## TASEP transport on networks: Theory for strongly connected networks

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In this document we present details of the analytical arguments for TASEP transport on general networks in the high connectivity limit (equations 7-9).

High connectivity and the notion of bottlenecks Our arguments are conjectured to be valid for networks with high connectivity, in the sense that any typical vertex becomes sufficiently connected to form a bottleneck. While still comprising the majority of networks, this condition is slightly stronger than simply asserting a high *average* connectivity c: it also requires the *typical* vertex to have a connectivity of order c. This implies that, as the average connectivity c increases, essentially all junctions will saturate at high occupancies and form bottlenecks. It is this observation which will allow us to formulate a theory for high connectivity.

As concrete examples of networks fulfilling this condition, we may of course cite Bethe networks  $(p_{deg}(k) = \delta_{k,c})$  and Poissonian networks  $(p_{deg}(k) = e^{-c} c^k / k!)$ , but also scale-free networks (for which  $p_{deg}(k) = k^{-\gamma} / (\sum_{k=c}^{\infty} k)$  for  $k \ge c$ , zero otherwise).

**Density distributions** As a starting point, it will be useful to recall that mean field equations for transport on any closed network are established in terms of the vertex occupancies  $\rho_v$  as

$$\frac{\partial \rho_v}{\partial t} = \sum_{v' \to v} J\left(\frac{\rho_{v'}}{k_{v'}^{\text{out}}}, 1 - \rho_v\right) - \sum_{v'' \leftarrow v} J\left(\frac{\rho_v}{k_v^{\text{out}}}, 1 - \rho_{v''}\right) \quad , \tag{1}$$

and that we are interested in steady-state solutions to this equation. These equations rely on the fact that the behaviour of any segment is *boundary-controlled*, i.e. the density  $\rho_s$  of a segment linking vertices v'and v'' is directly determined by the occupancies  $\rho_{v'}$ ,  $\rho_{v''}$  of these vertices. For any given instance of a random network, the vertex occupancies  $\rho_{v'}$  are given by equation (1). When sampling over independent network realisations, the variables  $(\rho_{v'}, \rho_{v''}, k)$  are statistically distributed, and we note their distribution  $W(\rho_{v'}\rho_{v''}, k)$ .

We wish to establish a relation for the distribution  $W_S(\rho_s)$  of segment densities  $\rho_s$ . To do so, we exploit the fact that the density on an edge is set by the adjacent vertices v' and v'', via the effective entry/exit rates. We first decompose into contributions from all vertex pairs, such that

$$W_S(\rho_s) = \sum_{k=0}^{\infty} \int_0^1 d\rho_{v'} \int_0^1 d\rho_{v''} W(\rho_{v'}, \rho_{v''}, k) \,\delta\left(\rho_s - \rho_{\pm}(v', v'')\right) \,, \tag{2}$$

where the sum in k runs over all possible out-degrees k [of vertex v']. In this expression the selection of high (low) density values  $\rho_+$  ( $\rho_-$ ) in the oriented segment connecting vertices v' and v'' is achieved, according to the occupancies of adjacent vertices, through the function

$$\rho_{\pm} = \begin{cases}
\rho_{-} = \frac{\rho_{v'}}{k_{v'}^{out}} & \text{if } \rho_{v'}/k_{v'}^{out} < 1 - \rho_{v''} \\
\rho_{+} = 1 - \rho_{v''} & \text{otherwise}
\end{cases}$$
(3)

In the following we make the assumption that the joint distribution of the random variables associated with  $(\rho_{v'}, \rho_{v''}, k)$  factorizes, for high connectivities, as

$$W(\rho_{v'}, \rho_{v''}, k) = W_V(\rho_{v'}) \ W_V(\rho_{v''}) \ \frac{p_{\text{deg}}(k) k}{c} .$$
(4)

Here  $p_{\text{deg}}(k_{out})$  is the distribution of out-degrees  $k_{out}$  of the vertices and  $p_{\text{deg}}(k) k/c$  is the corresponding degree distribution of vertices when picking an (oriented) *segment* at random. Although we cannot formally prove the factorization property, exact approaches on the study of dynamics of models on graphs indicate that this is valid in the high-connectivity limit [1, 2].

Mean-field theory: expansion close to saturation All strongly connected junctions form bottlenecks as soon as the necessary overall density is exceed. Their occupancies  $\rho_v$  will then be close to saturation, suggesting the expansion

$$\rho_v = 1 - r_v/c + \mathcal{O}\left(c^{-2}\right) , \qquad (5)$$

with  $r_v$  an expansion parameter taken to be independent of the typical connectivity c.

Substituting this into the argument of the delta function in equation (2) it becomes clear that, to leading order, both contributions can in fact be taken to be independent of  $\rho_{v'}$ :

$$\rho_{\pm} = \begin{cases} \rho_{-} = \frac{\rho_{v'}}{k} = \frac{1}{k} - \frac{r_{v'}}{kc} \simeq 1/k & \text{if } r_{v''} \gtrsim 1\\ \rho_{+} = \rho_{v''} \simeq 1 - \frac{r_{v''}}{c} & \text{if } r_{v''} \lesssim 1 \end{cases}$$
(6)

where we have used equation (3) and the expansion (5), and the fact that statistically relevant nodes have out-degree  $k_{out} \simeq c$ .

To leading order, we can therefore integrate out the term  $W_V(\rho_{v'})$  in equation (2), after factorization by equation (4), which yields

$$W_S(\rho_s) = \sum_{k=0}^{\infty} \frac{p_{\text{deg}}(k) \ k}{c} \int_0^1 d\rho_{v''} \ W_V(\rho_{v''}) \ \delta(\rho_s - \rho_{\pm}) \qquad , \tag{7}$$

which is the result announced in the main body of the paper, where we have chosen to express it in terms of the small parameter  $r_{v''}$ . In combination with equation (6) it implies a bimodal distribution, with a low density peak at around  $\rho_s \simeq 1/c$  and a high density peak at around  $\rho_s \simeq 1 - 1/c$ .

Infinitely connected limit Clearly, the limit of infinite connectivity is important to analyze. Taking  $c \to \infty$  yields  $\rho_{-} = 1/k \to 0$ , whereas  $\rho_{+} = 1 - r/c \to 1$ . Therefore the high-connectivity limit for the edge density distribution is

$$W_S(\rho_s) = (1 - \rho) \,\delta(\rho_s) + \rho \,\delta(\rho_s - 1) , \qquad (8)$$

where the prefactors follow from the conditions

$$\rho = \int_0^1 \rho_s W_S(\rho_s) \, d\rho_s \qquad \text{and} \qquad 1 = \int_0^1 W_S(\rho_s) \, d\rho_s \,, \tag{9}$$

expressing the definiton of the overall density as well as normalization of  $W_S$ . This establishes the bimodality of the distribution for the segment densities  $W_S(\rho_s)$ . As a consequence of this result, the fraction of high density segments is related to the overall density by  $n_{HD} = \rho$ . Equation (8) furthermore implies a simple relation for the variance of the distribution of segment densities,  $\sigma^2 = \rho (1-\rho)$ . We have numerically explored the solution to the mean field equations (1) for connectivities up to c = 50, see figure (1) confirming this limit for a Poissonian network, which thereby validates our approach.



Figure 1: The fraction of segments in HD phase  $n_{HD}$  and the variance  $\sigma^2$  of the segment density distribution  $W_S(\rho_s)$  as a function of overall density  $\rho$ , obtained for Poissonian networks. Different curves correspond to different average connectivities c. The predicted linear variation of  $n_{HD}$  with the overall density is robust, i.e. almost unsensitive to the connectivity c. The variance  $\sigma^2$  has a roughly parabolic shape for all connectivities, and asymptotically approaches the predicted limiting curve  $\sigma^2 = \rho (1 - \rho)$  for the infinitely connected limit.

Special case of regular networks The question arises whether the *bimodal* distribution of segment densities (equation (7)) is contradictory with the earlier mean-field results in the main article, stating a *unimodal* distribution for the Bethe network. A careful analysis shows how this apparent inconsistency is lifted. Indeed, for a Bethe network all vertex degrees are identical,  $p_{deg}(k) = \delta_{k,c}$  (using the Kronecker delta), and all vertices behave identically

$$W_S(\rho_s) = \delta \left(\rho_s - \rho_v\right) \quad \text{with} \quad \rho_v = const \quad .$$
 (10)

Furthermore all segments have identical densities, implying that the overall density is simply  $\rho = \rho_s = \rho_v$ .

Note that the condition discriminating the two regimes on a segment is  $r_v = 1$ , which translates to  $\rho_v^* = 1 - r_v^*/c \simeq 1 - 1/c$ . When compared to the analytical MF condition for having a HD phase in the Bethe network,  $\rho^* = c/(c+1)$ , a Taylor expansion for  $c \gg 1$  shows that these conditions are indeed consistent to leading order.

As to equation (8) for the bimodal distribution, it remains consistent in the limiting case, since  $\rho$  for a Bethe network reduces to either one or zero in the infinite connectivity limit; as is seen from the meanfield results, all intermediate densities involve shocks, for which the mean field equations are not valid by construction. For both  $\rho \to 0$  and  $\rho \to 1$  one of the peaks in equation (10) is eliminated, such that the result remains coherent also for regular networks.

## References

- [1] I. Neri and D. Bollé, J. Stat. Mech. (2009) P08009
- [2] E. Aurell and H. Mahmoudi, J. Stat. Mech. (2011) P04014