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Abstract. Consider a Boolean model Σ in $\mathbb{R}^d$. The centers are given by a homogeneous Poisson point process with intensity λ and the radii of distinct balls are i.i.d. with common distribution ν. The critical covered volume is the proportion of space covered by Σ when the intensity λ is critical for percolation. Previous numerical simulations and heuristic arguments suggest that the critical covered volume may be minimal when ν is a Dirac measure.

In this paper, we prove that it is not the case at least in high dimension. To establish this result we study the asymptotic behaviour, as d tends to infinity, of the critical covered volume. It appears that, in contrast to what happens in the constant radii case studied by Penrose, geometrical dependencies do not always vanish in high dimension.

1. Introduction and statement of the main results

Introduction. Consider a homogeneous Poisson point process on $\mathbb{R}^d$. At each point of this process, we center a ball with random radius, the radii of distinct balls being i.i.d. and independent of the point process. The union Σ of these random balls is called a Boolean model. This Boolean model only depends on three parameters : the intensity λ of the point process of centers, the common distribution ν of the radii of the balls and the dimension d.

Denote by $\lambda_d^c(\nu)$ the critical intensity for percolation in Σ. We then consider $c_d^c(\nu)$, the volumic proportion of space which is covered by Σ when $\lambda = \lambda_d^c(\nu)$. This quantity is called the critical covered volume. This quantity is scale invariant (see (3)). For example $c_d^c(\delta_1) = c_d^c(\delta_r)$ for any $r > 0$, where δ_r denotes the Dirac measure on r. Numerical simulations in low dimension and heuristic arguments in any dimension suggested that the critical covered volume may be minimal when ν is a Dirac measure, that is when all the balls have the same radius. We show that this is not true in high dimensions. This result is proved through the study of the following kind of asymptotics. Let μ be a probability measure on $]0, +\infty[$ such that $r^{-d}\mu(dr)$ is a finite measure for any $d \geq 1$. For any d, we then consider the probability measure $\widetilde{\mu}_d = Z_d^{-1} r^{-d}\mu(dr)$ where Z_d is a normalization constant. The r^{-d} normalization will be discussed and motivated below (7). We prove that, as soon as μ is non degenerate, one has :

$$c_d^c(\widetilde{\mu}_d) \ll c_d^c(\delta_1) \text{ as } d \to \infty. \tag{1}$$

This proves that, when d is large enough, the critical covered volume is not minimal in the constant radii case.

The constant radii case has been studied by Penrose [14]. He proved that the asymptotic behavior of $c_d^c(\delta_1)$ is given by the critical parameter of the associated Galton-Watson process. This is due to the fact that geometrical dependencies vanish in high dimension in that case. We first focus in this paper on the case where the radii take only two distinct values. We prove that, in that case, the asymptotic behavior of $c_d^c(\widetilde{\mu}_d)$ is given by a competition between genealogy effects (given by the associated two-type Galton-Watson process) and geometrical dependencies effects. This yields (1) in that case. The general non degenerate distribution of radii case follows.

The Boolean model. Let us give here a different – but equivalent – construction of the Boolean model. Let ν be a finite[1] measure on $(0, +\infty)$. We assume that the mass of ν is positive. Let $d \geq 2$ be an integer, $\lambda > 0$ be a real number and ξ be a Poisson point process on $\mathbb{R}^d \times (0, +\infty)$ whose intensity measure is the Lebesgue measure on $\mathbb{R}^d$ times $\lambda\nu$. We define a random subset of $\mathbb{R}^d$ as follows:

$$\Sigma(\lambda\nu) = \bigcup_{(c,r)\in\xi} B(c, r),$$

where $B(c, r)$ is the open Euclidean ball centered at $c \in \mathbb{R}^d$ and with radius $r \in (0, +\infty)$. The random subset $\Sigma(\lambda\nu)$ is a Boolean model driven by $\lambda\nu$.

We say that $\Sigma(\lambda\nu)$ percolates if the connected component of $\Sigma(\lambda\nu)$ that contains the origin is unbounded with positive probability. This is equivalent to the almost-sure existence of an unbounded connected component of $\Sigma(\lambda\nu)$. We refer to the book by Meester and Roy [11] for background on continuum percolation. The critical intensity is defined by:

$$\lambda_d^c(\nu) = \inf\{\lambda > 0 : \Sigma(\lambda\nu) \text{ percolates}\}.$$

One easily checks that $\lambda_d^c(\nu)$ is finite. In [6] it is proven that $\lambda_d^c(\nu)$ is positive if and only if

$$\int r^d \nu(dr) < +\infty. \tag{2}$$

We assume that this assumption is fulfilled.

By ergodicity, the Boolean model $\Sigma(\lambda\nu)$ has a deterministic natural density. This is also the probability that a given point belongs to the Boolean model and it is given by :

$$P(0 \in \Sigma(\lambda\nu)) = 1 - \exp\left(-\lambda \int v_d r^d \nu(dr)\right),$$

where v_d denotes the volume of the unit ball in $\mathbb{R}^d$. The critical covered volume $c_d^c(\nu)$ is the density of the Boolean model when the intensity is critical :

$$c_d^c(\nu) = 1 - \exp\left(-\lambda_d^c(\nu) \int v_d r^d \nu(dr)\right).$$

It is thus more convenient to study the critical covered volume through the normalized critical intensity:

$$\widetilde{\lambda}_d^c(\nu) = \lambda_d^c(\nu) \int v_d (2r)^d \nu(dr).$$

We then have $c_d^c(\nu) = 1 - \exp\left(-\frac{\widetilde{\lambda}_d^c(\nu)}{2^d}\right)$. The factor 2^d may seem arbitrary here. Its interest will appear in the statement of the next theorems.

We will now give two scaling relations which partly justify our preference for c_d^c or $\widetilde{\lambda}_d^c$ over λ_d^c. For all $a > 0$, define $H^a\nu$ as the image of ν under the map defined by $x \mapsto ax$. By scaling, we get:

$$\widetilde{\lambda}_d^c(H^a\nu) = \widetilde{\lambda}_d^c(\nu). \tag{3}$$

This is a consequence of Proposition 2.11 in [11], and it may become more obvious when considering the two following facts : a critical Boolean model remains critical when rescaling and the density is invariant by rescaling ; therefore the critical covered volume

1. There is no greater generality in considering finite measures instead of probability measures ; this is simply more convenient.

and then the normalized threshold are invariant. One also easily checks the following invariance:

$$\widetilde{\lambda}_d^c(a\nu) = \widetilde{\lambda}_d^c(\nu). \tag{4}$$

Critical intensity as a function of ν. It has been conjectured by Kertész and Vicsek [9] that the critical covered volume – or equivalently the normalized critical intensity – should be independent of ν, as soon as the support of ν is bounded. Phani and Dhar [4] gave a heuristic argument suggesting that the conjecture were false. A rigorous proof was then given by Meester, Roy and Sarkar in [12]. More precisely, they gave examples of measures ν with two atoms such that:

$$\widetilde{\lambda}_d^c(\nu) > \widetilde{\lambda}_d^c(\delta_1). \tag{5}$$

As a consequence of Theorem 1.1 in the paper by Menshikov, Popov and Vachkovskaia [13], we even get that $\widetilde{\lambda}_d^c(\nu)$ can be arbitrarily large. More precisely, if

$$\text{if } \nu(n,a) = \sum_{k=0}^{n-1} a^{dk}\delta_{a^{-k}}, \text{ then } \widetilde{\lambda}_d^c(\nu(n,a)) \to n\widetilde{\lambda}_d^c(\delta_1) \text{ as } a \to \infty. \tag{6}$$

Actually the result of [13] is the following much stronger statement: $\lambda_d^c(\nu(+\infty, a)) \to \lambda_d^c(\delta_1)$ when $a \to \infty$. The convergence (6) is implicit in the work of Meester, Roy and Sarkar in [12], at least when $n = 2$. There were also heuristics for such a result in [4].

By Theorem 2.1 in [6], we get the existence of a positive constant C_d, that depends only on the dimension d, such that:

$$\widetilde{\lambda}_d^c(\nu) \geq C_d.$$

To sum up, $\widetilde{\lambda}_d^c(\cdot)$ is not bounded from above but is bounded from below by a positive constant. In other words, the critical covered volume $c_d^c(\cdot) \in (0,1)$ can be arbitrarily close to 1 but is bounded from below by a positive constant. It is thus natural to seek the optimal distribution, that is the one which minimizes the critical covered volume.

In the physical literature, it is strongly believed that, at least when $d = 2$ and $d = 3$, the critical covered volume is minimum in the case of a deterministic radius, when the distribution of radius is a Dirac measure. This conjecture is supported by numerical evidence (to the best of our knowledge, the most accurate estimations are given in a paper by Quintanilla and Ziff [15] when $d = 2$ and in a paper by Consiglio, Baker, Paul and Stanley [2] when $d = 3$). On Figure 1, we plot the critical covered volume in dimension 2 as a function of α and for different values of ρ when $\nu = (1-\alpha)\delta_1 + \alpha\rho^{-2}\delta_\rho$. The data for finite values of ρ come from numerical estimations in [15], while the data for infinite ρ come from the study of the multi-scale Boolean model. See Section 1.4 in [8] for further references. The conjecture is also supported by some heuristic arguments in any dimension (see for example Dhar [3]). See also [1]. In the above cited paper [12], it is noted that the rigorous proof of (5) suggests that the deterministic case might be optimal for any $d \geq 2$.

In this paper we show on the contrary that for all d large enough the critical covered volume is not minimized by the case of deterministic radii.

Critical intensity in high dimension : the case of a deterministic radius. Assume here that the measure ν is a Dirac mass at 1, that is that the radii of the balls are all equal to 1. Penrose proved the following result in [14] :

Theorem 1.1 (Penrose)**.** $\lim_{d\to\infty} \widetilde{\lambda}_d^c(\delta_1) = 1.$

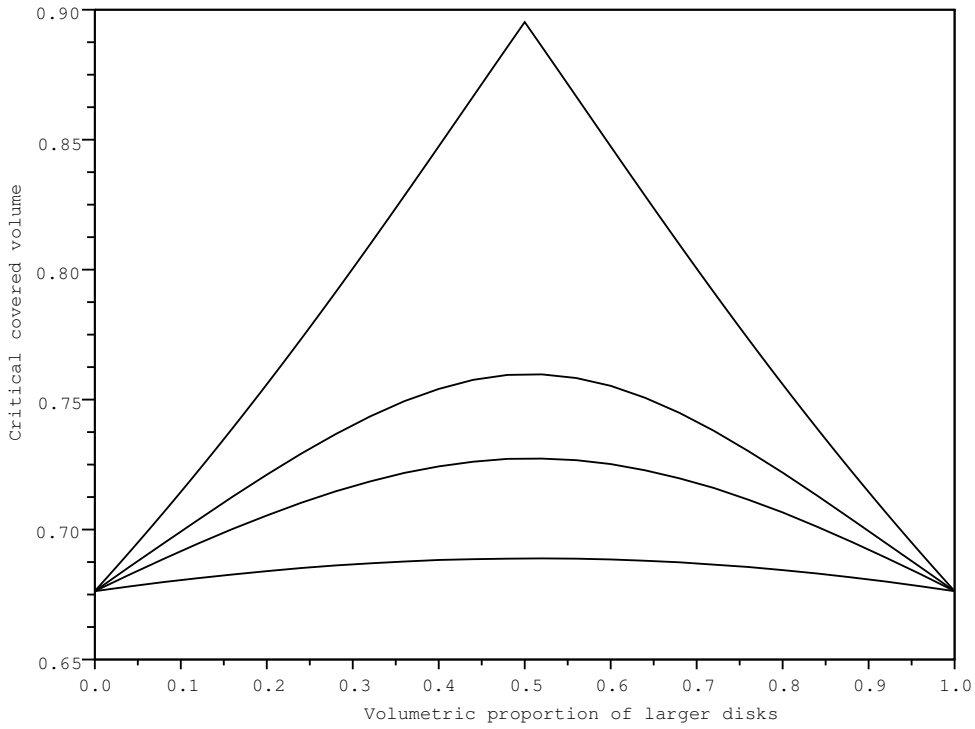

FIGURE 1. Critical covered volume as a function of α for different values of ρ. From bottom to top: $\rho = 2, \rho = 5, \rho = 10$ and the limit as $\rho \to \infty$.

With the scale invariance (3) of $\widetilde{\lambda}_d^c$, this limit can readily be generalized to any constant radius : for any $a > 0$,

$$\lim_{d\to\infty} \widetilde{\lambda}_d^c(\delta_a) = \lim_{d\to\infty} \widetilde{\lambda}_d^c(\delta_1) = 1.$$

Theorem 1.1 is the continuum analogue of a result of Kesten [10] for Bernoulli bond percolation on the nearest-neighbor integer lattice $\mathbb{Z}^d$, which says that the critical percolation parameter is asymptotically equivalent to $1/(2d)$.

Let us say a word about the ideas of the proof of Theorem 1.1.

The inequality $\widetilde{\lambda}_d^c(\delta_1) > 1$ holds for any $d \geq 2$. The proof is simple, and here is the idea. We consider the following natural genealogy. The deterministic ball $B(0,1)$ is said to be the ball of generation 0. The random balls of $\Sigma(\lambda\delta_1)$ that touch $B(0,1)$ are then the balls of generation 1. The random balls that touch one ball of generation 1 without being one of them are then the balls of generation 2 and so on. Let us denote by N_d the number of all balls that are descendants of $B(0,1)$. There is no percolation if and only if N_d is almost surely finite.

Now denote by m the Poisson distribution with mean $\lambda v_d 2^d$: this is the law of the number of balls of $\Sigma(\lambda\delta_1)$ that touch a given ball of radius 1. Therefore, if there were no interference between children of different balls, N_d would be equal to Z, the total population in a Galton-Watson process with offspring distribution m. Because of the interferences due to the fact that the Boolean model lives in $\mathbb{R}^d$, this is not true : in fact, N_d is only stochastically dominated by Z. Therefore, if $\lambda v_d 2^d \leq 1$, then Z is finite almost

surely, then N_d is finite almost surely and therefore there is no percolation. This implies

$$\widetilde{\lambda}_d^c(\delta_1) = v_d 2^d \lambda_d^c(\delta_1) > 1.$$

The difficult part of Theorem 1.1 is to prove that if d is large, then the interferences are small, then N_d is close to Z and therefore there is percolation for large d as soon as $v_d 2^d \lambda$ is a constant striclty larger than one.

To sum up, at first order, the asymptotic behavior of the critical intensity of the Boolean model with constant radius is given by the threshold of the associated Galton-Watson process, as in the case of Bernoulli percolation on $\mathbb{Z}^d$: roughly speaking, as the dimension increases, the geometrical constraints of the finite dimension space decrease and at the limit, we recover the non-geometrical case of the corresponding Galton-Watson process.

Critical intensity in high dimension : the case of random radii. If μ is a finite measure on $(0, +\infty)$ and if $d \geq 2$ is an integer, we define a measure μ_d on $(0, +\infty)$ by setting :

$$\mu_d(dr) = r^{-d}\mu(dr). \tag{7}$$

Note that, for any d, the assumption (2) is fulfilled by μ_d, and that $(\delta_1)_d = \delta_1$. Note also that μ_d is not necessarily a finite measure. However the definitions made above still make sense in this case and we still have $\lambda_d^c(\mu_d) \in (0, +\infty)$ thanks to Theorem 1.1 in [7].

We will study the behavior of $\widetilde{\lambda}_d^c(\mu_d)$ as d tends to infinity. Let us motivate the definition of μ_d with the following two related properties :

(1) Consider the Boolean model $\Sigma(\lambda\mu_d)$ on $\mathbb{R}^d$ driven by $\lambda\mu_d$ where $\lambda > 0$. For any $0 < s < t < \infty$, the number of balls of $\Sigma(\lambda\mu_d)$ with radius in $[s, t]$ that contains a given point is a Poisson random variable with intensity:

$$\int_{[s,t]} v_d r^d \lambda \mu_d(dr) = v_d \lambda \mu([s, t]).$$

Loosely speaking, this means that contrary to what happens in the Boolean model driven by $\lambda\mu$, the relative importance of radii of different sizes does not depend on the dimension d in the Boolean model driven by $\lambda\mu_d$.

(2) A closely related property is the following one. Consider for example the case $\mu = \alpha\delta_a + \beta\delta_b$. Then, a way to build $\Sigma(\lambda\mu_d)$ is to proceed as follows. Consider two independent Boolean model: Σ^A, driven by $\lambda\alpha\delta_1$, and Σ^B, driven by $\lambda\beta\delta_1$. Then set $\Sigma(\lambda\mu_d) = a\Sigma^A + b\Sigma^B$.

We prove the following result :

Theorem 1.2. *Let μ be a finite measure on $(0, +\infty)$. We assume that the mass of μ is positive and that μ is not concentrated on a singleton. Then :*

$$\limsup_{d\to+\infty} \frac{1}{d} \ln\left(\widetilde{\lambda}_d^c(\mu_d)\right) < 0.$$

As $(\delta_1)_d = \delta_1$, a straightforward consequence of Theorem 1.2 and Theorem 1.1 – or, actually, of the much weaker and easier convergence of $\ln(\widetilde{\lambda}_d^c(\delta_1))$ to 0 – is the following result:

Corollary 1.3. *Let μ be a finite mesure on $(0, +\infty)$. We assume that the mass of μ is positive and that μ is not concentrated on a singleton. Then, for any d large enough, we have:*

$$\widetilde{\lambda}_d^c(\mu_d) < \widetilde{\lambda}_d^c(\delta_1), \text{ or equivalently } c_d^c(\mu_d) < c_d^c(\delta_1).$$

In fact, Theorem 1.2 follows from the particular case of radii taking only two different values, for which we have more precise results.

Critical intensity in high dimension : the case of radii taking two values. To state the result, we need some further notations. Fix $\rho > 1$. Fix $k \geq 1$. Set $r_1 = r_{k+1} = 1 + \rho$, and for $i \in \{2, \dots, k\}$, $r_i = 2$. For $(a_i)_{2 \leq i \leq k+1} \in [0,1)^k$, we build an increasing sequence of distances $(d_i)_{1 \leq i \leq k+1}$ by setting $d_1 = 1 + \rho$ and, for every $i \in \{2, \dots, k+1\}$:

$$d_i^2 = d_{i-1}^2 + 2 r_i a_i d_{i-1} + r_i^2.$$

Note that the sequence $(d_i)_{1 \leq i \leq k+1}$ depends on ρ, k, and the a_i's.
We set $\mathcal{D}(a_2, \dots, a_{k+1}) = d_{k+1}$. Now set, for every $k \geq 1$,

$$\kappa_\rho^c(k) = \inf_{0 \leq a_2, \dots, a_{k+1} < 1} \max \left(\left(\frac{4\rho}{(1+\rho)^2 \sqrt{\prod_{2 \leq i \leq k+1} (1 - a_i^2)}} \right)^{\frac{1}{k+1}}, \frac{2\rho}{\mathcal{D}(a_2, \dots, a_{k+1})} \right). \tag{8}$$

Finally, let:

$$\kappa_\rho^c = \inf_{k \geq 1} \kappa_\rho^c(k). \tag{9}$$

We give some intuition on κ_ρ^c in Section 2.2. On Figure 2, we plot $\kappa_\rho^c(i)$, for $i \in \{1, 2, 3\}$. The data come from the formulas in Lemma 1.5 for $i = 1$ and from numerical estimations for $i \geq 2$.

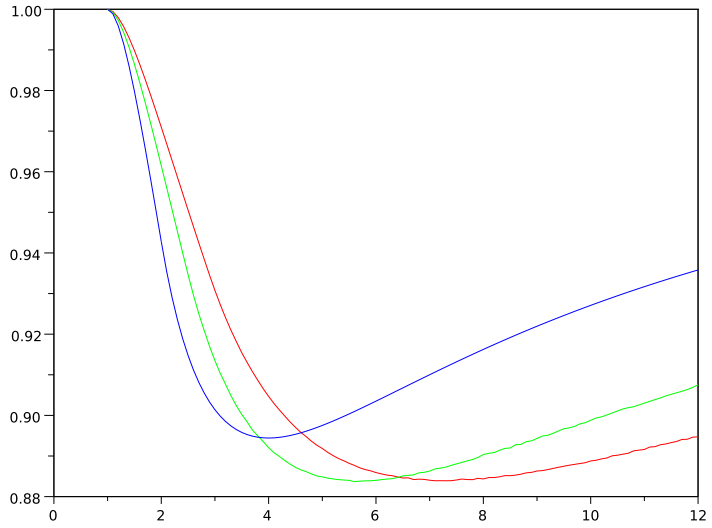

FIGURE 2. $\kappa_\rho^c(i)$, $i \in \{1, 2, 3\}$, from left to right.

This κ_ρ^c gives the asymptotic behaviour of $\widetilde{\lambda}_d^c(\mu_d)$ when μ charges two distinct points :

Theorem 1.4. *Let $b > a > 0$, $\alpha > 0$ and $\beta > 0$. Set $\mu = \alpha \delta_a + \beta \delta_b$ and $\rho = b/a > 1$. Then*

$$\lim_{d \to +\infty} \frac{1}{d} \ln \left(\widetilde{\lambda}_d^c(\mu_d) \right) = \ln \left(\kappa_\rho^c \right) < 0. \tag{10}$$

If one does not normalize the distribution one has [2] $\widetilde{\lambda}_d^c(\alpha\delta_a+\beta\delta_b) \to 1$ and thus $\widetilde{\lambda}_d^c(\alpha\delta_a+\beta\delta_b) \sim \widetilde{\lambda}_d^c(\delta_1)$. This behaviour is due to the fact that, without normalization, the influence of the small balls vanishes in high dimension.

In the next lemma we collect some properties of the $\kappa_\rho^c(k)$'s and κ_ρ^c. The only result needed for the proof of our main results is $0 < \kappa_\rho^c < 1$.

Lemma 1.5. *Let $\rho > 1$.*

- $0 < \kappa_\rho^c(1) < 1$. *More precisely :*

$$\text{If } 1 < \rho \leq 2 \text{ then } \kappa_\rho^c(1) = \frac{2\sqrt{\rho}}{1+\rho}, \qquad \text{while if } \rho \geq 2 \text{ then } \kappa_\rho^c(1) = \frac{\sqrt{4+\rho^2}}{1+\rho}.$$

- $0 < \kappa_\rho^c < 1$.
- *There exists $\rho_0 > 2$ such that if $\rho \leq \rho_0$, then $\kappa_\rho^c = \kappa_\rho^c(1)$. This implies*

$$\text{If } 1 < \rho \leq 2 \text{ then } \kappa_\rho^c = \frac{2\sqrt{\rho}}{1+\rho}, \qquad \text{while if } 2 \leq \rho \leq \rho_0 \text{ then } \kappa_\rho^c = \frac{\sqrt{4+\rho^2}}{1+\rho}.$$

- *As ρ goes to $+\infty$, $\kappa_\rho^c(k) = 1 - \dfrac{k}{\rho} + o(1/\rho)$. Thus one can not restrict the infimum in (9) to a finite number of k.*

Remember that in the case of a deterministic radius (assumptions of Theorem 1.1), the first order of the asymptotic behavior of the critical intensity in high dimension is given by the threshold of the associated Galton-Watson process.

In the case of two distinct radii (assumptions of Theorem 1.4), it is thus natural to compare κ_ρ^c with the critical parameter of the associated Galton-Watson process, which is now two-type, one for each radius.

Consider for example $\mu = \lambda(\delta_1 + \delta_\rho)$; then $\mu_d = \lambda\delta_1 + \frac{\lambda}{\rho^d}\delta_\rho$. Take $\lambda = \frac{\kappa^d}{v_d 2^d}$. Now consider the offspring distribution of type ρ of an individual of type 1. We define it to be the number of balls of a Boolean model directed by $\frac{\lambda}{\rho^d}\delta_\rho$ that intersects a given ball of radius 1. Therefore, this is a Poisson random variable with mean $\frac{\lambda}{\rho^d}v_d(1+\rho)^d$. The other offspring distribution are defined similarly. We moreover assume that the offspring of type 1 and ρ of a given individual are independent. The matrix of means of offspring distributions is thus given by:

$$M_d = \begin{pmatrix} \lambda v_d(1+1)^d & \frac{\lambda}{\rho^d}v_d(1+\rho)^d \\ \lambda v_d(1+\rho)^d & \frac{\lambda}{\rho^d}v_d(\rho+\rho)^d \end{pmatrix} = \kappa^d \begin{pmatrix} 1 & \left(\frac{1+\rho}{2\rho}\right)^d \\ \left(\frac{1+\rho}{2}\right)^d & 1 \end{pmatrix}.$$

Let r_d denotes the largest eigenvalue of M_d. The extinction probability of the two-type Galton-Watson process is 1 if and only if $r_d \leq 1$. We have:

$$r_d \sim \left(\frac{\kappa(1+\rho)}{2\sqrt{\rho}}\right)^d, \text{ and thus } \kappa = \frac{2\sqrt{\rho}}{1+\rho} \text{ is the critical parameter.}$$

With Theorem 1.4 and Lemma 1.5 we thus see that the comparison with the two-type Galton-Watson is asymptotically sharp on a logarithmic scale when $1 < \rho \leq 2$, but is not valid for $\rho > 2$. This contrasts with the case of a deterministic radius.

The proof of Theorem 1.1 (the constant radii case studied by Penrose) relies on the comparaison with a one-type Galton process which does not depend on d. In the two-values radii case, when κ is above or even slighlty below its critical parameter for the

2. The upper bound can be proven using $\lambda_d^c(\alpha\delta_a + \beta\delta_b) \leq \lambda_d^c(\beta\delta_b)$. The lower bound can be proven using the easy part of the comparison with a two-type Galton-Watson process.

Galton-Watson process, the mean number of children of type 1 of an individual of type ρ tends to infinity. This partly explains why geometrical dependencies can not be handled in the same way as in the constant radii case. This also partly explains why the critical value for percolation in not always given by the critical value for the Galton-Watson process.

The proofs of Theorem 1.4 and of Lemma 1.5 are given in Section 2. The main ideas of the proofs are given in Section 2.2. Theorem 1.2 is an easy consequence of Theorem 1.4. The proof is given in Section 3.

2. The case when the radii take two values

Before proving Theorem 1.4, we begin with the proof of Lemma 1.5:

2.1. **Proof of Lemma 1.5.**

• By definition, $\kappa_\rho^c(1) = \inf_{0\le a<1} \max(\phi_1(a), \phi_2(a))$, where $\phi_1, \phi_2 : [0,1) \to \mathbb{R}$ are defined by:

$$\phi_1(a) = \frac{2\sqrt{\rho}}{(1+\rho)(1-a^2)^{1/4}} \quad \text{and} \quad \phi_2(a) = \frac{\rho\sqrt{2}}{(1+\rho)\sqrt{1+a}}.$$

If $\rho \le 2$ then $\phi_1(0) \ge \phi_2(0)$. As ϕ_1 is increasing and ϕ_2 is decreasing, we get:

$$\kappa_\rho^c(1) = \inf_{0\le a<1} \phi_1(a) = \phi_1(0) = \frac{2\sqrt{\rho}}{1+\rho}.$$

Assume, on the contrary, $\rho \ge 2$. Set

$$a = \frac{\rho^2-4}{\rho^2+4} \in [0,1).$$

Then $\phi_1(a) = \phi_2(a)$. As ϕ_1 is increasing and ϕ_2 is decreasing, we get:

$$\kappa_\rho^c(1) = \phi_1(a) = \phi_2(a) = \frac{\sqrt{4+\rho^2}}{1+\rho}.$$

• Clearly we have, for every $k \ge 1$:

$$\kappa_\rho^c(k) \;\ge\; \inf_{0\le a_2,\dots,a_{k+1}<1} \left(\frac{4\rho}{(1+\rho)^2 \sqrt{\prod_{2\le i\le k+1}(1-a_i^2)}} \right)^{1/(k+1)} = \left(\frac{4\rho}{(1+\rho)^2} \right)^{1/(k+1)}.$$

Therefore, as $\kappa_\rho^c = \inf_{k\ge1} \kappa_\rho^c(k)$,

$$0 < \left(\frac{4\rho}{(1+\rho)^2} \right)^{1/2} \le \kappa_\rho^c \le \kappa_\rho^c(1) < 1.$$

• The last inequalities imply that $\kappa_\rho^c = \kappa_\rho^c(1)$ if $1 < \rho \le 2$. In fact, as soon as

$$\kappa_\rho^c(1) \le \left(\frac{4\rho}{(1+\rho)^2} \right)^{1/3}, \tag{11}$$

it is true that $\kappa_\rho^c(1) \le \kappa_\rho^c(k)$ for all $k \ge 2$ and therefore that $\kappa_\rho^c = \kappa_\rho^c(1)$. As the inequality in (11) is strict for $\rho = 2$, we obtain by continuity the existence of $\rho_0 > 2$ such that for every $\rho \in (1, \rho_0)$, $\kappa_\rho^c = \kappa_\rho^c(1)$.

• The minoration follows easily from the following observation: by construction, we have $\mathcal{D}(a_2, \dots, a_{k+1}) \le 2(\rho + k)$. This implies

$$\kappa_\rho^c(k) \ge \inf_{a_2,\dots a_{k+1}} \frac{2\rho}{\mathcal{D}(a_2,\dots,a_{k+1})} \ge \frac{\rho}{\rho+k} = 1 - \frac{k}{\rho} + o(1/\rho).$$

To obtain the majoration, fix $k \geq 1$. Take $\mu > 1/2$ and $\varepsilon > 0$ such that $\mu + (k-1)\varepsilon < 1$. Take, for $2 \leq i \leq k$, the specific values $a_i = \cos(\rho^{-\varepsilon}) = 1 + o(\rho^{-\varepsilon})$ and $a_{k+1} = \cos(\rho^{-\mu})$. Hence,

$$\frac{4\rho}{(1+\rho)^2\sqrt{\prod_{2\leq i\leq k+1}(1-a_i^2)}} \sim 4\rho^{-1+(k-1)\varepsilon+\mu}.$$

For $i \leq k-1$, we have $0 \leq d_{i+1} - d_i \leq d_{i+1} - a_i d_i \leq \dfrac{d_{i+1}^2 - (a_i d_i)^2}{d_{i+1} + a_i d_i} \leq \dfrac{2}{\rho+1}$.
By summation, we get $d_k = (1+\rho)(1 + \frac{2(k-1)}{1+\rho} + o(\rho^{-1}))$. Now,

$$d_{k+1}^2 = d_k^2 + 2(1+\rho)d_k\cos(\rho^{-\mu}) + (1+\rho)^2, \text{ and thus } \frac{2\rho}{d_{k+1}} = 1 - \frac{k-1}{\rho} + o(\rho^{-1}).$$

Finally, $\kappa_\rho^c(k) \leq \max\left(\left(\dfrac{4\rho}{(1+\rho)^2\sqrt{\prod_{2\leq i\leq k+1}(1-a_i^2)}}\right)^{\frac{1}{k+1}}, \dfrac{2\rho}{d_{k+1}}\right) \leq 1 - \dfrac{k-1}{\rho} + o(\rho^{-1})$.
This ends the proof. □

2.2. **Notations and ideas of the proof of Theorem 1.4.** In the whole proof, we fix $\rho > 1$ and $\kappa > 0$.

Once the dimension $d \geq 1$ is given, we consider two independent stationary Poisson point processes on $\mathbb{R}^d$: χ_1 and χ_ρ, with respective intensities

$$\lambda_1 = \frac{\kappa^d}{v_d 2^d} \quad \text{and} \quad \lambda_\rho = \frac{\kappa^d}{v_d 2^d \rho^d}.$$

With χ_1 and χ_ρ, we respectively associate the two Boolean models

$$\Sigma_1 = \bigcup_{x\in\chi_1} B(x,1) \quad \text{and} \quad \Sigma_\rho = \bigcup_{x\in\chi_\rho} B(x,\rho).$$

Note that Σ_ρ is an independent copy of $\rho\Sigma_1$. Note also that the expected number of balls of Σ_1 that touches a given ball of radius 1 is κ^d. Thus the expected number of balls of Σ_ρ that touches a given ball of radius ρ is also κ^d.

We focus on the percolation properties of the following two-type Boolean model

$$\Sigma = \Sigma_1 \cup \Sigma_\rho.$$

We begin by studying the existence of infinite k-alternating paths. For $k \geq 1$, an infinite k-alternating path is an infinite path made of balls such that the radius of the first ball is ρ, the radius of the next k balls is 1, the radius of the next ball is ρ and so on. For a fixed $k \geq 1$, we wonder whether infinite k-alternating paths exist and seek the critical threshold $\kappa_\rho^c(k)$ for their existence. A natural first step is to study the following quantities:

$$N_0 = \#\{x_1 \in \chi_\rho : \|x_1\| < 2\rho\}, \tag{12}$$

$$\text{and for } k \geq 1, \quad N_k = \#\left\{\begin{array}{l} x_{k+1} \in \chi_\rho : \exists (x_i)_{1\leq i\leq k} \in \chi_1 \text{ distinct such that} \\ \|x_1\| < 1+\rho,\ \forall i \in \{1,\dots,k-1\}\ \|x_{i+1} - x_i\| < 2, \\ \|x_{k+1} - x_k\| < 1+\rho \end{array}\right\}.$$

Fix $k \geq 1$. Remember that $\kappa_\rho^c(k)$ is defined in (8).

A lower bound for $\kappa_\rho^c(k)$. In Subsection 2.3, we obtain lower bounds for $\kappa_\rho^c(k)$ by looking for upper bounds for $E(N_k)$. On one side, a natural genealogy is associated to the definition of N_k (see also the comments below Theorem 1.1 and below Lemma 1.5). We start with an ancestor x_0 located at the origin. We then seek his children in $\chi_1 \cap B(x_0, 1+\rho)$: they constitute the first generation. If x_1 is one of those children, we then seek the children of x_1 in $\chi_1 \cap B(x_1, 2)$ to build the second generation and so on. On the other side, the process lives in $\mathbb{R}^d$ and the geometry induces dependences: if x_1 and x_1' are two individuals of the first generation, their children are a priori dependent. If we forget geometry and only consider genealogy, we get the following upper bound:

$$E(N_k) \leq \lambda_1 |B(\cdot, 1+\rho)| \left(\prod_{i=2}^{k} \lambda_1 |B(\cdot, 2)| \right) \lambda_\rho |B(\cdot, 1+\rho)|.$$

But the points of the last generation are in $B(0, 2\rho + 2k)$. So if we forget genealogy and only consider geometry we get the following upper bound:

$$E(N_k) \leq \lambda_\rho |B(0, 2\rho + 2k)|.$$

Expliciting the two previous bounds and combining them together, we get:

$$E(N_k) \leq \min \left(\frac{\kappa^{k+1}(1+\rho)^2}{4\rho}, \frac{\kappa(\rho+k)}{\rho} \right)^d.$$

In this upper bound, the first argument of the minimum is due to genealogy while the second one is due to geometry. To get the geometrical term, we considered the worst case: the one in which, at each generation i, x_i is as far from the origin as possible. This gives a very poor bound. To get a better bound, we proceed as follows. Fix $a_2, \ldots, a_{k+1} \in [0, 1)$. As before, we set $r_1 = r_{k+1} = 1 + \rho$, and for $i \in \{2, \ldots, k\}$, $r_i = 2$ and we build the increasing sequence of distances $(d_i)_{1 \leq i \leq k+1}$ by setting $d_1 = 1 + \rho$ and, for every $i \in \{2, \ldots, k+1\}$:

$$d_i^2 = d_{i-1}^2 + 2 r_i a_i d_{i-1} + r_i^2.$$

See Figure 3 for a better understanding of these distances d_i.

Denote by $\widetilde{N}_k(a_2, \ldots, a_{k+1})$ the number of points $x_{k+1} \in \chi_\rho$ for which there exists a path $x_1, \ldots, x_k$ fulfilling the same requirement as for N_k and such that $\|x_i\| \approx d_i$ for all i. Proceeding as before, we obtain the following upper bound:

$$E(\widetilde{N}_k(a_2, \ldots, a_{k+1})) \lesssim \min \left(\kappa^{k+1} \frac{(1+\rho)^2}{4\rho} \sqrt{\prod_{2 \leq i \leq k+1} (1-a_i)^2}, \frac{\kappa \mathcal{D}(a_2, \ldots, a_{k+1})}{2\rho} \right)^d.$$

Here again, the first argument of the minimum is due to genealogy while the second one is due to geometry [3]. Optimizing then on the a_i's, we get:

$$E(N_k) \lesssim \sup_{a_2, \ldots, a_{k+1}} \min \left(\kappa^{k+1} \frac{(1+\rho)^2}{4\rho} \sqrt{\prod_{2 \leq i \leq k+1} (1-a_i)^2}, \frac{\kappa \mathcal{D}(a_2, \ldots, a_{k+1})}{2\rho} \right)^d.$$

A precise statement is given in Lemma 2.4. The precise value of the threshold $\kappa_\rho^c(k)$ given in (8) is then the value such that the above upper bound converges to 0 when $\kappa < \kappa_\rho^c(k)$. This heuristic will be precised in Subsection 2.3: we will prove there that when $\kappa < \kappa_\rho^c(k)$,

3. There is essentially no geometrical constraint in generations 1 to k. Very roughly, this is due to the fact that, when i increases from 2 to k : there is more and more space (the d_i are increasing) ; the intensity of the relevant Poisson point process is the same ; the expected number of individuals in the i^{th} generation of the Galton-Watson process decreases.

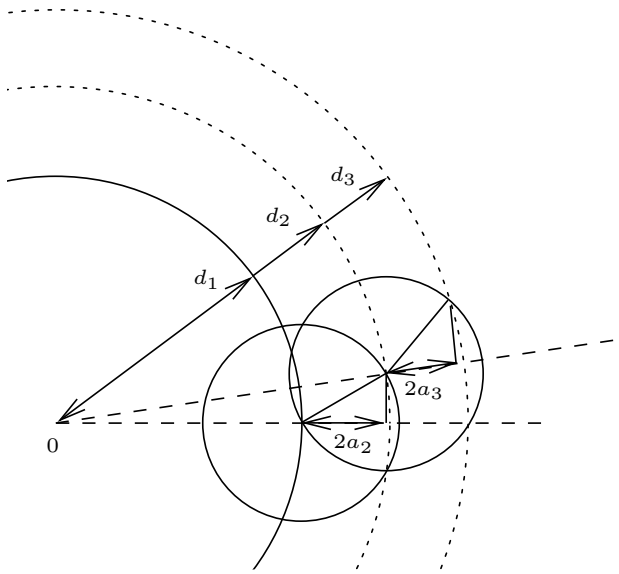

FIGURE 3. Definition of the distances d_i. Circles in plain line are of radius $1+\rho$ and 2.

$E(N_k)$ converges to 0 as d tends to infinity, and this will imply that there exists no infinite k-alternating path.

An upper bound for $\kappa_\rho^c(k)$. If, on the contrary, $\kappa > \kappa_\rho^c(k)$ then we will prove that $E(N_k)$ does not converge to 0. Actually, to prove that when $\kappa > \kappa_\rho^c(k)$ there exist infinite k-alternating path, we will show, in Subsection 2.4, the following stronger property : with a probability that converges to 1 as d tends to infinity, we can find a path which fulfills the requirements of the definition of N_k – or more precisely of $\widetilde{N}_k(a_2,\dots,a_{k+1})$ for some $a_2,\dots,a_{k+1}$ nearly optimal – and which fulfills some extra conditions on the positions of the balls. This is Proposition 2.8 and this is the main technical part of this paper. Those extra conditions provide independence properties between the existence of different paths of the same kind. We can then show the existence of many such paths and concatenate some of them to build an infinite k-alternating path. Technically, the last step is achieved by comparing our model with a supercritical oriented percolation process on $\mathbb{Z}^2$. In this comparison, an open bond in the oriented percolation process corresponds to one of the above paths in our model. This comparison with oriented percolation was already the last step in the paper of Penrose [14].

From infinite k-alternating paths to infinite paths. Recall $\kappa_\rho^c = \inf_{k\geq 1}\kappa_\rho^c(k)$. With the previous results, it is rather easy to show that there is no percolation for d large enough as

soon as $\kappa < \kappa_\rho^c$. When $\kappa > \kappa_\rho^c$ then $\kappa > \kappa_\rho^c(k)$ for a $k \geq 1$. Therefore there is k-alternating percolation and therefore there is percolation.

2.3. **Subcritical phase.** Let $\rho > 1$ be fixed. We consider, in $\mathbb{R}^d$, the two-type Boolean model Σ introduced in Subsection 2.2, with radii 1 and ρ and respective intensities

$$\lambda_1 = \frac{\kappa^d}{v_d 2^d} \text{ and } \lambda_\rho = \frac{\kappa^d}{v_d 2^d \rho^d}$$

depending on some $\kappa \in (0,1)$. The aim of this subsection is to prove the following proposition:

Proposition 2.1. *Let $\rho > 1$ be fixed. If $\kappa < \kappa_\rho^c$, then, as soon as the dimension d is large enough, percolation does not occur in the two-type Boolean model Σ.*

In the following of this subsection, we fix $\rho > 1$ and $0 < \kappa < \kappa_\rho^c$.

We start with an elementary upper bound, in which we do not take into account the geometrical constraints. We recall that the N_k have been introduced in (12).

Lemma 2.2. *$E(N_0) = \kappa^d$ and, for $k \geq 1$, $E(N_k) \leq \left(\dfrac{\kappa^{k+1}(1+\rho)^2}{4\rho} \right)^d$.*

Proof. The result for N_0 follows directly from the equality $E(N_0) = \lambda_\rho |B(0, 2\rho)|$.

Take now $k \geq 1$. We have :

$$E(N_k) \leq \lambda_1 |B(\cdot, 1+\rho)| \left(\prod_{i=2}^{k} \lambda_1 |B(\cdot, 2)| \right) \lambda_\rho |B(\cdot, 1+\rho)| \tag{13}$$

where $B(\cdot, r)$ stands for a ball with radius r and center unspecified. This can for instance be seen as follows :

$$\begin{aligned} E(N_k) &\leq E\left(\sum_{x_1,\ldots,x_k \in \chi_1 \text{ distinct},\, x_{k+1} \in \chi_\rho} 1_{x_1 \in B(0,1+\rho)} \ldots 1_{x_{k+1} \in B(x_k, 1+\rho)} \right) \\ &= \lambda_1^k \lambda_\rho \int_{\mathbb{R}^{d(k+1)}} dx_1 \ldots dx_{k+1} 1_{x_1 \in B(0,1+\rho)} \ldots 1_{x_{k+1} \in B(x_k, 1+\rho)}, \end{aligned}$$

which gives (13). The lemma follows. □

To give a more accurate upper bound for the N_k's, we are going to cut the balls into slices and to estimate which slices give the main contribution. For $x \in \mathbb{R}^d \setminus \{0\}$, $0 \leq a < b \leq 1$ and $r > 0$, we now define :

$$\text{If } a > 0 : \quad B(x, r, a, b) = \left\{ y \in \mathbb{R}^d : \|y - x\| \leq r \text{ and } ar < \left\langle y - x, \frac{x}{\|x\|} \right\rangle \leq br \right\},$$

$$\text{If } a = 0 : \quad B(x, r, a, b) = \left\{ y \in \mathbb{R}^d : \|y - x\| \leq r \text{ and } \left\langle y - x, \frac{x}{\|x\|} \right\rangle \leq br \right\}.$$

The next lemma gives asymptotics for the volume of these sets:

Lemma 2.3. *For $x \in \mathbb{R}^d \setminus \{0\}$, $0 \leq a < b \leq 1$ and $r > 0$,*

$$\lim_{d \to +\infty} \frac{1}{d} \ln \left(\frac{|B(x, r, a, b)|}{v_d} \right) = \ln \left(r\sqrt{1 - a^2} \right).$$

Actually we will only use :

$$\limsup_{d\to+\infty} \frac{1}{d} \ln\left(\frac{|B(x,r,a,b)|}{v_d}\right) \leq \ln\left(r\sqrt{1-a^2}\right).$$

Proof of Lemma 2.3. Note that it is sufficient to prove the lemma for $x = e_1$, first vector of the canonical basis, and $r = 1$.

First, if $a = 0$, the result follows directly from the inequality $v_d/2 \leq |B(e_1, 1, 0, b)| \leq v_d$.

Assume next that $a > 0$. On the one hand, $B(e_1, 1, a, b)$ is included in the cylinder

$$\{x = (x_1, \ldots, x_d) : x_1 \in [a, 1] \text{ and } \|(0, x_2, \ldots, x_d)\| \leq \sqrt{1-a^2}\},$$

which implies

$$|B(e_1, 1, a, b)| \leq v_{d-1}\sqrt{1-a^2}^{d-1}(1-a). \tag{14}$$

On the other end, by convexity, $B(e_1, 1, a, b)$ contains the following difference between two homothetical cones:

$$\left\{x = (x_1, \ldots, x_d) : x_1 \in [a, b] \text{ and } \|(0, x_2, \ldots, x_d)\| \leq \sqrt{1-a^2}\frac{1-x_1}{1-a}\right\},$$

which implies

$$\frac{v_{d-1}\sqrt{1-a^2}^{d-1}}{d}\big((1-a)-(1-b)\big) \leq |B(e_1, 1, a, b)|. \tag{15}$$

The lemma follows from (14) and (15). □

We can now improve the control given in Lemma 2.2:

Lemma 2.4. *For every $k \geq 1$,*

$$\begin{aligned}&\limsup_{d\to\infty} \frac{1}{d} \ln(E(N_k))\\ \leq\ & \ln\left(\sup_{0\leq a_2,\ldots,a_{k+1}<1} \min\left(\kappa^{k+1}\frac{(1+\rho)^2}{4\rho}\sqrt{\prod_{2\leq i\leq k+1}(1-a_i^2)}, \frac{\kappa\mathcal{D}(a_2,\ldots,a_{k+1})}{2\rho}\right)\right)\end{aligned}$$

Proof. • Fix $N \geq 1$. Note that the ball $B(x, r)$ is the disjoint union of the slices $B(x, r, n/N, (n+1)/N)$ for $n \in \{0, \ldots, N-1\}$. For any $n_2, \ldots, n_{k+1} \in \{0, \ldots, N-1\}$, we set

$$a_i = \frac{n_i}{N} \text{ and } a_i^+ = \frac{n_i+1}{N}.$$

We focus on the contribution of a specific product of slices :

$$N_k(n_2, \ldots, n_{k+1}) = \#\left\{\begin{array}{l} x_{k+1} \in \chi_\rho : \exists (x_i)_{1\leq i\leq k} \in \chi_1 \text{ distinct with} \\ \|x_1\| < 1+\rho,\ \forall i \in \{1, \ldots, k-1\}\, x_{i+1} \in B(x_i, 2, a_{i+1}, a_{i+1}^+), \\ x_{k+1} \in B(x_k, 1+\rho, a_{k+1}, a_{k+1}^+)\end{array}\right\}.$$

Then we have :

$$N_k \leq \sum N_k(n_2, \ldots, n_{k+1}), \tag{16}$$

where the sum is over $(n_2, \ldots, n_{k+1}) \in \{0, \ldots, N-1\}^k$.

• As we can check that the points contributing to $N_k(n_2, \ldots, n_{k+1})$ are in $B(0, \mathcal{D}(a_2^+, \ldots, a_{k+1}^+))$, we get :

$$E(N_k(n_2, \ldots, n_{k+1})) \leq \lambda_\rho v_d \mathcal{D}(a_2^+, \ldots, a_{k+1}^+)^d,$$

this leads to :

$$\limsup_{d\to+\infty} \frac{1}{d} \ln\left(E(N_k(n_2,\dots,n_{k+1}))\right) \leq \ln\left(\frac{\kappa \mathcal{D}(a_2^+,\dots,a_{k+1}^+)}{2\rho}\right). \tag{17}$$

• Besides, proceeding as in the proof of Lemma 2.2, we obtain :

$$E(N_k(n_2,\dots,n_{k+1})) \leq \lambda_1 |B(0,1+\rho)| \left(\prod_{i=2}^{k} \lambda_1 |B(\cdot,2,a_i,a_i^+)|\right) \lambda_\rho |B(\cdot,1+\rho,a_{k+1},a_{k+1}^+)|.$$

With Lemma 2.3, we deduce :

$$\limsup_{d\to\infty} \frac{1}{d} \ln E(N_k(n_2,\dots,n_{k+1})) \leq \ln\left(\kappa^{k+1} \frac{(1+\rho)^2}{4\rho} \sqrt{\prod_{2\leq i\leq k+1} (1-a_i^2)}\right). \tag{18}$$

• From (16), (17) and (18) we finally get :

$$\begin{aligned} &\limsup_{d\to+\infty} \frac{\ln(E(N_k))}{d} \\ \leq\ & \ln\left(\max_{a_2,\dots,a_{k+1}\in\{0,\dots,\frac{N-1}{N}\}} \min\left(\kappa^{k+1}\frac{(1+\rho)^2}{4\rho}\sqrt{\prod_{2\leq i\leq k+1}(1-a_i^2)}, \frac{\kappa\mathcal{D}(a_2^+,\dots,a_{k+1}^+)}{2\rho}\right)\right). \end{aligned}$$

As $\mathcal{D}$ is uniformly continuous on $[0,1]^k$, we end the proof by taking the limit when N goes to $+\infty$. □

The next step consists in taking into account all $k \geq 0$ simultaneously; we thus introduce

$$N = \#\left(\left\{ \begin{array}{l} y \in \chi_\rho : \exists k \geq 1, \exists (x_i)_{1\leq i\leq k} \in \chi_1 \text{ distinct with} \\ \|x_1\| < 1+\rho,\ \forall i \in \{1,\dots,k-1\}\ \|x_{i+1}-x_i\| < 2, \\ \|y - x_k\| < 1+\rho \end{array} \right\} \cup \{y \in \chi_\rho : \|y\| < 2\rho\}\right). \tag{19}$$

Lemma 2.5. *If $\kappa < \kappa_\rho^c$, then $\limsup_{d\to+\infty} \frac{1}{d}\ln(E(N)) < 0$.*

Proof. We have :

$$E(N) \leq \sum_{k\geq 0} E(N_k). \tag{20}$$

As $\kappa < \kappa_\rho^c$, Lemma 2.4 ensures that for every $k \geq 1$:

$$\limsup_{d\to+\infty} \frac{1}{d}\ln(E(N_k)) < 0. \tag{21}$$

Moreover, the assumption $\kappa < \kappa_\rho^c$ also implies, thanks to Lemma 1.5, that $\kappa < 1$. We can then choose k_0 large enough to have :

$$\frac{\kappa^{k_0+1}(1+\rho)^2}{4\rho} \leq \exp(-1).$$

With Lemma 2.2, we thus get :

$$E(N_0) + \sum_{k\geq k_0} E(N_k) \leq \kappa^d + \exp(-d) \sum_{k\geq 0} \kappa^{kd} = \kappa^d + \exp(-d)\frac{1}{1-\kappa^d}.$$

With (20) and (21), this ends the proof. □

The next lemma is elementary

Lemma 2.6. *Assume $\kappa < 1$.*
Then the connected components of $\bigcup_{x\in\chi_1} B(x,1)$ are bounded with probability 1.

Proof. For any integer $k \geq 0$, denote by M_k the number of balls with radius 1 linked to $B(0,1)$ by a chain of k distinct balls with radius 1. Proceeding as in the proof of Lemma 2.2, we get :

$$E(M_k) \leq \kappa^{d(k+1)}.$$

Now denote by M the number of balls with radius 1 linked to $B(0,1)$ by a chain of (perhaps no) balls with radius 1. Then :

$$E(M) \leq E\left(\sum_{k\geq 0} M_k\right) = \frac{\kappa^d}{1-\kappa^d} < +\infty.$$

Therefore, M is finite with probability 1. So the connected components that touch $B(0,1)$ are bounded with probability 1. So with probability 1, every connected component is bounded. □

Proof of Proposition 2.1. Remember that we proved in Lemma 1.5 that $\kappa_\rho^c < 1$. Take κ such that $0 < \kappa < \kappa_\rho^c < 1$.

Let ξ_1 be the set of random balls with radius ρ that can be connected to $B(0,\rho)$ through a chain of random balls with radius 1 (we consider the condition as fulfilled if the ball touches $B(0,\rho)$ directly). Let ξ_2 be the set of random balls with radius ρ that are not in ξ_1, but that can be connected to $B(0,\rho)$ through a path of random balls in which there is only one ball with radius ρ. We define similarly ξ_3, ξ_4 and so on and denote by ξ the disjoint union of all these sets.

We have $\#\xi_1 = N$. (Remember that N has been defined in (19).) By Lemma 2.5, we have :

$$\limsup_{d\to+\infty} \frac{1}{d}\ln(E(\xi_1)) < 0.$$

Take some $\mu > 0$ and assume from now on that d is large enough to have

$$\frac{1}{d}\ln(E(\#\xi_1)) \leq -\mu.$$

For every $k \geq 1$, we have then: $E(\#\xi_k) \leq (E(\#\xi_1))^k \leq \exp(-dk\mu)$.

As $\xi = \cup_{k\geq 1}\xi_k$, we deduce from the previous inequalities that ξ is finite with probability 1. So if an unbounded connected component of $\Sigma_1 \cup \Sigma_\rho$ touches $B(0,\rho)$ then there is an unbounded component in Σ_1. As $\kappa < 1$, Lemma 2.6 rules out the possibility of an unbounded connected component in Σ_1. So with probability 1, the connected components of $\Sigma_1 \cup \Sigma_\rho$ that touch $B(0,\rho)$ are bounded, which ends the proof. □

2.4. **Supercritical phase.** We fix here $\rho > 1$. We consider once again the two-type Boolean model Σ introduced in Subsection 2.2 and we fix an integer $k \geq 1$.

For every $n \geq 0$, we set $R_n = \rho$ if $k+1$ divides n and $R_n = 1$ otherwise. We say that percolation by k-alternation occurs if there exists an infinite sequence of distinct points $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ such that, for every $n \geq 0$:

– $x_n \in \chi_{R_n}$.
– $B(x_n, R_n) \cap B(x_{n+1}, R_{n+1}) \neq \emptyset$.

In other words, percolation by k-alternation occurs if there exists an infinite path along which k balls of radius 1 alternate with one ball of radius ρ, *ie* if there exists an infinite k-alternating path. The aim of this subsection is to prove the following proposition:

Proposition 2.7. *Let $\rho > 1$ and $k \geq 1$ be fixed. Assume that $\kappa \in (\kappa_\rho^c(k), 1)$. If the dimension d is large enough, then percolation by k-alternation occurs with probability one.*

As announced in Subsection 2.2, percolation by k-alternation of the two-type Boolean model in the supercritical case will be proved by embedding in the model a supercritical 2-dimensional oriented percolation process.

We thus specify the two first coordinates, and introduce the following notations. When $d \geq 3$, for any $x \in \mathbb{R}^d$, we write

$$x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{d-2}.$$

We write $B'(c, r)$ for the open Euclidean balls of $\mathbb{R}^2$ with center $c \in \mathbb{R}^2$ and radius $r > 0$. In the same way we denote by $B''(c, r)$ the open Euclidean balls of $\mathbb{R}^{d-2}$ with center $c \in \mathbb{R}^{d-2}$ and radius $r > 0$.

2.4.1. *One step in the 2-dimensional oriented percolation model.* The point here is to define the event that will govern the opening of the edges in the 2-dimensional oriented percolation process : it is naturally linked to the existence of a finite path composed of k balls of radius 1 and a ball of radius ρ, whose positions of centers are specified.

We define, for a given dimension d, the two following subsets of $\mathbb{R}^d$:

$$\begin{aligned} W &= d^{-1/2}\left((-1,1) \times (-1,0) \times \mathbb{R}^{d-2}\right), \\ W^+ &= d^{-1/2}\left((0,1) \times (0,1) \times \mathbb{R}^{d-2}\right). \end{aligned}$$

For $x_0 \in W$ we set :

$$\mathcal{G}^+(x_0) = \left\{ \begin{array}{c} \text{There exist distinct } x_1, \ldots, x_k \in \chi_1 \cap W^+ \text{ and } x_{k+1} \in \chi_\rho \cap W^+ \\ \text{such that } x_0, x_1, \ldots, x_{k+1} \text{ is a path} \end{array} \right\}. \quad (22)$$

Our goal here is to prove that the probability of occurrence of this event is asymptotically large :

Proposition 2.8. *Let $\rho > 1$ and $k \geq 1$ be fixed. Assume that $\kappa \in (\kappa_\rho^c(k), 1)$ and choose $p \in (0, 1)$. If the dimension d is large enough, then for every $x \in W$,*

$$P(\mathcal{G}^+(x)) \geq p.$$

Note already that by translation invariance, $P(\mathcal{G}^+(x))$ does not depend on x'', so we can assume without loss of generality that $x'' = 0$. In the sequel of this subsection, $\rho > 1$ and $k \geq 1$ are fixed.

We first recall the definitions of the $(d_i)_{1 \leq i \leq k+1}$ and of $\kappa_\rho^c(k)$ we give in the introduction. We set $r_1 = r_{k+1} = 1 + \rho$ and for $i \in \{2, \ldots, k\}$, $r_i = 2$. Then, for a given sequence $(a_i)_{2 \leq i \leq k+1} \in [0, 1)^k$, we build an increasing sequence $(d_i)_{1 \leq i \leq k+1}$ by setting $d_1 = 1 + \rho$ and, for every $i \in \{2, \ldots, k+1\}$:

$$d_i^2 = d_{i-1}^2 + 2 r_i a_i d_{i-1} + r_i^2.$$

Finally, we note $\mathcal{D}(a_2, \ldots, a_{k+1}) = d_{k+1}$ and we set

$$\kappa_\rho^c(k) = \inf_{0 \leq a_2, \ldots, a_{k+1} < 1} \max\left(\left(\frac{4\rho}{(1+\rho)^2 \sqrt{\prod_{2 \leq i \leq k+1}(1 - a_i^2)}} \right)^{1/(k+1)}, \frac{2\rho}{\mathcal{D}(a_2, \ldots, a_{k+1})} \right).$$

The first step consists in choosing a nearly optimal sequence $(a_i)_{2\leq i\leq k+1}\in[0,1)^k$ satisfying some extra inequalities :

Lemma 2.9. *We can choose $(a_i)_{2\leq i\leq k+1}\in[0,1)^k$ such that :*

$$1<\kappa^{k+1}\frac{(1+\rho)^2}{4\rho}\sqrt{\prod_{2\leq j\leq k+1}(1-a_j^2)}<\kappa\frac{d_{k+1}}{2\rho}. \tag{23}$$

Proof. As $\kappa_\rho^c(k)<\kappa$, we can choose $(a_i^0)_{2\leq i\leq k+1}\in(0,1)^k$ such that the two following conditions

$$\kappa \;>\; \left(\frac{4\rho}{(1+\rho)^2\sqrt{1-a_2^2}\dots\sqrt{1-a_{k+1}^2)}}\right)^{\frac{1}{k+1}}, \tag{24}$$

$$\kappa \;>\; \frac{2\rho}{d_{k+1}} \tag{25}$$

are fullfilled for $(a_i)_i=(a_i^0)_i$. We fix $(a_3,\cdots,a_{k+1})=(a_3^0,\cdots,a_{k+1}^0)$. Note that

$$f:\ a_2\mapsto\kappa\frac{d_{k+1}}{2\rho} \qquad \text{is continuous and increasing,}$$

$$g:\ a_2\mapsto\kappa^{k+1}\frac{(1+\rho)^2}{4\rho}\sqrt{\prod_{2\leq j\leq k+1}(1-a_j^2)} \qquad \text{is continuous and decreasing,}$$

and that $\lim_{a_2\to1}g(a_2)=0$. Moreover, Conditions (25) and (24) ensure that $f(a_2^0)>1$ and $g(a_2^0)>1$.

Thus if $f(a_2^0)>g(a_2^0)$ the proof is over. If $f(a_2^0)\leq g(a_2^0)$, we can take $a_2>a_2^0$ such that $1<g(a_2)<f(a_2^0)$: then $f(a_2)\geq f(a_2^0)>g(a_2)>1$ and the lemma is proved. □

Note that (23) implies (24) and (25).

As explained in Subsection 2.3, the main contribution to the number N_k of centers x_{k+1} of balls of radius ρ that are linked to a ball of radius ρ centered at the origin by a chain $(x_i)_{1\leq i\leq k}$ of k balls of radius 1 – see the precise definition (12) – is obtained for $\|x_i\|\sim d_i$, where the $d_i's$ are build from a (nearly) optimal sequence $(a_i)_{2\leq i\leq k+1}\in[0,1)^k$. So we fix a nearly optimal family $(a_i)_{2\leq i\leq k+1}\in(0,1)^k$ satisfying (23), we build the associated family of distances $(d_i)_{1\leq i\leq k+1}\in(0,1)^k$ and we are going to look for a good sequence of centers $(x_i)_{1\leq i\leq k+1}\in(0,1)^k$ with $\|x_i\|\sim d_i$.

We thus introduce the following subsets of $\mathbb{R}^2$:

$$\begin{aligned} D_0' &= \big(-d^{-1/2},d^{-1/2}\big)\times\big(-d^{-1/2},0\big),\\ \forall i\in\{1,\dots,k+1\}\quad D_i' &= \big(0,d^{-1/2}\big)^2, \end{aligned}$$

and the followining sets in $\mathbb{R}^{d-2}$

$$\begin{aligned} C_0'' &= \{0\},\\ \forall i\in\{1,\dots,k+1\}\quad C_i'' &= B''(0,d_i-2d^{-1})\setminus B''(0,d_i-3d^{-1}). \end{aligned}$$

Finally, for $i\in\{0,\dots,k+1\}$, we set $C_i=D_i'\times C_i''$. Note that for d large enough, these sets are disjoint. The next lemma controls the asymptotics in the dimension d of the volume of these sets

Lemma 2.10. *For every $i \in \{1, \dots, k+1\}$:*

$$\lim_{d\to+\infty} \frac{1}{d} \ln \frac{|C_i''|}{v_{d-2}} = \lim_{d\to+\infty} \frac{1}{d} \ln \frac{|C_i|}{v_d} = \ln d_i.$$

Proof. This can be proven by elementary computations. □

Each x_i will be taken in C_i, but we also have to ensure that the $(x_i)_{1\leq i\leq k+1}$ form a path. Note that for $i \in \{2, \dots, k+1\}$, we have $0 < d_{i-1} + a_i r_i < d_i$, which legitimates the following definition. See also Figure 3. For $i \in \{2, \dots, k+1\}$ and d large enough, we denote by θ_i the unique real number in $(0, \pi/2)$ such that

$$\cos\theta_i = \frac{d_{i-1} + a_i r_i}{d_i} + d^{-1/2}.$$

We introduce next, for $y \in C_{i-1}$, the following subset of $\mathbb{R}^{d-2}$:

$$D_i''(y'') = \{z'' \in C_i'' : \ \langle z'', y''\rangle \geq \|y''\|.\|z''\|.\cos\theta_i\}$$

We also set $D_0'' = C_0''$ and $D_1''(y'') = C_1''$ for every $y \in C_0$. Finally, we define for every $i \in \{1, \dots, k+1\}$ and $y \in C_{i-1}$:

$$D_i(y) = D_i' \times D_i''(y'') \subset C_i,$$

and $D_0 = D_0' \times D_0''$.

Lemma 2.11. • *If the dimension d is large enough, for every $i \in \{1, \dots, k+1\}$ and $y \in C_{i-1}$,*

$$D_i(y) \ \subset \ B(y, r_i) \cap C_i.$$

• *Let $x_0 \in D_0$. If there exist $X_1, \dots, X_k \in \chi_1$ and $X_{k+1} \in \chi_\rho$ such that $X_1 \in D_1(x_0), X_2 \in D_2(X_1), \dots, X_{k+1} \in D_{k+1}(X_k)$, then the event $\mathcal{G}_+(x_0)$ occurs.*

Proof. • The inclusion $D_i(y) \subset C_i$ is clear for every $i \in \{1, \dots, k+1\}$. Let $i \in \{2, \dots, k+1\}$, $y \in C_{i-1}$ and $z \in D_i(y)$. Then, as soon as d is large enough,

$$\begin{aligned}
\|z-y\|^2 &= \|z'-y'\|^2 + \|z''-y''\|^2 \\
&\leq \frac{2}{d} + \|y''\|^2 + \|z''\|^2 - 2 < y'', z'' > \\
&\leq \frac{2}{d} + (d_{i-1} - 2d^{-1})^2 + (d_i - 2d^{-1})^2 - 2(d_{i-1} - 3d^{-1})(d_i - 3d^{-1})\cos\theta_i \\
&\leq d_i^2 + d_{i-1}^2 - 2d_{i-1}(d_{i-1} + a_i r_i) - 2d^{-1/2} d_i d_{i-1} + O_i(d^{-1}) \\
&\leq r_i^2 - 2d^{-1/2} d_i d_{i-1} + O_i(d^{-1}) \leq r_i^2.
\end{aligned}$$

Let now $y \in C_0$ and $z \in D_1(y)$. As $d_1 = 1 + \rho = r_1 > 2$, we obtain, for d large enough :

$$\|z-y\|^2 = \|z'-y'\|^2 + \|z''-y''\|^2 \leq \frac{8}{d} + (d_1 - 2d^{-1})^2 \leq r_1^2.$$

• The second point is a simple consequence of the first point, of the fact that the sets $D_i(x_{i-1})$, as the sets C_i, are disjoint and of the definition of the event $\mathcal{G}_+$. □

Note that for $i \in \{1, \dots, k+1\}$, $|D_i(y)|$ and $|D_i''(y'')|$ do not depend on the choice of $y \in C_{i-1}$. We thus denote by $|D_i|$ and $|D_i''|$ these values. We now give asymptotic estimates for these values :

Lemma 2.12. *For every $i \in \{2, \dots, k+1\}$,*

$$\lim_{d\to+\infty} \frac{1}{d} \ln \frac{|D_i''|}{v_{d-2}} = \lim_{d\to+\infty} \frac{1}{d} \ln \frac{|D_i|}{v_d} = \ln(r_i\sqrt{1-a_i^2}).$$

Proof. We have, by homogeneity and isotropy:

$$|D_i''| = \left((d_i - 2d^{-1})^{d-2} - (d_i - 3d^{-1})^{d-2}\right) |S| \tag{26}$$

where $S = \{x = (x_1, \dots, x_{d-2}) \in B''(0,1) : x_1 \geq \|x\| \cos(\theta_i)\}$.
But S is included in the cylinder

$$\{(x_i)_{1\leq i\leq d-2} \in \mathbb{R}^{d-2} : x_1 \in [0,1], \|(x_2, \dots, x_{d-2})\| \leq \sin(\theta_i)\}$$

and S contains the cone

$$\{(x_i)_{1\leq i\leq d-2} \in \mathbb{R}^{d-2} : x_1 \in [0, \cos(\theta_i)], \|(x_2, \dots, x_{d-2})\| \leq x_1 \sin(\theta_i) \cos(\theta_i)^{-1}\}.$$

Therefore :

$$v_{d-3} \cos(\theta_i) \sin(\theta_i)^{d-3} (d-2)^{-1} \leq |S| \leq v_{d-3} \sin(\theta_i)^{d-3}. \tag{27}$$

From (26), (27), and the limits $\cos(\theta_i) \to (d_{i-1} + a_i r_i) d_i^{-1} \neq 0$ and $d_i \sin(\theta_i) \to r_i\sqrt{1-a_i^2}$, we get

$$\lim_{d\to+\infty} \frac{1}{d} \ln \left(\frac{|D_i''|}{v_{d-2}} \right) = \ln(r_i\sqrt{1-a_i^2}).$$

The lemma follows. Note that a direct calculus with spherical coordinates can also give the announced estimates. □

Everything is now in place to prove Proposition 2.8.

Proof of Proposition 2.8. Choose $p < 1$ and $x \in W$ such that $x'' = 0$.

• We start with a single individual, encoded by its position $\zeta_0 = \{x\} \subset C_0$, and we build, generation by generation, its descendance : we set, for $1 \leq i \leq k$,

$$\zeta_i = \chi_1 \cap \bigcup_{y\in\zeta_{i-1}} D_i(y) \subset C_i,$$

and for the $(k+1)$-th generation, we finally set

$$\zeta_{k+1} = \chi_\rho \cap \bigcup_{y\in\zeta_k} D_{k+1}(y) \subset C_{k+1}.$$

By Lemma 2.11, if $\zeta_{k+1} \neq \varnothing$ then the event $\mathcal{G}^+(x)$ occurs. To bound from below the probability that $\zeta_{k+1} \neq \varnothing$, we now build a simpler process ξ, stochastically dominated by ζ.

• We set $\alpha_i = \lambda_1 |D_i|$ for $i \in \{1, \dots, k\}$ and $\alpha_{k+1} = \lambda_\rho |D_{k+1}|$: thus, α_i is the mean number of children of a point of the $(i-1)$-th generation. Let $X_0 = x$ be the position of the first individual.

Consider a random vector $X = (X_0, X_1, \dots, X_{k+1})$ of points in $\mathbb{R}^d$ defined as follows : X_0 is defined by $X_0 = x$, X_1 is taken uniformly in $D_1(X_0)$, then X_2 is taken uniformly in $D_2(X_1)$, and so on. We think of X as a pontential single branch of descendance of x. Let then $(X^j)_{j\geq 1}$ be independent copies of X. Let now N be an independent Poisson random variable with parameter α_1 : this random variable will be the number of children of X_0. We will use the N first X^j, one for each child of X_0.

We now take into account the fact that some individuals may have no children. We shall deal with geometric dependencies later. Note that in our new process each individual of any generation i, $i \geq 1$, has at most one child. We made that choice in order to handle

more easily geometric dependencies. Let $Y = (Y_i^j)_{2\le i\le k+1, j\ge 1}$ be an independent family of independent random variables, such that Y_i^j follows the Bernoulli law with parameter $1-\exp(-\alpha_i)$, which is the probability that a Poisson random variable with parameter α_i is different from 0. We set $J_1 = \{1,\dots,N\}$ and, for every $i \in \{2,\dots,k+1\}$:

$$J_i = \{1 \le j \le N : Y_2^j = \cdots = Y_i^j = 1\}.$$

Thus the random set J_i gives the exponents of the branches that are, among the N initial branches, still alive at the i-th generation in a process with no dependecies due to geometry.

Until now, we did not take into account the geometrical constraints between individuals. For every $i \in \{2,\dots,k+1\}$ and every $j \ge 1$, we set

$$Z_i^j = 1 \text{ if } X_i^j \notin \bigcup_{j'\in J_{i-1}\setminus\{j\}} D_i(X_{i-1}^{j'}) \qquad \text{and } Z_i^j = 0 \text{ otherwise.}$$

We will reject an individual X_i^j and its descendance as soon as $Z_i^j = 0$. Recall that, when building generation i from generation $i-1$, we explore the Poisson point processes in the area $\bigcup_{j\in J_{i-1}} D_i(X_{i-1}^j)$. By construction of the C_i, these areas are distinct for different generations. Therefore, one can check that, for every $i \in \{2,\dots,k+1\}$, the set

$$\xi_i = \{X_i^j : j \in J_i \text{ and } Z_2^j = \cdots = Z_i^j = 1\}$$

is stochastically dominated by ζ_i. Thus to prove Proposition 2.8, we now need to bound from below the probability that ξ_{k+1} is not empty.

• Let T be the smallest integer j such that $Y_2^j = \cdots = Y_{k+1}^j = 1$: in other words, T is the smallest exponent of a branch that lives till generation $k+1$. To ensure that $\xi_{k+1} \neq \varnothing$, it is sufficient that $T \le N$ and that $Z_2^T = \cdots = Z_{k+1}^T = 1$. So :

$$\begin{aligned}
1 - P(\mathcal{G}^+(x)) &\le P(\xi_{k+1} = \emptyset) \\
&\le P(\#J_{k+1} = 0) + P\left(\{T \le N\} \cap \bigcup_{2\le i\le k+1} \{Z_i^T = 0\}\right) \\
&\le P(\#J_{k+1} = 0) + \sum_{2\le i\le k+1} P(T \le N \text{ and } Z_i^T = 0).
\end{aligned}$$

For every $2 \le i \le k+1$, we have by construction :

$$\begin{aligned}
P(T \le N \text{ and } Z_i^T = 0) &= P\left(T \le N,\ \exists j \in J_{i-1}\setminus\{T\} \text{ such that } X_i^T \in D_i(X_{i-1}^j)\right) \\
&\le \sum_{j\ge 1} P\left(T \le N \text{ and } j \in J_{i-1}\setminus\{T\} \text{ and } X_i^T \in D_i(X_{i-1}^j)\right) \\
&= \sum_{j\ge 1} E\left(1_{T\le N} 1_{j\in J_{i-1}\setminus\{T\}} P\left(X_i^T \in D_i(X_{i-1}^j) \,|\, Y, N\right)\right) \\
&= \sum_{j\ge 1} E\left(1_{T\le N} 1_{j\in J_{i-1}\setminus\{T\}}\right) P\left(X_i^1 \in D_i(X_{i-1}^2)\right) \\
&\le E(\#J_{i-1}) P\left(X_i^1 \in D_i(X_{i-1}^2)\right).
\end{aligned}$$

Besides, as $(X_i^1)''$ is uniformly distributed on C_i'' and is independent of $(X_{i-1}^2)''$,

$$P\left(X_i^1 \in D_i(X_{i-1}^2)\right) = P\left((X_i^1)'' \in D_i''((X_{i-1}^2)'')\right) = \frac{|D_i''|}{|C_i''|}. \tag{28}$$

This leads to

$$1 - P(\mathcal{G}^+(x)) \leq P(\#J_{k+1} = 0) + \sum_{i=2}^{k+1} E(\#J_{i-1}) \frac{|D_i''|}{|C_i''|}. \tag{29}$$

- For $1 \leq i \leq k+1$, the cardinality of J_i follows a Poisson law with parameter

$$\eta_i = \alpha_1 \prod_{i'=2}^{i} (1 - \exp(-\alpha_{i'})).$$

Remember that $\alpha_i = \lambda_1 |D_i|$ for $i \in \{1, \ldots, k\}$ and $\alpha_{k+1} = \lambda_\rho |D_{k+1}|$. By Lemma 2.12, we have the following limits:

$$\begin{aligned}
\lim_{d\to+\infty} \frac{1}{d} \ln \alpha_1 &= \ln \frac{\kappa(1+\rho)}{2} > 0, \\
\lim_{d\to+\infty} \frac{1}{d} \ln \alpha_i &= \ln(\kappa\sqrt{1-a_i^2}) < 0 \text{ for } 2 \leq i \leq k, \\
\lim_{d\to+\infty} \frac{1}{d} \ln \alpha_{k+1} &= \ln(\kappa\sqrt{1-a_{k+1}^2}\frac{1+\rho}{2\rho}) < 0.
\end{aligned}$$

To see the signs of the limits, note that $\kappa < 1$, that $\frac{1+\rho}{2\rho} < 1$ and that (23) implies that

$$\kappa > \kappa^{k+1} > \frac{4\rho}{(1+\rho)^2\sqrt{1-a_2^2}\ldots\sqrt{1-a_{k+1}^2}} > \frac{2}{1+\rho}.$$

Consequently, we first see that

$$\begin{aligned}
\lim_{d\to+\infty} \frac{1}{d} \ln(\eta_{k+1}) &= \lim_{d\to+\infty} \frac{1}{d} \ln(\alpha_1 \ldots \alpha_{k+1}) \\
&= \ln\left(\kappa^{k+1} \frac{(1+\rho)^2}{4\rho} \sqrt{\prod_{2\leq j\leq k+1} (1-a_j^2)}\right) > 0 \quad \text{with (23);} \\
\text{therefore,} \quad & \lim_{d\to+\infty} P(\#J_{k+1} = 0) = 0. \qquad (30)
\end{aligned}$$

Similarly, for $2 \leq i \leq k+1$, we have

$$\lim_{d\to+\infty} \frac{1}{d} \ln(\eta_{i-1}) = \ln\left(\kappa^{i-1} \frac{1+\rho}{2} \sqrt{\prod_{2\leq i'\leq i-1} (1-a_{i'}^2)}\right).$$

Lemmas 2.10 and 2.12 ensure that :

$$\lim_{d\to+\infty} \frac{1}{d} \ln\left(\frac{|D_i''|}{|C_i''|}\right) = \ln\left(\frac{r_i\sqrt{1-a_i^2}}{d_i}\right).$$

Thus, for $2 \leq i \leq k+1$, we have :

$$\limsup_{d\to+\infty} \frac{1}{d} \ln\left(E(\#J_{i-1}) \frac{|D_i''|}{|C_i''|}\right) \leq \ln\left(\frac{r_i(1+\rho)\kappa^{i-1}}{2d_i} \sqrt{\prod_{2\leq i'\leq i} (1-a_{i'}^2)}\right).$$

Now,

$$\text{for } 2 \le i \le k, \quad \limsup_{d\to+\infty} \frac{1}{d} \ln\left(E(\#J_{i-1}) \frac{|D_i''|}{|C_i''|} \right) \le \ln\left(\frac{1+\rho}{d_i} \right) < 0, \tag{31}$$

$$\limsup_{d\to+\infty} \frac{1}{d} \ln\left(E(\#J_k) \frac{|D_{k+1}''|}{|C_{k+1}''|} \right) \le \ln\left(\frac{(1+\rho)^2 \kappa^k}{2d_{k+1}} \sqrt{\prod_{2\le i'\le k+1} (1-a_{i'}^2)} \right) < 0 \tag{32}$$

with (23). To end the proof, we put estimates (30), (31) and (32) in (29). □

2.4.2. *Several steps in the* 2*-dimensional oriented percolation model.* We prove here Proposition 2.7 by building the supercritical 2-dimensional oriented percolation process embedded in the two-type Boolean Model.

Proof of Proposition 2.7. We first define an oriented graph in the following manner: the set of sites is

$$S = \{(a,n) \in \mathbb{Z}\times\mathbb{N} : \ |a| \le n, \ a+n \text{ is even } \};$$

from any point $(a,n) \in S$, we put an oriented edge from (a,n) to $(a+1,n+1)$, and an oriented edge from (a,n) to $(a-1,n+1)$. We denote by $\vec{p}_c(2) \in (0,1)$ the critical parameter for Bernoulli percolation on this oriented graph – see Durrett [5] for results on oriented percolation in dimension 2.

For any $(a,n) \in S$, we define the following subsets of $\mathbb{R}^d$

$$\begin{aligned} W_{a,n} &= d^{-1/2}\left( ]a-1,a+1[\times]n-1,n[\times\mathbb{R}^{d-2} \right), \\ W_{a,n}^- &= d^{-1/2}\left( ]a-1,a[\times]n,n+1[\times\mathbb{R}^{d-2} \right), \\ W_{a,n}^+ &= d^{-1/2}\left( ]a,a+1[\times]n,n+1[\times\mathbb{R}^{d-2} \right). \end{aligned}$$

Note that the $(W_{a,n})_{(a,n)\in S}$ are disjoint and that $W_{a,n}^+ \cup W_{a+2,n}^- \subset W_{a+1,n+1}$.

We now fix $k \ge 1$ and $\kappa \in (\kappa_\rho^c(k), 1)$, and for $x_0 \in W_{a,n}$, we introduce the events :

$$\mathcal{G}_{a,n}^+(x_0) = \left\{ \begin{array}{c} \text{There exist distinct } x_1,\dots,x_k \in \chi_1 \cap W_{a,n}^+ \text{ and } x_{k+1} \in \chi_\rho \cap W_{a,n}^+ \\ \text{such that } x_0,x_1,\dots,x_{k+1} \text{ is a path} \end{array} \right\},$$

$$\mathcal{G}_{a,n}^-(x_0) = \left\{ \begin{array}{c} \text{There exist distinct } x_1,\dots,x_k \in \chi_1 \cap W_{a,n}^- \text{ and } x_{k+1} \in \chi_\rho \cap W_{a,n}^- \\ \text{such that } x_0,x_1,\dots,x_{k+1} \text{ is a path} \end{array} \right\}.$$

Note that $\mathcal{G}_{0,0}^+(x)$ is exactly the event $\mathcal{G}^+(x)$ introduced in (22), and that the other events are obtained from this one by symmetry and/or translation.

Next we choose $p \in (\vec{p}_c(2), 1)$. With Proposition 2.8, and by translation and symmetry invariance, we know that for every large enough dimension d, for every $(a,n) \in S$, for every $x \in W_{a,n}$:

$$P(\mathcal{G}_{a,n}^\pm(x)) \ge p. \tag{33}$$

We fix then a dimension d large enough to satisfy (33). We can now construct the random states, open or closed, of the edges of our oriented graph. We denote by ∞ a virtual site.

Definition of the site on level 0**.** Almost surely, $\chi_\rho \cap W_{0,0} \neq \varnothing$. We take then some $x(0,0) \in \chi_\rho \cap W_{0,0}$.

Definition of the edges between levels n **and** $n+1$**.** Fix $n \ge 0$ and assume we built a site $x(a,n) \in W_{a,n} \cup \{\infty\}$ for every a such that $(a,n) \in S$. Consider $(a,n) \in S$:

– If $x(a,n) = \infty$: we decide that each of the two edges starting from (a,n) is open with probability p and closed with probability $1-p$, independently of everything else; we set $z^-(a,n) = z^+(a,n) = \infty$.

– Otherwise :
  – Edge to the left-hand side :
    – if the event $\mathcal{G}^-_{a,n}(x(a,n))$ occurs : we take for $z^-(a,n)$ some point $x_{k+1} \in W^-_{a,n} \subset W_{a-1,n+1}$ given by the occurrence of the event, and we open the edge from (a,n) to $(a-1,n+1)$;
    – otherwise : we set $z^-(a,n) = \infty$ and we close the edge from (a,n) to $(a-1,n+1)$.
  – Edge to the right-hand side :
    – if the event $\mathcal{G}^+_{a,n}(x(a,n))$ occurs : we take for $z^+(a,n)$ some point $x_{k+1} \in W^+_{a,n} \subset W_{a+1,n+1}$ given by the occurrence of the event, and we open the edge from (a,n) to $(a+1,n+1)$;
    – otherwise : we set $z^+(a,n) = \infty$ and we close the edge from (a,n) to $(a+1,n+1)$.

For (a,n) outside S, we set $z^\pm(a,n) = \infty$.

Definition of the sites at level $n+1$. Fix $n \geq 0$ and assume we determined the state of every edge between levels n and $n+1$. Consider $(a,n+1) \in S$:
– If $z^+(a-1,n) \neq \infty$: set $x(a,n+1) = z^+(a-1,n) \in W_{a,n+1}$.
– Otherwise :
  – if $z^-(a+1,n) \neq \infty$: set $x(a,n+1) = z^-(a+1,n) \in W_{a,n+1}$,
  – otherwise : set $x(a,n+1) = \infty$.

Assume that there exists an open path of length n starting from the origin in this oriented percolation : we can check that the leftmost open path of length n starting from the origin gives a path in the two-type Boolean model with n alternating sequences of k balls with radius 1 and one ball with radius ρ. Thus, percolation in this oriented percolation model implies percolation by k-alternation in the two-type Boolean model. Let us check that percolation occurs indeed with positive probability.

For every n, denote by $\mathcal{F}_n$ the σ-field generated by the restrictions of the Poisson point processes χ_1 and χ_ρ to the set

$$d^{-1/2}\left(\mathbb{R} \times (-\infty, n) \times \mathbb{R}^{d-2}\right).$$

By definition of the events $\mathcal{G}$ – remember that the $(W_{a,n})_{(a,n)\in S}$ are disjoint – and by (33), the states of the different edges between levels n and $n+1$ are independent conditionally to $\mathcal{F}_n$. Moreover, conditionally to $\mathcal{F}_n$, each edge between levels n and $n+1$ has a probability at least p to be open. Therefore, the oriented percolation model we built stochastically dominates Bernoulli oriented percolation with parameter p. As $p > \vec{p}_c(\mathbb{Z}^2)$, with positive probability, there exists an infinite open path in the oriented percolation model we built; this ends the proof of Proposition 2.7. □

2.5. **Proof of Theorem 1.4.** We first prove how Propositions 2.1 and 2.7 give (10) when $a = 1$, $b > 1$ and $\alpha = \beta = 1$, and then we see how we can deduce the general case by scaling and coupling.

When $a = 1$, $b > 1$ and $\alpha = \beta = 1$. Set $\rho = b$. In this case, $\nu = \delta_1 + \delta_\rho$, so $\nu_d = \delta_1 + \dfrac{1}{\rho^d}\delta_\rho$.

Note then that the two-type Boolean model Σ introduced in Subsection 2.2 and whose intensities depend on $\kappa \in (0,1)$ coincides with the Boolean model directed by the measure

$$\frac{\kappa^d}{v_d 2^d}\nu_d$$

as defined in the introduction.

If $\kappa < \kappa_\rho^c$ then, by Proposition 2.1, there is no percolation for d large enough. Therefore, for any such κ and for any large enough d we have:

$$\lambda_d^c(\nu_d) \geq \frac{\kappa^d}{v_d 2^d} \quad \text{and then} \quad \widetilde{\lambda}_d^c(\nu_d) = \lambda_d^c(\nu_d) v_d 2^d \int r^d \nu_d(dr) \geq 2\kappa^d.$$

Letting d goes to $+\infty$ and then κ goes to κ_ρ^c, we then obtain

$$\liminf_{d\to+\infty} \frac{1}{d} \ln\left(\lambda_d^c(\nu_d)\right) \geq \ln\left(\kappa_\rho^c\right). \tag{34}$$

As $\kappa_\rho^c < 1$ by Lemma 1.5, choose now κ such that $\kappa_\rho^c < \kappa < 1$. Then, there exists $k \geq 1$ such that $\kappa_\rho^c(k) < \kappa$. Therefore, by Proposition 2.7, there is percolation for d large enough in Σ; by coupling, this remains true for larger κ. Therefore, for any $\kappa > \kappa_\rho^c$ and for any large enough d we have, as before:

$$\lambda_d^c(\nu_d) \leq \frac{\kappa^d}{v_d 2^d} \quad \text{and then} \quad \widetilde{\lambda}_d^c(\nu_d) \leq 2\kappa^d.$$

Letting d goes to $+\infty$ and then κ goes to κ_ρ^c, we then obtain

$$\limsup_{d\to+\infty} \frac{1}{d} \ln\left(\lambda_d^c(\nu_d)\right) \leq \ln\left(\kappa_\rho^c\right). \tag{35}$$

Bringing (34) and (35) together, we get (10) when $a = 1$, $b = \rho > 1$ and $\alpha = \beta = 1$.

When $b > a > 0$ and $\alpha = \beta = 1$. Set $\rho = b/a$. Here, $\nu = \delta_a + \delta_b$; set $\mu = \delta_1 + \delta_\rho$. With the notation of the introduction,

$$\nu_d = \frac{1}{a^d}(\delta_a + \frac{1}{\rho^d}\delta_b) = \frac{1}{a^d} H^a.(\delta_1 + \frac{1}{\rho^d}\delta_\rho) = \frac{1}{a^d} H^a \mu_d.$$

By the scaling relations (3) and (4), we obtain

$$\widetilde{\lambda}_d^c(\nu_d) = \widetilde{\lambda}_d^c(\mu_d).$$

The result when $b > a > 0$ and $\alpha = \beta = 1$ follows then from the previous case.

When $b > a > 0$ and $\alpha, \beta > 0$. Here $\nu = \alpha\delta_a + \beta\delta_b$. Set $\mu = \delta_a + \delta_b$, $m = \min(\alpha, \beta)$ and $M = \max(\alpha, \beta)$. Then $m\mu_d \leq \nu_d \leq M\mu_d$ and so

$$\begin{aligned} m \int r^d d\mu_d(r) \leq \quad & \int r^d d\nu_d(r) \quad \leq M \int r^d d\mu_d(r), \\ \frac{1}{M}\lambda_d^c(\mu_d) = \lambda_d^c(M\mu_d) \leq \quad & \lambda_d^c(\nu_d) \quad \leq \lambda_d^c(m\mu_d) = \frac{1}{m}\lambda_d^c(\mu_d). \end{aligned}$$

The two previous inequalities give:

$$\frac{m}{M}\widetilde{\lambda}_d^c(\mu_d) \leq \widetilde{\lambda}_d^c(\nu_d) \leq \frac{M}{m}\widetilde{\lambda}_d^c(\mu_d),$$

and the theorem follows from the previous case. □

3. Proof of Theorem 1.2

Theorem 1.2 follows from Theorem 1.4 by coupling and scaling. By assumption, μ is a measure on $(0,+\infty)$ whose support is not a singleton. We can therefore choose $b' > a' > 0$ in the support, set $\rho = b'/a'$ and then take a small enough $\varepsilon > 0$ such that

$$a'(1+\varepsilon) < b'(1-\varepsilon), \qquad \mu([a'(1-\varepsilon), a'(1+\varepsilon)] > 0,$$
$$\mu([b'(1-\varepsilon), b'(1+\varepsilon)] > 0, \qquad (1+\varepsilon)(1-\varepsilon)^{-1}\kappa_\rho^c < 1.$$

Set $a = a'(1-\varepsilon), b = b'(1-\varepsilon)$ and $\tau = (1+\varepsilon)(1-\varepsilon)^{-1} > 1$. We have

$$a\tau < b,\ \mu([a,a\tau]) > 0,\ \mu([b,b\tau]) > 0 \text{ and } \tau\kappa_\rho^c < 1.$$

Set $\nu = \mu([a,a\tau])\delta_a + \mu([b,b\tau])\delta_b$ and $S = [a,a\tau] \cup [b,b\tau]$. For all $d \geq 1$ we have

$$\tau^{-d}\nu_d(\{a\}) \ = \ \mu([a,a\tau])(a\tau)^{-d} \leq \int_{[a,a\tau]} r^{-d}\mu(dr) = \mu_d([a,a\tau])$$

and, similarly, $\tau^{-d}\nu_d(\{b\}) \leq \mu_d([b,b\tau])$. By coupling, this implies that $\lambda_d^c(1_S\mu_d) \leq \lambda_d^c(\tau^{-d}\nu_d)$, and then that

$$\lambda_d^c(\mu_d) \leq \lambda_d^c(1_S\mu_d) \leq \lambda_d^c(\tau^{-d}\nu_d) = \tau^d\lambda_d^c(\nu_d).$$

But $\widetilde{\lambda}_d^c(\mu_d) = \lambda_d^c(\mu_d)2^d v_d\mu((0,+\infty))$ and, similarly, $\widetilde{\lambda}_d^c(\nu_d) = \lambda_d^c(\nu)2^d v_d\nu((0,+\infty))$, which leads to

$$\widetilde{\lambda}_d^c(\mu_d) \leq \tau^d \frac{\mu((0,+\infty))}{\nu((0,+\infty))}\widetilde{\lambda}_d^c(\nu_d).$$

But by Theorem 1.4 we have

$$\lim_{d\to+\infty}\frac{1}{d}\ln\left(\widetilde{\lambda}_d^c(\nu_d)\right) = \ln(\kappa_\rho^c), \text{ and then } \limsup_{d\to+\infty}\frac{1}{d}\ln\left(\widetilde{\lambda}_d^c(\mu_d)\right) \leq \ln(\tau\kappa_\rho^c) < 0,$$

which ends ce proof. □

Note that as a byproduct of the proof, we obtain the following upper bound :

$$\limsup_{d\to+\infty}\frac{1}{d}\ln\left(\widetilde{\lambda}_d^c(\mu_d)\right) \leq \inf_{0<a<b<+\infty,\ a,b\in\mathrm{Supp}(\mu)}\ln(\kappa_{b/a}^c).$$

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