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► **To cite this version:**

Nathalie Saint-Geours, Christian Lavergne, Jean-Stéphane Bailly, Frédéric Grelot. Change of support effects in spatial variance-based sensitivity analysis. *Mathematical Geosciences*, 2012, 44 (8), pp.945-958. 10.1007/s11004-012-9406-5 . hal-00618017v1

**HAL Id: hal-00618017**

**<https://hal.science/hal-00618017v1>**

Submitted on 1 Dec 2011 (v1), last revised 6 Dec 2012 (v2)

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1 **Change of support effects in spatial variance-based sensitivity anal-**  
2 **ysis**<sup>1</sup>

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18 **Abstract** Variance-based global sensitivity analysis (GSA) aims at study-  
19 ing how uncertainty in the output of a model can be apportioned to different  
20 sources of uncertainty in its inputs. GSA is an essential ingredient in model  
21 building: it helps to identify model inputs that account for most of model  
22 output variability. Yet this approach is not really appropriate for spatial  
23 models, as it cannot describe how uncertainty interacts with another key  
24 issue in spatial modeling: the issue of model upscaling and change of spa-  
25 tial support. In many environmental models, the end-user is interested in  
26 the spatial average or sum of model output over a given spatial unit (e.g.  
27 the average porosity of a geological block). Under a change of spatial sup-  
28 port, the relative contribution of uncertain model inputs to the variance of  
29 aggregated model output may change. We propose in this paper a simple for-  
30 malism to discuss this question within GSA framework by defining *point* and  
31 *block* sensitivity indices. We show that the relative contribution of an un-  
32 certain spatially distributed model input increases with its covariance range  
33 and decreases with the size of the spatial unit considered for model output  
34 aggregation. Results are briefly illustrated by a simple example.

35 **Keywords** Sensitivity analysis · Sobol' indices · Model upscaling · Change  
36 of support · Regularization theory · Spatial model

## 37 1 Introduction

38 Variance-based global sensitivity analysis (GSA) aims at studying how un-  
39 certainty in the output of a model can be apportioned to different sources of  
40 uncertainty in its inputs. Here *model* denotes any computer code in which a  
41 response variable is calculated as a deterministic function of input variables.  
42 Born in the 1990ies (Sobol' 1993), GSA is now recognized as an essential in-  
43 gredient in model building (European Commission 2009; US Environmental  
44 Protection Agency 2009) and is widely used in different fields (Cariboni et  
45 al 2007; Tarantola et al 2002). It is based on the decomposition of model  
46 output variance into conditional variances. So-called *first-order sensitivity*  
47 *indices* measure the main effect contribution of each uncertain model input  
48 to the model output variance. Based on these sensitivity indices, ranking of  
49 model inputs helps 1) identifying inputs that account for most of model out-  
50 put variability and need to be better scrutinized with extra data-gathering  
51 2) simplifying model by identifying model inputs that have little influence on  
52 model output variance.

53 Although GSA was initially designed for models with scalar inputs and scalar  
54 output only, some work has been recently carried out on extending it to en-  
55 vironmental models where both inputs and output are spatially distributed

56 over a 2D domain (Lilburne and Tarantola 2009, for a review). In these  
57 works, spatial model inputs are usually maps derived from sampled field-  
58 data (e.g. digital elevation model, landuse map). They are uncertain due to  
59 measurement errors, lack of knowledge or aleatory variability (Brown and  
60 Heuvelink 2007; Refsgaard et al 2007), and are modeled as random fields.  
61 Model output is also spatially distributed (e.g. a flood map, a pollution map).  
62 Authors use geostatistical simulation to include spatially distributed model  
63 inputs into GSA approach (Ruffo et al 2006; Saint-Geours et al 2010)  
64 and display efficient estimation procedures to compute sensitivity indices in  
65 a spatial context, either with respect to the spatial average of model out-  
66 put (Lilburne and Tarantola 2009), or with respect to the values of model  
67 output at each site of a study area (Heuvelink et al 2010; Marrel et al 2011;  
68 Pettit and Wilson 2010).

69 Yet, to date none of these studies has reported on a key issue: the link  
70 between uncertainty propagation and model upscaling/downscaling. *Model*  
71 *upscaling* consists in the application of a spatial model, developed at a specific  
72 scale, to a larger scale. In many environmental models, physical quantities  
73 considered are additive (e.g. porosity, evapotranspiration), i.e. their large  
74 scale properties derive from small scale properties by simple averaging. In this  
75 case, model end-user is usually interested in the spatial linear average or sum

76 of spatial output over a given spatial unit (e.g. the average porosity of a block,  
77 the total evapotranspiration over a plot of land) and model upscaling is thus  
78 reduced to a *change of support* problem. [Heuvelink \(1998\)](#) pointed out that  
79 under a change of spatial support of model output, the relative contribution  
80 of uncertain model inputs to the variability of aggregated model output may  
81 change. Exploring how sensitivity analysis results interact with change of  
82 support issue is thus of great importance to check the robustness of model-  
83 based environmental impact assessment studies and give better confidence in  
84 their results. It would allow the modeler to answer the following questions:  
85 what are the most influential model inputs over a given spatial support? For  
86 which support size does a spatially distributed model input contribute the  
87 most to model output uncertainty? Does spatial structure of uncertainty of  
88 a model input influence its contribution to model output variability?

89 Change of support effect has been extensively discussed in geostatistics with  
90 the regularization theory ([Journel and Huijbregts 1978](#), p.77). Hence, we try  
91 in this paper to bring together regularization theory and variance-based GSA  
92 frameworks. Our idea is to define *site sensitivity indices* and *block sensitivity*  
93 *indices*, in order to i) provide a simple formalism to extend variance-based  
94 GSA to spatial models, when the modeler's interest is in spatial average or  
95 sum of model output over a given spatial support (section 2); ii) discuss the

96 influence of model upscaling on the relative contribution of uncertain model  
97 inputs to the variance of model output (section 3). We limit our study to  
98 point-based models, i.e. models where the computation of model output at  
99 some location uses the value of spatial inputs at that same location only  
100 (Heuvelink, Brus, and Reinds 2010). An example is used throughout the  
101 paper to illustrate formal definitions and properties. We finally discuss the  
102 limits of our approach and its links with related work in section 4.

## 103 2 Variance-based sensitivity indices for a spatial model

### 104 2.1 Description of spatial model $\mathcal{M}$

105 Let  $\mathcal{D} \subset \mathbb{R}^2$  denote a 2D spatial domain,  $\mathbf{x} \in \mathcal{D}$  a site,  $\mathbf{h}$  the lag vector  
106 between two sites  $\mathbf{x}$  and  $\mathbf{x}'$ , and  $v \subset \mathcal{D}$  some spatial support of area  $|v|$ . For  
107 sake of clarity, we consider a spatial model  $Y = \mathcal{M}(\mathbf{U}, Z)$  with  $n$  uncertain  
108 scalar inputs  $U_1, \dots, U_n$  that are grouped into a vector  $\mathbf{U} \in \mathbb{R}^n$  of joint  
109 pdf  $p(\mathbf{U})$ , and a single uncertain spatial input  $\{Z(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ , that we will  
110 often simply denote by  $Z(\mathbf{x})$ .  $Z(\mathbf{x})$  is supposed to be a strictly stationary  
111 random field (SRF) of mean  $\mu$  with finite variance and covariogram  $C(\cdot)$ . Let  
112 assume its covariance structure, characterized by range  $a \in \mathbb{R}$  and nugget

113 effect  $\eta \in [0; 1[$ , is isotropic and of the form:

$$C(\mathbf{h}) = \begin{cases} C(0) & \text{if } \mathbf{h} = \mathbf{0} \\ (1 - \eta) \cdot C(0) \cdot \rho_a(\|\mathbf{h}\|) & \text{if } \mathbf{h} \neq \mathbf{0} \end{cases} \quad (1)$$

114 with  $\rho_a(\cdot)$  some valid correlogram (Cressie 1993, p.67). Model output is a  
 115 2D random field  $\{Y(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ , that we will simply denote by  $Y(\mathbf{x})$ . As  
 116 discussed in the introduction, we limit our study to *point-based* models, hence  
 117 we assume there exists a mapping  $\psi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  such that:

$$\forall \mathbf{x} \in \mathcal{D}, \quad Y(\mathbf{x}) = \psi[\mathbf{U}, Z(\mathbf{x})] \quad (2)$$

118 We also assume that  $\psi$  is such that for any site  $\mathbf{x} \in \mathcal{D}$ , random variables  $Y(\mathbf{x})$ ,  
 119  $\mathbb{E}[Y(\mathbf{x}) | Z(\mathbf{x})]$  and  $\mathbb{E}[Y(\mathbf{x}) | \mathbf{U}]$  have finite expectation and finite variance.

120 Sensitivity analysis of model  $\mathcal{M}$  must be performed with respect to a scalar  
 121 quantity of interest derived from spatially distributed model output  $Y(\mathbf{x})$ .

122 Here we consider two different outputs of interest: the value  $Y(\mathbf{x}^*)$  at some  
 123 specific site  $\mathbf{x}^* \in \mathcal{D}$  and the aggregated value  $Y_v = 1/|v| \int_v Y(\mathbf{x}) d\mathbf{x}$  over  
 124 support  $v$ . As model inputs  $\mathbf{U}$  and  $Z(\mathbf{x})$  are uncertain,  $Y(\mathbf{x}^*)$  and  $Y_v$  are both  
 125 random variables: sensitivity analysis will discuss the relative contribution  
 126 of uncertain model inputs  $\mathbf{U}$  and  $Z(\mathbf{x})$  to their respective variances.



## 127 2.2 Site sensitivity indices and block sensitivity indices

128 Before defining sensitivity indices for spatial model  $\mathcal{M}$ , we very briefly re-  
129 view the mathematical basis of variance-based GSA. Consider a model  $Y =$   
130  $f(X_1, \dots, X_n)$  where  $X_i$  are independent random variables. First-order sen-  
131 sitivity index  $S_i$  of model input  $X_i$  is defined by:

$$S_i = \frac{\text{Var}[E(Y|X_i)]}{\text{Var}(Y)} \quad (3)$$

132  $S_i \in [0; 1]$  measures the main effect contribution of uncertain model input  $X_i$   
133 to the variance of model output  $Y$ . It is the expected part of output variance  
134  $\text{Var}(Y)$  that could be reduced if input  $X_i$  was perfectly known. Sensitivity  
135 indices can be used to identify the model inputs that account for most of  
136 model output variability (model inputs  $X_i$  with high first order indices  $S_i$ ).  
137 Sum of  $S_i$  is lower than 1: difference  $1 - \sum_i S_i$  accounts for the contribution  
138 of interactions between model inputs  $X_i$  to model output variance  $\text{Var}(Y)$ .  
139 If needed, [Saltelli et al \(2008\)](#) will provide more details on GSA theory and  
140 on estimation of sensitivity indices.

141 To extend GSA to spatial model  $\mathcal{M}$ , we propose to use different types of  
142 sensitivity indices to describe the relative contribution of uncertain model  
143 inputs  $\mathbf{U}$  and  $Z(\mathbf{x})$  to the variability of model output: at site scale (i.e. with  
144 respect to output of interest  $Y(\mathbf{x}^*)$ ) and at larger scale (i.e. with respect

145 to output of interest  $Y_v$ ). First-order sensitivity indices of model inputs  
 146 with respect to  $Y(\mathbf{x}^*)$  are called *site sensitivity indices*. Under stationary  
 147 hypothesis on SRF  $Z(\mathbf{x})$ , they don't depend on site  $\mathbf{x}^*$ , thus will simply be  
 148 denoted by  $S_{\mathbf{U}}$  and  $S_Z$ :

$$S_{\mathbf{U}} = \frac{\text{Var}[\mathbb{E}(Y(\mathbf{x}^*) | \mathbf{U})]}{\text{Var}[Y(\mathbf{x}^*)]} \quad ; \quad S_Z = \frac{\text{Var}[\mathbb{E}(Y(\mathbf{x}^*) | \{Z(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\})]}{\text{Var}[Y(\mathbf{x}^*)]} \quad (4)$$

149 First-order sensitivity indices of model inputs with respect to block average  
 150  $Y_v$  are called *block sensitivity indices* and are denoted by  $S_{\mathbf{U}}(v)$  and  $S_Z(v)$ :

$$S_{\mathbf{U}}(v) = \frac{\text{Var}[\mathbb{E}(Y_v | \mathbf{U})]}{\text{Var}[Y_v]} \quad ; \quad S_Z(v) = \frac{\text{Var}[\mathbb{E}(Y_v | \{Z(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\})]}{\text{Var}[Y_v]} \quad (5)$$

151 Ratio  $S_Z(v)/S_{\mathbf{U}}(v)$  gives the relative contribution of uncertain model in-  
 152 puts  $Z(\mathbf{x})$  and  $\mathbf{U}$  to the variability of the output of interest  $Y_v$ : when  
 153  $S_Z(v)/S_{\mathbf{U}}(v) > 1$ , variance of  $Y_v$  is mainly explained by uncertainty on 2D  
 154 input field  $Z(\mathbf{x})$ ; when  $S_Z(v)/S_{\mathbf{U}}(v) < 1$ , it's the uncertainty on non-spatial  
 155 input  $\mathbf{U}$  that accounts for most of  $\text{Var}(Y_v)$ .

### 156 2.3 Illustrative example

157 The proposed formalism for spatial GSA is illustrated through the following  
 158 example. A model  $Y = \mathcal{M}(\mathbf{U}, Z)$  is used for economic assessment of flood  
 159 risk.  $Z(\mathbf{x})$  is the map of maximal water levels (m) reached during a flood

160 event over a given area  $\mathcal{D}$ . It is assumed to be a Gaussian random field  
 161 with mean  $\mu = 50$  and exponential covariance  $C(\mathbf{h})$  with  $C(0) = 100$ , range  
 162  $a = 5$  and nugget  $\eta = 0.1$ .  $\mathbf{U}$  is a set of three economic parameters  $U_1$ ,  
 163  $U_2$  and  $U_3$ , assumed to be independent random variables following Gaussian  
 164 distributions  $\mathcal{N}(1.5, 0.5)$ ,  $\mathcal{N}(55, 5)$  and  $\mathcal{N}(10, 10)$  respectively. Model output  
 165  $Y(\mathbf{x})$  is the map of expected economic damages due to the flood over the  
 166 area, which depend on  $\mathbf{U}$  and  $Z(\mathbf{x})$  through mapping  $\psi$ :

$$\forall \mathbf{x} \in \mathcal{D}, \quad Y(\mathbf{x}) = \psi[\mathbf{U}, Z(\mathbf{x})] = U_1 \cdot Z(\mathbf{x}) - U_2 \cdot e^{-0.036 \cdot Z(\mathbf{x})} - U_3 \quad (6)$$

167 Stakeholders are interested in two outputs: flood damage  $Y(\mathbf{x}^*)$  on a specific  
 168 building  $\mathbf{x}^* \in \mathcal{D}$ , and total damage  $|v| \cdot Y_v$  over a district  $v$  (here a disc of ra-  
 169 dius  $r = 50$ ). Here, analytical expression of model  $\mathcal{M}$  may be simple enough  
 170 to derive exact values of sensitivity indices, but it is usually not the case in  
 171 real applications. Thus we considered model  $\mathcal{M}$  as a *black box* and estimated  
 172 site sensitivity indices and block sensitivity indices with a sampling-based ap-  
 173 proach (Lilburne and Tarantola 2009), using  $N = 4096$  model runs (Table 1).  
 174 All calculations were carried out in R (R Development Core Team 2009).  
 175 It appears that at site scale, uncertainty on the map of water levels explains  
 176 most of the variance of  $Y(\mathbf{x}^*)$ :  $S_Z = 0.89$ . At a larger scale, uncertainty  
 177 on total flood damage  $|v| \cdot Y_v$  is mainly due to the uncertainty on economic  
 178 parameters  $U_1$ ,  $U_2$  and  $U_3$ :  $S_{\mathbf{U}}(v) = 0.86$ . To improve accuracy of damage

179 estimation on a specific building, uncertainty should thus be first reduced on  
 180 the map of water levels  $Z(\mathbf{x})$ ; on the contrary, to improve accuracy of total  
 181 damage estimation over a large district  $v$ , modeler should focus on reducing  
 182 uncertainty on economic parameters  $U_1$ ,  $U_2$  and  $U_3$ .

### 183 3 Change of support effect on block sensitivity indices

184 We assess in this section how the ranking of uncertain model inputs based  
 185 on their block sensitivity indices vary under a change of support  $v$ .

#### 186 3.1 Relation between site sensitivity indices and block sensitivity indices

187 Site sensitivity indices and block sensitivity indices are related. Let  $\mathbb{E}_Z Y(\mathbf{x})$   
 188 denote the conditional expectation of  $Y(\mathbf{x})$  knowing  $Z(\mathbf{x})$ , that is:

$$\forall \mathbf{x} \in \mathcal{D}, \quad \mathbb{E}_Z Y(\mathbf{x}) = \mathbb{E}[Y(\mathbf{x}) \mid Z(\mathbf{x})] \quad (7)$$

189  $\mathbb{E}_Z Y(\mathbf{x})$  is the transform of input stationary random field  $Z(\mathbf{x})$  through func-  
 190 tion  $\bar{\psi}(z) = \int \psi(\mathbf{u}, z)p(\mathbf{u})d\mathbf{u}$  [Eq. (2)]. Under our assumptions on  $\psi$ ,  $\mathbb{E}_Z Y(\mathbf{x})$   
 191 is a SRF with finite variance. Let  $C^*(\cdot)$  denote its covariogram,  $\sigma^2 = C^*(0)$   
 192 its variance and  $\sigma_v^2$  its block variance over support  $v$ , that is, the variance  
 193 of block average  $1/|v| \int_v \mathbb{E}_Z Y(\mathbf{x})d\mathbf{x}$ . Block variance  $\sigma_v^2$  is equal to the mean  
 194 value of  $C^*(\mathbf{h})$  when the two extremities of lag vector  $\mathbf{h}$  describe support  $v$ ,

195 which we denote by  $\overline{C^*}(v, v)$  (Journal and Huijbregts 1978, p.78). Using  
 196 these notations, it follows from Equations (4) and (5) that site sensitivity  
 197 indices and block sensitivity indices are related by (see Appendix A for a  
 198 proof):

$$\frac{S_Z(v)}{S_U(v)} = \frac{S_Z}{S_U} \cdot \frac{\sigma_v^2}{\sigma^2} = \frac{S_Z}{S_U} \cdot \frac{\overline{C^*}(v, v)}{C^*(0)} \quad (8)$$

### 199 3.2 Change of support effect

200 Consider now that model  $\mathcal{M}$  was initially developed to study spatial average  
 201  $Y_v$  over support  $v$ , and that after model upscaling the modeler is interested in  
 202 spatial average  $Y_V$  over support  $V$  such that  $V \gg v$ . We know from Krige's  
 203 relation (Journal and Huijbregts 1978, p.67) that block variance  $\sigma_v^2$  decreases  
 204 with the size of support:  $\sigma_V^2 \leq \sigma_v^2$ . It follows from Equation (8) that:

$$\frac{S_Z(V)}{S_U(V)} \leq \frac{S_Z(v)}{S_U(v)} \quad (9)$$

205 The relative contribution of input random field  $Z(\mathbf{x})$  to the variability of  
 206 aggregated model output - compared to the contribution of uncertain scalar  
 207 input  $\mathbf{U}$  - is thus smaller on support  $V$  than on support  $v$ . Model upscal-  
 208 ing results in a lower influence of uncertain spatial input  $Z(\mathbf{x})$  on model  
 209 output uncertainty. More specifically, let suppose that covariogram  $C^*(\cdot)$   
 210 of  $\mathbb{E}_Z Y(\mathbf{x})$  random field has finite range and that support  $v$  is large with

211 respect to this range. To a first approximation, block variance  $\sigma_v^2$  is of the  
 212 form  $\sigma_v^2 \simeq \sigma^2 A/|v|$  where  $A$  is the so-called *integral range* of  $C^*(\cdot)$  defined  
 213 by  $A = 1/\sigma^2 \int C^*(\mathbf{h})d\mathbf{h}$  (Chilès and Delfiner 1999, p.73). It follows from  
 214 Equation (8) that:

$$\frac{S_Z(v)}{S_U(v)} \simeq \frac{|v|_{\text{lim}}}{|v|} \quad \text{with} \quad |v|_{\text{lim}} = A \cdot \frac{S_Z}{S_U} \quad (10)$$

215 Equation (10) shows that ratio  $|v|_{\text{lim}}/|v|$  drives the relative contribution of  
 216 model inputs  $Z(\mathbf{x})$  and  $\mathbf{U}$  to output variance  $\text{Var}(Y_v)$ . The larger this ratio,  
 217 the larger the part of output variance  $\text{Var}(Y_v)$  explained by the uncertainty  
 218 on input random field  $Z(\mathbf{x})$ . For a low ratio (i.e. when the area of support  
 219  $v$  is large compared to critical size  $|v|_{\text{lim}}$ ), variability of  $Z(\mathbf{x})$  is mainly *local*,  
 220 and spatial correlation of  $Z(\mathbf{x})$  over support  $v$  is weak. This local variabil-  
 221 ity averages over support  $v$  when aggregated model output  $Y_v$  is computed:  
 222 hence the uncertainty on input 2D-field  $Z(\mathbf{x})$  has a small influence on output  
 223 variance  $\text{Var}(Y_v)$ . On the contrary, for a greater ratio (i.e. when the area of  
 224 support  $v$  is small compared to critical size  $|v|_{\text{lim}}$ ), spatial correlation of  $Z(\mathbf{x})$   
 225 over  $v$  is strong. The averaging-out effect is weaker, hence the uncertainty  
 226 on model input  $Z(\mathbf{x})$  has a larger influence on output variance  $\text{Var}(Y_v)$ .

### 227 3.3 Influence of covariance range and nugget effect

228 Critical size  $|v|_{\text{lim}} = A \cdot \frac{S_Z}{S_U}$  depends on the covariance structure  $C^*(\cdot)$  of  
 229  $\mathbb{E}_Z Y(\mathbf{x})$  random field, which is itself driven by covariogram  $C(\cdot)$  of input  
 230 SRF  $Z(\mathbf{x})$ . Let us assume now that  $Z(\mathbf{x})$  is a Gaussian random field (GRF).  
 231 As  $\mathbb{E}_Z Y(\mathbf{x})$  is a transform of GRF  $Z(\mathbf{x})$  through function  $\bar{\psi}$  [Eq. (7)], it can  
 232 be decomposed into an Hermitian expansion and its covariogram  $C^*(\cdot)$  can  
 233 be written as (Chilès and Delfiner 1999, p.396-399; see Appendix B for a  
 234 proof):

$$C^*(\mathbf{h}) = \sum_{k=0}^{\infty} \lambda_k^2 \cdot [C(\mathbf{h})]^k \quad (11)$$

235 For most usual transition covariogram models (e.g. spherical, exponential,  
 236 Gaussian models), covariance  $C(\mathbf{h})$  is a monotonically increasing function of  
 237 range parameter  $a$ . In this case, it follows from Equation (11) that integral  
 238 range  $A = 1/\sigma^2 \int C^*(\mathbf{h}) d\mathbf{h}$  also grows with range  $a$ . An increase in range  
 239 parameter  $a$  thus leads to a growth of critical size  $|v|_{\text{lim}}$ , and ratio of block  
 240 sensitivity indices  $S_Z(v)$  and  $S_U(v)$  verifies [Eq. (10)]:

$$\frac{\partial}{\partial a} \left[ \frac{S_Z(v)}{S_U(v)} \right] \geq 0 \quad (12)$$

241 The relative contribution of uncertain model input  $Z(\mathbf{x})$  to the variance of  
 242 output of interest  $Y_v$  gets larger when covariance range of  $Z(\mathbf{x})$  increases.  
 243 Indeed, when range  $a$  increases, the averaging-out effect that occurs when

244 model output is aggregated over spatial support  $v$  gets weaker, thus the  
 245 influence of input random field  $Z(\mathbf{x})$  on output variance  $\text{Var}(Y_v)$  grows.

246 The impact of nugget effect on block sensitivity indices can be interpreted  
 247 the same way. Nugget parameter  $\eta$  controls the relative part of *pure noise* in  
 248 input random field  $Z(\mathbf{x})$  [Eq. (1)]. The lower  $\eta$ , the weaker the averaging-out  
 249 effect will be when block average  $Y_v$  is computed over support  $v$ , hence the  
 250 larger the part of output variance  $\text{Var}(Y_v)$  that will be explained by  $Z(\mathbf{x})$ .  
 251 Critical size  $|v|_{\text{lim}}$  is thus a decreasing function of nugget parameter  $\eta$ , and  
 252 ratio of block sensitivity indices  $S_Z(v)$  and  $S_{\mathbf{U}}(v)$  verifies [Eq. (1), (8), (11)]:

$$\frac{\partial}{\partial \eta} \left[ \frac{S_Z(v)}{S_{\mathbf{U}}(v)} \right] \leq 0 \quad (13)$$

### 253 3.4 Illustrative example

254 To illustrate change of support effects on sensitivity analysis results, we per-  
 255 formed spatial GSA on our numerical example in the following settings: vary-  
 256 ing disc-shaped support  $v$  of increasing size (Fig. 2); varying range from  $a = 1$   
 257 to  $a = 10$  (Fig. 3); varying nugget parameter from  $\eta = 0$  to  $\eta = 0.9$  (Fig. 4).  
 258 For each setting, we computed estimates of output variance  $\text{Var}(Y_v)$ , block  
 259 sensitivity indices  $S_{\mathbf{U}}(v)$ ,  $S_Z(v)$  and ratio  $S_Z(v)/S_{\mathbf{U}}(v)$  over  $N = 4096$  model  
 260 runs. The whole process was repeated 100 times and mean values with 95%



261 confidence interval are shown for each estimate. In accordance with Equa-  
 262 tions (9), (12) and (13), it appears that block sensitivity index  $S_Z(v)$  (i)  
 263 decreases when support  $v$  gets larger (Fig. 2b) (ii) increases with covariance  
 264 range  $a$  (Fig. 3b) (iii) decreases with nugget parameter  $\eta$  (Fig. 4b), while  
 265 sensitivity index  $S_{\mathbf{U}}(v)$  has the opposite behaviour. Change of support effect  
 266 is clearly highlighted in Figure 2b: contribution of economic parameters  $U_1$ ,  
 267  $U_2$  and  $U_3$  to the variability of total flood damage  $|v| \cdot Y_v$  exceeds the contri-  
 268 bution of water level map  $Z(\mathbf{x})$  when the radius  $r$  of support  $v$  is greater than  
 269  $r_{\text{lim}} \simeq 18$ ; for radius  $r < r_{\text{lim}}$ , uncertainty on total flood damage over support  
 270  $v$  is mainly due to the uncertainty on water levels  $Z(\mathbf{x})$ . Finally, Figure 2c  
 271 illustrates that ratio  $S_Z(v)/S_{\mathbf{U}}(v)$  is proportional to  $1/|v|$  when support  $v$  is  
 272 large enough. Theoretical curve  $S_Z(v)/S_{\mathbf{U}}(v) = |v|_{\text{lim}}/|v|$  [Eq. 10] was fitted  
 273 (least squares -  $R^2 = 0.99$ ) on data points (for  $r \geq 20$  only), yielding an  
 274 estimate of critical size  $|v|_{\text{lim}} \simeq 1068$ .

## 275 4 Discussion

276 Our first goal was to provide a formalism to extend variance-based GSA  
 277 approach to spatial models when the modeler is mainly interested in the  
 278 linear average or the sum of model output  $Y(\mathbf{x})$  over some spatial unit  $v$ .  
 279 Our approach is strongly inspired from previous related works. Other authors

280 had already computed site sensitivity indices (Marrel et al 2011; Pettit and  
281 Wilson 2010) and block sensitivity indices (Lilburne and Tarantola 2009),  
282 but without naming them nor exploring their analytical properties or their  
283 relationship. Our paper is an attempt to do so. Equation (8) gives an exact  
284 relation between site and block sensitivity indices: it may prove useful in the  
285 case of a model with a simple enough analytical expression.

286 Our research also sought to account for change of support effects in the prop-  
287 agation of uncertainty through spatial models, within variance-based GSA  
288 framework. We proved that the relative influence of spatially distributed  
289 model input  $Z(\mathbf{x})$  decreases under model upscaling: when support  $v$  is large  
290 enough, the ratio of block sensitivity index of spatially distributed input with  
291 block sensitivity index of scalar inputs is proportional to  $|v|_{\text{lim}}/|v|$ . Critical  
292 size  $|v|_{\text{lim}}$  depends on the covariance structure of input SRF  $Z(\mathbf{x})$ : it usually  
293 grows with an increase of range parameter  $a$  or a decrease of nugget parame-  
294 ter  $\eta$ . These findings are translation into GSA formalism of the averaging-out  
295 effect clearly portrayed by Journel and Huijbregts (1978) in the regulariza-  
296 tion theory. Our contribution is to discuss this issue from the point of view  
297 of GSA practitioners. Formalizing the effect of a change of support on sen-  
298 sitivity analysis results may well help modelers when they consider model  
299 upscaling: it will orientate future data gathering by identifying model inputs

300 that will have major influence on model output variability over a new spatial  
301 support. It also promotes an increased awareness on the issue of choosing the  
302 appropriate accuracy of spatial model inputs for a given scale: when block  
303 sensitivity index of spatial input  $Z(\mathbf{x})$  is small ( $S_Z(v) < 0.1$ ), it may mean  
304 that it is over-accurate compared to the uncertainty of other model inputs,  
305 and that costs associated with field data gathering could have been avoided.

306 It should be noted that our approach is based on conditions that may not  
307 be met in some practical cases. First, we considered a simple model  $\mathcal{M}$  with  
308 a single spatially distributed input  $Z(\mathbf{x})$ . In real applications, modelers of-  
309 ten have to deal with several spatial inputs  $Z_1(\mathbf{x}), \dots, Z_m(\mathbf{x})$ , with different  
310 covariance structures  $C_i(\cdot)$ , ranges  $a_i$  and nuggets  $\eta_i$ . In this case, it can be  
311 shown that Equation (8) still holds separately for each spatial input  $Z_i(\mathbf{x})$ .  
312 Yet no conclusion can be drawn *a priori* on how a change of support im-  
313 pacts the relative ranking of two spatial inputs  $Z_i(\mathbf{x})$  and  $Z_j(\mathbf{x})$ : ratio of  
314 their block sensitivity indices  $S_{Z_i}(v)/S_{Z_j}(v)$  will depend on the ratio of block  
315 variances  $\sigma_{v,i}^2/\sigma_{v,j}^2$ . Second, some environmental models are not point-based  
316 and involve spatial interactions (e.g. erosion, groundwater flow models). In  
317 this case, it still may be possible to build a point-based surrogate model as  
318 a coarse approximation of the original model; if not, then change of support  
319 properties discussed in section 3 may not hold. Third, we assumed input

320 random field  $Z(\mathbf{x})$  to be stationary: if not, site sensitivity indices depend on  
321 site  $\mathbf{x}^*$  [Eq. (4)]. It is then possible to compute maps of these indices (Marrel  
322 et al 2011; Pettit and Wilson 2010) to discuss the spatial variability of  
323 model inputs sensitivities.

324 Finally, we focused on the case where the modeler's interest is in the spatial  
325 linear average or the sum of model output  $Y(\mathbf{x})$  over support  $v$ . As dis-  
326 cussed by Lilburne and Tarantola (2009), other outputs of interest may be  
327 considered, such as maximum value of  $Y(\mathbf{x})$  over  $v$  (e.g. maximal pollutant  
328 concentration over a zone), some quantile of  $Y(\mathbf{x})$  over  $v$  (Heuvelink et al  
329 2010), the percentage of  $v$  where  $Y(\mathbf{x})$  exceeds a certain threshold, etc. Up to  
330 our knowledge, no investigation has been done on the properties of sensitivity  
331 indices computed with respect to such outputs of interest.

## 332 5 Conclusion

333 This paper provides a formalism to apply variance-based global sensitivity  
334 analysis to spatial models when the modeler's interest is in the average or the  
335 sum of model output  $Y(\mathbf{x})$  over a given spatial unit  $v$ . Site sensitivity indices  
336 and block sensitivity indices allow us to discuss how a change of support  
337 influence the relative contribution of uncertain model inputs to the variance  
338 of the output of interest. We give an analytical relationship between these

339 two types of sensitivity indices. Our results show that block sensitivity index  
340 of input random field  $Z(\mathbf{x})$  increases with ratio  $|v|_{\text{lim}}/|v|$ , where  $|v|$  is the area  
341 of spatial support  $v$  and critical size  $|v|_{\text{lim}}$  depends on the covariance structure  
342 of  $Z(\mathbf{x})$ . Our formalization is made with a view towards promoting the use  
343 of sensitivity analysis in model-based spatial decision support systems. Yet  
344 further research is needed to explore the case of non point-based models and  
345 extend our study to outputs of interest other than the average value of model  
346 output over zone  $v$ .

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399 **Appendix A: proof for relation between site sensitivity indices and**  
 400 **block sensitivity indices**

401 As mentioned in section 2, we assume that mapping  $\psi$  is such that for  
 402 any site  $\mathbf{x} \in \mathcal{D}$ , random variables  $Y(\mathbf{x})$ ,  $\mathbb{E}[Y(\mathbf{x}) | Z(\mathbf{x})]$  and  $\mathbb{E}[Y(\mathbf{x}) | \mathbf{U}]$   
 403 have finite expectation and finite variance. Ratio of block sensitivity indices  
 404 writes [Eq. (5)]:

$$\frac{S_Z(v)}{S_{\mathbf{U}}(v)} = \frac{\text{Var}(\mathbb{E}[Y_v | \{Z(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}])}{\text{Var}(\mathbb{E}[Y_v | \mathbf{U}])} \quad (14)$$

Conditional expectation of block average  $Y_v$  knowing  $\{Z(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$  writes:

$$\begin{aligned} \mathbb{E}[Y_v | Z] &= \mathbb{E}\left[1/|v| \int_v Y(\mathbf{x}) d\mathbf{x} \mid \{Z(\mathbf{x}') : \mathbf{x}' \in \mathcal{D}\}\right] \quad (\text{definition of } Y_v) \\ &= 1/|v| \int_v \mathbb{E}[Y(\mathbf{x}) \mid \{Z(\mathbf{x}') : \mathbf{x}' \in \mathcal{D}\}] d\mathbf{x} \\ &= 1/|v| \int_v \mathbb{E}[Y(\mathbf{x}) \mid Z(\mathbf{x})] d\mathbf{x} \quad (\text{for a point-based model}) \\ &= 1/|v| \int_v \mathbb{E}_Z Y(\mathbf{x}) d\mathbf{x} \quad (\text{definition of } \mathbb{E}_Z Y(\mathbf{x})) \end{aligned}$$

Thus we have  $\text{Var}(\mathbb{E}[Y_v | Z]) = \text{Var}(1/|v| \int_v \mathbb{E}_Z Y(\mathbf{x}) d\mathbf{x}) = \sigma_v^2$  (definition of  $\sigma_v^2$ ). Moreover, conditional expectation of block average  $Y_v$  knowing scalar input  $\mathbf{U}$  writes:

$$\begin{aligned} \mathbb{E}[Y_v | \mathbf{U}] &= \mathbb{E}\left[1/|v| \int_v Y(\mathbf{x}) d\mathbf{x} \mid \mathbf{U}\right] \quad (\text{definition of } Y_v) \\ &= 1/|v| \int_v \mathbb{E}[Y(\mathbf{x}) \mid \mathbf{U}] d\mathbf{x} \end{aligned}$$



405  $\mathbb{E}[Y(\mathbf{x}) | \mathbf{U}]$  does not depend on site  $\mathbf{x}$  under stationarity of SRF  $Z(\mathbf{x})$ ,  
 406 thus we have in particular  $\mathbb{E}[Y_v | \mathbf{U}] = \mathbb{E}[Y(\mathbf{x}^*) | \mathbf{U}]$ , and  $\text{Var}(\mathbb{E}[Y_v | \mathbf{U}]) =$   
 407  $\text{Var}(\mathbb{E}[Y(\mathbf{x}^*) | \mathbf{U}])$ . Combining these expressions with Equation (14) yields:

$$\frac{S_Z(v)}{S_U(v)} = \frac{\sigma_v^2}{\text{Var}(\mathbb{E}[Y(\mathbf{x}^*) | \mathbf{U}])} \quad (15)$$

408 Ratio of site sensitivity indices writes [Eq. (4)]:

$$\frac{S_Z}{S_U} = \frac{\text{Var}(\mathbb{E}[Y(\mathbf{x}^*) | \{Z(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}])}{\text{Var}(\mathbb{E}[Y(\mathbf{x}^*) | \mathbf{U}])} \quad (16)$$

We notice that for point-based models,  $\text{Var}[\mathbb{E}(Y(\mathbf{x}^*) | \{Z(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\})] =$   
 $\text{Var}[\mathbb{E}_Z Y(\mathbf{x}^*)] = \sigma^2$  (definition of  $\mathbb{E}_Z Y(\mathbf{x})$  [Eq. (7)]). It finally follows from  
 Equations (15) and (16) that:

$$\frac{S_Z(v)}{S_U(v)} = \frac{S_Z}{S_U} \cdot \frac{\sigma_v^2}{\sigma^2}$$

#### 409 **Appendix B: Hermitian expansion of random field $\mathbb{E}_Z Y(\mathbf{x})$**

$\mathbb{E}_Z Y(\mathbf{x})$  random field can be written [Eq. (2), (7)] as a transformation of  
 $Z(\mathbf{x})$  Gaussian random field through function  $\bar{\psi} : z \mapsto \int_{\mathbb{R}^n} \psi(\mathbf{u}, z) \cdot p(\mathbf{u}) d\mathbf{u}$ :

$$\mathbb{E}_Z Y = \bar{\psi}(Z)$$

We previously assumed that for any site  $\mathbf{x} \in \mathcal{D}$ ,  $\mathbb{E}_Z Y(\mathbf{x})$  has finite expectation  
 and finite variance. Under this condition, function  $\bar{\psi}$  belongs to Hilbert space

$L^2(\mathcal{G})$  of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  which are square-integrable with respect to Gaussian density  $g(\cdot)$ . Hence  $\bar{\psi}$  can be expanded on the sequence of Hermite polynomials  $(\chi_k)_{k \in \mathbb{N}}$ , which forms an orthonormal basis of  $L^2(\mathcal{G})$  (Chilès and Delfiner 1999, p.399):

$$\bar{\psi} = \sum_{k=0}^{\infty} \alpha_k \cdot \chi_k \quad \text{with} \quad \chi_k(z) = \frac{1}{\sqrt{k!}} \cdot \frac{1}{g(z)} \cdot \frac{\partial^k}{\partial z^k} g(z)$$

where coefficients  $\alpha_k$  are given by:  $\alpha_k = \int_{\mathbb{R}} \chi_k(z) \bar{\psi}(z) g(z) dz$ . It follows that  $\mathbb{E}_Z Y(\mathbf{x})$  can be written as an infinite expansion of polynomials of  $Z(\mathbf{x})$ :

$$\forall \mathbf{x} \in \mathcal{D}, \quad \mathbb{E}_Z Y(\mathbf{x}) = \sum_{k=0}^{\infty} \alpha_k \cdot \chi_k [Z(\mathbf{x})]$$

Its covariance then writes (Chilès and Delfiner 1999, p.396 Eq. (6.23) and p.399 Eq. (6.25)):

$$\text{Cov}(\mathbb{E}_Z Y(\mathbf{x}), \mathbb{E}_Z Y(\mathbf{x} + \mathbf{h})) = \sum_{k=0}^{\infty} \alpha_k^2 \cdot \left[ \frac{C(\mathbf{h})}{C(0)} \right]^k = \sum_{k=0}^{\infty} \lambda_k^2 \cdot [C(\mathbf{h})]^k$$

410 where  $C(\mathbf{h})$  is the covariogram of GRF  $Z(\mathbf{x})$  and  $\lambda_k = \alpha_k \cdot C(0)^{-k/2}$ .

411 **Captions of tables**

412 **Tab. 1** Sensitivity analysis results over  $N = 4096$  model runs with respect  
413 to the outputs of interest  $Y(\mathbf{x}^*)$  and  $|v| \cdot Y_v$ . Mean values with  $\pm$ s.d. over  
414 100 replicas.

415

TABLE 1

Output of interest	Mean	Variance	Type of indices	Sensitivity indices
$Y(\mathbf{x}^*)$	$66.5 \pm 4.2$	$1393 \pm 188$	Site S.I.	$S_{\mathbf{U}} = 0.09 \pm 0.02$ $S_{\mathbf{Z}} = 0.89 \pm 0.02$
$ v  \cdot Y_v$	$68.6 \pm 0.4$	$145 \pm 4$	Block S.I.	$S_{\mathbf{U}}(v) = 0.86 \pm 0.01$ $S_{\mathbf{Z}}(v) = 0.12 \pm 0.02$

416 **Captions of figures**

417 **Fig. 1** Spatial model with uncertain inputs  $\mathbf{U}$  and  $Z(\mathbf{x})$  and spatial output  
418  $Y(\mathbf{x})$ . The modeler is interested in the block average of  $Y(\mathbf{x})$  over some  
419 spatial unit  $v$

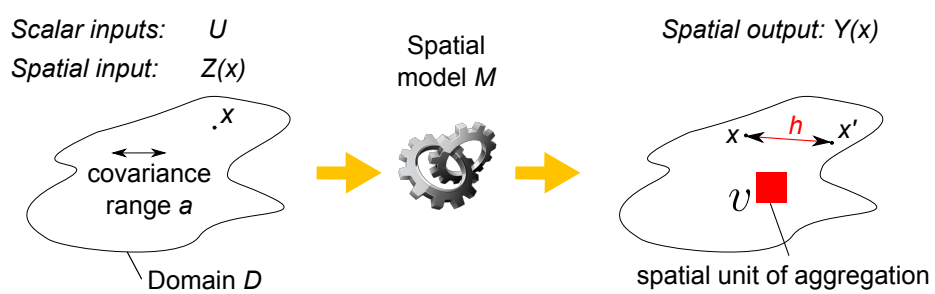
420 **Fig. 2** GSA results depending on the size of disc-shaped support  $v$  (with  
421 radius  $r$  and area  $|v| = \pi r^2$ ), for  $a = 5$ ,  $\eta = 0.1$ : **(a)** total variance of  
422 output of interest  $Y_v$ , **(b)** block sensitivity indices  $S_{\mathbf{U}}(v)$  (*solid line*) and  
423  $S_Z(v)$  (*dashed line*), **(c)** ratio  $S_Z(v)/S_{\mathbf{U}}(v)$  with fitted curve  $S_Z(v)/S_{\mathbf{U}}(v) =$   
424  $|v|_{\text{lim}}/|v|$  (*dashed line*). Error bars show 95 % confidence interval over 100  
425 replicas

426 **Fig. 3** GSA results depending on covariance range  $a$ , for  $\eta = 0.1$  and a  
427 disc-shaped support  $v$  of radius  $r = 50$ : **(a)** total variance of  $Y_v$ , **(b)** block  
428 sensitivity indices  $S_{\mathbf{U}}(v)$  (*solid line*) and  $S_Z(v)$  (*dashed line*). Error bars  
429 show 95 % confidence interval over 100 replicas

430 **Fig. 4** GSA results depending on covariance nugget  $\eta$ , for  $a = 5$  and a  
431 disc-shaped support  $v$  of radius  $r = 50$ : **(a)** total variance of  $Y_v$ , **(b)** block  
432 sensitivity indices  $S_{\mathbf{U}}(v)$  (*solid line*) and  $S_Z(v)$  (*dashed line*). Error bars  
433 show 95 % confidence interval over 100 replicas

434

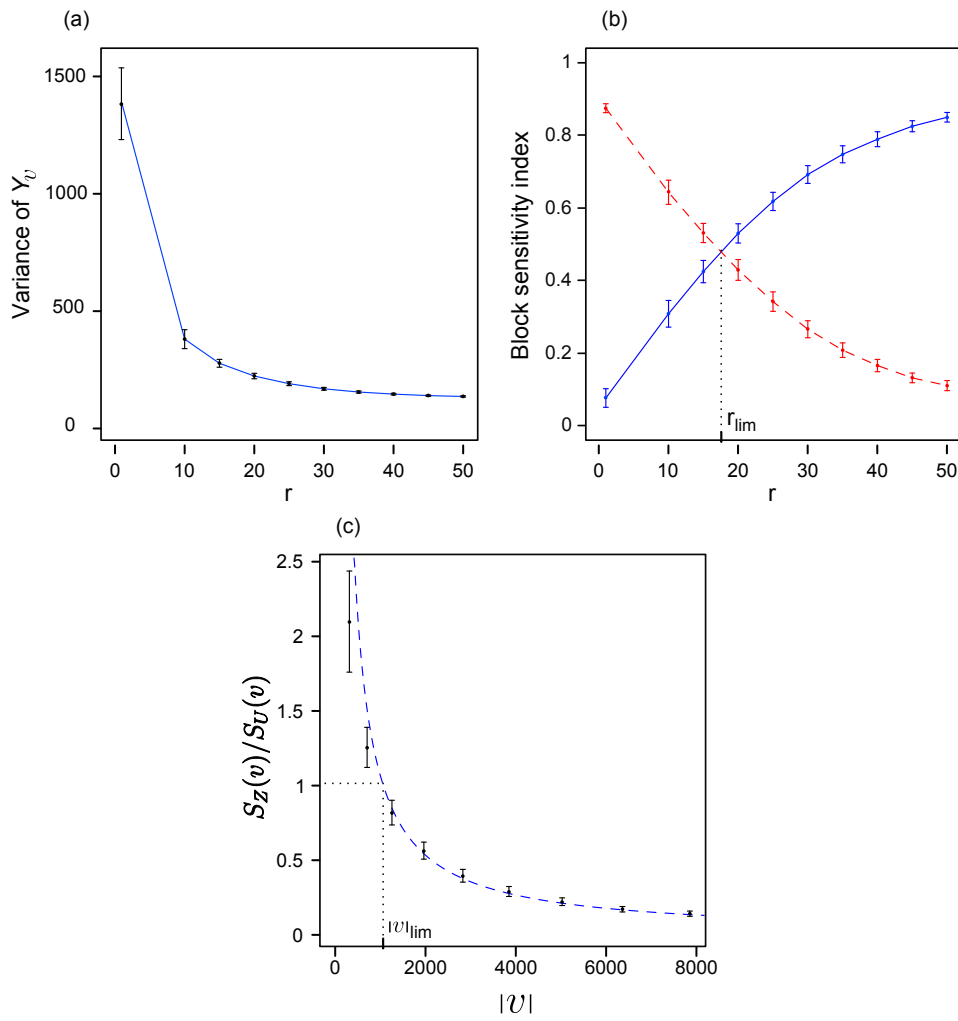
FIGURE 1



435

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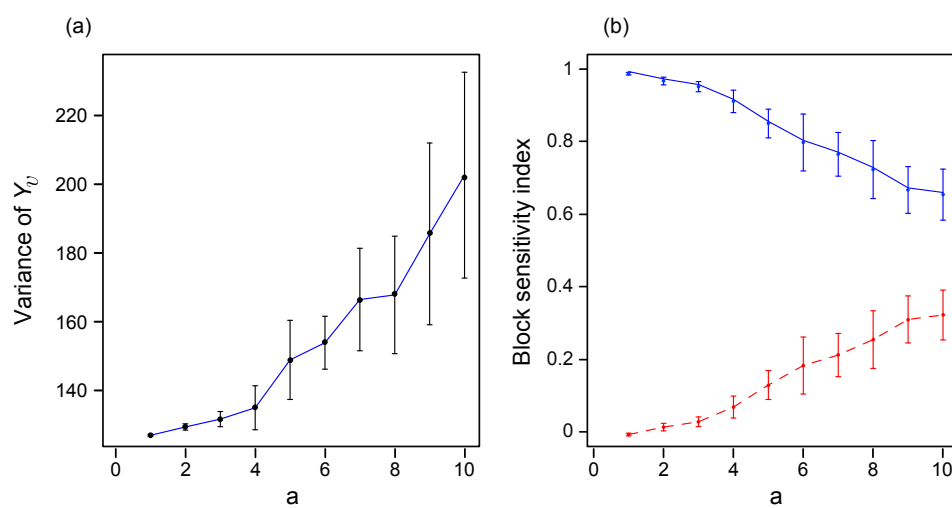
FIGURE 2



437

438

FIGURE 3

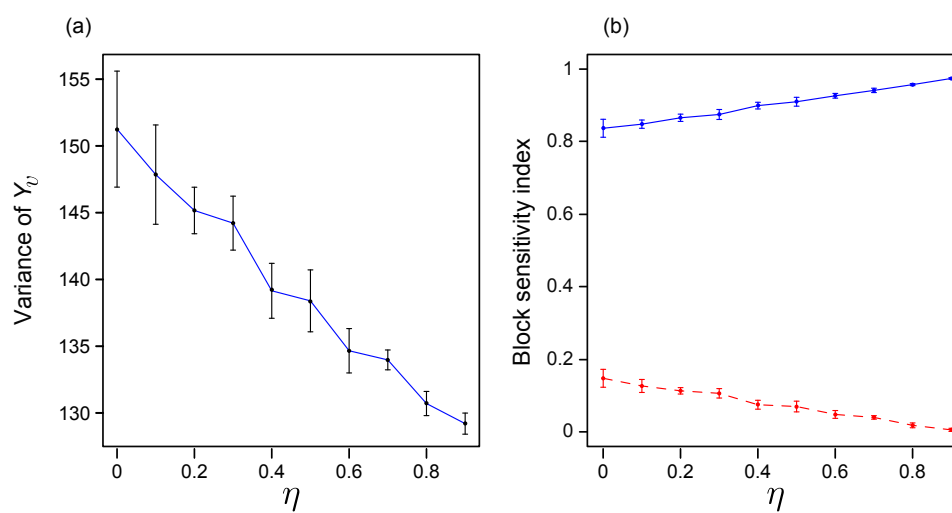


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440

FIGURE 4



441