$ by some unitary $\mathbb{K}$-algebra containing $\mathbb{K}(X)$, such as $\mathbb{K}[[X]]$, and introduce then the notion of Xd/dX-module of finite type over $\mathbb{K}[[X]]$.

An important result by G. Chudnovsky [29, 30], one that relies on Siegel's lemma[7], asserts that, if A is a (μ, μ) matrix with coefficients in $\mathbb{K}[X]$ such that the differential system

$$(d/dX - A)[Y] = 0 \tag{4.11}$$

admits a solution Y_0 in $(\mathbb{K}[[X]])^\mu$ with $\mathbb{K}(X)$-linearly independent components, then the *size* of the corresponding Xd/dX-module $\mathcal{M}_A$ is bounded from above by $C(\Gamma)\, h(Y_0)$, where $h(Y_0)$ denotes the maximum of the heights of the coefficients of Y_0, the height being understood here as the height of a formal power series with coefficients in $\mathbb{K}$ (see [3]). In particular, $\mathcal{M}_A$ satisfies the Galochkin condition when the differential system admits a solution with $\mathbb{K}(X)$-linearly independent components, which are all G-functions (see [3] for various definitions[8] of such an arithmetic notion). Note that G. Chudnovsky's theorem has been extended to the several variable context by L. di Vizio in [32].

4.3 A second ingredient for the proof of Theorem 4.2 : a theorem by N. Katz

Here again, one keeps to the one variable context. A differential operator with coefficients in $\mathcal{M}_{\mu,\mu}(\mathbb{C}[X])$

$$\mathcal{L} = \sum_0^L A_l(X)(d/dX)^q,$$

it is called *fuschian* if all its singularities $a \in \mathbb{C} \cup \{\infty\}$ are *regular* ones. That is, are such that

$$\min_{l<L}(\mathrm{val}_a(A_l) - l) \geq \mathrm{val}_a(A_L) - L.$$

A theorem by N. Katz [44] asserts that any Xd/dX-module over $\mathbb{K}(X)$ ($\mathbb{K}$ being a number field) which satisfies Galochkin condition is necessarily fuschian.

This result has also an extension to the context of several variables ([32]). Such an extension can be combined with Chudnovsky's theorem in higher dimension, as formulated in geometric terms also in ([32]).

The proof of Theorem 4.2 ([5]) follows from such a combination between Chudnovsky's and Katz's theorems. It relies on the elementary proof proposed in [23], which bypasses the p-adic methods based on the Bézivin-Robba criterion that were previously introduced in [24].

[7]See e.g. [33], Chapter VIII, for a pedestrian presentation and a proof.

[8]To say it briefly, a G-function is a formal power series in $\overline{\mathbb{Q}}[[X]]$ which is in the kernel of some element in $\overline{\mathbb{Q}}[X, d/dX]$ and, at the same time, has a finite logarithmic height, when considered as a power series in $\overline{\mathbb{Q}}[[X]]$ (see [3] for the notion of logarithmic height for a power series).

4.4 The $\mathcal{D}$-modules approach

Let us start here with a few observations about division questions in multivariate complex analysis. This approach is reminiscent of pseudo-Wiener deconvolution methods that involve as deconvolutors filters with transfer functions

$$\omega \in \mathbb{R}^n \longmapsto \frac{\overline{F_j(\omega)}}{\|F(\omega)\|^2 + \epsilon^2},$$

where the F_j, $j = 1, ..., N$, are the transfer functions of the convolutor filters, and $\epsilon^2 << 1$ stands here for a signal to noise ratio.

Let $F_1, ..., F_N$ be N elements in the Paley-Wiener algebra $\widehat{\mathcal{E}'(\mathbb{R}^n)}$. Consider the holomorphic map $z \mapsto F(z) := (F_1(z), ..., F_N(z))$ as an holomorphic section of the trivial bundle $\mathbb{C}^n \times \mathbb{C}^N \to \mathbb{C}^n$, equipped with its canonical basis. Let

$$\sigma(z) = \frac{\sum_{j=1}^{N} \overline{F_j(z)} \otimes e_j}{\|F(z)\|^2}, \quad z \in \mathbb{C}^n \setminus F^{-1}(0).$$

It can be shown that there are bundle-valued currents P_F and R_F in $\mathbb{C}^n$ defined by the formulas

$$\begin{aligned} P_F &:= \Big[\|F(z)\|^{2\lambda} \sum_{r=1}^{n} \frac{\sigma(z) \wedge (\overline{\partial}[\sigma(z)])^{r-1}}{(2i\pi)^r}\Big]_{\lambda=0} \\ R_F &:= \Big[\overline{\partial}\,[\|F(z)\|^{2\lambda}] \wedge \sum_{r=1}^{n} \frac{\sigma(z) \wedge (\overline{\partial}\,[\sigma(z)])^{r-1}}{(2i\pi)^r}\Big]_{\lambda=0}. \end{aligned} \tag{4.12}$$

That is, one analytically continues the complex parameter λ from $\{\mathrm{Re}\,\lambda >> 1\}$ to some half-plane $\{\mathrm{Re}\,\lambda > -\eta\}$ for some $\eta > 0$. Note that $\mathrm{Supp}\, R_F \subset F^{-1}(0)$ and that P_F and R_F are related by $((2i\pi)\rfloor_F - \overline{\partial}) \circ P_F = 1 - R_F$, where $\rfloor_F$ denotes the interior product with F.

In order to justify such a construction, one takes a log resolution $\pi : \widetilde{\mathbb{C}^n} \to \mathbb{C}^n$ for the subvariety $\{F_1 = \cdots = F_N = 0\}$. Such a log resolution factorizes through the normalized blow-up of $\mathbb{C}^n$ along the coherent ideal sheaf $(F_1, ..., F_N)\,\mathcal{O}_{\mathbb{C}^n}$. When $N \leq n$ and $F_1, ..., F_N$ define a complete intersection in $\mathbb{C}^n$, the current R_F reduces to its $(0, N)$ component, which coincides in this case with the current realized in a neighborhood of $\bigcup_1^N F_j^{-1}(0)$ as the value at $\lambda_1 = \cdots = \lambda_N = 0$ of the analytically continued current-valued holomorphic map

$$(\lambda_1, ..., \lambda_N) \in \{\mathrm{Re}\,\lambda_1 >> 1, ..., \mathrm{Re}\,\lambda_N >> 1\} \longmapsto \frac{1}{(2i\pi)^N} \bigwedge_{j=N}^{1} \overline{\partial}\Big(\frac{|F_j|^{2\lambda_j}}{F_j}\Big). \tag{4.13}$$

When $F_1, ..., F_N$ are polynomials (that is Fourier transforms of distributions with support $\{0\}$), all distribution coefficients of the current P_F belong to $\mathcal{S}'(\mathbb{C}^n \simeq \mathbb{R}^{2n})$, in which case the ideal $(F_1, ..., F_N)$ is of course closed in the Paley-Wiener algebra. The current P_F is said *to have Paley-Wiener growth* in $\mathbb{C}^n$ if and only if all its distribution coefficients T satisfy the weaker condition

$$\begin{aligned} &\exists\, p \in \mathbb{N},\ , \exists\, A > 0, \exists\, C > 0, \quad \text{such that} \\ &|\langle T, \varphi\rangle| \leq C \sup_{|\underline{l}|+|\underline{m}|\leq p} \sup_{\mathbb{C}^n} \left[(1+\|z\|)^p e^{A\|\mathrm{Im}\, z\|} \left| \frac{\partial^{\underline{l}+\underline{m}}\,[\varphi]}{\partial \zeta^{\underline{l}}\, \partial \overline{\zeta}^{\underline{m}}}\,(z) \right| \right]. \end{aligned} \tag{4.14}$$

If P_F has Paley-Wiener growth, so has R_F, since $((2i\pi)\rfloor_F - \overline{\partial}) \circ P_F = 1 - R_F$. Division methods such as developped in [19, 20, 1, 2], show that, if P_F (hence R_F) has Paley-Wiener growth in $\mathbb{C}^n$,

$$\Big([I(F_1, ..., F_N)]_{\mathrm{loc}} \Big)^{\min(n,N)} \subset I(F_1, ..., F_N). \tag{4.15}$$

In the particular case where $N \leq n$ and $(F_1, ..., F_N)$ define a complete intersection in $\mathbb{C}^n$, the fact that P_F (hence R_F) has Paley-Wiener growth in $\mathbb{C}^n$ implies that $I(F_1, ..., F_N)$ is closed in the Paley-Wiener algebra (one can replace the exponent $\min(n, N)$ by 1 in (4.15)). When $(F_1, ..., F_N)$ have no common zeroes in $\mathbb{C}^n$, it is therefore equivalent to say that $I(F_1, ..., F_N)$ is closed in the Paley-Wiener algebra or to say that P_F has Paley-Wiener growth (here $R_F \equiv 0$ since $F^{-1}(0) = \emptyset$). Conjecture 1.2 suggests then the following conjecture.

Conjecture 4.1 *Let $F_1, ..., F_N$ be N exponential polynomials such as in Conjecture 1.2. The current P_F (hence also R_F) has Paley-Wiener growth.*

Remark 4.1 *Conjecture 4.1 implies Conjecture 1.4 : when $n = 1$, take N large enough and $F_1, ..., F_N$ the list of successive derivatives of the exponential polynomial $f\ : z \mapsto \sum_{k=0}^{M} b_k(z)\, e^{i\alpha_k z}$ (see e.g. [15]).*

In order to rephrase Conjecture 4.1 in more algebraic terms, let us recall the following trick. If $\mathrm{Re}\,\beta > 0$, and $t_1, \ldots, t_N$ are N strictly positive numbers, then one has, for any $(\gamma_1, \ldots, \gamma_{N-1}) \in]0, \infty[^{N-1}$ such that $\gamma_1 + \cdots + \gamma_{N-1} < \mathrm{Re}\,\beta$,

$$\begin{aligned} &(t_1 + \cdots + t_N)^{-\beta} \\ &= \frac{1}{(2i\pi)^{N-1}\Gamma(\beta)} \int_{\gamma_1 + i\mathbb{R}} \cdots \int_{\gamma_{N-1}+i\mathbb{R}} \Gamma_N^*(\zeta)\, t_1^{-\zeta_1} \cdots t_{N-1}^{-\zeta_{N-1}} t_N^{\zeta^*}\, d\zeta_1 \cdots d\zeta_{N-1}, \end{aligned} \tag{4.16}$$

where

$$\Gamma_N^*(\zeta) = \Gamma(\zeta_1) \cdots \Gamma(\zeta_{N-1}) \Gamma(\beta - \zeta_1 - \cdots - \zeta_{N-1})\,,\ \zeta^* = \sum_{k=1}^{N-1} \zeta_k - \beta\,.$$

Formula (4.16) allows the transformation of the *additive* operation between the t_j (namely $(t_1+\cdots+t_N)^{-\beta}$) into a *multiplicative* one (namely $t_1^{-\zeta_1}\cdots t_{N-1}^{-\zeta_{N-1}}t_N^{\zeta^*}$, once in the integrand). One can view it as a continuous version of the binomial formula (with negative exponent). Taking for example $t_j = |F_j(z)|^2$, $j = 1, ..., N$, it follows that one way then to tackle Conjecture 4.1 could be to study (first formally, then numerically in $\mathbb{C}^n$, pairing antiholomorphic coordinates with holomorphic ones in order to recover positivity) the analytic continuation of

$$\lambda = (\lambda_1, ..., \lambda_N) \longmapsto \prod_{j=1}^{N} (F_j(z_1, ..., z_n))^{\lambda_j}. \tag{4.17}$$

When $F_1, ..., F_N$ are polynomials in $\mathbb{K}[X_1, ..., X_n] = \mathbb{K}[X]$, where $\mathbb{K}$ is a number field, one may consider the $\mathbb{K}(\lambda)\langle X, d/dX\rangle$-module $\mathfrak{M}(F)$ freely generated by a single generator (formally denoted as $\mathfrak{F}^\lambda = \mathfrak{F}_1^{\lambda_1} \otimes \cdots \otimes \mathfrak{F}_N^{\lambda_N}$), namely

$$\mathfrak{M}(F) = \mathbb{K}(\lambda)[X]\left[\frac{1}{F_1}, ..., \frac{1}{F_N}\right] \cdot \mathfrak{F}^\lambda.$$

This $\mathbb{K}(\lambda)\langle X, d/dX\rangle$-module is holonomic (i.e. $\dim \mathfrak{M}(F) = n$). A noetherianity argument (see e.g. [35]) implies then that there exists a set of *global Bernstein-Sato algebraic relations*

$$\mathcal{Q}_j(\lambda, X, d/dX)\big[F_j \cdot \mathfrak{F}^\lambda\big] = \mathcal{B}(\lambda) \cdot \mathfrak{F}^\lambda,\ j = 1, ..., N, \tag{4.18}$$

where $\mathcal{B} \in \mathbb{K}[\lambda]$ and $\mathcal{Q}_j \in \mathbb{K}[\lambda]\langle X, d/dX\rangle$, $j = 1, ..., N$. Such a set of algebraic relations (4.18) can be used in order to express (via (4.16) with $t_j = |F_j(z)|^2$, $t_=1, ..., N$) the current P_F as a current with coefficients in $\mathcal{S}'(\mathbb{C}^n)$.

Local analytic analogs of global Bernstein-Sato algebraic relations (4.18) indeed exist. When $f_1, ..., f_N$ are N elements in $\mathcal{O}_{\mathbb{C}^n,0}$ and $\mathfrak{t}$ is an holonomic distribution about the origin in $\mathbb{C}^n$ (for example, a distribution coefficient of some integration current $[V]$, or of some Coleff-Herrera current, see [27]), then there exists a set of *local Bernstein-Sato analytic equations*

$$q_{\mathfrak{t},j}(\lambda, \zeta, \partial/\partial\zeta)\big[f_j \cdot \mathfrak{f}^\lambda \otimes \mathfrak{t}\big] = b_{\mathfrak{t}}(\lambda) \cdot \mathfrak{f}^\lambda \otimes \mathfrak{t},\ j = 1, ..., N, \tag{4.19}$$

where $q_{\mathfrak{t},j}$ denotes a germ at the origin of a holomorphic differential operator with coefficients analytic in ζ and polynomial in λ, and $b_{\mathfrak{t}}$ is a finite product of affine forms $\kappa_0 + \kappa_1\lambda_1 + \cdots + \kappa_n\lambda_n$, with $\kappa_0 \in \mathbb{N}^*$, $(\kappa_1, ..., \kappa_M) \in \mathbb{N}^M \setminus \{0\}$ ([25, 55, 42, 26]). Unfortunately, such a local result does not provide any algebraic information about the $q_{\mathfrak{t},j}$, when for example the f_j's represent the germs at the origin of exponential polynomials of the form (1.3), as in Conjecture 1.2 or Conjecture 1.3.

One intermediate way to proceed in this case is to consider the case of formal power series. For example, let us suggest an approach to tackle Conjecture

1.4 for exponential sums. Consider an exponential sum

$$f\ : \zeta \in \mathbb{C} \longmapsto \sum_{k=0}^{M} b_k e^{i\alpha_k \zeta},$$

with algebraic coefficients b_k, and purely imaginary algebraic distinct frequencies $i\alpha_k$. Let $\mathbb{K}$ be the number field generated by the b_k's, the α_k's, and i. Let $n \geq 1$ be the rank of the subgroup $\Gamma(f) = \mathbb{Z}\alpha_0 + \cdots + \mathbb{Z}\alpha_M$, and $(\gamma_1, ..., \gamma_n)$ be a basis of $\Gamma(f)$. For each $j = 1, ..., M$, let $P_j \in \mathbb{K}[X_1, ..., X_n]$ such that

$$\frac{d^{j-1} f}{d\zeta^{j-1}}(z) = P_j(e^{i\gamma_1 z}, ..., e^{i\gamma_n z}), \quad \forall\, z \in \mathbb{C},$$

and $P := (P_1, ..., P_M)\ : \mathbb{C}^n \to \mathbb{C}^M$. Let $N = M + n - 1$, and the exponential polynomials $F_1, ..., F_N$ be defined as follows:

- for $j = 1, ..., M$, F_j is the exponential sum in n variables, with coefficients in $\mathbb{K}$,
$$(z_1, ..., z_n) \longmapsto F_j(z) = P_j(e^{iz_1}, ..., e^{iz_n})\ ;$$
- for $j = 1, ..., n-1$, F_{M+j} is the linear form, also with coefficients in $\mathbb{K}$,
$$(z_1, ..., z_n) \longmapsto \gamma_n\, z_j - \gamma_j\, z_n.$$

Let ξ be a point in $\mathbb{C}^n$, such that $e^{i\xi} \in \mathbb{K}^n \cap \{P = 0\}$. The Taylor developments of $F_1, ..., F_M$ at ξ correspond to power series $\mathfrak{f}_{1,\xi}, ..., \mathfrak{f}_{M,\xi}$ in $\mathbb{K}[[X_1, ..., X_n]]$, while the Taylor developpements at ξ of $F_{M+1}, ..., F_N$ correspond to the affine power series

$$\mathfrak{f}_{M+j,\xi}\ : X = (X_1, ..., X_n) \longmapsto \mathfrak{u}_j + (\gamma_n\, X_j - \gamma_j X_n),\ j = 1, ..., n-1,$$

where $\mathfrak{u}_j = \gamma_n\, \xi_j - \gamma_j \xi_n$ is a linear combination of logarithms of algebraic numbers with algebraic coefficients. Here $\mathfrak{u}_1, ..., \mathfrak{u}_{n-1}$ can be interpreted as parameters. Inspired by [11], one could conjecture[9] the existence of a set of global formal generic Bernstein-Sato relations:

$$\mathfrak{Q}_{\xi,j}(\lambda, X, \mathfrak{u}_1, ..., \mathfrak{u}_{n-1}, d/dX)\left[\mathfrak{f}_{j,\xi} \cdot \mathfrak{F}_\xi^\lambda\right] = \mathfrak{g}_\xi(\mathfrak{u}_1, ..., \mathfrak{u}_{n-1})\, \mathfrak{b}_\xi(\lambda) \cdot \mathfrak{F}_\xi^\lambda,\ j = 1, ..., N, \tag{4.20}$$

where $\mathfrak{F}_\xi^\lambda = \mathfrak{f}_{1,\xi}^{\lambda_1} \otimes \ldots \mathfrak{f}_{N,\xi}^{\lambda_N}$, $\mathfrak{Q}_{\xi,j}$ is a differential operator with coefficients in $\mathbb{K}[\lambda]\,[[\mathfrak{u}, X]]$, $\mathfrak{g}_\xi \in \mathbb{K}[[\mathfrak{u}]]$, $\mathfrak{b}_\xi \in \mathbb{K}[\lambda]$. Moreover, an argument based on Siegel's method (and principle), as that developed by L. Ehrenpreis[10] in [37], could be then used in order to ensure then that the formal power series coefficients

[9] The lines which follow intend just to sketch what could be a conjectural approach to Conjecture 1.4 for exponential sums f such that $\Gamma(f)$ has small rank.

[10] Note that this work of L. Ehrenpreis appeared in the Lecture Notes volume where appeared also the important results by G. Chudnovsky [29, 30].

(in $X, \mathfrak{u}$) of the $\mathfrak{Q}_j$ (considered as polynomials in λ and d/dX) have indeed a radius of convergence which is bounded from below by $\rho > 0$, independently of ξ, provided $e^{i\xi}$ belongs to a compact subset of $(\mathbb{C}^*)^n$. Then (4.20) would provide a *semi-global Bernstein-Sato set of relations*. The results quoted in section 4, which rely on Siegel's lemma (see e.g. the proof of Chudnovsky's theorem in [33], or the approach to Gelfand-Shidlovsky theorem as in [21]) give indeed some credit to the conjectural existence of such a collection (indexed by ξ, with $e^{i\xi} \in \mathbb{K}^n \cap P^{-1}(0)$) of Berntein-Sato sets of semi-global relations $\mathcal{B}_\xi$ as (4.20). One could then identify terms with lower degree in $\mathfrak{u}$ in (4.20) and thus assume, in each set of relations $\mathcal{B}_\xi$ such as (4.20), that $\mathfrak{g}_\xi$ is homogeneous in $\mathfrak{u}$. In the particular case $n = 3$ (where we recall almost nothing is known concerning Conjecture 1.4, see section 2), one could thus assume that $\mathfrak{g}_\xi$ factorizes as a product of linear factors $\beta_{\xi,1}\mathfrak{u}_1 + \beta_{\xi,2}\mathfrak{u}_2$, where $\beta_{\xi,1}$ and $\beta_{\xi,2}$ belong to $\mathbb{K}$. Combining this with A. Baker's theorem (take $(\mathfrak{u}_1, \mathfrak{u}_2) = (\log \xi_1 + 2ik_1\pi, \log \xi_2 + 2ik_2\pi)$, $(k_1, k_2) \in \mathbb{Z}^2$), one would get (with (4.20)) some way to control the analytic continuation procedure (4.17), leading to the conjectural lower estimates

$$\sum_{j=1}^{M} |P_j(e^{\gamma_1 z}, ..., e^{\gamma_n z})| = \sum_{1}^{M} \Big| \frac{d^{j-1} f}{d\zeta^{j-1}}(z) \Big| \geq c\, \frac{e^{-A|\mathrm{Im}\, z|}}{(1+|z|)^p},$$

that ensure (1.5) (see [15]).

The conjectural approach proposed above can be seen as an attempt to take into account the intrinsic arithmetic rigidity of such problems that the results quoted in section 4 suggest.

Another approach, one that would seem more direct, would be to try to mimic the algebraic construction that leads to the construction of a global set of Bernstein-Sato relations such as (4.18) when $F_1, ..., F_N$ belong to $\mathbb{K}[X_1, ..., X_n]$. That is, let $F_1, ..., F_N$ be N exponential polynomials of the form

$$F_j(z) = P_j(z_1, ..., z_n, e^{i\,\gamma_{1,1}\,z_1}, ..., e^{i\,\gamma_{1,N_1}\,z_1}, ..., e^{i\,\gamma_{n,1}\,z_n}, ..., e^{i\,\gamma_{n,N_n}\,z_n}),$$
$$j = 1, ..., N,$$

where $P_j \in \mathbb{K}[X_1, ..., X_n, Y_{1,1}, ..., Y_{1,N_1}, ..., Y_{n,1}, ..., Y_{n,N}]$, the $\gamma_{j,k}$ being also elements in $\mathbb{K}$ such that $\gamma_{j,1}, ..., \gamma_{j,N_j}$ are $\mathbb{Q}$-linearly independent for any $j = 1, ..., n$. Instead of the Weyl algebra $\mathbb{K}(\lambda)\langle X, d/dX\rangle$, one could introduce a non commutative algebra such as

$$\mathbb{K}(\lambda_1, ..., \lambda_n)\Big\langle X_1, ..., X_n, Y_{1,1}, ..., Y_{1,N_1}, ..., Y_{n,1}, ..., Y_{n,N_n}, \partial_1, ..., \partial_n \Big\rangle,$$

with the following commutation rules: for any $j, k \in \{1, ..., n\}$, for any $l \in \{1, ..., N_j\}$,

$$[\partial_k, X_j] = -\delta_{jk}\,, \quad [X_k, Y_{j,l}] = 0\,, \quad [\partial_k, Y_{j,l}] = -\,\gamma_{j,l}\,\delta_{kl}\,Y_{j,l}.$$

One may consider, as in the Weyl algebra case, the $\mathbb{K}(\lambda)\langle X, Y, \partial\rangle$-module

$$\mathfrak{M}(F) = \mathbb{K}(\lambda)[X, Y, \partial]\left[\frac{1}{F_1}, ..., \frac{1}{F_N}\right] \cdot \mathfrak{F}^{\lambda}.$$

Nœtheriannity arguments based on the concept of *dimension*[11] for such a module lead (inspired by the argument described by F. Ehlers in [35]) to the existence, in some very particular cases, of what would be a substitute for a set of global Bernstein-Sato relations such as (4.18) (see [18]). Unfortunately, the results obtained here cover only situations basically quite close of that of Conjecture 1.4 when $\operatorname{rank}\Gamma(f) \leq 2$. Here are the results obtained that way :

- the current P_F attached to any system $F = (F_1, ..., F_N)$, $F_j(z_1, ..., z_n) = P_j(z_1, ..., z_n, e^{i\,z_n})$, $j = 1, ..., N$, where $P_j \in \mathbb{C}[X_1, ..., X_n, Y]$, has Paley-Wiener growth in $\mathbb{C}^n$;
- the current P_F attached to any system $F = (F_1, ..., F_N)$, $F_j(z_1, ..., z_n) = P_j(z_1, ..., z_{n-1}, e^{i\,z_n}, e^{i\,\gamma\,z_n})$, $j = 1, ..., N$, where the polynomials P_j belong to $\overline{\mathbb{Q}}[X_1, ..., X_{n-1}, Y_1, Y_2]$ and $\gamma \in (\overline{\mathbb{Q}} \cap \mathbb{R}) \setminus \mathbb{Q}$, has Paley-Wiener growth in $\mathbb{C}^n$.

Note that only the second situation carries an arithmetic structure. The methods developed in [18] failed, at least for their intended purpose of making progress towards Conjectures 1.2 or even 1.4. For example, they do not seem to be of any help towards Conjecture 1.4, when $\operatorname{rank}(\Gamma(f)) = 2$ and f is a true exponential polynomial (not an exponential sum). The main reason for such a failure is that these methods take into account only the concept of *dimension*, and ignore that of *logarithmic size*. On the other hand, the conjectural approach towards Conjecture 1.4 when $\operatorname{rank}\Gamma(f) = 3$ (such as sketched above) was taking into account such concepts, basically through Siegel's lemma. It is natural to ask the following question: can some argument based on a filtration with respect to the size lead to what would be a substitute for a set of global Bernstein-Sato relations such as (4.18) or (4.20) ? That would indeed provide a decisive step towards all conjectures mentioned here.

5 Some other miscellaneous approaches

This paper is intended to give brief, up-to-date discussions of the fascinating conjectures arising from arithmetic considerations added to L. Ehrenpreis's contributions to the study of the "*slowly decreasing condition*" in the Paley-Wiener algebra. One should add that recent developments in amœba theory [49, 50, 43], in relation with tropical geometry, might also be of some interest for such conjectures. Unfortunately, they usually are more adapted to the

[11] That is on concepts of algebraic, not really arithmetic, nature, though arihmetics is deeply involved.

case of complex frequencies[12] than to the most delicate so-called "neutral case" where all frequencies are purely imaginary as in the questions discussed here. The most serious stumbling block is that, from the combinatorics point of view, when dealing with "algebraic" cones in $\mathbb{R}^n$, one is missing Gordon's lemma. One needs then to bypass such a difficulty; see, for example, [12] for the construction of toric varieties associated to non rational fans. In this connection, we mention some references that might inspire ideas for deciding such conjectures about exponential sums [38, 43, 41, 45, 46, 49, 50, 51, 54, 56]. Unfortunately, most of them do not really take into account the arithmetic constraints, and are more in the spirit of C. Moreno's papers [47, 48].

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