Flexibility of affine cones over del Pezzo surfaces of degree 4 and 5

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FLEXIBILITY OF AFFINE CONES OVER DEL PEZZO SURFACES OF DEGREE 4 AND 5

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ABSTRACT. We prove that the action of the special automorphism group on affine cones over del Pezzo surfaces of degree 4 and 5 is infinitely transitive.

1. INTRODUCTION

An affine algebraic variety X defined over an algebraically closed field $\mathbb{K}$ of characteristic zero is called *flexible* if the tangent space of X at any smooth point is spanned by the tangent vectors to the orbits of one-parameter unipotent group actions [1]. In this paper we establish flexibility of affine cones over del Pezzo surfaces of degree 4 and 5.

It is well known that every effective action of one-dimensional unipotent group $\mathbb{G}_a = \mathbb{G}_a(\mathbb{K})$ on X defines a locally nilpotent derivation $\delta \in \mathrm{LND}(\mathbb{K}[X])$ of the algebra of regular functions on X. All such actions generate a subgroup of *special automorphisms* $\mathrm{SAut}\, X \subset \mathrm{Aut}\, X$.

A group G is said to act on a set S *infinitely transitively* if it acts transitively on the set of m-tuples of pairwise distinct points in S for any $m \in \mathbb{N}$.

The following theorem explains the significance of the flexibility concept.

Theorem 1 ([1, Theorem 0.1]). *Let X be an affine algebraic variety of dimension ≥ 2. Then the following conditions are equivalent:*

(1) *The variety X is flexible;*
(2) *the group $\mathrm{SAut}\, X$ acts transitively on the smooth locus X_{reg} of X;*
(3) *the group $\mathrm{SAut}\, X$ acts infinitely transitively on X_{reg}.*

Three classes of flexible affine varieties are described in [2], namely affine cones over flag varieties, non-degenerate toric varieties of dimension ≥ 2, and suspensions over flexible varieties. Note that affine cones over del Pezzo varieties of degree ≥ 6 are toric, thereby they are flexible.

In this paper we consider cases of degree 4 and 5. In case of degree 5 we prove flexibility of affine cones corresponding to polarizations defined by arbitrary very ample divisors, whereas for degree 4 we prove flexibility only for certain very ample divisors, the anticanonical one included. As for del Pezzo surfaces of degree ≤ 3, it is proven the non-existence of any $\mathbb{G}_a$-actions on the affine cones over plurianticanonical embeddings, see [3, Theorem 1.1] for the case of degree 3 and [7, Corollary 1.8] for the case of degree ≤ 2.

In the proof we use the construction from [6], which allows to associate a regular $\mathbb{G}_a$-action on an affine cone over a projective variety Y to every open cylindrical subset of Y of a special form. In Theorem 5 we provide a criterion of flexibility of an affine cone over a projective variety in terms of a transversal cover by such cylinders. We apply it to del Pezzo surfaces.

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2. FLEXIBILITY OF AFFINE CONES

Let Y be a projective variety and H be a very ample divisor on Y. A polarization of Y by H provides an embedding $Y \hookrightarrow \mathbb{P}^n$. Consider an affine cone $X = \mathrm{AffCone}_H\, Y \subset \mathbb{A}^{n+1}$ with vertex at the origin $0 \in \mathring{A}^{n+1}$ corresponding to this embedding. There is a natural homothety

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action of the multiplicative group $\mathbb{G}_m = \mathbb{G}_m(\mathbb{K})$ of the field $\mathbb{K}$ on X. It defines a grading on the algebra $\mathbb{K}[X]$. A derivation on $\mathbb{K}[X]$ is called *homogeneous* if it sends homogeneous elements into homogeneous ones. A subset of all homogeneous locally nilpotent derivations is denoted by $\mathrm{HLND}(\mathbb{K}[X])$.

Definition 2 ([6, Definitions 3.1.5, 3.1.7])**.** We say that an open subset U of a variety Y is a *cylinder* if $U \cong Z \times \mathbb{A}^1$, where Z is a smooth variety with $\operatorname{Pic} Z = 0$. Given a divisor $H \subset Y$, we say that a cylinder U is *H-polar* if $U = Y \setminus \operatorname{supp} D$ for some effective divisor $D \in |dH|$, where $d > 0$.

Definition 3. We call a subset $W \subset Y$ *invariant* with respect to a cylinder $U = Z \times \mathbb{A}^1$ if $W \cap U = \pi_1^{-1}(\pi_1(W))$, where $\pi_1 \colon U \to Z$ is the first projection of the direct product. In other words, every $\mathbb{A}^1$-fiber of the cylinder is either contained in W or does not meet W.

Definition 4. We say that a variety Y is *transversally covered* by cylinders U_i, $i = 1, \ldots, s$, if $Y = \bigcup U_i$ and there is no proper subset $W \subset Y$ invariant with respect to all U_i.

Clearly, every cylinder U_i is smooth. Thus, a singular variety Y does not admit a transversal covering by cylinders. It is also clear that $\dim Y \geq 1$. The following theorem gives a criterion of flexibility for the affine cone over a projective embedding $Y \hookrightarrow \mathbb{P}^n$ corresponding to the polarization by H.

Theorem 5. *If for some very ample divisor H on a smooth projective variety Y there exists a transversal covering by H-polar cylinders, then the affine cone $X = \operatorname{AffCone}_H Y$ is flexible.*

Proof. The statement is obvious for $X = \mathbb{A}^{n+1}$. Thus, we may suppose that the vertex of the cone is the only singular point.

By [6, Theorem 3.1.9] for every cylinder on Y there corresponds a homogeneous $\mathbb{G}_a$-action on X. Note from the explicit construction in [6, Proposition 3.1.5] that the projection $\pi \colon X^\times = X \setminus \{0\} \to Y$ sends the orbits of this action to fibers of the cylinder on Y, and the subset of fixed points on X is the preimage of the cylinder complement.

Let $G \subset \operatorname{SAut} X$ be a subgroup generated by corresponding $\mathbb{G}_a$-actions. Consider an orbit Gx of some point $x \in X^\times$. The image $\pi(Gx) \subset Y$ is invariant w.r.t. all covering cylinders. The transversality condition implies $\pi(Gx) = Y$. Since the group G is generated by homogeneous actions, the natural $\mathbb{G}_m$-action on X by homotheties normalizes the G-action on X and sends G-orbits to G-orbits. Thus, $X^\times$ is a union of G-orbits, which projections coincide with Y. Hence $X^\times = \bigcup_{\lambda \in \mathbb{G}_m} \lambda Gx$, where all G-orbits are closed in $X^\times$.

Let us show that there exists the only open G-orbit $Gx = X^\times$. Assume the contrary. Then $\dim Gx = \dim Y$ and the stabilizer $S \subset \mathbb{G}_m$ of the orbit Gx is finite. Moreover, since the action of S on Gx is free, for any point $x' \in Gx$ the intersection $Gx \cap \mathbb{G}_m x'$ is an S-orbit consisting of $|S|$ distinct points. The blow up of X in the origin is the total space of the linear bundle $[-H]$ on Y. It has a natural completion — a $\mathbb{P}^1$-bundle $\hat{X} \to Y$. For a general point $x' \in Gx$ the intersection $\overline{Gx} \cap \overline{\mathbb{G}_m x'}$, where $\overline{Z}$ denotes the closure of $Z \subset X^\times$ in $\hat{X}$, coincides with the orbit Sx. So, the intersection index $\overline{Gx} \cdot \overline{\mathbb{G}_m x'}$ equals $|S|$. Since the intersection index is constant, for any point $x' \in Gx$ there holds $\overline{Gx} \cap \overline{\mathbb{G}_m x'} = Sx' \subset X^\times$. Therefore, a quasi-affine variety $X^\times$ contains a projective one $\overline{Gx}$, which is a contradiction. So, the group G acts on $X^\times$ transitively. □

3. Del Pezzo surface of degree 5

Let Y be a del Pezzo surface of degree 5. It is obtained by blowing up the projective plane $\mathbb{P}^2$ in four points $P_1, \ldots, P_4$, no three of which are collinear [9, Theorem IV.2.5]. Since the automorphism group of the projective plane acts transitively on such 4-tuples of points, such a surface is unique up to isomorphism.

Theorem 6. *Let H be an arbitrary very ample divisor on the del Pezzo surface Y of degree 5. Then the corresponding affine cone $\operatorname{AffCone}_H Y$ is flexible.*

The proof proceeds in several steps, see Sections 3.1 and 3.2. We let E_i denote the exceptional divisor (i.e. the (-1)-curve), which is the preimage of the blown up point P_i. Let e_0 be the divisor class of a line, which contains none of the points P_i, and let e_i $(i = 1, \dots, 4)$ be a divisor class of E_i. These classes generate a Picard group $\operatorname{Pic} Y = \langle e_0, \dots, e_4 \rangle_{\mathbb{Z}} \cong \mathbb{Z}^5$. The intersection index defines a symmetric bilinear form on the Picard group such that the basis $\{e_0, \dots, e_4\}$ is orthogonal, $e_0^2 = 1$ and $e_i^2 = -1$. Exceptional divisor classes are e_i and $e_0 - e_i - e_j$ for distinct $i, j \neq 0$.

By Kleiman's ampleness criterion [8, Theorem 1.4.9] the closure of the ample cone $\operatorname{Ample} Y$ is dual to the cone of effective divisors $\overline{\mathrm{NE}}(Y)$. In case of a del Pezzo surface of degree < 8 the cone $\overline{\mathrm{NE}}(Y)$ is generated by exceptional divisors [4, Theorem 8.2.19]. Therefore, the ample cone is defined by inequalities

$$x_0 > 0,\ x_i < 0 \qquad i = 1, \dots, 4, \tag{1}$$

$$x_0 + x_i + x_j > 0, \qquad 0 \neq i \neq j \neq 0, \tag{2}$$

where $(x_0, \dots, x_4) \in \operatorname{Pic} Y$. It has the following ten extremal rays

$$e_0,\ e_0 - e_j,\ 2e_0 - \sum_{i \neq 0} e_i,\ 2e_0 - \sum_{i \neq 0, j} e_i \quad \text{where } j = 1, \dots, 4. \tag{3}$$

For five of them the corresponding orthogonal facet of the effective cone contains four non-intersecting (-1)-curves. They define the contraction $Y \to \mathbb{P}^2$ corresponding to the chosen extremal ray.

Any other ray defines a pencil of quadrics on Y. More precisely, an orthogonal complement to the ray contains three pairs of intersecting (-1)-curves which define the degenerate fibers of the pencil. Herewith, the class of the pencil fiber belongs to the chosen ray.

3.1. **Cylinders.** Let us fix a blowdown $\varphi \colon Y \to \mathbb{P}^2$ of four pairwise disjoint (-1)-curves $E_1, \dots, E_4$ into points $P_1, \dots, P_4$ using the notation as above. Let $l_{ij} \subset \mathbb{P}^2$ be the line passing through the points P_i and P_j. Consider the open subset $U_1 = \varphi^{-1}(\mathbb{P}^2 \setminus (l_{12} \cup l_{34})) \subset Y$. This is a cylinder defined by the pencil of lines passing through the base point $\mathrm{Bs}(U_1) = l_{12} \cap l_{34}$. We have $U_1 \cong \mathbb{A}^1_* \times \mathbb{A}^1$, where $\mathbb{A}^1_* = \mathbb{A}^1 \setminus \{0\}$. Similarly let $U_2 = \varphi^{-1}(\mathbb{P}^2 \setminus (l_{13} \cup l_{24}))$ and $U_3 = \varphi^{-1}(\mathbb{P}^2 \setminus (l_{14} \cup l_{23}))$, see fig. 1. Furthermore, consider the blowings down of other 4-tuples of non-intersecting (-1)-curves on Y. There are five of them as shown on fig. 2. For every blowing down we define three cylinders in a similar way. Note that these cylinders are in one-to-one correspondence with the intersection points of the (-1)-curves, and the automorphism group $\operatorname{Aut} Y \cong S_5$ acts transitively on them.

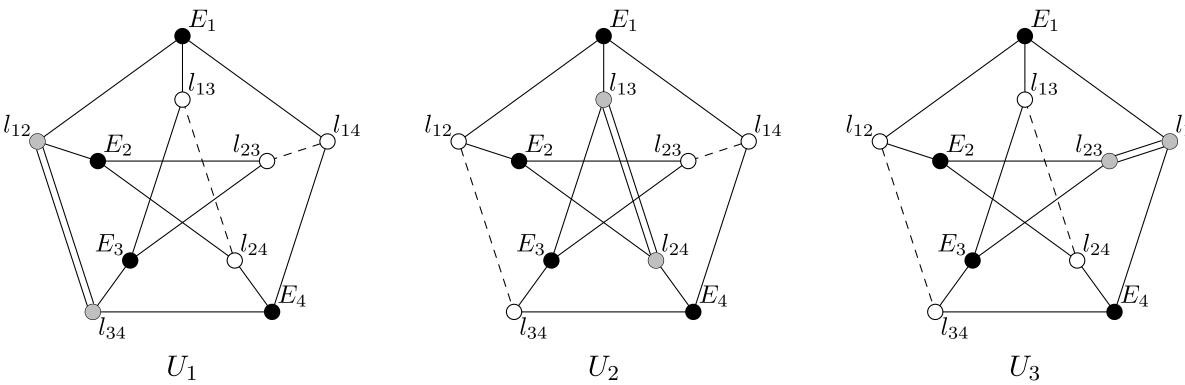
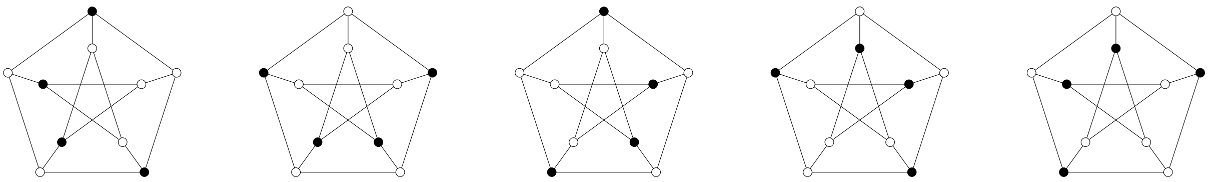


FIGURE 1. Arrangement of cylinders on the incidence graph of (-1)-curves on the del Pezzo surface of degree 5. The gray and the black vertices correspond to (-1)-curves forming the complement to a cylinder. The dashed edges correspond to (-1)-curve intersections contained in the cylinder. The double edge corresponds to the base point of the cylinder.



U_1, U_2, U_3 U_4, U_5, U_6 U_7, U_8, U_9 U_{10}, U_{11}, U_{12} U_{13}, U_{14}, U_{15}

FIGURE 2. Black vertices correspond to 4-tuples of (-1)-curves. Every blowing down defines three cylinders similarly as on fig. 1.

Thus we have cylinders $U_1, U_2, \ldots, U_{15}$ as shown on Figures 1 and 2. It is easy to check that every intersection of (-1)-curves is contained in some cylinder, hence $\bigcup U_i = Y$. We claim that there is no proper subset $W \subset Y$, which is invariant with respect to all 15 cylinders. Assume that there exists such a subset W. Let us fix an arbitrary point of W. It is contained in a fiber S of some cylinder, hence W contains S. Without loss of generality S is a fiber of U_1. Then the line $l = \overline{\varphi(S)} \subset \mathbb{P}^2$ passes through the base point $\mathrm{Bs}(U_1)$. Since the points $\mathrm{Bs}(U_1), \mathrm{Bs}(U_2)$, and $\mathrm{Bs}(U_3)$ do not lie on the same line, one of them does not belong to l. Suppose $\mathrm{Bs}(U_2) \notin l$. Then the fiber S intersects almost every fiber of the cylinder U_2, and W contains them. So, W is dense in Y. The complement $Y \setminus W$ is also invariant with respect to all cylinders, and by the same reason it is dense in Y, a contradiction.

3.2. **Polarity condition.** In this subsection we show that for any ample divisor H on Y all the 15 cylinders U_i are H-polar. Consider the set of effective divisors $\{\alpha_i E_i + \beta_1 l_{12} + \beta_3 l_{34} \mid \alpha_i, \beta_i > 0\}$ whose support is the complement to U_1. The image of this set in the Picard group is an open cone C, whose extremal rays are $e_1,\ e_2,\ e_3,\ e_4,\ e_0 - e_1 - e_2$, and $e_0 - e_3 - e_4$. It is easy to check that the primitive vectors of the ample cone (3) can be expressed as linear combinations with non-negative rational coefficients of the primitive vectors of the cone C. Therefore the cylinder U_1 is H-polar for any ample divisor H. By automorphisms $\mathrm{Aut}\, Y$ we may translate U_1 to any cylinder, hence the cylinders U_i are H-polar for any ample divisor H. Using Theorem 5 we obtain the assertion. Now Theorem 6 is proved.

4. Del Pezzo surfaces of degree 4

Every del Pezzo surface of degree 4 is isomorphic to a blowing up of a projective plane $\mathbb{P}^2$ in five points, where no three are collinear. Such surfaces form a two-parameter family.

By E_i we denote the (-1)-curve which is the preimages of the blown up point P_i. As before, let e_0 be the divisor class of a line which does not contain the blown up points, and e_i $(i = 1, \ldots, 5)$ be the divisor class of E_i. A set $\{e_0, \ldots, e_5\}$ forms an orthogonal basis of the Picard group $\mathrm{Pic}\, Y \cong \mathbb{Z}^6$, and $e_0^2 = 1,\ e_i^2 = -1$. The classes of (-1)-curves are $e_i,\ e_0 - e_i - e_j,\ 2e_0 - \sum_{k \neq 0} e_k$ for any pair of distinct indices $i, j \neq 0$. The ample cone is defined by inequalities

$$x_0 > 0,\ x_i < 0 \qquad i = 1, \ldots, 5, \tag{4}$$

$$x_0 + x_i + x_j > 0, \qquad 0 \neq i \neq j \neq 0, \tag{5}$$

$$2x_0 + x_1 + \ldots + x_5 > 0, \tag{6}$$

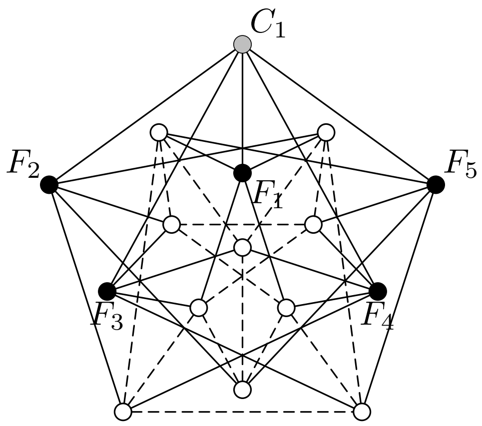
where $(x_0, \ldots, x_5) \in \mathrm{Pic}\, Y$. Its extremal rays are

$$e_0,\ e_0 - e_j,\ 2e_0 - \sum_{k \neq 0, i} e_k,\ 2e_0 - \sum_{k \neq 0, i, j} e_k,\ \text{and } 3e_0 - \sum_{k \neq 0} e_k - e_i \tag{7}$$

for any pair of distinct indices $i, j \in \{1, \ldots, 5\}$. Similarly to the case of del Pezzo surface of degree 5, sixteen extremal rays correspond to blowings down $Y \to \mathbb{P}^2$, and ten rays correspond to pencils of quadrics on Y.

4.1. **Cylinders.** Let us fix some (-1)-curve C_1 and consider the blowing down $\sigma_1 \colon Y \to \mathbb{P}^2$ of five (-1)-curves $F_1, \ldots, F_5$ that meet C_1, see fig. 3. This blowing down is well defined since the contracted divisors do not intersect. The image $\sigma_1(C_1)$ is a smooth quadric c passing through the blown down points $Q_1, \ldots, Q_5$. Take an arbitrary line $l \subset \mathbb{P}^2$ which is tangent to c at a point different from $Q_1, \ldots, Q_5$. A quadric pencil in $\mathbb{P}^2$ generated by divisors c and $2l$ determines a cylinder $U \cong \mathbb{A}^1_* \times \mathbb{A}^1 \subset Y$ whose complement is the complete preimage of the support of the divisor $c + 2l$ on $\mathbb{P}^2$. Denote by $\mathcal{U}_{C_1}$ the family of all such cylinders in Y for all such tangents l. Note that $Y \setminus \bigcup_{U \in \mathcal{U}_{C_1}} U$ is a union of C_1 and the exceptional divisors F_i $(i = 1, \ldots, 5)$. Apply this construction to the (-1)-curves $C_2, \ldots, C_5$, which form a 5-cycle along with C_1 on the incidence graph as shown on fig. 3. Overall we obtain five cylinder families $\mathcal{U}_{C_1}, \ldots, \mathcal{U}_{C_5}$. It is easy to see that their union covers Y.

Let W be a proper subset of Y which is invariant with respect to the cylinders of all families, and let $w \in W$ be an arbitrary point. We may suppose that w belongs to a cylinder of the family $\mathcal{U}_{C_1}$. Then the image $\sigma_1(W) \subset \mathbb{P}^2$ is invariant with respect to the cylinder family $\{\sigma_1(U) \mid U \in \mathcal{U}_{C_1}\}$. Note that every cylinder of this family is a complement to the quadric c and its tangent line. It is well known that given a quadric and two points outside it we can find a quadric passing through these two points and tangent to the given quadric. Therefore, for almost every point $x \in \mathbb{P}^2 \setminus c$ there exists a fiber of some cylinder which contains x and $\sigma_1(w)$. Namely, x must not lie on the tangent line to c passing through $\sigma_1(w)$ as well as on the quadrics which are tangent to c at blown down points and contain $\sigma_1(w)$. Thus W is dense in Y. Similarly, $Y \setminus W$ is dense in Y, a contradiction. Finally, the families $\mathcal{U}_{C_1}, \ldots, \mathcal{U}_{C_5}$ form a transversal cover of Y.



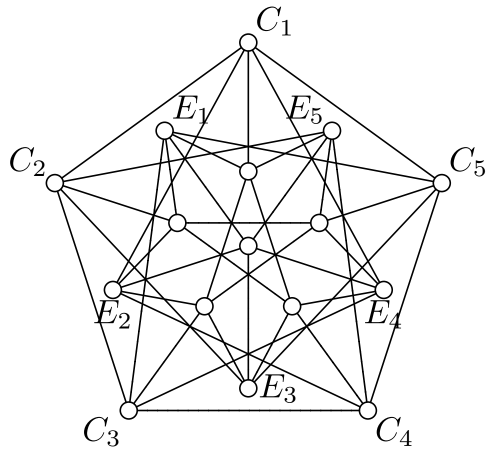


FIGURE 3. The incidence graph of (-1)-curves on a del Pezzo surface of degree 4. On the left the gray vertex corresponds to the quadric preimage C_1 and black vertices correspond to the contracted (-1)-curves. The dashed edges correspond to (-1)-curve intersections contained in the cylinders of a family. Four other families corresponding to $C_2, \ldots, C_5$ are obtained symmetrically by the graph rotations.

4.2. **Polarity condition.** Ample divisors H such that cylinders of the family $\mathcal{U}_{C_1}$ are H-polar, are exactly the ample divisors in the open cone $\operatorname{Ample} Y \cap \{\alpha_1 F_1 + \ldots + \alpha_5 F_5 + \alpha_6 C_1 + \alpha_7 \sigma_1^{-1}(l) \mid \alpha_j > 0\}$ in $\operatorname{Pic} Y$. We define such a cone for every C_i, $i = 1, \ldots, 5$ and denote it by $\operatorname{Ample}(C_i, Y)$. It does not depend on a choice of a tangent line l since it does not contain blown up points by definition. Then the set of divisors H such that cylinders in $\bigcup_i \mathcal{U}_{C_i}$ are H-polar is an open cone $\bigcap_i \operatorname{Ample}(C_i, Y)$. A computation shows that it has exactly 72 extremal rays, which can be expressed as

$$e_0, \qquad\qquad 9e_0 - 5e_{i_1} - e_{i_2} - 2e_{i_3} - 4e_{i_4} - 3e_{i_5},$$

$$
\begin{array}{ll}
4e_0 - 2e_{i_1} - 2e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5}, & 9e_0 - 4e_{i_1} - 4e_{i_2} - 4e_{i_3} - 2e_{i_4} - 2e_{i_5}, \\
5e_0 - 2e_{i_1} - 2e_{i_2} - e_{i_3} - 3e_{i_4} - e_{i_5}, & 11e_0 - 6e_{i_1} - 2e_{i_2} - 2e_{i_3} - 4e_{i_4} - 4e_{i_5}, \\
5e_0 - 2e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4}, & 11e_0 - 6e_{i_1} - 4e_{i_2} - 4e_{i_3} - 2e_{i_4} - 2e_{i_5}, \\
5e_0 - 2e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5}, & 11e_0 - 6e_{i_1} - 2e_{i_2} - 4e_{i_3} - 4e_{i_4} - 4e_{i_5}, \\
6e_0 - 2e_{i_1} - 2e_{i_2} - 3e_{i_3} - e_{i_4} - 3e_{i_5}, & 11e_0 - 6e_{i_1} - 4e_{i_2} - 4e_{i_3} - 4e_{i_4} - 2e_{i_5}, \\
7e_0 - 4e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5}, & 15e_0 - 8e_{i_1} - 2e_{i_2} - 4e_{i_3} - 6e_{i_4} - 6e_{i_5}, \\
9e_0 - 5e_{i_1} - 3e_{i_2} - 4e_{i_3} - 2e_{i_4} - 1e_{i_5}, & 15e_0 - 8e_{i_1} - 6e_{i_2} - 6e_{i_3} - 4e_{i_4} - 2e_{i_5},
\end{array}
$$

where the tuple $(i_1, \ldots, i_5)$ runs over all cyclic permutations of $(1, 2, 3, 4, 5)$.

It is easy to see that the anticanonical divisor $(-K_Y)$ is contained in $\bigcap_i \operatorname{Ample}(C_i, Y)$. Similarly to Theorem 6 we obtain the following result.

Theorem 7. *Let Y be a del Pezzo surface of degree 4, and H be a very ample divisor in the open cone $\bigcap_{i=1}^{5} \operatorname{Ample}(C_i, Y)$. Then the affine cone $\operatorname{AffCone}_H Y$ is flexible. In particular, this holds for the anticanonical divisor $H = -K_Y$.*

We have identified a subcone of the ample cone such that the very ample divisors contained in this subcone define a flexible affine cone. However, this subcone is strictly contained in the ample cone. For example, the ample divisor class $8e_0 - 2e_1 - 4e_2 - e_3 - e_4 - 3e_5$ lies outside of that subcone. Thus the flexibility problem for the affine cone over the polarization of a del Pezzo surface of degree 4 by any very ample divisor remains open.

References

[1] I. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch, M. Zaidenberg, *Flexible varieties and automorphism groups*, Duke Math. J. **162** (2013), no. 4, 767–823.

[2] I.V. Arzhantsev, K. Kuyumzhiyan, M. Zaidenberg, *Flag varieties, toric varieties, and suspensions: three instances of infinite transitivity*, Sbornik: Math. **203** (2012), no. 7, 923–949.

[3] I. Cheltsov, J. Park, and J. Won, *Affine cones over smooth cubic surfaces*, arXiv:1303.2648.

[4] I.V. Dolgachev, *Classical Algebraic Geometry: A Modern View*, Cambridge University Press, 2012.

[5] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.

[6] T. Kishimoto, Yu. Prokhorov, M. Zaidenberg, *Group actions on affine cones*, arXiv:0905.4647v2, 41p.; in: Affine Algebraic Geometry, CRM Proc. and Lecture Notes, vol. 54, Amer. Math. Soc., Providence, RI, 2011, 123–163.

[7] T. Kishimoto, Yu. Prokhorov, and M. Zaidenberg. *Unipotent Group Actions on Del Pezzo Cones*, Algebr. Geom. (to appear); arXiv:1212.4479, 11p.

[8] R. Lazarsfeld, *Positivity in Algebraic Geometry I. Classical setting: line bundles and linear series*, Ergebnisse der Mathematik **48**, Springer, 2004.

[9] Yu. I. Manin, *Cubic forms: algebra, geometry, arithmetic*, Nauka, Moscow, 1972 (Russian); English transl. North–Holland, Amsterdam, 1974.

[10] W. A. Stein et al., Sage Mathematics Software (Version 4.6.1). The Sage Development Team, 2011, http://www.sagemath.org.

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