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## ► To cite this version:

Alexander Perepechko. Flexibility of affine cones over del Pezzo surfaces of degree 4 and 5. Functional Analysis and Its Applications, 2013, 47 (4), pp.45-52. hal-00617627v2

**HAL Id: hal-00617627**

**<https://hal.science/hal-00617627v2>**

Submitted on 17 Nov 2013

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# FLEXIBILITY OF AFFINE CONES OVER DEL PEZZO SURFACES OF DEGREE 4 AND 5

A.YU. PEREPECHKO

ABSTRACT. We prove that the action of the special automorphism group on affine cones over del Pezzo surfaces of degree 4 and 5 is infinitely transitive.

## 1. INTRODUCTION

An affine algebraic variety  $X$  defined over an algebraically closed field  $\mathbb{K}$  of characteristic zero is called *flexible* if the tangent space of  $X$  at any smooth point is spanned by the tangent vectors to the orbits of one-parameter unipotent group actions [1]. In this paper we establish flexibility of affine cones over del Pezzo surfaces of degree 4 and 5.

It is well known that every effective action of one-dimensional unipotent group  $\mathbb{G}_a = \mathbb{G}_a(\mathbb{K})$  on  $X$  defines a locally nilpotent derivation  $\delta \in \text{LND}(\mathbb{K}[X])$  of the algebra of regular functions on  $X$ . All such actions generate a subgroup of *special automorphisms*  $\text{SAut } X \subset \text{Aut } X$ .

A group  $G$  is said to act on a set  $S$  *infinitely transitively* if it acts transitively on the set of  $m$ -tuples of pairwise distinct points in  $S$  for any  $m \in \mathbb{N}$ .

The following theorem explains the significance of the flexibility concept.

**Theorem 1** ([1, Theorem 0.1]). *Let  $X$  be an affine algebraic variety of dimension  $\geq 2$ . Then the following conditions are equivalent:*

- (1) *The variety  $X$  is flexible;*
- (2) *the group  $\text{SAut } X$  acts transitively on the smooth locus  $X_{\text{reg}}$  of  $X$ ;*
- (3) *the group  $\text{SAut } X$  acts infinitely transitively on  $X_{\text{reg}}$ .*

Three classes of flexible affine varieties are described in [2], namely affine cones over flag varieties, non-degenerate toric varieties of dimension  $\geq 2$ , and suspensions over flexible varieties. Note that affine cones over del Pezzo varieties of degree  $\geq 6$  are toric, thereby they are flexible.

In this paper we consider cases of degree 4 and 5. In case of degree 5 we prove flexibility of affine cones corresponding to polarizations defined by arbitrary very ample divisors, whereas for degree 4 we prove flexibility only for certain very ample divisors, the anticanonical one included. As for del Pezzo surfaces of degree  $\leq 3$ , it is proven the non-existence of any  $\mathbb{G}_a$ -actions on the affine cones over plurianticanonical embeddings, see [3, Theorem 1.1] for the case of degree 3 and [7, Corollary 1.8] for the case of degree  $\leq 2$ .

In the proof we use the construction from [6], which allows to associate a regular  $\mathbb{G}_a$ -action on an affine cone over a projective variety  $Y$  to every open cylindrical subset of  $Y$  of a special form. In Theorem 5 we provide a criterion of flexibility of an affine cone over a projective variety in terms of a transversal cover by such cylinders. We apply it to del Pezzo surfaces.

The author is grateful to M.G. Zaidenberg for posing the problem and numerous discussions, to I.V. Arzhantsev for useful remarks, and to the reviewer for corrections.

## 2. FLEXIBILITY OF AFFINE CONES

Let  $Y$  be a projective variety and  $H$  be a very ample divisor on  $Y$ . A polarization of  $Y$  by  $H$  provides an embedding  $Y \hookrightarrow \mathbb{P}^n$ . Consider an affine cone  $X = \text{AffCone}_H Y \subset \mathbb{A}^{n+1}$  with vertex at the origin  $0 \in \mathbb{A}^{n+1}$  corresponding to this embedding. There is a natural homothety

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This work was supported by The Ministry of education and science of Russian Federation, project 8214, in part by the Simons Foundation, and by RFBR grant nos. 12-01-00704 and 12-01-31342mol a.

action of the multiplicative group  $\mathbb{G}_m = \mathbb{G}_m(\mathbb{K})$  of the field  $\mathbb{K}$  on  $X$ . It defines a grading on the algebra  $\mathbb{K}[X]$ . A derivation on  $\mathbb{K}[X]$  is called *homogeneous* if it sends homogeneous elements into homogeneous ones. A subset of all homogeneous locally nilpotent derivations is denoted by  $\text{HLND}(\mathbb{K}[X])$ .

**Definition 2** ([6, Definitions 3.1.5, 3.1.7]). We say that an open subset  $U$  of a variety  $Y$  is a *cylinder* if  $U \cong Z \times \mathbb{A}^1$ , where  $Z$  is a smooth variety with  $\text{Pic } Z = 0$ . Given a divisor  $H \subset Y$ , we say that a cylinder  $U$  is *H-polar* if  $U = Y \setminus \text{supp } D$  for some effective divisor  $D \in |dH|$ , where  $d > 0$ .

**Definition 3.** We call a subset  $W \subset Y$  *invariant* with respect to a cylinder  $U = Z \times \mathbb{A}^1$  if  $W \cap U = \pi_1^{-1}(\pi_1(W))$ , where  $\pi_1: U \rightarrow Z$  is the first projection of the direct product. In other words, every  $\mathbb{A}^1$ -fiber of the cylinder is either contained in  $W$  or does not meet  $W$ .

**Definition 4.** We say that a variety  $Y$  is *transversally covered* by cylinders  $U_i$ ,  $i = 1, \dots, s$ , if  $Y = \bigcup U_i$  and there is no proper subset  $W \subset Y$  invariant with respect to all  $U_i$ .

Clearly, every cylinder  $U_i$  is smooth. Thus, a singular variety  $Y$  does not admit a transversal covering by cylinders. It is also clear that  $\dim Y \geq 1$ . The following theorem gives a criterion of flexibility for the affine cone over a projective embedding  $Y \hookrightarrow \mathbb{P}^n$  corresponding to the polarization by  $H$ .

**Theorem 5.** *If for some very ample divisor  $H$  on a smooth projective variety  $Y$  there exists a transversal covering by  $H$ -polar cylinders, then the affine cone  $X = \text{AffCone}_H Y$  is flexible.*

*Proof.* The statement is obvious for  $X = \mathbb{A}^{n+1}$ . Thus, we may suppose that the vertex of the cone is the only singular point.

By [6, Theorem 3.1.9] for every cylinder on  $Y$  there corresponds a homogeneous  $\mathbb{G}_a$ -action on  $X$ . Note from the explicit construction in [6, Proposition 3.1.5] that the projection  $\pi: X^\times = X \setminus \{0\} \rightarrow Y$  sends the orbits of this action to fibers of the cylinder on  $Y$ , and the subset of fixed points on  $X$  is the preimage of the cylinder complement.

Let  $G \subset \text{SAut } X$  be a subgroup generated by corresponding  $\mathbb{G}_a$ -actions. Consider an orbit  $Gx$  of some point  $x \in X^\times$ . The image  $\pi(Gx) \subset Y$  is invariant w.r.t. all covering cylinders. The transversality condition implies  $\pi(Gx) = Y$ . Since the group  $G$  is generated by homogeneous actions, the natural  $\mathbb{G}_m$ -action on  $X$  by homotheties normalizes the  $G$ -action on  $X$  and sends  $G$ -orbits to  $G$ -orbits. Thus,  $X^\times$  is a union of  $G$ -orbits, which projections coincide with  $Y$ . Hence  $X^\times = \bigcup_{\lambda \in \mathbb{G}_m} \lambda Gx$ , where all  $G$ -orbits are closed in  $X^\times$ .

Let us show that there exists the only open  $G$ -orbit  $Gx = X^\times$ . Assume the contrary. Then  $\dim Gx = \dim Y$  and the stabilizer  $S \subset \mathbb{G}_m$  of the orbit  $Gx$  is finite. Moreover, since the action of  $S$  on  $Gx$  is free, for any point  $x' \in Gx$  the intersection  $Gx \cap \mathbb{G}_m x'$  is an  $S$ -orbit consisting of  $|S|$  distinct points. The blow up of  $X$  in the origin is the total space of the linear bundle  $[-H]$  on  $Y$ . It has a natural completion — a  $\mathbb{P}^1$ -bundle  $\hat{X} \rightarrow Y$ . For a general point  $x' \in Gx$  the intersection  $\overline{Gx} \cap \overline{\mathbb{G}_m x'}$ , where  $\overline{Z}$  denotes the closure of  $Z \subset X^\times$  in  $\hat{X}$ , coincides with the orbit  $Sx$ . So, the intersection index  $\overline{Gx} \cdot \overline{\mathbb{G}_m x'}$  equals  $|S|$ . Since the intersection index is constant, for any point  $x' \in Gx$  there holds  $\overline{Gx} \cap \overline{\mathbb{G}_m x'} = Sx' \subset X^\times$ . Therefore, a quasi-affine variety  $X^\times$  contains a projective one  $\overline{Gx}$ , which is a contradiction. So, the group  $G$  acts on  $X^\times$  transitively.  $\square$

### 3. DEL PEZZO SURFACE OF DEGREE 5

Let  $Y$  be a del Pezzo surface of degree 5. It is obtained by blowing up the projective plane  $\mathbb{P}^2$  in four points  $P_1, \dots, P_4$ , no three of which are collinear [9, Theorem IV.2.5]. Since the automorphism group of the projective plane acts transitively on such 4-tuples of points, such a surface is unique up to isomorphism.

**Theorem 6.** *Let  $H$  be an arbitrary very ample divisor on the del Pezzo surface  $Y$  of degree 5. Then the corresponding affine cone  $\text{AffCone}_H Y$  is flexible.*

The proof proceeds in several steps, see Sections 3.1 and 3.2. We let  $E_i$  denote the exceptional divisor (i.e. the  $(-1)$ -curve), which is the preimage of the blown up point  $P_i$ . Let  $e_0$  be the divisor class of a line, which contains none of the points  $P_i$ , and let  $e_i$  ( $i = 1, \dots, 4$ ) be a divisor class of  $E_i$ . These classes generate a Picard group  $\text{Pic } Y = \langle e_0, \dots, e_4 \rangle_{\mathbb{Z}} \cong \mathbb{Z}^5$ . The intersection index defines a symmetric bilinear form on the Picard group such that the basis  $\{e_0, \dots, e_4\}$  is orthogonal,  $e_0^2 = 1$  and  $e_i^2 = -1$ . Exceptional divisor classes are  $e_i$  and  $e_0 - e_i - e_j$  for distinct  $i, j \neq 0$ .

By Kleiman's ampleness criterion [8, Theorem 1.4.9] the closure of the ample cone  $\text{Ample } Y$  is dual to the cone of effective divisors  $\overline{\text{NE}}(Y)$ . In case of a del Pezzo surface of degree  $< 8$  the cone  $\overline{\text{NE}}(Y)$  is generated by exceptional divisors [4, Theorem 8.2.19]. Therefore, the ample cone is defined by inequalities

$$(1) \quad x_0 > 0, \quad x_i < 0 \quad i = 1, \dots, 4,$$

$$(2) \quad x_0 + x_i + x_j > 0, \quad 0 \neq i \neq j \neq 0,$$

where  $(x_0, \dots, x_4) \in \text{Pic } Y$ . It has the following ten extremal rays

$$(3) \quad e_0, e_0 - e_j, 2e_0 - \sum_{i \neq 0} e_i, 2e_0 - \sum_{i \neq 0, j} e_i \quad \text{where } j = 1, \dots, 4.$$

For five of them the corresponding orthogonal facet of the effective cone contains four non-intersecting  $(-1)$ -curves. They define the contraction  $Y \rightarrow \mathbb{P}^2$  corresponding to the chosen extremal ray.

Any other ray defines a pencil of quadrics on  $Y$ . More precisely, an orthogonal complement to the ray contains three pairs of intersecting  $(-1)$ -curves which define the degenerate fibers of the pencil. Herewith, the class of the pencil fiber belongs to the chosen ray.

**3.1. Cylinders.** Let us fix a blowdown  $\varphi: Y \rightarrow \mathbb{P}^2$  of four pairwise disjoint  $(-1)$ -curves  $E_1, \dots, E_4$  into points  $P_1, \dots, P_4$  using the notation as above. Let  $l_{ij} \subset \mathbb{P}^2$  be the line passing through the points  $P_i$  and  $P_j$ . Consider the open subset  $U_1 = \varphi^{-1}(\mathbb{P}^2 \setminus (l_{12} \cup l_{34})) \subset Y$ . This is a cylinder defined by the pencil of lines passing through the base point  $\text{Bs}(U_1) = l_{12} \cap l_{34}$ . We have  $U_1 \cong \mathbb{A}_*^1 \times \mathbb{A}^1$ , where  $\mathbb{A}_*^1 = \mathbb{A}^1 \setminus \{0\}$ . Similarly let  $U_2 = \varphi^{-1}(\mathbb{P}^2 \setminus (l_{13} \cup l_{24}))$  and  $U_3 = \varphi^{-1}(\mathbb{P}^2 \setminus (l_{14} \cup l_{23}))$ , see fig. 1. Furthermore, consider the blowings down of other 4-tuples of non-intersecting  $(-1)$ -curves on  $Y$ . There are five of them as shown on fig. 2. For every blowing down we define three cylinders in a similar way. Note that these cylinders are in one-to-one correspondence with the intersection points of the  $(-1)$ -curves, and the automorphism group  $\text{Aut } Y \cong S_5$  acts transitively on them.

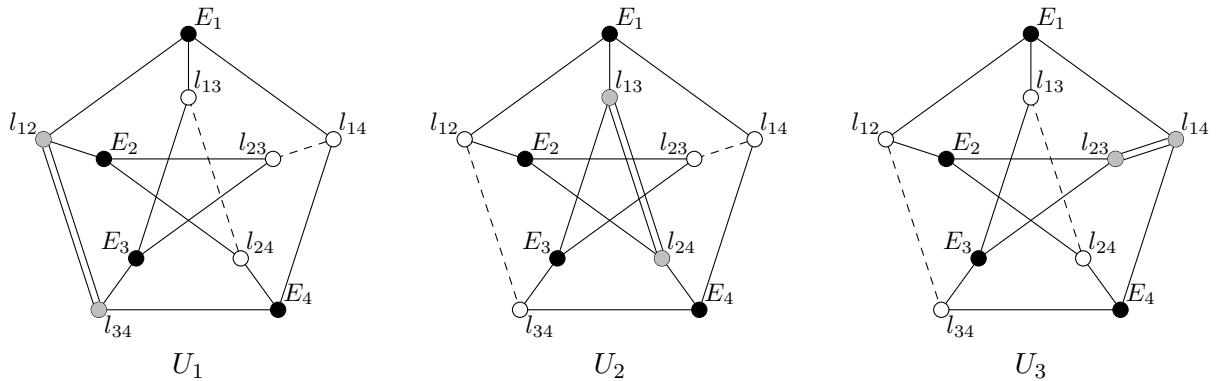


FIGURE 1. Arrangement of cylinders on the incidence graph of  $(-1)$ -curves on the del Pezzo surface of degree 5. The gray and the black vertices correspond to  $(-1)$ -curves forming the complement to a cylinder. The dashed edges correspond to  $(-1)$ -curve intersections contained in the cylinder. The double edge corresponds to the base point of the cylinder.

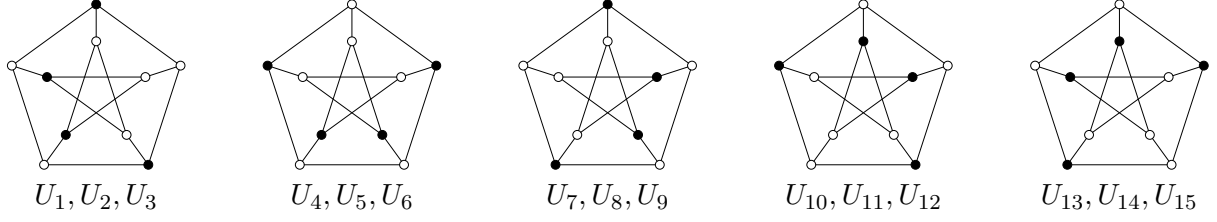


FIGURE 2. Black vertices correspond to 4-tuples of  $(-1)$ -curves. Every blowing down defines three cylinders similarly as on fig. 1.

Thus we have cylinders  $U_1, U_2, \dots, U_{15}$  as shown on Figures 1 and 2. It is easy to check that every intersection of  $(-1)$ -curves is contained in some cylinder, hence  $\bigcup U_i = Y$ . We claim that there is no proper subset  $W \subset Y$ , which is invariant with respect to all 15 cylinders. Assume that there exists such a subset  $W$ . Let us fix an arbitrary point of  $W$ . It is contained in a fiber  $S$  of some cylinder, hence  $W$  contains  $S$ . Without loss of generality  $S$  is a fiber of  $U_1$ . Then the line  $l = \varphi(S) \subset \mathbb{P}^2$  passes through the base point  $\text{Bs}(U_1)$ . Since the points  $\text{Bs}(U_1)$ ,  $\text{Bs}(U_2)$ , and  $\text{Bs}(U_3)$  do not lie on the same line, one of them does not belong to  $l$ . Suppose  $\text{Bs}(U_2) \notin l$ . Then the fiber  $S$  intersects almost every fiber of the cylinder  $U_2$ , and  $W$  contains them. So,  $W$  is dense in  $Y$ . The complement  $Y \setminus W$  is also invariant with respect to all cylinders, and by the same reason it is dense in  $Y$ , a contradiction.

**3.2. Polarity condition.** In this subsection we show that for any ample divisor  $H$  on  $Y$  all the 15 cylinders  $U_i$  are  $H$ -polar. Consider the set of effective divisors  $\{\alpha_i E_i + \beta_1 l_{12} + \beta_3 l_{34} \mid \alpha_i, \beta_i > 0\}$  whose support is the complement to  $U_1$ . The image of this set in the Picard group is an open cone  $C$ , whose extremal rays are  $e_1, e_2, e_3, e_4, e_0 - e_1 - e_2$ , and  $e_0 - e_3 - e_4$ . It is easy to check that the primitive vectors of the ample cone (3) can be expressed as linear combinations with non-negative rational coefficients of the primitive vectors of the cone  $C$ . Therefore the cylinder  $U_1$  is  $H$ -polar for any ample divisor  $H$ . By automorphisms  $\text{Aut } Y$  we may translate  $U_1$  to any cylinder, hence the cylinders  $U_i$  are  $H$ -polar for any ample divisor  $H$ . Using Theorem 5 we obtain the assertion. Now Theorem 6 is proved.

#### 4. DEL PEZZO SURFACES OF DEGREE 4

Every del Pezzo surface of degree 4 is isomorphic to a blowing up of a projective plane  $\mathbb{P}^2$  in five points, where no three are collinear. Such surfaces form a two-parameter family.

By  $E_i$  we denote the  $(-1)$ -curve which is the preimages of the blown up point  $P_i$ . As before, let  $e_0$  be the divisor class of a line which does not contain the blown up points, and  $e_i$  ( $i = 1, \dots, 5$ ) be the divisor class of  $E_i$ . A set  $\{e_0, \dots, e_5\}$  forms an orthogonal basis of the Picard group  $\text{Pic } Y \cong \mathbb{Z}^6$ , and  $e_0^2 = 1, e_i^2 = -1$ . The classes of  $(-1)$ -curves are  $e_i, e_0 - e_i - e_j, 2e_0 - \sum_{k \neq 0} e_k$  for any pair of distinct indices  $i, j \neq 0$ . The ample cone is defined by inequalities

$$(4) \quad x_0 > 0, x_i < 0 \quad i = 1, \dots, 5,$$

$$(5) \quad x_0 + x_i + x_j > 0, \quad 0 \neq i \neq j \neq 0,$$

$$(6) \quad 2x_0 + x_1 + \dots + x_5 > 0,$$

where  $(x_0, \dots, x_5) \in \text{Pic } Y$ . Its extremal rays are

$$(7) \quad e_0, e_0 - e_j, 2e_0 - \sum_{k \neq 0, i} e_k, 2e_0 - \sum_{k \neq 0, i, j} e_k, \text{ and } 3e_0 - \sum_{k \neq 0} e_k - e_i$$

for any pair of distinct indices  $i, j \in \{1, \dots, 5\}$ . Similarly to the case of del Pezzo surface of degree 5, sixteen extremal rays correspond to blowings down  $Y \rightarrow \mathbb{P}^2$ , and ten rays correspond to pencils of quadrics on  $Y$ .

**4.1. Cylinders.** Let us fix some  $(-1)$ -curve  $C_1$  and consider the blowing down  $\sigma_1: Y \rightarrow \mathbb{P}^2$  of five  $(-1)$ -curves  $F_1, \dots, F_5$  that meet  $C_1$ , see fig. 3. This blowing down is well defined since the contracted divisors do not intersect. The image  $\sigma_1(C_1)$  is a smooth quadric  $c$  passing through the blown down points  $Q_1, \dots, Q_5$ . Take an arbitrary line  $l \subset \mathbb{P}^2$  which is tangent to  $c$  at a point different from  $Q_1, \dots, Q_5$ . A quadric pencil in  $\mathbb{P}^2$  generated by divisors  $c$  and  $2l$  determines a cylinder  $U \cong \mathbb{A}_*^1 \times \mathbb{A}^1 \subset Y$  whose complement is the complete preimage of the support of the divisor  $c + 2l$  on  $\mathbb{P}^2$ . Denote by  $\mathcal{U}_{C_1}$  the family of all such cylinders in  $Y$  for all such tangents  $l$ . Note that  $Y \setminus \bigcup_{U \in \mathcal{U}_{C_1}} U$  is a union of  $C_1$  and the exceptional divisors  $F_i$  ( $i = 1, \dots, 5$ ). Apply this construction to the  $(-1)$ -curves  $C_2, \dots, C_5$ , which form a 5-cycle along with  $C_1$  on the incidence graph as shown on fig. 3. Overall we obtain five cylinder families  $\mathcal{U}_{C_1}, \dots, \mathcal{U}_{C_5}$ . It is easy to see that their union covers  $Y$ .

Let  $W$  be a proper subset of  $Y$  which is invariant with respect to the cylinders of all families, and let  $w \in W$  be an arbitrary point. We may suppose that  $w$  belongs to a cylinder of the family  $\mathcal{U}_{C_1}$ . Then the image  $\sigma_1(W) \subset \mathbb{P}^2$  is invariant with respect to the cylinder family  $\{\sigma_1(U) \mid U \in \mathcal{U}_{C_1}\}$ . Note that every cylinder of this family is a complement to the quadric  $c$  and its tangent line. It is well known that given a quadric and two points outside it we can find a quadric passing through these two points and tangent to the given quadric. Therefore, for almost every point  $x \in \mathbb{P}^2 \setminus c$  there exists a fiber of some cylinder which contains  $x$  and  $\sigma_1(w)$ . Namely,  $x$  must not lie on the tangent line to  $c$  passing through  $\sigma_1(w)$  as well as on the quadrics which are tangent to  $c$  at blown down points and contain  $\sigma_1(w)$ . Thus  $W$  is dense in  $Y$ . Similarly,  $Y \setminus W$  is dense in  $Y$ , a contradiction. Finally, the families  $\mathcal{U}_{C_1}, \dots, \mathcal{U}_{C_5}$  form a transversal cover of  $Y$ .

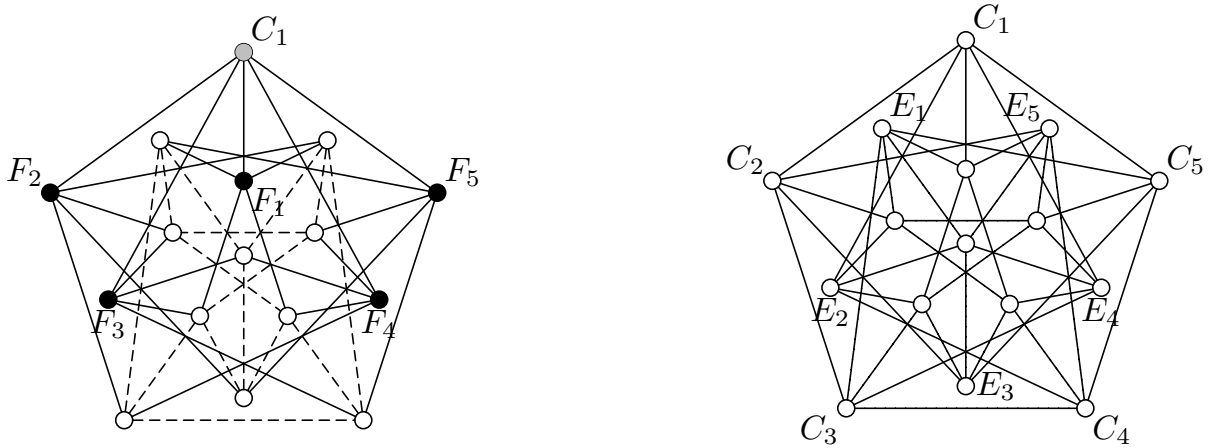


FIGURE 3. The incidence graph of  $(-1)$ -curves on a del Pezzo surface of degree 4. On the left the gray vertex corresponds to the quadric preimage  $C_1$  and black vertices correspond to the contracted  $(-1)$ -curves. The dashed edges correspond to  $(-1)$ -curve intersections contained in the cylinders of a family. Four other families corresponding to  $C_2, \dots, C_5$  are obtained symmetrically by the graph rotations.

**4.2. Polarity condition.** Ample divisors  $H$  such that cylinders of the family  $\mathcal{U}_{C_1}$  are  $H$ -polar, are exactly the ample divisors in the open cone  $\text{Ample } Y \cap \{\alpha_1 F_1 + \dots + \alpha_5 F_5 + \alpha_6 C_1 + \alpha_7 \sigma_1^{-1}(l) \mid \alpha_j > 0\}$  in  $\text{Pic } Y$ . We define such a cone for every  $C_i$ ,  $i = 1, \dots, 5$  and denote it by  $\text{Ample}(C_i, Y)$ . It does not depend on a choice of a tangent line  $l$  since it does not contain blown up points by definition. Then the set of divisors  $H$  such that cylinders in  $\bigcup_i \mathcal{U}_{C_i}$  are  $H$ -polar is an open cone  $\bigcap_i \text{Ample}(C_i, Y)$ . A computation shows that it has exactly 72 extremal rays, which can be expressed as

$$e_0, \quad 9e_0 - 5e_{i_1} - e_{i_2} - 2e_{i_3} - 4e_{i_4} - 3e_{i_5},$$

$$\begin{array}{ll}
4e_0 - 2e_{i_1} - 2e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5}, & 9e_0 - 4e_{i_1} - 4e_{i_2} - 4e_{i_3} - 2e_{i_4} - 2e_{i_5}, \\
5e_0 - 2e_{i_1} - 2e_{i_2} - e_{i_3} - 3e_{i_4} - e_{i_5}, & 11e_0 - 6e_{i_1} - 2e_{i_2} - 2e_{i_3} - 4e_{i_4} - 4e_{i_5}, \\
5e_0 - 2e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4}, & 11e_0 - 6e_{i_1} - 4e_{i_2} - 4e_{i_3} - 2e_{i_4} - 2e_{i_5}, \\
5e_0 - 2e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5}, & 11e_0 - 6e_{i_1} - 2e_{i_2} - 4e_{i_3} - 4e_{i_4} - 4e_{i_5}, \\
6e_0 - 2e_{i_1} - 2e_{i_2} - 3e_{i_3} - e_{i_4} - 3e_{i_5}, & 11e_0 - 6e_{i_1} - 4e_{i_2} - 4e_{i_3} - 4e_{i_4} - 2e_{i_5}, \\
7e_0 - 4e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5}, & 15e_0 - 8e_{i_1} - 2e_{i_2} - 4e_{i_3} - 6e_{i_4} - 6e_{i_5}, \\
9e_0 - 5e_{i_1} - 3e_{i_2} - 4e_{i_3} - 2e_{i_4} - 1e_{i_5}, & 15e_0 - 8e_{i_1} - 6e_{i_2} - 6e_{i_3} - 4e_{i_4} - 2e_{i_5},
\end{array}$$

where the tuple  $(i_1, \dots, i_5)$  runs over all cyclic permutations of  $(1, 2, 3, 4, 5)$ .

It is easy to see that the anticanonical divisor  $(-K_Y)$  is contained in  $\bigcap_i \text{Ample}(C_i, Y)$ . Similarly to Theorem 6 we obtain the following result.

**Theorem 7.** *Let  $Y$  be a del Pezzo surface of degree 4, and  $H$  be a very ample divisor in the open cone  $\bigcap_{i=1}^5 \text{Ample}(C_i, Y)$ . Then the affine cone  $\text{AffCone}_H Y$  is flexible. In particular, this holds for the anticanonical divisor  $H = -K_Y$ .*

We have identified a subcone of the ample cone such that the very ample divisors contained in this subcone define a flexible affine cone. However, this subcone is strictly contained in the ample cone. For example, the ample divisor class  $8e_0 - 2e_1 - 4e_2 - e_3 - e_4 - 3e_5$  lies outside of that subcone. Thus the flexibility problem for the affine cone over the polarization of a del Pezzo surface of degree 4 by any very ample divisor remains open.

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